UNIVERSITÉ DE NANTES

Ordinary differential equation, transport theory and Sobolev spaces

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Chapter 1

Introduction

In this paper, we make a report on Di perna Lions works(1989) and Levy [7] and Guillaume first [11] published in 2016, second [12] published in 2019, on Ordinary differential equation, transport theory and sobolev spaces:

$$\partial_t u - b \cdot \nabla_x u = cu \qquad \text{in } (0,T) \times \mathbb{R}^N,$$

On each document, we discuss about existence, uniqueness and regularity of the weak solution, and we reproduced their proves following the steps in the documents [7],[11] and [12] but with more details the skip in their publication. The general idea here, is that when the vector field b, and and the real function c verify some less regularity conditions that we will eventually state, when we are given an initial datum $u_0 \in L^p$, at time $t = 0$, then a solution corresponding to the initial datum u_0 verifies

$$\|u(t)\|_{L^p(\mathbb{R}^N)} \leq K_p(t) \|u_0\|^p_{L^p(\mathbb{R}^N)}$$

where K_p will be determined later in the existence section of the Di perna's approach. Later with some high regularity on b, we will also show that there is uniqueness of such a solution u corresponding to u_0. Levy also did the same thing but with one more which is the diffusion term in the equation.

we are going to write with rigor, the regularity conditions which should carry on c, b and $\nabla.b$ which allows the existence, estimation of the norm and the uniqueness of the solution corresponding to an initial data u_0, and some techniques of proves.

Chapter 2

Linear transport equation following Di Perna and Lion's approach (1989)

2.1 Definition and some claims

definition 1. Let $(E; \|\cdot\|_E)$ be any normed space. Let define a function $f : (t, x) \in (\mathbb{R}, E) \to \mathbb{R}$. We say that $f \in L^p(\mathbb{R}; \|\cdot\|_E)$ for $0 < p \leq \infty$, if

$$\int_{\mathbb{R}} \|f(t)\|_E^p dt < \infty.$$

definition 2. Let Ω be an any open subset on $\mathbb{R}^N$, $p \in [1, \infty]$ and m positive intiger. We define the sobolev space $W^{m,p}$ by

$$W^{m,p} = \left\{ u \in L^p(\Omega) | \forall \alpha \text{ such that } |\alpha| \leq m, D^\alpha u \in L^p(\Omega) \right\}$$

where α is a multi-indice, $D^\alpha u$ is a weak partial derivative of u in weak sense. The space $W^{m,p}$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \left\{ \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} \quad 1 \leq p < \infty \text{ or } \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty} \text{ if } p = +\infty \right\}$$

In case p is finite, Meyers-serrin theorem give an equivalent definition, by completion of the space $u \in C^\infty(\Omega) | \|u\|$.

What follow is a result we will be using in the all of the work.
Let us defined a function $a : \mathbb{R}^N \longrightarrow \mathbb{R}^N$. Now what does it take so that $\int_{\mathbb{R}^N} \nabla \cdot a = 0$?
Claim: It does take two condition which are

$$\nabla \cdot a \in L^1$$

and

$$\int_{\mathbb{R}^N} \frac{|a(x)|}{1 + |x|} \, \mathrm{d}x < \infty.$$

Proof.

$$\int_{\mathbb{R}^N} \nabla \cdot a \, \mathrm{d}x = \lim_{R \to \infty} \int_{\mathbb{R}^N} (\nabla \cdot a) \chi \, \mathrm{d}x$$

$$\int_{\mathbb{R}^N} (\nabla \cdot a) \chi \, \mathrm{d}x = -\int_{\mathbb{R}^N} a \cdot \chi_R \, \mathrm{d}x = -\frac{1}{R} \int_{\mathbb{R}^N} a \cdot (\nabla \chi_R) \, \mathrm{d}x.$$

Taking the absolute value, we may have

$$\left|\int_{\mathbb{R}^N} \nabla \cdot a \,\mathrm{d}x\right| \leq C \int_{1<\frac{|x|}{R}<2} \frac{|a(x)|}{1+|x|} \,\mathrm{d}x \underset{R\to 0}{\longrightarrow} 0,$$

as

$$\int_{\mathbb{R}^N} \frac{|a(x)|}{1+|x|} \,\mathrm{d}x < \infty$$

and $\mathbb{R}^N = \underset{k}{\cup}[2^k < |x| < 2^{k+1}]$. □

2.2 Existence

In this part, our aim is to discus under which conditions the equation of a linear transport have solution, and later discuss some other properties as uniqueness and the regularization and some estimate. Let us begin with a simple existence result for the following linear transport equation

$$\partial_t u - b \cdot \nabla_x u = cu \qquad \text{in } (0,T) \times \mathbb{R}^N, \tag{2.1}$$

where $T > 0$ is given and we will always assume that $b,\ c$ satisfy at least

$$b \in L^1(0,T;(L^1_{loc}(\mathbb{R}^N))^N), \quad c \in L^1(0,T;L^1_{loc}(\mathbb{R}^N)). \tag{2.2}$$

For all test function ϕ on $(0,T) \times \mathbb{R}^N$, we have

$$\mathrm{div}(b\phi) = (\mathrm{div} b)\phi + b \cdot \nabla\phi.$$

This leads us to defined

$$b \cdot \nabla u := \mathrm{div}(bu) - (\mathrm{div})u,$$

which at prior does not make sense.

Our aim is to build a solution of equation (2.1) with an initial condition u^0 in $L^p(\mathbb{R}^N)$ where $p \in [1,\infty]$, u^0 is given. Let us recall that the equation can be understood in distribution sense that is for instance for all test functions $\phi \in C^\infty([0,T] \times \mathbb{R}^N)$ with compact support in $[0,T) \times \mathbb{R}^N$, as

$$-\int_0^T \int_{\mathbb{R}^N} u\frac{\partial\phi}{\partial t} \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^N} u^0\phi(0,x)\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^N} u\{div(b\phi) - c\phi\}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{2.3}$$

which makes sense when we assume that and can be only obtain if

$$-c + div\ b \in L^1(0,T;L^q_{loc}(\mathbb{R}^N)), \quad b \in L^1(0,T;(L^q_{loc}(\mathbb{R}^N))^N) \tag{2.4}$$

where q and p are conjugates$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

definition 3. Let us consider the problem

$$\begin{cases} \partial_t u - b \cdot \nabla_x u = cu \qquad \text{in } (0,T) \times \mathbb{R}^N \\ u(t=0,x) = u^0(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \end{cases} \tag{2.5}$$

We call a weak solution of the above equation, a function $u(t,x) \in L^1_{\mathrm{loc}}$ that verifies

$$-\int_0^T \int_{\mathbb{R}^N} u(t,x)(\partial_t\varphi - \mathrm{div}(b\phi))(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} c(t,x)u(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} u^0(x)\phi(0,x)\,\mathrm{d}x,$$

for all test function $\phi \in \mathcal{D}\big((0,T) \times \mathbb{R}^N\big)$. This definition gives place to the distributional solution we mention above, which is our goal.

In the rest of the problem, we will always consider $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \rho = 1$, and let us define $\rho_\epsilon = \frac{1}{\epsilon^N}\rho\big(\frac{\cdot}{\epsilon}\big)$ and Supp$\rho \subset B(0,1)$ And a smooth function with compact support $0 \leq \chi \leq 1$ equal to 1 in the neighborhood of origin, and consider $\chi_\epsilon : x \longmapsto \chi\big(\frac{x}{\epsilon}\big)$. An approximation by χ is called truncation.

Lemma 4. *Let $p \in [0, p[$ and $f \in L^p(\mathbb{R}^N)$, then $\rho_\epsilon * f$ and $\chi_\epsilon(\rho * f)$ converges to f in $L^p(\mathbb{R}^N)$. In particular $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in $\big(L^p(\mathbb{R}^N), \|\|_{L^p(\mathbb{R}^N)}\big)$. From there, one can also deduce that $C_0^\infty(\mathbb{R}^N)$ is dense in $L^p_{loc}(\mathbb{R}^N)$. The same thing happens in $L^p_{loc}(\mathbb{R}^N)$.*

Theorem 5 (Existence)**.** *Let $p \in [1, \infty]$, $u^0 \in L^p(\mathbb{R}^N)$, assume the condition (2.4) (at page 5) and*

$$\begin{cases} c - \frac{1}{p} div\ b \in L^1(0,T; L^\infty_{loc}(\mathbb{R}^N)) & if\ \ p > 1 \\ \\ c,\ div\ b \in L^1(0,T; L^\infty(\mathbb{R}^N)) & if\ \ p = 1 \end{cases} . \tag{2.6}$$

Then, there exists a weak solution u of (2.1) in $L^\infty(0,T; L^p(\mathbb{R}^N))$ corresponding to the initial condition u^0.

We make an observation that the Theoreme will remain true if we add at the right hand side another term $f \in L^1(0,T; L^p(\mathbb{R}^N))$ Before we prove this, let us recall some know result that we will recall without prove them.

definition 6. Let $\mathcal{F}$ be a family of functions. We say that $\mathcal{F}$ is uniformly integrable in $L^1(E)$ if for very $\epsilon > 0$, $\exists\delta$, $\mu(E) \Rightarrow \forall f \in \mathcal{F}$, $\int_E |f| \ < \epsilon$, this is also equivalent to say that: $\exists\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\frac{\phi(t)}{t} \to +\infty$ and $\sup\limits_{f\in\mathcal{F}}\bigg(\int_E \phi(|E|)\bigg) < \infty$

Theorem 7. *(Bounded sequence have weak limits) Let $\Omega \in \mathbb{R}^N$ be measurable set and consider $L^p(\Omega)$ with $1 < p < \infty$. Let (f_n) be a sequence of functions, bounded in $L^P(\Omega)$. Then there exists a subsequence $(f_{\varphi(n)})$ that converges weakly in $L^p(\Omega)$.*

Theorem 8. *(Banach-Alaoglu-Bourbaki) The closed unit ball*

$$B_E^* = \{f \in E^* : |f| \leq 1\}$$

is compact in the weak topology $\sigma(E^*, E)$.*

Lemma 9. *Assume that X is reflexive, and let A be a bounded subset in $L^1(\mu, X)$. Then A is weakly relatively compact if and only if A is uniformely integrable.*

We subdivided the proof in to tow parts. The first part we will show the estimate, which infer that the solution if it exists, will belong to $L^\infty(0,T; L^p(\mathbb{R}^N))$.

The proof of the Existence Theorem 5. (i) A prior estimate

First consider $p = \infty$, we have

$$u(t,x) = u^0(x) + \int_0^t \frac{\partial u}{\partial s}(s,x)\,\mathrm{d}s$$

we get applying absolute value

$$\begin{aligned} |u(t,x)| &\leq \left|u^0(x)\right| + \int_0^t \left|\frac{\partial u}{\partial s}(s,x)\right| \mathrm{d}s \\ &\leq \left|u^0(x)\right| + \int_0^t |b \cdot \nabla_x u(s,x)|\,\mathrm{d}s + \int_0^t |cu(s,x)|\,\mathrm{d}s \end{aligned}$$

this implies taking the L^∞ norm which respect to $x \in \mathbb{R}^N$, that

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \left\|u^0\right\|_{L^\infty(\mathbb{R}^N)} + \int_0^t \|b \cdot \nabla_x u(s)\|_{L^\infty(\mathbb{R}^N)}\,\mathrm{d}s + \int_0^t \|cu(s)\|_{L^\infty(\mathbb{R}^N)}\,\mathrm{d}s.$$

But, based on the reasoning we did in part A.2 in the appendix, we may have this approximation

$$\int_0^t \|b \cdot \nabla_x u(s)\|_{L^\infty(\mathbb{R}^N)} \,\mathrm{d}s \simeq 0.$$

We then get

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \left\|u^0\right\|_{L^\infty(\mathbb{R}^N)} + \int_0^t \|cu(s)\|_{L^\infty(\mathbb{R}^N)} \,\mathrm{d}s,$$

applying the Gronwall inequality to the above inequality, we get

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq K\left\|u^0\right\|_{L^\infty(\mathbb{R}^N)} \quad \text{for all} \quad t \in (0,T), \tag{2.7}$$

where $K = \exp\bigl(\int_0^T \|c(s)\|_{L^\infty(\mathbb{R}^N)} \,\mathrm{d}s\bigr)$.

In other hand, if $p < \infty$, multiplying equation (2.1) by pu^{p-1}, (here we assume that $b,\ u$ are regular enough) we get

$$pu^{p-1}\partial_t u - pu^{p-1} b \cdot \nabla_x u = pcu^p,$$

integrating this we get

$$\partial_t u^p - b \cdot \nabla_x u^p = pcu^p. \tag{2.8}$$

Now let consider the function $\beta_\epsilon(s) = \sqrt{\epsilon^2 + s^2} - \epsilon$, which is continuous and β' is uniformly bounded which respect to ϵ. by multiplying the above equation by $\beta'_\epsilon(u^p)$, and using Chain Rule to the above equation, we have

$$\partial_t \beta_\epsilon(u^p) + b \cdot \nabla_x \beta_\epsilon(u^p) = pcu^p \beta'_\epsilon(u^p).$$

One will observe that the above result is True when applying to equation (2.8) a regularization property which we will state later.

But one observes that as $\epsilon \to 0$, $\beta_\epsilon(u^p) \to |u|^p$ and that $\beta'_\epsilon(u^p) \to \pm 1$, and we can finally write

$$\partial_t |u|^p - b \cdot \nabla_x |u|^p = pc\, |u|^p\,,$$

which is equal to

$$\partial_t |u|^p - \nabla \cdot (|u|^p b) + |u|^p \nabla \cdot b = pc\, |u|^p$$

integrating this over $\mathbb{R}^N$ (it makes sense to integrate this, in distribution sense, since b and c are regular enough), we get

$$\partial_t \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla \cdot (|u|^p b) \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^p \nabla \cdot b \,\mathrm{d}x = \int_{\mathbb{R}^N} pc\, |u|^p \,\mathrm{d}x,$$

also adding more condition that b is truncated, we may have in distribution sense

$$0 =< \nabla 1, |u|^p b >= - < 1, \nabla \cdot (|u|^p b) >= - \int_{\mathbb{R}^N} \nabla \cdot (|u|^p b) \,\mathrm{d}x.$$

Hence, we will have

$$\partial_t \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x + \int_{\mathbb{R}^N} |u|^p \nabla \cdot b \,\mathrm{d}x = \int_{\mathbb{R}^N} pc\, |u|^p \,\mathrm{d}x$$

since the condition in (2.6) (at page 6) we may have the estimate

$$\partial_t \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x \leq \|(pc - div\ b)(t)\|_\infty \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x$$

Now solving this, we get

$$\|u(t)\|^p_{L^p(\mathbb{R}^N)} \leq K_p^p(t)\|u^0\|^p_{L^p(\mathbb{R}^N)} \quad \text{for all} \quad t \in (0,T)$$

where $K_p(t) = \exp\big(\int_0^t \|(c - \frac{1}{p}div\ b)(s)\|_\infty \,\mathrm{d}s\big)$. Taking the entire above inequality to power $\frac{1}{p}$, we got

$$\|u(t)\|_p \leq K_p\|u^0\|_p \quad \text{for all} \quad t \in (0,T). \tag{2.9}$$

Hence we have proved that if the solution exists and regular enough as well as b and c, it belongs to $L^\infty(0,T;L^p(\mathbb{R}^N))$.

(ii) Approached regularized problem

To prove the existence, we will use regularization by an approximation by identity, and also truncation. Indeed let us approximate considering $b_\epsilon = b * \rho_\epsilon, \quad c_\epsilon = c * \rho_\epsilon$ and $u^0_\epsilon = u^0 * \rho_\epsilon$., we consider them being truncated by χ defined at somewhere above. From lemma 4 $b_\epsilon,\ c_\epsilon$ and u^0_ϵ converges respectively to $b,\ c$ and u^0 as $\epsilon \to 0$ and by the same Lemma, we may assume that $b_\epsilon \in L^1\big(0,T;C^1_b(\mathbb{R}^N)\big)$ and $c_\epsilon \in L^1\big(0,T;C^1_b(\mathbb{R}^N)\big)$ since b_ϵ and c_ϵ are in $C^\infty_0(\mathbb{R}^N)$ (we explain that the are also smooth which respect to the t variable). Hence by Cauchy-Lipschitz Theorem 37, the equation

$$\partial_t u_\epsilon - b_\epsilon \cdot \nabla_x u_\epsilon = c_\epsilon u_\epsilon \qquad \text{in } (0,T) \times \mathbb{R}^N \tag{2.10}$$

has a unique solution $u_\epsilon \in C\big(0,T;C^1_b(\mathbb{R}^N)\big)$. From Equations (2.7) and (2.9), u_ϵ is uniformely bounded in with respect toϵ, in $L^p(\mathbb{R}^N)$ (since $\|u^0 * \rho_\epsilon\|_p \leq \|u\|_p\|\rho\|_1$ using Young's convolution Inequality and $K_{p,\epsilon} \leq K_p$). Hence if $p > 1$ by Theorem 7, we may extract a subsequences if necessary that converges weakly in $L^\infty(0,T;L^p(\mathbb{R}^N))$ and if $p = \infty$ by Theorem 8 subsequences that converges weakly* to some u. Hence u will be the solution of the problem (2.1) (We show this in the third part).

Now when $p = 1$, we do the same thing but this time, we have to show that u_ϵ is weakly relatively compact in $L^\infty(0,T;L^1_{loc}(\mathbb{R}^N))$. We consider $u^0_n \in \mathcal{D}(\mathbb{R}^N)$ that converges to u^0 in $L^1(\mathbb{R}^N)$. And we denotes as well $u_{n,\epsilon}$ an approximated solution of (2.10) with initial condition $u^0_{n,\epsilon}$. Now as what we did above using equation (2.9), we will have

$$\|u_{n,\epsilon}\|_{L^\infty(0,T;L^p(\mathbb{R}^N))} \leq C(n,p) = K_p\big\|u^0_n\big\|_{L^p(\mathbb{R}^N)} \quad \forall p > 1 \Leftrightarrow \int |u_{n,\epsilon}|^p \leq C_{n,p},$$

this implies that there exists a positive real value function ϕ with $\frac{\phi(t)}{t} \to +\infty$ as t goes to ∞, such that

$$\int \phi\big(|u_{n,\epsilon}|\big) \leq C_n,$$

and also $u_{n,\epsilon}$ is bounded in $L^1(\mathbb{R}^N)$ independently of ϵ and n, from the same Equation (2.9), with $p = 1$. Moreover $\forall\ \epsilon > 0,\ u_{n,\epsilon} \underset{n}{\to} u_\epsilon$ in L^1 fort,

$$\|u_{n,\epsilon} - u_\epsilon\|_1 \leq K_1\|u^0_{n,\epsilon} - u^0_\epsilon\|_1 \leq \|u^0_n - u^0\|_1 \underset{n}{\to} 0,$$

from the previous argument, there must exists $\tilde{\phi}$ such that $\int \tilde{\phi}\big(|u_\epsilon|\big) < C$, which is equivalent to say that u_ϵ is uniformly integrable, hence call on Lemma 9, u_ϵ is relatively compact in $L^\infty(0,T;L^1_{loc}(\mathbb{R}^N))$.

(iii) Convergence of the approached solutions and passage to the limit in the equation

Let ϕ be a test function, that is $\phi \in \mathcal{D}\big((0,T) \times \mathbb{R}^N\big)$. Multiplying equation (2.10) by ϕ, and integrating over time and space, we have

$$\int_0^T \int_{\mathbb{R}^N} \phi(t,x)\partial_t u_\epsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^N} \phi(t,x)(b_\epsilon \cdot \nabla_x u_\epsilon)(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} \phi c_\epsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t u_\epsilon.$$

But by the higher regularity of u_ϵ, b_ϵ and ϕ we have

$$\operatorname{div}(\phi b_\epsilon u_\epsilon) = \phi b_\epsilon \cdot \nabla u_\epsilon + u_\epsilon b_\epsilon \cdot \nabla\phi + \phi u_{\epsilon \nabla \cdot b_\epsilon},$$

we then deduce that

$$-\int_0^T \int_{\mathbb{R}^N} \phi(t,x)(b_\epsilon \cdot \nabla_x u_\epsilon)(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} u_\epsilon(t,x) b_\epsilon(t,x) \cdot (\nabla\phi)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} \phi(t,x) u_\epsilon(t,x) \nabla \cdot b_\epsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Also by integration by part with respect to the time variable, we get

$$\int_0^T \int_{\mathbb{R}^N} \phi(t,x)\partial_t u_\epsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^N} \phi(0,x) u_\epsilon^0(x)\,\mathrm{d}x - \int_0^T \int_{\mathbb{R}^N} u_\epsilon(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

We then have

$$-\int_0^T \int_{\mathbb{R}^N} u_\epsilon(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} u_\epsilon(t,x) b_\epsilon(t,x) \cdot (\nabla\phi)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} \phi(t,x) u_\epsilon(t,x) \nabla \cdot b_\epsilon(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{R}^N} c(t,x) u_\epsilon(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} \phi(0,x) u_\epsilon^0(x)\,\mathrm{d}x.$$

Applying the definition of weak convergence of u_ϵ we discussed above, entire above equation converges to

$$-\int_0^T \int_{\mathbb{R}^N} u(t,x)\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} u(t,x) b(t,x) \cdot (\nabla\phi)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} \phi(t,x) u(t,x) \nabla \cdot b(t,x)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{R}^N} c(t,x) u(t,x)\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} \phi(0,x) u^0(x)\,\mathrm{d}x.$$

hence the weak limit u of u_ϵ, is a weak solution of (2.1). So, we have proved Theorem 5.

□

Now we want show that the uniqueness of the solution under some regularization. But first of all, let us elaborate some tools theorems before.

2.3 Commutator

Theorem 10. *Let $p \in [1,\infty]$, let $u \in L^\infty(0,T; L^p(\mathbb{R}^N))$ be solution of the problem (2.1) and consider $b \in L^\infty(0,T; W^{1,\alpha}_{loc},(\mathbb{R}^N))$, $c \in L^\infty(0,T; L^p_{loc}(\mathbb{R}^N))$ for some $\alpha \geq q$, where q is the conjugate of p. Then, if we consider $u_\epsilon = u * \rho_\epsilon$, u_ϵ satisfies*

$$\frac{\partial}{\partial t} u_\epsilon - b \cdot \nabla_x u_\epsilon = c u_\epsilon + r_\epsilon \tag{2.11}$$

where r_ϵ converges to 0 in $L^1(0,T; L^\beta_{loc}(\mathbb{R}^N))$ as ϵ goes to 0, and β is given by $\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}$.

We make an observation that the Theorem will remain true if we add at the right hand side another term $f \in L^1(0,T; L^\beta_{loc}(\mathbb{R}^N))$.

Lemma 11.

- *Let $B \in \left(W^{1,\alpha}_{loc}(\mathbb{R}^N)\right)^N$, $w \in L^p_{loc}(\mathbb{R}^N)$ with $p \in [1,\infty]$, $\alpha \geq q$. Then*

$$(B \cdot \nabla w) * \rho_\epsilon - B \cdot \nabla(w * \rho_\epsilon) \underset{\epsilon}{\to} 0 \quad \text{in } L^\beta_{loc}(\mathbb{R}^N),$$

where β is given in the same way as in Theorem 10, and again $B \cdot \nabla w = \operatorname{div}(Bw) + w\nabla \cdot B$.

- *Let $B \in L^1\big(0,T;W^{1,\alpha}_{loc}(\mathbb{R}^N)\big)$, $w \in L^\infty\big(0,T;L^p_{loc}(\mathbb{R}^N)\big)$ with $p \in [1,\infty]$, $\alpha \geq q$. Then*

$$(B \cdot \nabla w) * \rho_\epsilon - B \cdot \nabla(w * \rho_\epsilon) \underset{\epsilon}{\to} 0 \quad \textit{in } L^1\big(0,T;L^\beta_{loc}(\mathbb{R}^N)\big).$$

Proof. By some computation, we showed that

$$(B \cdot \nabla w) * \rho_\epsilon - B \cdot \nabla(w * \rho_\epsilon) = \nabla \cdot (Bw) * \rho_\epsilon - (\nabla \cdot B) w * \rho_\epsilon - B \cdot w * \nabla \rho_\epsilon.$$

we have

$$(B \cdot w * \nabla\rho_\epsilon)(x) = \int_{\mathbb{R}^N} B(x) w(y) \nabla\rho_\epsilon(x-y)\,\mathrm{d}y$$

and by distribution argument,

$$(\nabla \cdot (Bw) * \rho_\epsilon)(x) = \int_{\mathbb{R}^N} (\nabla \cdot (Bw))(y)\rho_\epsilon(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^N} B(y)w(y) \cdot \nabla\rho_\epsilon(x-y)\,\mathrm{d}y.$$

we then have

$$((B \cdot \nabla w) * \rho_\epsilon - B \cdot \nabla(w * \rho_\epsilon))(x) = \int_{\mathbb{R}^N} w(y)\{\big(B(y)-B(x)\big) \cdot \nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y - ((w div\ B) * \rho_\epsilon)(x).$$

Let us denote $Q_\epsilon(w,B)$ the first term and $T_\epsilon(w,B)$ the second term, both of the right hand side of the previous equality. We have $T_\epsilon(w,B) \underset{\epsilon}{\to} w\ \mathrm{div}B$ in $L^\beta_{loc}(\mathbb{R}^N)$. Now let us estimate the norm of $Q_\epsilon(w,B)$ in $L^\beta_{loc}(\mathbb{R}^N)$ for a small ϵ. Indeed we have

$$\begin{aligned}\left\|\int_{\mathbb{R}^N} w(y)\{(B(y)-B(x)) \cdot \nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y\right\|^\beta_{L^\beta(B_R)} &\leq \int_{B_R}\left|\int_{\mathbb{R}^N} w(y)\{(B(y)-B(x)) \cdot \nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y\right|^\beta \mathrm{d}x \\ &\leq \int_{B_R}\int_{B(x,\epsilon)}\left|w(y)\frac{(B(y)-B(x))}{\epsilon} \cdot \epsilon\nabla\rho_\epsilon(x-y)\right|^\beta \mathrm{d}y\,\mathrm{d},x\end{aligned}$$

for $x \in B_R = B(0,R)$ and Supp $\rho \subset B(0,1)$, $|x-y| \leq \epsilon$ implies $y \in B(x,\epsilon)$.

Since $\frac{1}{p} + \frac{1}{\alpha} = \frac{1}{\beta}$ and then $\frac{1}{p} + \frac{1}{\alpha} + \frac{1}{\beta'} = 1$, where β' is Hölder conjugate of β, we have by Hölder

$$\begin{aligned}\int_{\mathbb{R}^N}\left|w(y)\frac{(B(y)-B(x))}{\epsilon} \cdot \epsilon\nabla\rho_\epsilon(x-y)\right|^\beta \mathrm{d}y \leq \|w\|^\beta_{L^p(B(x,\epsilon))}\left(\int_{B(x,\epsilon)}\left|\frac{(B(y)-B(x))}{\epsilon}\right|^\alpha \mathrm{d}y\right)^{\frac{\beta}{\alpha}} \\ \left(\int_{B(x,\epsilon)}|\epsilon\nabla\rho_\epsilon(x-y)|^{\beta'}\,\mathrm{d}y\right)^{\frac{\beta}{\beta'}}.\end{aligned}$$

We can then write

$$\begin{aligned}\left\|\int_{\mathbb{R}^N} w(y)\{(B(y)-B(x)) \cdot \nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y\right\|^\beta_{L^\beta(B_R)} \leq \|w\|^\beta_{L^p(B_{R+1})}\times \\ \int_{B_R}\left[\left(\int_{B(x,\epsilon)}\left|\frac{(B(y)-B(x))}{\epsilon}\right|^\alpha \mathrm{d}y\right)^{\frac{\beta}{\alpha}}\left(\int_{B(x,\epsilon)}|\epsilon\nabla\rho_\epsilon(x-y)|^{\beta'}\,\mathrm{d}y\right)^{\frac{\beta}{\beta'}}\right]\mathrm{d}x,\end{aligned}$$

But

$$\epsilon\nabla\rho_\epsilon(x) = \epsilon^{-N}\nabla\rho(\epsilon^{-1}x),$$

we have

$$\begin{aligned}
\int_{B(x,\epsilon)} |\epsilon\nabla\rho_\epsilon(x-y)|^{\beta'}\,\mathrm{d}y &= \frac{1}{\epsilon^N}\int_{B(x,\epsilon)} \left|\nabla\rho(\epsilon^{-1}(x-y))\right|^{\beta'}\mathrm{d}y \\
&\leq \operatorname*{Sup}_{y\in B(x,\epsilon)} \left|\nabla\rho(\epsilon^{-1}(x-y))\right|^{\beta'} \frac{1}{\epsilon^N}\int_{B(x,\epsilon)}\mathrm{d}y \\
&\leq \operatorname*{Sup}_{y\in B(0,1)} |\nabla\rho(y)|^{\beta'} \frac{1}{\epsilon^N}\int_{B(0,\epsilon)}\mathrm{d}y.
\end{aligned}$$

Let us pose

$$C = \operatorname*{Sup}_{y\in B(0,1)} |\nabla\rho(y)|^{\beta'} \frac{1}{\epsilon^N}\int_{B(0,\epsilon)}\mathrm{d}y = \operatorname*{Sup}_{y\in B(0,1)} |\nabla\rho(y)|^{\beta'} \frac{1}{\epsilon^N}\mathrm{Vol}(B(0,\epsilon)),$$

C in a constant independent from ϵ since the $\frac{1}{\epsilon^N}$ canceled out with ϵ^N in the expression of the volume.

We have then

$$\begin{aligned}
\left\|\int_{\mathbb{R}^N} w(y)\{(B(y)-B(x))\cdot\nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y\right\|_{L^\beta(B_R)}^\beta &\leq C^\beta\|w\|_{L^p(B_{R+1})}^\beta \int_{B_R}\left(\int_{B(x,\epsilon)}\left|\frac{(B(y)-B(x))}{\epsilon}\right|^\alpha \mathrm{d}y\right)^{\frac{\beta}{\alpha}}\mathrm{d}x \\
&\leq C^\beta\|w\|_{L^p(B_{R+1})}^\beta \left(\int_{B_R}\int_{B(x,\epsilon)}\left|\frac{(B(y)-B(x))}{\epsilon}\right|^\alpha \mathrm{d}y\,\mathrm{d}x\right)^{\frac{\beta}{\alpha}},
\end{aligned}$$

The second inequality is obtain by Jesen's inequality for concave function $x^{\frac{\beta}{\alpha}}$, since $\frac{\beta}{\alpha} = \frac{p}{p+\alpha} < 1$. We have then

$$\begin{aligned}
\left\|\int_{\mathbb{R}^N} w(y)\{(B(y)-B(x))\cdot\nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y\right\|_{L^\beta(B_R)} &\leq C\|w\|_{L^p(B_{R+1})} \left(\int_{B_R}\int_{B(x,\epsilon)}\left|\frac{(B(y)-B(x))}{\epsilon}\right|^\alpha \mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{\alpha}} \\
&= C\|w\|_{L^p(B_{R+1})} \\
&\left(\int_{B_{R+1}}\int_{|z|\leq c}\left(\int_0^1 |\nabla B(x+t\epsilon z)|^\alpha\,\mathrm{d}t\right)\mathrm{d}z\,\mathrm{d}x\right)^{\frac{1}{\alpha}} \\
&\leq C\|w\|_{L^p(B_{R+1})}\|\nabla B\|_{L^\alpha(B_{R+2})},
\end{aligned}$$

the last inequality is obtain by using Jesen's inequality with the convex function x^α, since $\alpha \geq 1$.

In other word

$$\|Q_\epsilon(w,B)\|_{L^\beta(B_R)} \leq C\|w\|_{L^p(B_{R+1})}\|\nabla B\|_{L^\alpha(B_{R+2})},$$

meaning that Q_ϵ is a bounded or continuous operator. We need to prove that $Q_\epsilon(w,B) \underset{\epsilon}{\to} w\mathrm{div}$. If w and B are smooth

$$Q_\epsilon(w,B)(x) = \int w(y)\{\big(B(y)-B(x)\big)\cdot\nabla\rho_\epsilon(x-y)\}\,\mathrm{d}y$$

$$Q_\epsilon(w,B) = \mathrm{div}(wB)*\rho_\epsilon - B\cdot\nabla(w*\rho_\epsilon) \underset{\epsilon}{\to} w\ \mathrm{div}B \ \text{ in } \ L^\beta_{loc}.$$

Now in general, given any w, B that only verify the hypothesis in the Theorem 10, we use density $L^p_{loc}(\mathbb{R}^N)$ to show the convergence of $Q_\epsilon(w,B)$ to w div$b$. Indeed by density in Lemma 4, there exists a sequence $\left(w_n\right)_n$ in C^1

or C^∞ such that $w_n \to w$ in $L^p_{loc}(\mathbb{R}^N)$, hence the convergence also in $L^\beta_{loc}(\mathbb{R}^N)$ since $\beta \le p$. Since Q_ϵ is continuous as bounded operator, we have $Q_\epsilon(w_n.\cdot) \underset{n}{\to} Q_\epsilon(w,\cdot)$, the same for $Q_\epsilon(\cdot, B_n)$ in L^β_{loc}. We then have

$$Q_\epsilon(w,B) - (\text{div}B)w = \big[Q_\epsilon(w,B) - Q_\epsilon(w_n,B_n)\big] + \big[Q_\epsilon(w_n,B_n) - (\text{div}B_n)w_n\big] + \big[(\text{div}B_n)w_n - (\text{div}B)w\big] \underset{\epsilon}{\to} 0$$

in $L^\beta_{loc}(\mathbb{R}^N)$. Hence we have proved the first part of The Lemma **??**. The proof of the second one is infer follow from the one we have proved, so we skip it.

□

prove of theorm 10. Convolving the all equation (2.1) ρ_ϵ, we get

$$\partial_t u_\epsilon - \rho_\epsilon * (b \cdot \nabla u_\epsilon) = (cu) * \rho_\epsilon$$

we have adding some term subtracting then back,

$$\partial_t u_\epsilon - b \cdot \nabla u_\epsilon + b \cdot \nabla u_\epsilon - \rho_\epsilon * (b \cdot \nabla u_\epsilon) = cu_\epsilon + (cu) * \rho_\epsilon - cu_\epsilon$$

this is

$$\partial_t u_\epsilon - b \cdot \nabla u_\epsilon = cu_\epsilon + r_{1,\epsilon} + r_{2,\epsilon},$$

$r_{1,\epsilon} = b \cdot \nabla u_\epsilon - \rho_\epsilon * (b \cdot \nabla u_\epsilon)$ and $r_{2,\epsilon} = (cu)\rho_\epsilon - cu_\epsilon$ and we pose $r_\epsilon = r_{1,\epsilon} + r_{2,\epsilon} \longrightarrow$, r_ϵ converges to 0 in $L^1(0,T; L^\beta_{loc}(\mathbb{R}^N))$ as ϵ goes to 0 after lemma 11.

□

In the following, we are going to discuss about the uniqueness, which is possible when the vector field verifies some conditions.

2.4 Uniqueness

Theorem 12 (Uniqueness)**.** *Let $u \in L^\infty(0,T; L^p(\mathbb{R}^N))$ for $p \in [1,\infty]$, be a solution of the problem* (2.1) *(at page 5) under an initial condition $u_0 = 0$, and we assume that some other condition which are:*

1. *The function c and the vector field b are such that c,* div $b \in L^1(0,T; L^\infty(\mathbb{R}^N))$, $b \in L^1(0,T; W^{1,q}_{\text{loc}}(\mathbb{R}^N))$
2. *and*
$$\frac{b}{1+|x|} \in L^1(0,T; L^1(\mathbb{R}^N)) + L^1(0,T; L^\infty(\mathbb{R}^N)), \tag{2.12}$$
let us say that
$$\frac{b}{1+|x|} = \frac{b_1}{1+|x|} + \frac{b_2}{1+|x|},$$
where
$$\frac{b_1}{1+|x|} \in L^1(0,T; L^1(\mathbb{R}^N))$$
and
$$\frac{b_2}{1+|x|} \in L^1(0,T; L^\infty(\mathbb{R}^N)).$$

Then, $u = 0$.

Proof. Observing well the conditions on b and c, they verify the condition in Theorem 10 with $\alpha = q$ and $\beta = 1$, so applying that theorem we get

$$\frac{\partial}{\partial t} u_\epsilon - b \cdot \nabla_x u_\epsilon = c u_\epsilon + r_\epsilon,$$

where r_ϵ converges to 0 in $L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^N))$ as ϵ goes to 0. Now if we consider a real function β such that $\beta \in C^1(\mathbb{R})$, β' is bounded on $\mathbb{R}$, then multiplying the above equality by $\beta'(u_\epsilon)$ and using the Chain Rule we get

$$\frac{\partial}{\partial t} \beta(u_\epsilon) - b \cdot \nabla_x \beta(u) = c u_\epsilon \beta'(u_\epsilon) + r_\epsilon \beta'(u_\epsilon).$$

Letting ϵ go to 0, r_ϵ goes to 0 in L^1_{loc} and $\beta'(u_\epsilon)$ is bounded in L^∞, we obtain

$$\frac{\partial}{\partial t} \beta(u) - b \cdot \nabla_x \beta(u) = c u \beta'(u) \quad \text{in}(0,T) \times \mathbb{R}^N. \tag{2.13}$$

Now, let us consider a function $\phi_R = \phi\big(\frac{\cdot}{R}\big)$ for $R \geq 1$ where ϕ is a test function on $\mathbb{R}^N$ with Supp $\phi \in B_2$, $\phi = 0$ on B_1(B_1 and B_2 are balls of radius respectively 1 and 2). Multiplying by ϕ_R and integrating over space we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \beta(u)\phi_R \,\mathrm{d}x = \int c u \beta'(u)\phi_R \,\mathrm{d}x - \int \beta(u)\phi_R \nabla \cdot b \,\mathrm{d}x \,\mathrm{d}x - \int \beta(u) b \cdot \nabla \phi_R \,\mathrm{d}x,$$

this is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \beta(u)\phi_R \,\mathrm{d}x = \int c u \beta'(u)\phi_R \,\mathrm{d}x - \int (\nabla \cdot b)\beta(u)\phi_R \,\mathrm{d}x - \frac{1}{R} \int \beta(u) b \cdot (\nabla \phi)_R \,\mathrm{d}x,$$

Let $M \in (0, \infty)$, and then choose $\beta(t) = (|t| \wedge M)^p$ which is the minimum function knew as a Lipschitz on $\mathbb{R}$. We can easily see that

$$\int (|u| \wedge M)^p \,\mathrm{d}x \to \int |u|^p \,\mathrm{d}x \quad \text{as } M \text{ goes to infinity.}$$

Hence we can then write $(|u| \wedge M)^p \in L^\infty(0,T; L^1 \cap L^\infty(\mathbb{R}^N))$. Taking the absolute value both sides of the above equation with the β defined, also considering the L^∞ condition on c and the characteristic of ϕ and another approximation (u$\beta'(u) \approx \beta(u)$) we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (|u| \wedge M)^p \,\mathrm{d}x \leq C_0 \int (|u| \wedge M)^p \,\mathrm{d}x + C_1 \int \nabla \cdot b \,\mathrm{d}x + \frac{C_2}{R} \int_{R \leq |x| \leq 2R} (|u| \wedge M)^p |b(t,x)| \,\mathrm{d}x,$$

where C_0, C_1 and C_2 are respectively $\underset{x \in \mathbb{R}^N}{\text{Sup}} |c\beta'|$, $\underset{x \in \mathbb{R}^N}{\text{Sup}} |\phi_R \beta|$ and $\underset{x \in \mathbb{R}^N}{\text{Sup}} |(\nabla \phi)|_R$. But one can also shows that

$$\int_{\mathbb{R}^N} \nabla \cdot b \,\mathrm{d}x = 0.$$

We have then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (|u| \wedge M)^p \,\mathrm{d}x \leq C_0 \int (|u| \wedge M)^p \,\mathrm{d}x + \frac{C_2}{R} \int_{R \leq |x| \leq 2R} (|u| \wedge M)^p |b(t,x)| \,\mathrm{d}x,$$

Now since $|x| \leq 2R$ and $R \geq 1$ we can write $|x| + 1 \leq 3R$ and then we have

$$\frac{|b(t,x)|}{3R} 1_{R \leq |x| \leq 2R} \leq \frac{|b(t,x)|}{1 + |x|} 1_{R \leq |x| \leq 2R},$$

but we since, $\{x : \quad R \leq |x| \leq 2R\} \subset \{x : \quad R \leq |x|\}$, we have

$$\frac{1}{3R} \int_{R \leq |x| \leq 2R} (|u| \wedge M)^p |b(t,x)| \,\mathrm{d}x \leq \int_{|x| \geq R} (|u| \wedge M)^p \frac{|b(t,x)|}{1 + |x|} \,\mathrm{d}x.$$

There for

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (|u|\wedge M)^p \,\mathrm{d}x \le C_0 \int (|u|\wedge M)^p \,\mathrm{d}x + C_2 \int_{|x|\ge R} (|u|\wedge M)^p \frac{|b(t,x)|}{1+|x|}\,\mathrm{d}x.$$

Therefore, we deduce form condition (2.12) (12)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (|u|\wedge M)^p \,\mathrm{d}x \le C_0 \int (|u|\wedge M)^p \,\mathrm{d}x + C_2 h(t)\int_{|x|\ge R} (|u|\wedge M)^p \,\mathrm{d}x + C_2 \int_{|x|\ge R} \frac{|b_1(t,x)|}{1+|x|}\,\mathrm{d}x,$$

where $h(t) = \left\| b_2/(1+|x|)\right\|_\infty$. Letting R goes to infinity, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (|u|\wedge M)^p \,\mathrm{d}x \le C \int (|u|\wedge M)^p \,\mathrm{d}x.$$

This implies

$$\int (|u|\wedge M)^p \phi_R \,\mathrm{d}x \le \exp(Ct) \int (|u^0|\wedge M)^p \,\mathrm{d}x = 0,$$

for $u^0 = 0$. Hence $|u|\wedge M = 0$, when $p<\infty$(since the function $(|u|\wedge M)^p$ is continuous). In the case when $p=\infty$, we use some further more arguments. For instance if $u \in L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^N))$ the proof still holds. But it is simple to use a duality argument that includes all the cases, a method that we will only explain here, since we will properly deal with it in a general case in the rest of the work. It consists in proving that

$$\int_0^T \int_{\mathbb{R}^N} uf \,\mathrm{d}x\,\mathrm{d}t = 0,$$

for any test function $f \in \mathcal{D}((0,T)\times\mathbb{R}^N)$. To do this, we consider φ the solution of the backward equation problem

$$\frac{\partial\varphi}{\partial t} - b\cdot\nabla_x\varphi = -(c+\mathrm{div} b)\varphi + f \qquad \text{in } (0,T)\times\mathbb{R}^N, \quad \varphi|_{t=T}= 0 \text{ on } \mathbb{R}^N.$$

By Theorem 5 φ exists, unique by the Theorem 12 and also it belongs to at least to $L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^N))$, since fact that $f \in \mathcal{D}((0,T)\times\mathbb{R}^N)$. Using regularization result in Theorem 10, we infer that

$$\frac{\partial}{\partial t}u_\epsilon - b\cdot\nabla_x u_\epsilon = cu_\epsilon + r_\epsilon \qquad \text{in } (0,T)\times\mathbb{R}^N, \quad u_\epsilon|_{t=0}= 0 \text{ on } \mathbb{R}^N.$$

$$\frac{\partial\varphi_\epsilon}{\partial t} - b\cdot\nabla_x\varphi_\epsilon = -(c+\mathrm{div} b)\varphi_\epsilon + f_\epsilon + g_\epsilon \qquad \text{in } (0,T)\times\mathbb{R}^N, \quad \varphi_\epsilon|_{t=T}= 0 \text{ on } \mathbb{R}^N,$$

multiplying the first equation by $\varphi_\epsilon\phi_R$ where $\phi_R = \phi\big(\frac{\cdot}{R}\big)$ with $R\ge 1$, and ϕ as in the proof of Theorem 12 and integrating over time and space, doing integration by part on time, considering the fact $u_\epsilon|_{t=0}$ and $\varphi_\epsilon|_{t=T}= 0$, and using the second equation, we obtain

$$-\int_0^T\int_{\mathbb{R}^N} u_\epsilon\phi_R(f_\epsilon+g_\epsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^N}\Big(\varphi_\epsilon\phi_R b\cdot\nabla_x u_\epsilon + u_\epsilon\phi_R b\cdot\nabla_x\varphi_\epsilon \,\mathrm{d}x\,\mathrm{d}t + u_\epsilon\phi_R\varphi_\epsilon \mathrm{div} b\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^N} r_\epsilon\phi_R\varphi_\epsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Using Chan Rule we get (as b is $W^{1,1}_{\mathrm{loc}}$ and the three other factors are regular)

$$\mathrm{div}(\varphi_\epsilon u_\epsilon\phi_R b) - \varphi_\epsilon u_\epsilon b\cdot\nabla\phi_R = \varphi_\epsilon\phi_R b\cdot\nabla_x u_\epsilon + u_\epsilon\phi_R b\cdot\nabla_x\varphi_\epsilon + u_\epsilon\phi_R\varphi_\epsilon$$

and since Suppϕ is compact, we have

$$\int_{\mathbb{R}^N}\mathrm{div}(\varphi_\epsilon u_\epsilon\phi_R b)\,\mathrm{d}x = 0.$$

Therefore we deduce from the above equation that

$$\int_0^T \int_{\mathbb{R}^N} u_\epsilon \phi_R (f_\epsilon + g_\epsilon)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^N} \varphi_\epsilon u_\epsilon b \cdot \nabla \phi_R \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^N} r_\epsilon \phi_R \varphi_\epsilon \,\mathrm{d}x\,\mathrm{d}t,$$

When we let ϵ goes to 0, we get

$$\int_0^T \int_{\mathbb{R}^N} u \phi_R f \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^N} \varphi u b \cdot \nabla \phi_R \,\mathrm{d}x\,\mathrm{d}t,$$

this is equal to

$$\int_0^T \int_{\mathbb{R}^N} u \phi_R f \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{R}\int_0^T \int_{\mathbb{R}^N} \varphi u b \cdot (\nabla f)_R \,\mathrm{d}x\,\mathrm{d}t.$$

Applying the absolute value to this and using what we did above regarding the condition (2.12) (at page 12) on b and using the fact that $(\nabla\phi)_R$ belongs to $\mathcal{D}(\mathbb{R}^N)$ and the fact that $|u||\varphi| \in L^\infty(0,T;L^1 \cap L^\infty(\mathbb{R}^N))$ we have

$$\Big|\int_0^T \int_{\mathbb{R}^N} u \phi_R f \,\mathrm{d}x\,\mathrm{d}t\Big| \leq C \int_0^T \int_{|x|\geq R} \frac{|b(t,x)|}{1+|x|}\,\mathrm{d}x.$$

Letting R goes to infinity, the second hand side tends to 0 and we have then

$$\int_0^T \int_{\mathbb{R}^N} u f \,\mathrm{d}x\,\mathrm{d}t = 0,$$

for any f. Hence we deduce that $u = 0$.

□

Corollary 12.1. *Let us assume equation* (2.1) *wi th:* c, div $b \in L^1(0,T;L^\infty(\mathbb{R}^N))$, $b \in L^1(0,T;W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N))$ *and Condition* (2.12) *(at page 12) and an initial condition* $u^0 \in L^p(\mathbb{R}^N)$ *for* $p \in [1,\infty]$. *Then:*

1. *there exists a unique solution* $u \in L^\infty(0,T;L^p(\mathbb{R}^N))$ *to the Cauchy problem (Equation* (2.1) *together with* u^0 *).*
2. *Furthermore,* $u \in C(0,T;L^p(\mathbb{R}^N))$ *if* $p < \infty$
3. *Also we have*

$$\frac{\partial}{\partial t}\beta(u) - b \cdot \nabla_x \beta(u) = cu\beta'(u) \quad \text{in } (0,T)\times\mathbb{R}^N \tag{2.14}$$

for all real variable function $\beta \in C^1(\mathbb{R})$ *that vanishes at* 0, *such that* $|\beta'(t)| \leq C|t|$.

Proof. From Theorem 5, the problem does have a solution. Since the equation is linear, so if we suppose that u_1 and u_2 are two solutions, then $u_1 - u_2$ is going to be solution of the same equation corresponding to the initial condition $u^0 - u^0 = 0$. Therefore from Theorem 12 we have $u_1 - u_2 = 0$, thus $u_1 = u_2$.

we have the function $u : [0,T] \to L^p$, verifies:

i) bounded; $\mathrm{Sup}\|u(t)\|_p < \infty$

ii) $t \mapsto \|u(t)\|_p$ is continuous

iii) For all $\phi \in C_c^\infty(\mathbb{R}^N)$, $t \mapsto \int_{\mathbb{R}^N} \varphi u(t)\,\mathrm{d}x$ is continuous
then,

1. $\forall\, v \in L^q$, $t \mapsto \int_{\mathbb{R}^N} v u(t)\,\mathrm{d}x$ is continuous ie in u is continuous in L^p- weak
2. i) ii) et 1) imply that u is continuous in L^p:

$$\|u(t+h) - u(t)\|_p \to 0 \quad \text{as } h \to 0.$$

For the second statement, we have $1 < p < \infty$ so q the Hölder conjugate of p verifies $1 < p < \infty$.

□

2.5 Duality

As we said before the result in this subsection, known as Duality is an important tool to prove the uniqueness of the transport equation. With duality, it is easy to show that if the initial condition u^0, then the corresponding solution $u = 0$.

Theorem 13. *Let* $(b,\ c)$ *satisfies* (2.12) *(at page 12) and* (5.1) *(at page 33) which is:* $c,\ \mathrm{div} b \in L^1(0,T; L^\infty(\mathbb{R}^N))$ *and* $b \in L^1(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N))$. *Let* $u \in L^\infty(0,T; L^p(\mathbb{R}^N))$ *be renormalized solution of equation* (2.1)

$$\partial_t u - b \cdot \nabla_x u = cu \qquad \text{in } (0,T) \times \mathbb{R}^N,$$

and $v \in L^\infty(0,T; L^q(\mathbb{R}^N))$ *a renormalized solution of*

$$\frac{\partial v}{\partial t} - b \cdot \nabla_x v = -(c - \mathrm{div} b)v + f \quad \text{in} \quad [0,T] \times \mathbb{R}^N,$$

where $f \in L^1(0,T; L^q(\mathbb{R}^N))$, *and* q *is Hölder conjuguate od* p $(\frac{1}{p} + \frac{1}{q} = 1)$. *Then the following holds*

$$\int_{\mathbb{R}^N} u(T)v(T)\,\mathrm{d}x - \int_{\mathbb{R}^N} u(0)v(0)\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^N} fu\,\mathrm{d}x\,\mathrm{d}t. \tag{2.15}$$

Proof. Using the regularization results in Theorem 10, we deduce

$$\frac{\partial}{\partial t}\tilde{u}_\epsilon - b \cdot \nabla_x \tilde{u}_\epsilon = cu\beta'(u) + r_\epsilon \quad \text{in } (0,T) \times \mathbb{R}^N \tag{2.16}$$

$$\frac{\partial \tilde{v}_\epsilon}{\partial t} - b \cdot \nabla_x \tilde{v}_\epsilon = -(c - \mathrm{div} b)v\beta'(v) + f + s_\epsilon \qquad \text{in } (0,T) \times \mathbb{R}^N, \tag{2.17}$$

where $\tilde{u} = \beta(u)$, $\tilde{v} = \beta(v)$, $\tilde{u_\epsilon} = \tilde{u} * \rho_\epsilon$, $\tilde{v}_\epsilon = \tilde{v} * \rho_\epsilon$ and $r_\epsilon,\ s_\epsilon \to\in L^1(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^N))$, and $\beta \in C^1(\mathbb{R})$ and β' is bounded. Let us consider the function ϕ_R as in the proof of Theorem 12, now multiplying equation (2.16) by $\tilde{v}_\epsilon\phi_R$ we get

$$\tilde{v}_\epsilon\phi_R\frac{\partial}{\partial t}\tilde{u}_\epsilon - \tilde{v}_\epsilon\phi_R b \cdot \nabla_x \tilde{u}_\epsilon = \tilde{v}_\epsilon\phi_R cu\beta'(u) + \tilde{v}_\epsilon\phi_R r_\epsilon.$$

but we have by integration by part

$$\int_0^T \tilde{v}_\epsilon\phi_R\frac{\partial}{\partial t}\tilde{u}_\epsilon\,\mathrm{d}t = \tilde{v}_\epsilon\phi_R\tilde{u}_\epsilon|_0^T - \int_0^T \tilde{u}_\epsilon\phi_R\frac{\partial}{\partial t}\tilde{v}_\epsilon\,\mathrm{d}t,$$

using equation (2.17) we deduce that

$$\int_0^T \tilde{v}_\epsilon\phi_R\frac{\partial}{\partial t}\tilde{u}_\epsilon\,\mathrm{d}t = \tilde{v}_\epsilon(T)\phi_R\tilde{u}_\epsilon(T) - \tilde{v}_\epsilon(0)\phi_R\tilde{u}_\epsilon(0) - \int_0^T \Big(b \cdot \nabla_x\tilde{v}_\epsilon - (c - \mathrm{div} b)v\beta'(v) + f + s_\epsilon\Big)\tilde{u}_\epsilon\phi_R\,\mathrm{d}t.$$

We then deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}\Big(\tilde{v}_\epsilon(T)\phi_R\tilde{u}_\epsilon(T) - \tilde{v}_\epsilon(0)\phi_R\tilde{u}_\epsilon(0)\Big)\,\mathrm{d}x = \int_0^T\int_{\mathbb{R}^N}(f + s_\epsilon)\tilde{u}_\epsilon\phi_R\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N}\int_0^T \tilde{v}_\epsilon\phi_R r_\epsilon\,\mathrm{d}x\,\mathrm{d}t+ \\ \int_0^T\int_{\mathbb{R}^N}\Big(\tilde{v}_\epsilon\phi_R b \cdot \nabla_x\tilde{u}_\epsilon + \tilde{u}_\epsilon\phi_R b \cdot \nabla_x\tilde{v}_\epsilon\Big)\,\mathrm{d}x\,\mathrm{d}t+ \\ \int_0^T\int_{\mathbb{R}^N}\tilde{v}_\epsilon\phi_R cu\beta'(u)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{R}^N}(c - \mathrm{div} b)v\tilde{u}_\epsilon\phi_R\beta'(v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By chain rule we have,

$$\tilde{v_\epsilon}\phi_R b\cdot\nabla_x\tilde{u_\epsilon}+\tilde{u_\epsilon}\phi_R b\cdot\nabla_x\tilde{v_\epsilon}=\mathrm{div}(\tilde{u_\epsilon}\tilde{v_\epsilon}\phi_R b)-\tilde{u_\epsilon}\tilde{v_\epsilon}b\cdot\nabla\phi_R-\tilde{u_\epsilon}\tilde{v_\epsilon}\phi_R\mathrm{div}\ b.$$

Hence we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}\tilde{v_\epsilon}(T)\phi_R\tilde{u_\epsilon}(T)\,\mathrm{d}x-\int_{\mathbb{R}^N}\tilde{v_\epsilon}(0)\phi_R\tilde{u_\epsilon}(0)\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^N}(f+s_\epsilon)\tilde{u_\epsilon}\phi_R\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^N}\int_0^T\tilde{v_\epsilon}\phi_R r_\epsilon\,\mathrm{d}x\,\mathrm{d}t+\\ -\int_0^T\int_{\mathbb{R}^N}\tilde{u_\epsilon}\tilde{v_\epsilon}b\cdot\nabla\phi_R\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}\tilde{u_\epsilon}\tilde{v_\epsilon}\phi_R\mathrm{div}\ b\,\mathrm{d}x\,\mathrm{d}t+\\ \int_0^T\int_{\mathbb{R}^N}\tilde{v_\epsilon}\phi_R cu\beta'(u)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}(c-\mathrm{div}b)v\tilde{u_\epsilon}\phi_R\beta'(v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Letting ϵ goes to 0 and R goes to infinity, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\tilde{v}(T)\tilde{u}(T)\,\mathrm{d}x-\int_{\mathbb{R}^N}\tilde{v}(0)\tilde{u}(0)\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^N}f\tilde{u}\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}\tilde{u}\tilde{v}(\mathrm{div}b)\,\mathrm{d}x\,\mathrm{d}t+\\ \int_0^T\int_{\mathbb{R}^N}\tilde{v}cu\beta'(u)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}(c-\mathrm{div}b)v\tilde{u}\beta'(v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From here, we consider a family of the (β_n) of those such that

$$|\beta_n(s)|\leq|t|\quad\text{and }\beta_n\underset{n}{\rightarrow}s\text{ uniformely on compact set of }\mathbb{R}.$$

We will have base on conditions $\tilde{u}\in L^\infty(0,T;L^p(\mathbb{R}^N))$ and $\tilde{v}\in L^\infty(0,T;L^q(\mathbb{R}^N))$. We have

$$\begin{aligned}\int_{\mathbb{R}^N}\beta_n(v(T))\beta_n(u(T))\,\mathrm{d}x-\int_{\mathbb{R}^N}\beta_n(v(0))\beta_n(u(0))\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^N}f\beta_n(u)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}\beta_n(u)\beta_n(v)(\mathrm{div}b)\,\mathrm{d}x\,\mathrm{d}t+\\ \int_0^T\int_{\mathbb{R}^N}cu\beta_n(v)\beta_n'(u)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}(c-\mathrm{div}b)v\beta_n(u)\beta_n'(v)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Taking the limit of the above equation as n goes to 0, we get equation (2.15).

□

Corollary 13.1. *Let u_n be a renormalized solution of equation* (2.1) *with* (b,c) *replace by* $(b_n,0)$ *with initial condition u_n^0, where we assume that b_n satisfies condition* (2.12) *(at page 12) and that u_n^0. converges weakly u^0 in $L^p(\mathbb{R}^N)$ for some $p\in(1,+\infty]$. The sequence u_n converges weakly in $L^\infty(0,T;L^p(\mathbb{R}^N))$ to u, the renormalized solution of* (2.1) *with $c=0$ for the initial condition u^0.*

Proof. From the previous result, we have u_n is bounded in $L^\infty(0,T;L^p(\mathbb{R}^N))$ since $u_n^0\in L^p(\mathbb{R}^N)$ as converges weakly in to u^0 in $L^p(\mathbb{R}^N)$, then from a well known result in functional analysis, we deduce that u_n^0 is bounded in $L^p(\mathbb{R}^N)$, hence using the estimate result in (2.9) we deduce that u_n is bounded as well. Now let us consider the solution of the Cauchy problem

$$\frac{\partial v_n}{\partial t}-b_n\cdot\nabla_x v_n=-\mathrm{div}b_n v_n+f\quad\text{in}\quad(0,T)\times,\mathbb{R}^N\quad v_n|_{t=T}=0\text{ on }\mathbb{R}^N,\tag{2.18}$$

where $f\in\mathcal{D}((0,T)\times\mathbb{R}^N)$ and $v_n(0)$ converges weakly to $v(0)$ in $L^q(\mathbb{R}^N)$. Applying this theorem 13, to v_n and u_n, we get

$$\int_{\mathbb{R}^N}u_n(T)v_n(T)\,\mathrm{d}x-\int_{\mathbb{R}^N}u_n(0)v_n(0)\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^N}fu_n\,\mathrm{d}x\,\mathrm{d}t,$$

considering the fact that $v_n|_{t=T} = 0$, we may write

$$-\int_{\mathbb{R}^N} u_n^0 v_n(0)\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^N} f u_n \,\mathrm{d}x\,\mathrm{d}t. \tag{2.19}$$

Let us consider v and u a weak solution respectively of

$$\frac{\partial v}{\partial t} - b\cdot\nabla_x v = -\mathrm{div} bv + f \quad \text{in} \quad (0,T)\times,\mathbb{R}^N \quad v|_{t=T} = 0 \text{ on } \mathbb{R}^N.$$

$$\frac{\partial u}{\partial t} - b\cdot\nabla_x u = 0 \quad \text{in} \quad (0,T)\times,\mathbb{R}^N \quad u|_{t=0} = u^0 \text{ on } \mathbb{R}^N.$$

Applying Theorem 13 to those two last Equations, we have

$$-\int_{\mathbb{R}^N} u^0 v(0)\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^N} f u \,\mathrm{d}x\,\mathrm{d}t.$$

Using Equation (2.19) and the fact that u_n^0 converges weakly to u^0 in $L^p(\mathbb{R}^N)$ we have

$$\int_0^T \int_{\mathbb{R}^N} f u_n \,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^N} u_n^0 v_n(0)\,\mathrm{d}x \underset{n}{\longrightarrow} -\int_{\mathbb{R}^N} u^0 v(0)\,\mathrm{d}x = \int_0^T \int_{\mathbb{R}^N} f u \,\mathrm{d}x\,\mathrm{d}t.$$

Hence, emphasizing the fact that $f \in L^1(0,T;L^q(\mathbb{R}^N))$, we conclude that u_n converges weakly to u in $L^\infty(0,T;L^p(\mathbb{R}^N))$.
□

Chapter 3

Uniqueness for a rough transport-diffusion equation (Levy)

In this section we deal with uniqueness for a transport-diffusion equation with rough coefficients, which can be viewed somehow as a generalization of the previous results but it does oppose it since there is some different integrability on the vector field and it is supposed to be divergence free.

definition 14. Let s be in $\mathbb{R}$. The homogeneous space $\dot{H}^s(\mathbb{R}^N)$ or $\dot{H}^s$ is the space of tempered distributions u over $\mathbb{R}^N$, with the Fourier transform $\dot{u} \in L^2_{\text{loc}}(\mathbb{R}^N)$ and satisfies

$$\|u\|^2_{\dot{H}^s} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi.$$

Proposition 15. *Let $s_0 \leq s \leq s_1$. Then $s = (1-\theta)s_0 + \theta s_1$. Then $\dot{H}^{s_0} \cap \dot{H}^{s_1}$ is included in $\dot{H}^s$ and we have*

$$\|u\|_{\dot{H}^s} \leq \|u\|^{1-\theta}_{\dot{H}^{s_0}} \|u\|^{\theta}_{\dot{H}^{s_1}} \text{ with } s = (1-\theta)s_0 + \theta s_1.$$

Proposition 16. *Let k be a positive integer. The space $\dot{H}^{-k}(\mathbb{R}^N)$ consists of distributions which are the sums of derivatives of order k of $L^2(\mathbb{R}^N)$ functions.*

Lemma 17. *If s is in $[0, \frac{N}{2}[$, then the space $\dot{H}^s$ is continuously embedded in $L^{\frac{2d}{d-2s}}(\mathbb{R}^N)$.*

Corollary 17.1. *if p belongs to $]1, 2]$, then $L^p(\mathbb{R}^N)$ embeds continuously in $\dot{H}^s(\mathbb{R}^N)$ with $s = \frac{N}{2} - \frac{N}{p}$.*

Lemma 18. *Let $(u_n)_n$ and $(v_n)_n$ be two sequences converging respectively strongly in L^{m_1} and weakly in L^{m_2}, with $\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{m}$. Then the product sequence $(w_n)_n$ where $w_n = u_n v_n$ for all n, converges weakly in L^m.*

3.1 Commutator and estimation on the smooth solution of the equation with regular vector field

Theorem 19. *Given ν be a divergence free vector field i.e $\operatorname{div} \nu = 0$ in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and φ_0 a smooth function in $\mathcal{D}(\mathbb{R}^3)$, there exists a distributional solution of the Cauchy problem*

$$(c') \begin{cases} \partial_t \varphi - \nu \cdot \nabla_x \varphi - \Delta \varphi = 0 \\ \varphi(0) = \varphi_0 \end{cases}$$

with the estimates

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \tag{3.1}$$

and

$$\|\partial_j\varphi(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla_x\partial_j\varphi(t)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \le \|\partial_j\varphi_0\|^2_{L^2(\mathbb{R}^3)} + \|\varphi_0\|^2_{L^\infty(\mathbb{R}^3)}\|\partial_j\nu\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^3)} \tag{3.2}$$

for $j = 1,\ 2,\ 3$ and for any time t.

Proof. To prove this theorem, we consider some mollifying kernel $\rho = \rho(t,x)$ and let us defined $\nu_\delta := \rho_\delta * \nu$, where $\rho_\delta(t,x) := \delta^{-4}\rho(\frac{t}{\delta},\frac{x}{\delta})$. Now let (c'_δ) be the cauchy problem (c'), where we replaced ν by ν_δ. By Heat kernel together with a heat kernel estimates and Banach fixed point theorem , (c'_δ) has a smooth solution that we denotes φ^δ(we skech this later) We have then

$$\partial_t\varphi^\delta - \nu_\delta\cdot\nabla_x\varphi^\delta - \Delta\varphi^\delta = 0$$

Multiplying this equation by $\varphi^\delta|\varphi^\delta|^{p-2}$ for $p\ge 2$ we have

$$\varphi^\delta|\varphi^\delta|^{p-2}\partial_t\varphi^\delta - \varphi^\delta|\varphi^\delta|^{p-2}\nu_\delta\cdot\nabla_x\varphi^\delta - \varphi^\delta|\varphi^\delta|^{p-2}\Delta\varphi^\delta = 0,$$

the above equation is equivalent to

$$\frac{1}{p}\partial_t|\varphi^\delta|^p - \varphi^\delta|\varphi^\delta|^{p-2}\nu_\delta\cdot\nabla_x\varphi^\delta - \varphi^\delta|\varphi^\delta|^{p-2}\Delta\varphi^\delta = 0$$

integrating over time and space we got

$$\frac{1}{p}\int_0^t\int_{\mathbb{R}^3}\partial_s|\varphi^\delta(s)|^p\,\mathrm{d}s\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^3}\varphi^\delta(s,x)|\varphi^\delta(s,x)|^{p-2}\Delta\varphi^\delta(s,x)\,\mathrm{d}s\,\mathrm{d}x = 0,$$

we do not include the term involving $\varphi^\delta|\varphi^\delta|^{p-2}\nu_\delta\cdot\nabla_x\varphi^\delta = \frac{1}{p}\nabla\cdot(\nu_\delta|\varphi|^p)$ because its integral over space vanishes as ν_δ is of divergence free. But we have

$$|\varphi^\delta(t)|^p - |\varphi^\delta_0|^p = \int_0^t\partial_s|\varphi^\delta(s)|^p\,\mathrm{d}s,$$

replacing this in the above equation, we got

$$\frac{1}{p}\|\varphi^\delta(t)\|^p_{L^p(\mathbb{R}^3)} - \int_0^t\int_{\mathbb{R}^3}\varphi^\delta(s,x)|\varphi^\delta(s,x)|^{p-2}\Delta\varphi^\delta(s,x)\,\mathrm{d}s\,\mathrm{d}x = \frac{1}{p}\|\varphi^\delta_0\|^p_{L^p(\mathbb{R}^3)}$$

But

$$\varphi^\delta|\varphi^\delta|^{p-2}\Delta\varphi^\delta = \mathrm{div}(\nabla\varphi^\delta)\times\mathrm{sign}\big(\varphi^\delta\big)|\varphi^\delta|^{p-1} = \mathrm{div}\Big(\nabla\varphi^\delta\times|\varphi^\delta|^{p-1}\Big) - (p-1)\nabla\varphi^\delta\cdot\nabla\varphi^\delta\times|\varphi^\delta|^{p-2},$$

when we integrate this, the divergence term vanishes and we have

$$-\int_0^t\int_{\mathbb{R}^3}\varphi^\delta(s,x)|\varphi^\delta(s,x)|^{p-2}\Delta\varphi^\delta(s,x)\,\mathrm{d}s\,\mathrm{d}x = (p-1)\int_0^t\int_{\mathbb{R}^3}\nabla\varphi^\delta(s,x)\cdot\nabla\varphi^\delta(s,x)\times|\varphi^\delta(s,x)|^{p-2}\,\mathrm{d}s\,\mathrm{d}x,$$

this is the same as

$$-\int_0^t\int_{\mathbb{R}^3}\varphi^\delta(s,x)|\varphi^\delta(s,x)|^{p-2}\Delta\varphi^\delta(s,x)\,\mathrm{d}s\,\mathrm{d}x = (p-1)\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{p-2}{2}}\Big\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s.$$

We deduce that

$$\frac{1}{p}\|\varphi^\delta(t)\|^p_{L^p(\mathbb{R}^3)} + (p-1)\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{p-2}{2}}\Big\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s = \frac{1}{p}\|\varphi^\delta_0\|^p_{L^p(\mathbb{R}^3)} \tag{3.3}$$

for $p\ge 2$. We deduce that

$$\frac{1}{p}\|\varphi^{\delta}(t)\|^{p}_{L^{p}(\mathbb{R}^3)} \leq \frac{1}{p}\|\varphi^{\delta}_0\|^{p}_{L^{p}(\mathbb{R}^3)} \Rightarrow \|\varphi^{\delta}(t)\|^{p}_{L^{p}(\mathbb{R}^3)} \leq \|\varphi^{\delta}_0\|^{p}_{L^{p}(\mathbb{R}^3)},$$

tending p to infinity, we got the first estimate

$$\|\varphi^{\delta}(t)\|_{L^{\infty}(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^{\infty}(\mathbb{R}^3)}.$$

For the second estimate, let us derivate the equation which respect to x_j for $1 \leq j \leq 3$. Doing so, we got

$$\partial_j\partial_t\varphi^{\delta} - \nu_{\delta}\cdot\partial_j\nabla\varphi^{\delta} - \partial_j\Delta\varphi^{\delta} = \partial_j\nu_{\delta}\cdot\nabla\varphi^{\delta},$$

because φ^{δ} is regular we can swap the partial derivative, and then the above equation is the same as

$$\partial_t\partial_j\varphi^{\delta} - \nu\cdot\nabla\partial_j\varphi^{\delta} - \Delta\partial_j\varphi^{\delta} = \partial_j\nu_{\delta}\cdot\nabla\varphi^{\delta}.$$

By multiplying this by $\partial_j\varphi^{\delta} = \partial_j\varphi^{\delta}\times|\partial_j\varphi^{\delta}|^{p-2}$, with $p=2$, and integrating over space and time and using the estimate (3.3), we got

$$\frac{1}{2}\|\partial_j\varphi^{\delta}(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla\partial_j\varphi^{\delta}(s)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s = \frac{1}{2}\|\partial_j\varphi^{\delta}_0\|^2_{L^2(\mathbb{R}^3)} + \int_0^t\int_{\mathbb{R}^3}\partial_j\nu_{\delta}(s,x)\cdot\nabla\varphi^{\delta}(s,x)\partial_j\varphi^{\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x.$$

But we have

$$\partial_j\varphi^{\delta}\partial_j\nu_{\delta}\cdot\nabla\varphi^{\delta} = \partial_j(\varphi^{\delta}\nabla\varphi^{\delta})\cdot\partial_j\nu_{\delta} - \varphi^{\delta}\partial_j\nu_{\delta}\cdot\nabla\partial_j\varphi^{\delta},$$

but

$$\begin{aligned}
2\partial_j(\varphi^{\delta}\nabla\varphi^{\delta})\cdot\partial_j\nu_{\delta} &= \partial_j\left(\nabla(\varphi^{\delta})^2\right)\cdot\partial_j\nu_{\delta}\varphi^{\delta} \\
&= \nabla\partial_j(\varphi^{\delta})^2\cdot\partial_j\nu_{\delta} \\
&= \mathrm{div}(\partial_j\nu_{\delta}\partial_j(\varphi^{\delta})^2) - \mathrm{div}(\partial_j\nu_{\delta})\cdot\nabla(\varphi^{\delta})^2 \\
&= \mathrm{div}(\partial_j\nu_{\delta}\partial_j(\varphi^{\delta})^2) - \partial_j\mathrm{div}(\nu_{\delta})\cdot\nabla(\varphi^{\delta})^2 \\
&= \mathrm{div}(\partial_j\nu_{\delta}\partial_j(\varphi^{\delta})^2),
\end{aligned}$$

because $\mathrm{div}(\nu_{\delta}) = 0$. We have then

$$\partial_j\varphi^{\delta}\partial_j\nu_{\delta}\cdot\nabla\varphi^{\delta} = \frac{1}{2}\mathrm{div}(\partial_j\nu_{\delta}\partial_j(\varphi^{\delta})^2) - \varphi^{\delta}\partial_j\nu_{\delta}\cdot\nabla\partial_j\varphi^{\delta}.$$

Hence we have

$$\int_0^t\int_{\mathbb{R}^3}\partial_j\nu_{\delta}(s,x)\cdot\nabla\varphi^{\delta}(s,x)\partial_j\varphi^{\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x = -\int_0^t\int_{\mathbb{R}^3}\varphi^{\delta}(s,x)\partial_j\nu_{\delta}(s,x)\cdot\nabla\partial_j\varphi^{\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x$$

. Base on the previous steps, we can now write

$$\frac{1}{2}\|\partial_j\varphi^{\delta}(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \left\|\nabla\partial_j\varphi^{\delta}(s)\right\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s = \frac{1}{2}\|\partial_j\varphi^{\delta}_0\|^2_{L^2(\mathbb{R}^3)} - \int_0^t\int_{\mathbb{R}^3}\varphi^{\delta}(s,x)\partial_j\nu_{\delta}(s,x)\cdot\nabla\partial_j\varphi^{\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x.$$

Now let us pose

$$g(t) = -\int_0^t\int_{\mathbb{R}^3}\varphi^{\delta}(s,x)\partial_j\nu_{\delta}(s,x)\cdot\nabla\partial_j\varphi^{\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x,$$

by the first estimate in the theorem, we have

$$g(t) \leq \|\varphi_0^\delta\|_{L^\infty(\mathbb{R}^3)} \int_0^t \int_{\mathbb{R}^3} \left|\partial_j \nu_\delta(s,x) \cdot \nabla \partial_j \varphi^\delta(s,x)\right| \mathrm{d}s\, \mathrm{d}x,$$

using Hölder inequality we obtain

$$g(t) \leq \|\varphi_0^\delta\|_{L^\infty(\mathbb{R}^3)} \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)} \|\partial_j \nu_\delta(s)\|_{L^2(\mathbb{R}^3)} \,\mathrm{d}s.$$

Now recall that $(a-b)^2 \geq 0$ which gives $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. Posing

$$a = \|\varphi_0^\delta\|_{L^\infty(\mathbb{R}^3)} \|\partial_j \nu_\delta(s)\|_{L^2(\mathbb{R}^3)}$$

and

$$b = \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)},$$

we obtain

$$g(t) \leq \frac{1}{2}\int_0^t \|\nabla \partial_j \varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s + \frac{1}{2}\|\varphi_0^\delta\|^2_{L^\infty(\mathbb{R}^3)} \int_0^t \|\partial_j \nu_\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s.$$

We deduce that

$$\frac{1}{2}\|\partial_j \varphi^\delta(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s \leq \frac{1}{2}\|\partial_j \varphi_0^\delta\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\int_0^t \|\nabla \partial_j \varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s + \frac{1}{2}\|\varphi_0^\delta\|^2_{L^\infty(\mathbb{R}^3)} \int_0^t \|\partial_j \nu_\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s$$

hence,

$$\|\partial_j \varphi^\delta(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s \leq \|\partial_j \varphi_0^\delta\|^2_{L^2(\mathbb{R}^3)} + \|\varphi_0^\delta\|^2_{L^\infty(\mathbb{R}^3)} \int_0^t \|\partial_j \nu_\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s,$$

Thus

$$\|\partial_j \varphi^\delta(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|^2_{L^2(\mathbb{R}^3)} \,\mathrm{d}s \leq \|\partial_j \varphi_0\|^2_{L^2(\mathbb{R}^3)} + \|\varphi_0\|^2_{L^\infty(\mathbb{R}^3)} \|\partial_j \nu(s)\|^2_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)}.$$

From these approximations, we deduce using Fourier transform and the definition of Homogenous Sobolev space that the family $(\varphi^\delta)_\delta$ is bounded in $L^\infty(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. We can then assume up to some extraction that $(\varphi^\delta)_\delta$ converge weakly in $L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3))$ and weakly-* in $L^\infty(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ to some φ. We also deduce that

$$\nabla \varphi^\delta \rightharpoonup \nabla \varphi^\delta \text{ in } L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^N)) \text{ and } L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N))$$

Now using interpolation results, we find that $\nabla \varphi^\delta \rightharpoonup \nabla \varphi$ weakly in $L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$. Now considering the fact that $\nu_\delta \to \nu$ in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ by definition, together with the fact that $\nabla \varphi^\delta \rightharpoonup \nabla \varphi$ weakly in $L^2(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, we have:

$$\Delta \varphi^\delta \rightharpoonup \Delta \varphi \text{ in } L^2(\mathbb{R}_+ \times \mathbb{R}^3)$$

and

$$\nu_\delta \cdot \nabla \varphi^\delta,\ \partial_t \varphi^\delta \rightharpoonup \nu \cdot \nabla \varphi, \partial_t \varphi \ \text{ in } L^{\frac{4}{3}}(\mathbb{R}_+, L^2(\mathbb{R}^3)),$$

where the second convergence is obtain applying on lemma 18.

Hence φ will be a solution of the Cauchy problem (c').

□

We will state the following lemma that will be a tool for proving the uniqueness of an important result that will state. But we can see that this lemma is nothing new, it is basically the same result as in equation (11).

Lemma 20. *Let ν be a divergence-free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and let φ be a function in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$. Let this time around the function $\rho \in \mathcal{D}(\mathbb{R}^3)$ and supported inside the unit ball of $\mathbb{R}^3$ and $\rho := \epsilon^{-3}\rho(\frac{\cdot}{\epsilon})$. Let us defined the commutator c^ϵ by*

$$c^\epsilon(t,x) := \nu(t,x)\cdot(\nabla\rho_\epsilon * \varphi(t))(x) - (\nabla\rho_\epsilon * (\nu(t)\varphi(t)))(x) = (\nu\cdot\nabla(\varphi*\rho_\epsilon) - (\nabla\cdot(\nu\varphi))*\rho_\epsilon)(t,x)$$

and $\nabla\cdot(\nu\varphi) = \nu\cdot\nabla\varphi$. Then

$$\|c^\epsilon\|_{L^2(\mathbb{R}_+\times\mathbb{R}^3)} \leq \|\nabla\nu\|_{L^2(\mathbb{R}_+,\mathbb{R}^3)}\|\varphi^\delta\|_{L^\infty(\mathbb{R}_+,\mathbb{R}^N)},$$

and c^ϵ converges to 0 in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$.

This Lemma is obviously not new when we get a look at the lemma 11, and it serve the same purpose.

Here we discus about the existence of solution to the problem

3.2 The result on the uniqueness of Levy

Theorem 21. *Let ν in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ be a divergence-free vector field, that is,* $\operatorname{div}\nu = 0$, *and let $a \in L^2(\mathbb{R}_+\times\mathbb{R}^3)$ distributional solution of the Cauchy problem*

$$(c)\begin{cases} \partial_t a - \nabla\cdot(a\nu) - \Delta a = 0 \\ a(0) = 0 \end{cases}. \tag{3.4}$$

Then a is identically zero on $\mathbb{R}_+ \times \mathbb{R}^3$.

Proof. Let consider ρ and ρ_ϵ as in the previous Lemma 20. Convolving the equation (c) with ρ_ϵ, we have

$$\rho_\epsilon * \partial_t a + \rho_\epsilon * \nabla\cdot(a\nu) - \rho_\epsilon * \Delta a = 0$$

which is equivalent to

$$\partial_t(\rho_\epsilon * a) + \rho_\epsilon * \nabla\cdot(a\nu) - \Delta(a*\rho_\epsilon) = 0,$$

now posing $a_\epsilon = a*\rho_\epsilon$, and adding $\nabla\cdot(a_\epsilon\nu)$ both sides of the previous equation, we got

$$\partial_t a_\epsilon + \nabla\cdot(a_\epsilon\nu) - \Delta a_\epsilon = \nabla\cdot(a_\epsilon\nu) - \rho_\epsilon * \nabla\cdot(a\nu). \tag{3.5}$$

Now let φ^δ be solution of the backward Cauchy problem, meaning we reverse by starting from from a time $T > 0$ instead of 0, the above equation (is the above equation with $t \mapsto T - t$)

$$(-c'_\delta)\begin{cases} -\partial_t\varphi^\delta - \nu_\delta\cdot\nabla\varphi^\delta - \Delta\varphi^\delta = 0 \\ \varphi^\delta(T) = \varphi^\delta_T \quad \text{and } \varphi^\delta_0 = 0 \end{cases}$$

Multiplying equation (3.5) by φ^δ and rearranging we have

$$\varphi^\delta\partial_t a_\epsilon = -\varphi^\delta\nabla\cdot(a_\epsilon\nu) + \varphi^\delta\Delta a_\epsilon + \varphi^\delta\nabla\cdot(a_\epsilon\nu) - \varphi^\delta\rho_\epsilon * \nabla(a\nu),$$

integration over time and space

$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\partial_t a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x = -\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\nabla\cdot(a_\epsilon(s,x)\nu(s,x))\,\mathrm{d}s\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\Delta a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x +$$
$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\nabla\cdot(a_\epsilon(s,x)\nu(s,x)) - \varphi^\delta(s,x)\rho_\epsilon(x) * \nabla(a(s,x)\nu(s,x))\,\mathrm{d}s\,\mathrm{d}x.$$

But one observes that

$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\nabla\cdot(a_\epsilon(s,x)\nu(s,x))-\varphi^\delta(s,x)\rho_\epsilon(x)*\nabla(a(s,x)\nu(s,x))\,\mathrm{d}s\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^3}a(s,x)c^{\epsilon,\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x,$$

we then deduce that

$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\partial_t a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x=-\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\nabla\cdot(a_\epsilon(s,x)\nu(s,x))\,\mathrm{d}s\,\mathrm{d}x+$$
$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\Delta a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^3}a(s,x)c^{\epsilon,\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x.$$

But by integration by part with respect to the time variable, we have(as $a_\epsilon|_{t=0}=0$)

$$\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\partial_t a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s=\int_{\mathbb{R}^3}[a_\epsilon(s,x)\varphi^\delta(s,x)]_0^T\,\mathrm{d}x-\int_0^T\int_{\mathbb{R}^3}a_\epsilon(s,x)\partial_t\varphi^\delta(s,x)\,dx\,\mathrm{d}s$$
$$=<a_\epsilon(T),\varphi^\delta(T)>_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}-\int_0^T\int_{\mathbb{R}^3}a_\epsilon(s,x)\partial_t\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Comparing the last two equations, We can then deduce that

$$<a_\epsilon(T),\varphi^\delta(T)>_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}=\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\Delta a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^3}a(s,x)c^{\epsilon,\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x+$$
$$\int_0^T\int_{\mathbb{R}^3}a_\epsilon(s,x)\partial_t\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s-\int_0^T\int_{\mathbb{R}^3}\varphi^\delta(s,x)\nabla\cdot(a_\epsilon(s,x)\nu(s,x))\,\mathrm{d}s\,\mathrm{d}x.$$

Doing integration by part over space variable and the Chain Rule for divergence the above equation become

$$<a_\epsilon(T),\varphi^\delta(T)>_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}=\int_0^T\int_{\mathbb{R}^3}a(s,x)c^{\epsilon,\delta}(s,x)\,\mathrm{d}s\,\mathrm{d}x+$$
$$\int_0^T\int_{\mathbb{R}^3}a_\epsilon(s,x)\Big(\partial_t\varphi^\delta(s,x)+\nu_\delta(s,x)\cdot\nabla\varphi^\delta(s,x)+\Delta\varphi^\delta(s,x)\Big)\,\mathrm{d}x\,\mathrm{d}s,$$

where $c^{\epsilon,\delta}:=\nu\cdot\nabla\big(\varphi^\delta*\rho_\epsilon\big)-\big(\nabla\cdot\big(\nu\varphi^\delta\big)\big)*\rho_\epsilon$.

From lemma 20, $(c^{\epsilon,\delta})_{\epsilon,\delta}$ is bounded in $L^2(\mathbb{R}_+\times\mathbb{R}^3)$, up to extracting subsequence, it must be weakly convergent in $L^2(\mathbb{R}_+\times\mathbb{R}^3)$ as δ goes to 0. But since $\nu\cdot\nabla\varphi^\delta\rightharpoonup\nu\cdot\nabla\varphi$ in $L^{\frac{4}{3}}(\mathbb{R}_+,L^2(\mathbb{R}^3))$ as $\delta\to0$, then the only weak limit in $L^2(\mathbb{R}_+\times\mathbb{R}^3)$ of $(c^{\epsilon,\delta})_{\epsilon,\delta}$ as $\delta\to0$ is c^ϵ, this lead taking the limit as $\delta\to0$ in the last equation(one observe that the second term in the right hand side gives 0 because φ solves (c')), we get

$$<a_\epsilon(T),\varphi(T)>_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}=\int_0^T\int_{\mathbb{R}^3}a(s,x)c^\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x.$$

Since $a\in L^2\big(\mathbb{R}_+,\mathbb{R}^3\big)$ and $C^\epsilon\underset{\epsilon\to0}{\to}0$ in $L^2\big(\mathbb{R}_+,\mathbb{R}^3\big)$ by lemma 20,

taking the limit $\epsilon\longrightarrow0$, we finally get

$$<a(T),\varphi(T)>_{\mathcal{D}'(\mathbb{R}^3),\mathcal{D}(\mathbb{R}^3)}=0,$$

for any test function $\varphi(T)$. Hence $a(T)$ is the zero distribution and final $a\equiv0$. □

Chapter 4

Extension of the uniqueness, Levy(2019)

Here, we established some new results, which compare to those we have seen so far in Di Perna and Lions [7] approach to the transport equation first, and second to the Levy 2016 for a dimension higher dimension ($N \geq 3$), look a generalization if we may say so, but in they reality is not since for example we lack proof of some assumption we made and we ignore their veracity. We will emphasize on these clearly in the rest of the work.

4.1 Extension on the Di Perna Lions result

Lemma 22. *Let $T > 0$, v be a divergence free vector field in $L^{p'}(\mathbb{R}_+, \dot{W}^{1,q'}(\mathbb{R}^3))$, and a be a fixed function in $L^p(\mathbb{R}_+, L^q(\mathbb{R}^N))$, with p' and q' respectively the Hölder conjuguate of p and q. Let ρ be a smooth positive compactly supported function with unit L^1 norm on $\mathbb{R}^N$ and define $\rho_\epsilon := \epsilon^{-N}\rho(\frac{\cdot}{\epsilon})$ of the function ρ. Let us define the commutator C^ϵ by $C^\epsilon(t,x) := v(t,x) \cdot (\nabla\rho_\epsilon * a(t))(x) - (\nabla\rho_\epsilon * (v(t)a(t)))$. Then, as $\epsilon \to 0$,*

$$\|C^\epsilon\|_{L^1(\mathbb{R}_+ \times \mathbb{R}^N)} \to 0.$$

As I was saying before, this look like the commutator in Di Perna and Lions's publication we have seen so far but here instead of $p = \infty$, we rather take any $p \geq 1$. This also look like the lemma (20).

Proof. By definition We have

$$\begin{aligned} C^\epsilon(t,x) &= \int_{\mathbb{R}^N} v(t,x) \cdot \nabla\rho_\epsilon(x-y)a(t,y)\,\mathrm{d}y - \int_{\mathbb{R}^N} \nabla\rho_\epsilon(x-y) \cdot v(t,y)a(t,y)\,\mathrm{d}y \\ &= \frac{1}{\epsilon^N}\int_{\mathbb{R}^N} a(t,y)\nabla\rho\left(\frac{x-y}{\epsilon}\right) \cdot \frac{v(t,x)-v(t,y)}{\epsilon}\,\mathrm{d}y, \end{aligned}$$

doing the change of variable $y = x - \epsilon z,\ \mathrm{d}y = -\epsilon^N\,\mathrm{d}z$

$$C^\epsilon(t,x) = \int_{\mathbb{R}^N} a(t,x-\epsilon z)\nabla\rho(z) \cdot \frac{v(t,x)-v(t,x-\epsilon z)}{\epsilon}\,\mathrm{d}z.$$

We have

$$v(t,x) - v(t,x-\epsilon z) = \int_0^1 \nabla v(t,x-h\epsilon z) \cdot (\epsilon z)\,\mathrm{d}h,$$

which is true for smooth functions and extends to $\dot{W}^{1,q'}(\mathbb{R}^N)$. Using fubini we have

$$\begin{aligned} C^\epsilon(t,x) &= \int_0^1 \int_{\mathbb{R}^N} a(t,x-\epsilon z)\nabla v(t,x-h\epsilon z) : (\nabla\rho(z) \otimes z)\,\mathrm{d}z\,\mathrm{d}h \\ &= \int_0^1 \int_{\mathbb{R}^N} a(t,x-\epsilon)\sum_{j,k}\partial_j v_k(t,x-h\epsilon z)\partial_k\rho(z)z_j\,\mathrm{d}z\,\mathrm{d}y \end{aligned}$$

Now, let us define

$$\tilde{C}^\epsilon(t,x) := -\int_0^1 \int_{\mathbb{R}^N} a(t,x-h\epsilon z)\nabla v(t,x-h\epsilon z) : (\nabla\rho(z)\otimes z)\,\mathrm{d}z\,\mathrm{d}h.$$

Now we claim that $\|C^\epsilon - \tilde{C}^\epsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^N)} \to 0$. Indeed we have

$$\begin{aligned}\|C^\epsilon - \tilde{C}^\epsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^N)} &\leq \int_0^\infty \int_{\mathbb{R}^N}\int_0^1\int_{\mathbb{R}^N} |a(t,x-\epsilon z)-a(t,x-h\epsilon z)||\nabla v(t,x-h\epsilon z)||\nabla\rho(z)\otimes z|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}h\,\mathrm{d}x\\ &\leq \int_0^\infty \int_{\mathbb{R}^N}\int_0^1 \|a(t,-\epsilon z)-a(t,-h\epsilon z)\|_{L^q(\mathbb{R}^N)}\|\nabla v(t)\|_{L^{q'}}|\nabla\rho(z)\otimes z|\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}h\end{aligned}$$

since $a\in L^p(\mathbb{R}_+, L^q(\mathbb{R}^N))$ and $q<\infty$, for all $t\in\mathbb{R}_+$, for all $z\in\mathbb{R}^N$and $h\in[0,1]$, we have

$$\|a(t,-\epsilon z)-a(t,-h\epsilon z)\|_{L^q(\mathbb{R}^N)} \underset{n}{\longrightarrow} 0 \quad \text{using dominated convergence (uniformely with respect to } h\in[0,1]).$$

Indeed, we clearly see that

$$\|a(t,x-\epsilon z)-a(t,x-h\epsilon z)\|_{L^q(\mathbb{R}^N)}\|\nabla v(t)\|_{L^{q'}}|\nabla\rho(z)\otimes z| \leq 2\|a(t)\|_{L^q(\mathbb{R}^N)}\|\nabla v(t)\|_{L^{q'}}|\nabla\rho(z)\otimes z|,$$

and now calling on dominated convergence, we get the claim.

Now let us denote $M(t,x)=a(t,x)\nabla v(t,x)$, M is fixed in $L^1(\mathbb{R}_+\times\mathbb{R}^N)$ (as $\nabla v\in L^{p'}(L^{q'}), a\in L^p(L^q)$), then

$$\tilde{C}^\epsilon(t,x) := -\int_0^1\int_{\mathbb{R}^N} M(t,x-h\epsilon z) : (\nabla\rho(z)\otimes z)\,\mathrm{d}z\,\mathrm{d}h.$$

By integration by part we have,

$$-\int_{\mathbb{R}^N}\nabla\rho(z)\otimes z dz = \Big(\int_{\mathbb{R}^N}\rho(z)\,\mathrm{d}z\Big) I_d = I_d,$$

where I_d is the N dimensional identity matrix. From this we may write

$$\begin{aligned}\tilde{C}^0(t,x) &= -\int_0^1\int_{\mathbb{R}^N} M(t,x) : (\nabla\rho(z)\otimes z)\,\mathrm{d}z\,\mathrm{d}h\\ &= -M(t,x) : \int_{\mathbb{R}^N}\nabla\rho(z)\otimes z\,\mathrm{d}z\\ &= M(t,x) : I_d\\ &= a(t,x)\nabla v(t,x) : I_d\\ &= a(t,x)\mathrm{div}\ v(t,x)\\ &= 0 \quad \text{(for v is of divergence free).}\end{aligned}$$

Hence we deduce that $\|C^\epsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^N)}\to 0$, as $\epsilon\to 0$.

□

The following result is almost the same as in the previous chapter, only that here the dimension space is considered to be $N\geq 3$ and the presence of the viscosity coefficient.

Theorem 23. *Let $\nu\geq 0$ be a positive real number, and $v=v(t,x)$ be a fixed divergence free vector field in $L^{p'}(\mathbb{R}_+, \dot{W}^{q'}(\mathbb{R}^N))$, where p' and q' are the Hölder conjugate of p and q. Let φ_0 be a smooth, compactly supported function in $\mathbb{R}^N$. There exists a unique $\varphi\in L^\infty(\mathbb{R}_+\times\mathbb{R}^N)$, solution of the cauchy problem*

$$(c_1')\begin{cases}\partial_t\varphi-\nabla\cdot(\varphi v)-\nu\Delta\varphi=0\\ \\ \varphi(0)=\varphi_0\end{cases}\tag{4.1}$$

and satisfies

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)}\leq\|\varphi_0\|_{L^\infty(\mathbb{R}^3)}.$$

The proof of this theorem follows the same steps as the proof of Theorem (19) but we only retains the needed estimate, so the terms involving ν are not considered.

We can also define from the above Cauchy problem its corresponding backward problem defined as

$$(-c_1')\begin{cases}-\partial_t\varphi-\nabla\cdot(\varphi v)-\nu\Delta\varphi=0\\ \\ \varphi(T)=\varphi_0\end{cases},$$

which is obtain by composing the entire Cauchy problem (c_1') by a change in time variable define as $t:\mapsto T-t$.

Proof. Let us choose some mollifying Kernel $\rho=\rho(x)$, and let us define $v_\delta:=\rho*v$, where $\rho_\delta(x)=\delta^{-N}\rho\big(\frac{\cdot}{\delta}\big)$. Let $(c_{1,\delta}')$ be the cauchy problem (c_1') where v is is replaced by v_δ. The existence of a smooth solution φ^δ to $(c_{1,\delta}')$ is obtained thanks to, for example to the Friedrichs method combined with heat kernel estimates. By the same thing we did before multiplying the equation

$$\partial_t\varphi^\delta-\nabla\cdot(\varphi^\delta v_\delta)-\nu\Delta\varphi^\delta=0$$

by $\varphi^\delta|\varphi^\delta|^{p-2}$ for $p\geq 2$, and integrating by in space and time, we get

$$\frac{1}{p}\|\varphi^\delta(t)\|^p_{L^p(\mathbb{R}^N)}+\nu(p-1)\int_0^t\left\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{p-2}{2}}\right\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}s=\frac{1}{p}\|\varphi_0^\delta\|^p_{L^p(\mathbb{R}^N)},$$

discarding the gradient term, taking p-th root on both sides and letting p goes to infinity, we obtain

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^3)}\leq\|\varphi_0\|_{L^\infty(\mathbb{R}^3)}.$$

Thus the family $(\varphi^\delta)_\delta$ is bounded in $L^\infty(\mathbb{R}_+\times\mathbb{R}^N)$, and up to an extraction, $(\varphi^\delta)_\delta$ convergence weakly* to some φ in $L^\infty(\mathbb{R}_+\times\mathbb{R}^N)$. Also since $v\in L^{p'}(\mathbb{R}_+,\dot{W}^{q'}(\mathbb{R}^N))$, we may say that $v\in L^1_{\text{loc}}(\mathbb{R}_+\times\mathbb{R}^N)$ and by definition, we may say that $v_\delta\to v$ strongly in $L^1_{\text{loc}}(\mathbb{R}_+\times\mathbb{R}^N)$ as $\delta\to 0$, and as consequence

$$\Delta\varphi^\delta\rightharpoonup^*\Delta\varphi\quad\text{in }L^\infty\Big(\mathbb{R}_+,\dot{W}^{-2,\infty}\Big)$$

and

$$\varphi^\delta v_\delta\rightharpoonup\varphi v\quad\text{in }L^1_{\text{loc}}(\mathbb{R}_+\times\mathbb{R}^N),\text{using (18)}$$

□

Multiplying the equation by a text function $\phi(\phi\in\mathcal{D}([0,T]\times\mathbb{R}^N))$
and taking integration over time ans space, we have

$$\int_0^T\int_{\mathbb{R}^N}\phi(t,x)\partial_t\varphi^\delta(t,x)\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^N}\nabla\cdot\big((\varphi^\delta v_\delta)\big)(t,x)\,\mathrm{d}x\,\mathrm{d}t-\nu\int_0^T\int_{\mathbb{R}^N}\phi(t,x)\nabla\varphi^\delta(t,x)\,\mathrm{d}x\,\mathrm{d}t=0$$

doing integral by part with respect to the time variable and space variable, with, using the high regularity of functions involved, the fact that v_δ is divergence free and the decreasing at at of φ, we may have

$$-\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(t,x)(\partial_t\phi-\nabla\cdot(\phi v_\delta)-\nu\phi)(t,x)\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}^N}\varphi_0^\delta\phi(0,x)\,\mathrm{d}x$$

taking the limit and using the previous reasoning about the weak convergence of φ^δ, we may get

$$-\int_0^T \int_{\mathbb{R}^N} \varphi(t,x)(\partial_t \phi - \nabla \cdot (\phi v) - \nu \phi)(t,x) \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{R}^N} \varphi_0 \phi(0,x) \,\mathrm{d}x.$$

Hence φ is a weak solution of (c_1').

4.2 Existence of a solution to the problem (4.1)

We want to solve the equation

$$\partial_t \varphi - \nabla \cdot (\varphi v) - \nu \Delta \varphi = 0 \tag{4.2}$$

For doing that, we are going to find a solution to the heat equation related to (4.2) with a source term a function of time and space g, that is

$$\partial_t \varphi - \nu \Delta \varphi = g, \tag{4.3}$$

yes of course with the initial condition $\varphi(0,x) = \varphi_0(x)$.

4.2.1 Step1:

Taking the Fourier transform of the Equation with the space variable, of (4.3) we get the equation,

$$\partial_t \mathcal{F}\varphi(t,\xi) + \nu |\xi|^2 \mathcal{F}\varphi(t,\xi) = \mathcal{F}g(t,\xi), \tag{4.4}$$

this equation is a first ordinary differential equation.

We can show that the homogeneous solution, which is the solution of the Equation

$$\partial_t \mathcal{F}\varphi(t,\xi) + \nu |\xi|^2 \mathcal{F}\varphi(t,\xi) = 0,$$

is

$$\mathcal{F}\varphi_H(t,\xi) = \mathcal{F}\varphi_0(\xi) \times \exp\Big(-\nu t |\xi|^2\Big),$$

and as we did in the appendix, we use the constant variation method to deduce a particular solution to the equation (4.4), which is

$$\mathcal{F}\varphi_p(t,\xi) = \exp\Big(-t\nu|\xi|^2\Big) \times \int_0^t \Big(\exp\Big(s\nu|\xi|^2\Big) \times \mathcal{F}g(s,\xi)\Big) \,\mathrm{d}s = \int_0^t \Big(\exp\Big(\nu(t-s)|\xi|^2\Big) \times \mathcal{F}g(s,\xi)\Big) \,\mathrm{d}s.$$

We then have a general solution to the Equation (4.3) as the summation of $\mathcal{F}\varphi_H + \mathcal{F}\varphi_p$,

$$\mathcal{F}\varphi_G(t,\xi) = \mathcal{F}\varphi_0(\xi) \times \exp\Big(-\nu t|\xi|^2\Big) + \int_0^t \Big(\exp\Big(\nu(t-s)|\xi|^2\Big) \times \mathcal{F}g(s,\xi)\Big) \,\mathrm{d}s.$$

4.2.2 Step2: The solution of (4.3) with Fourier

Taking the Fourier inverse of $\mathcal{F}\varphi_H$, we obtain

$$\varphi_H(t,x) = \varphi_0(x) * \mathcal{F}^{-1}h(t,x),$$

where

$$h(t,\xi) := \exp\Big(-\nu t|\xi|^2\Big).$$

One can showed that

$$\mathcal{F}^{-1}h(t,x) = \theta(t,x) := \frac{1}{(4\nu t\pi)^{\frac{N}{2}}} \times \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Now changing all those quantity in the expression of φ_H, we get

$$\varphi_H(t,x) = \varphi_0(x) * \theta(t,x), \tag{4.5}$$

with

$$\theta(t,x) = \frac{1}{(4\nu t\pi)^{\frac{N}{2}}} \times \exp\left(-\frac{|x|^2}{4\nu t}\right).$$

Actually, θ is a solution to the homogeneous heat equation i.e the homogeneous equation associated to Equation (4.3), which is

$$\partial_t\varphi - \nu\Delta\varphi = 0, \tag{4.6}$$

with an initial condition $\varphi_0(x) = \delta_0(x)$, the Dirac mass at 0. One can show that

$$\int_{\mathbb{R}^N} \theta(t,x)\,\mathrm{d}x = 1 \quad \text{for } t > 0.$$

This function is called the elementary solution of Equation (4.6), and any other solution corresponding to an initial condition φ_0, in general, is given by the formula (4.5).

By the same computation, and taking use of Theorem (39), we show that

$$\varphi_p(t) = \int_0^t (g(s) * \theta(t-s))\,\mathrm{d}s.$$

We then deduce a general solution of (4.3), as summation of φ_H and φ_p, which is

$$\varphi(t,x) = (G(t)\varphi_0)(x) + \int_0^t (G(t-s)g(s))(x)\,\mathrm{d}s, \tag{4.7}$$

where G is an operator defined as

$$(G(t)\varphi(s)(x)) := (\varphi(s) * \theta(t))(x).$$

4.2.3 Step3: Now show the existence of a solution to the problem (4.1)

From the previous steps, taking $g = (v\cdot\nabla\varphi)$ we may define the sequence

$$\varphi_{n+1}(t) = G(t)\varphi_0 + \int_0^t G(t-s)(v\cdot\nabla\varphi_n)(s)\,\mathrm{d}s,$$

for which the limit will give us of course a solution to the Problem (4.1). Our job now is to prove that the limit does exist. In the following, we are going to prove that the limit really exits basing on some well known results that we will eventually refereed to.

Let defined

$$\begin{aligned} F : L^{q'}(\mathbb{R}^N) &\longrightarrow L^{q'}(\mathbb{R}^N) \\ (g(t))_{t\geq 0} &\longmapsto (G(t-s)v\cdot\nabla g(s))_{t\geq 0} \end{aligned}$$

Proposition 24. *We can choose:*

- *An open neighborhood $\mathcal{U}$ of $(t_0 = 0, \varphi_0)$ in $\Big(\mathbb{R}_+, L^{q'}(\mathbb{R}^N)\Big)$ on which F is bounded and Lipschitz in φ, we may posed L the lipschitz constant of F and M an upper bound of the norm of F, that is*

$$\sup_{(t,\varphi)\in\mathcal{U}} \|F(t,\varphi)\|_{L^{q'}(\mathbb{R}^N)} \leq M < \infty,$$

  and

$$\|F(t,\varphi_1) - F(t,\varphi_2)\|_{L^{q'}(\mathbb{R}^N)} \leq L\|\varphi_1(t) - \varphi_2(t)\|_{L^{q'}(\mathbb{R}^N)};$$

- *An real $\tau > 0$, small enough so that $\bar{I}_\tau \times \bar{B}_{M\tau} := [0,\tau] \times \bar{B}(\varphi_0; M\tau) \subset \mathcal{U}$, and $L\tau \leq 1$, where $\bar{I}_\tau = [0;\tau]$ and $\bar{B}_{M\tau} := \bar{B}(\varphi_0; M\tau)$.*

Let defined

$$\mathcal{A} = \Big\{\varphi : \bar{I}_\tau \to \bar{B}_{M\tau} \quad \text{continue}\Big\} \subset C^0(\bar{I}_\tau; L^{q'}(\mathbb{R}^N)).$$

We recall that $\mathcal{A}$ is closed subset of a banach space, therefore is a complete metric space together with the distance

$$d_{\mathcal{A}}(\varphi_1,\varphi_2) := \sup_{t\in\bar{I}_\tau} \|\varphi_1(t) - \varphi_2(t)\|_{L^{q'}(\mathbb{R}^N)}.$$

Let defined the function

$$\begin{aligned} \varGamma : \mathcal{A} &\longrightarrow C^0(\bar{I}_\tau, L^{q'}(\mathbb{R}^N)) \\ \varphi &\longmapsto \varGamma\varphi(t) = G(t)\varphi_0 + \int_0^t F(s,\varphi(s))\,\mathrm{d}s. \end{aligned}$$

Lemma 25. *We have $\varGamma : \mathcal{A} \longrightarrow \mathcal{A}$, that is $\varGamma(\mathcal{A}) \subset \mathcal{A}$.*

Proof. We have

$$\begin{aligned} \|\varGamma\varphi(t) - \varphi_0\|_{L^{q'}(\mathbb{R}^N)} &= \|G(t)\varphi_0 - \varphi_0\|_{L^{q'}(\mathbb{R}^N)} + \left\|\int_0^t F(s,\varphi(s))\,\mathrm{d}s\right\|_{L^{q'}(\mathbb{R}^N)} \\ &< \epsilon + \int_0^t \|F(s,\varphi(s))\|_{L^{q'}(\mathbb{R}^N)}\,\mathrm{d}s \\ &< \epsilon + M\int_0^t \mathrm{d}s \\ &< \epsilon + M\tau. \end{aligned}$$

hence $\varGamma\varphi\big(\bar{I}_\tau\big) \subset \bar{B}_{M\tau}$.

□

Lemma 26. *The function $\varGamma$ is $L\tau-$ Lipschitz, using the $L^\infty\Big(\bar{\mathcal{I}}, L^{q'}(\mathbb{R}^N)\Big)$ on x and t, that is*

$$\|\varGamma\varphi_1(t) - \varGamma\varphi_2(t)\|_{L^\infty\left(\bar{\mathcal{I}},L^{q'}(\mathbb{R}^N)\right)} \leq L\tau\|\varphi_1(t) - \varphi_2(t)\|_{L^\infty\left(\bar{\mathcal{I}},L^{q'}(\mathbb{R}^N)\right)},$$

for all $\varphi_1, \varphi_2 \in \mathcal{A}$. In other word, $\varGamma$ is a contracting application since $L\tau < 1$.

Proof.

$$\begin{aligned}\|\Gamma\varphi_1-\Gamma\varphi_2\|_{L^\infty\left(\bar{\mathcal{I}},L^{q'}(\mathbb{R}^N)\right)} &= \sup_{t\in\bar{I}_\tau}\|\Gamma\varphi_1(t)-\Gamma\varphi_2(t)\|_{L^{q'}(\mathbb{R}^N)}\\ &\le \sup_{t\in\bar{I}_\tau}\int_0^t L\|\varphi(s)-\varphi(s)\|_{L^{q'}(\mathbb{R}^N)}\,\mathrm{d}s\\ &\le L\|\varphi-\varphi\|_{L^\infty\left(\bar{\mathcal{I}},L^{q'}(\mathbb{R}^N)\right)}\sup_{t\in\bar{I}_\tau}\int_0^t \mathrm{d}s\\ &\le L\tau\|\varphi_1-\varphi_2\|_{L^\infty\left(\bar{\mathcal{I}},L^{q'}(\mathbb{R}^N)\right)}.\end{aligned}$$

□

Now that we prove that Γ is a contracting application, we can now call on the Banach fixed point theorem 40, that there exists $\varphi^*\in\mathcal{A}$ so that $\Gamma\varphi^*=\varphi^*$. We deduce that the sequence (φ_n) admits an unique limit which is φ^*

Hence we have proved the existence of a solution to equation (4.1). that is the local version, the global existence is obtain by controlling the norm, replacing the norm $\sup_{t\ge0}\|\varphi(t)\|_{L^{q'}(\mathbb{R}^N)}$ by $\sup_{t\ge0}e^{-\lambda t}\|\varphi(t)\|_{L^{q'}(\mathbb{R}^N)}$, for λ large enough and so the can work with all $t\ge0$ instead of the interval $\bar{\mathcal{I}}$. This finish the prove of the existence.

4.3 extension of the uniqueness (Levy 2019)

Here is the extension of the uniqueness result.

Theorem 27. *Let $N\ge0$ be an integer. Let $\nu\ge0$ be a positive parameter. Let $1\le p,q\le\infty$ be real numbers with respectively Hölder conjugates p' and q' i.e $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let $v=v(t,x)$ be a fixed divergence free vector field in $L^{p'}(\mathbb{R}_+,\dot{W}^{q'}(\mathbb{R}^N))$. Let consider a real valued function $a\in L^p([0,T],L^q(\mathbb{R}^N))$ for a given real number $T>0$. Assume that the function a is a distributional solution of the Cauchy problem*

$$(c_1)\begin{cases}\partial_t a-\nabla\cdot(av)-\nu\Delta a=0\\ \\ a(0)=0\end{cases}$$

with the initial condition understood in the sense of $C^0([0,T],\mathcal{D}'(\mathbb{R}^N))$, that is for any function $\varphi\in\mathcal{D}(\mathbb{R}_+\times\mathbb{R}^N)$ and any $T>0$, the following hold

$$\int_{[0,T]\times\mathbb{R}^N}a(t,x)(\partial_t\varphi(t,x)+v(t,x)\cdot\nabla\varphi(t,x)+\nu\Delta\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}^N}a(T,x)\varphi(T,x)\,\mathrm{d}x.$$

Then a is identically zero on $[0,T]\times\mathbb{R}^N$.

Comment: This is somehow a generalization of result in Theorems 12 and 21, but here we still not sure about that kind of solution, since there is no prove provided yet. However this does not make the theorem inapplicable in practice. For example when dealing with the Navier-stokes equation, the vorticity of a leray solution belongs to $L^2(\mathbb{R}_+\times\mathbb{R}^N)$.

Proof. Let $\rho=\rho(x)$ be a radial mollifying kernel and define $\rho_\epsilon(x)=\epsilon^{-N}\rho\big(\frac{\cdot}{\epsilon}\big)$. Convolving the Equation in c_1 on a by ρ_ϵ, and applying the same technique we applied in the proof of Theorem 21 we obtain (multiplying by φ^δ, adding and subtracting and integrating over space and time, we get)

and we then have

$$\begin{aligned}< a_\epsilon(T),\varphi_0>_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)}=&\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)c^{\epsilon,\delta}(s,x)\,\mathrm{d}x\,\mathrm{d}s\\ &-\int_0^T\int_{\mathbb{R}^N}a_\epsilon(s,x)\big(-\partial_t\varphi^\delta(s,x)-\nabla\cdot(v(s,x)\varphi^\delta(s,x))-\nu\Delta\varphi^\delta(s,x)\big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

But by the definition of φ^δ, we know that

$$\partial_t \varphi^\delta + \nabla \cdot (\varphi^\delta v_\delta) + \nu \Delta \varphi^\delta = 0,$$

adding $-\nabla \cdot (\varphi^\delta v)$ both side, we get after doing some arranging

$$\partial_t \varphi^\delta + \nabla \cdot (\varphi^\delta v) + \nu \Delta \varphi^\delta = -\nabla \cdot (\varphi^\delta (v^\delta - v)).$$

We have then

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} a(s,x) c^{\epsilon,\delta}(s,x) \, \mathrm{d}x \, \mathrm{d}s$$
$$- \int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x) \nabla \cdot (\varphi^\delta(s,x)(v_\delta - v)(s,x)) \, \mathrm{d}x \, \mathrm{d}s.$$

We know that $C^{\epsilon,\delta} \to C^{\epsilon,0} = C^\epsilon$ strongly in $L^1(\mathbb{R}_+ \times \mathbb{R}^N)$ to 0, and since C^∞ functions are dense in L^p spaces, and φ^∞, we deduce that

$$\int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x) c^{\epsilon,\delta}(s,x) \, \mathrm{d}x \, \mathrm{d}s \longrightarrow \int_0^T \int_{\mathbb{R}^N} a(s,x) c^\epsilon(s,x) \, \mathrm{d}x \, \mathrm{d}s,$$

as $\delta \to 0$. Also $v_\delta - v \in L^{p'}(\mathbb{R}_+, L^{q'}(\mathbb{R}^N))$ and $\|v_\delta - v\|_{L^{p'}(\mathbb{R}_+, L^{q'}(\mathbb{R}^N))} \longrightarrow 0$.

Since $v_\delta - v \to 0$ strongly in $L^{p'}(\mathbb{R}_+, L^{q'}(\mathbb{R}^N))$ and $\varphi^\delta \rightharpoonup \varphi$ in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^N)$(as bounded, it has a subsequence that converge ewakly, as abuse of notation, let call that φ^δ) then

$$(v_\delta - v) \cdot \varphi^\delta \rightharpoonup 0 \quad \text{in } L^{p'}(\mathbb{R}_+, L^{q'}(\mathbb{R}^N)).$$

And since $\nabla a_\epsilon \in L^p(\mathbb{R}_+, L^q(\mathbb{R}^N))$, we apply the definition of weak convergence to deduce that

$$\int_0^T \varphi^\delta (v_\delta - v) \cdot \nabla a_\epsilon(s,x) \, \mathrm{d}x \, \mathrm{d}s \to 0,$$

as $\delta \to 0$.

From the previous reasoning, we conclude that

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} a(s,x) c^\epsilon(s,x) \, \mathrm{d}x \, \mathrm{d}s.$$

Doing the same reasoning as above in the previous chapter in Theorem 21 and lemma 20, when $\epsilon \to 0$, the right hand side goes to 0. we then have

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = 0,$$

for all test function φ_0, this implies that $a(T)$ is the zero distribution. hence $a = 0$.

□

Chapter 5

Other results

5.1 Renormalization and stability of Di Perna and Lions

In this part, we will require less intelligibility conditions on the vector field b, and we will always consider trough out the rest of this first part that:

$$c,\ \mathrm{div} b \in L^1(0,T;L^\infty(\mathbb{R}^N)) \quad \text{and} \quad b \in L^1(0,T;W^{1,1}_{\text{loc}}(\mathbb{R}^N)) \tag{5.1}$$

and the Condition (2.12).

definition 28. We will call the space L^0 the set of all measurable functions u with value in $\bar{\mathbb{R}}$ such that

$$\text{meas}\{|u| > \lambda\} < \infty, \quad \text{for all } \lambda > 0.$$

We will also use some specific functions called an admissible functions, that is $\beta \in C^1(\mathbb{R})$ such that β and $|t|\beta'(t)$ are bounded and β vanishes near 0. We can observe that for such functions β, we have $\beta(u) \in L^1 \cap L^\infty(\mathbb{R}^N)$, and $\beta(u)$, $u\beta'(u) \in L^1(\mathbb{R}^N)$.

We say that a sequence u_n converges to u in L^0 if the image sequence $\beta(u_n)$ converges to $\beta(u)$ in $L^1(\mathbb{R}^N)$, and that u_n is bounded in $L^0(\mathbb{R}^N)$ if $\beta(u_n)$ is bounded in $L^1(\mathbb{R}^N)$ for all β.

definition 29. We say that $u \in L^\infty(0,T;L^0)$ is a re normalized solution of (2.1) with an initial condition $u_0 \in L^0$ if all admissible functions $\beta(u)$ solves the Equation (2.14) :

$$\frac{\partial}{\partial t}\beta(u) - b \cdot \nabla_x \beta(u) = cu\beta'(u) \quad \text{in } (0,T) \times \mathbb{R}^N$$

in weak sense, with the initial condition $\beta(u_0)$.

Let us consider $a \in L(0,T;L^\infty(\mathbb{R}^N))$. Let us pose the Problem

$$\frac{\partial f}{\partial t} - b \cdot \nabla_x f = -a \quad \text{on} \quad [0,T] \times \mathbb{R}^N,\ f|_{t=0} = 0. \tag{5.2}$$

That problem does have unique solution from the previous results. We will be using it to prove the following lemma.

Lemma 30. *Let us consider the Equation (2.1) under the conditions (2.12) and (5.1) with an initial condition $u^0 \in L^0$. Then $u \in L(0,T;L^0(\mathbb{R}^N))$ is a renormalized solution of (2.1) corresponding to the initial condition u^0 if and only if $e^{-f}u$ is a renormalized solution with (b,c) replaced by $(b, a+c)$ for the same initial condition u^0.*

Proof. We assume that $u \in L(0,T;L^0(\mathbb{R}^N))$ is a renormalization solution of (2.1) corresponding to u^0 and β and γ be two admissible function. We will show that $\gamma(e^{-f}\beta(u)) = h$ solves

$$\frac{\partial h}{\partial t} - b \cdot \nabla_x h = e^{-f}\gamma'(e^{-f}\beta(u))\Big(cu\beta'(u) + a\beta(u)\Big), \tag{5.3}$$

on $[0,T]\times\mathbb{R}^N$ with initial condition $h|t=0=\gamma\circ\beta(u_0)$.

Using the regularization result in theorem 10 we have

$$\frac{\partial f_\epsilon}{\partial t} - b\cdot\nabla_x f_\epsilon = -a+\psi_\epsilon$$

and

$$\frac{\partial v_\epsilon}{\partial t} - b\cdot\nabla_x v_\epsilon = cu\beta'(u)+r_\epsilon$$

where $v=\beta(u)$, $v_0=\beta(u^0)$, $f_\epsilon = f*\rho_\epsilon$, $v_\epsilon^0 = v*\rho_\epsilon$ and $\psi_\epsilon,\ r_\epsilon\to 0$ in $L(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$.

Now let $h_\epsilon=\gamma(e^{-f_\epsilon}v_\epsilon)$, using Cain Rule we have

$$\frac{\partial h_\epsilon}{\partial t} = \gamma'(e^{-f_\epsilon})\Big(\frac{\partial v_\epsilon}{\partial t}e^{-f_\epsilon} - \frac{\partial f_\epsilon}{\partial t}v_\epsilon e^{-f_\epsilon}\Big)$$

using the two equations above and after doing some ordering and simplification, we get

$$\frac{\partial h_\epsilon}{\partial t} = e^{-f_\epsilon}\gamma'(e^{-f_\epsilon})\Big(b\cdot\nabla_x v_\epsilon - v_\epsilon b\cdot\nabla_x f_\epsilon\Big) + e^{-f_\epsilon}\gamma'(e^{-f_\epsilon})\Big(cu\beta'(u)+av_\epsilon+r_\epsilon-\psi_\epsilon v_\epsilon\Big),$$

but we remark that

$$b\cdot\nabla_x h_\epsilon = e^{-f_\epsilon}\gamma'(e^{-f_\epsilon})\Big(b\cdot\nabla_x v_\epsilon - v_\epsilon b\cdot\nabla_x f_\epsilon\Big),$$

hence we deduce that

$$\frac{\partial h_\epsilon}{\partial t} - b\cdot\nabla_x h_\epsilon = e^{-f_\epsilon}\gamma'(e^{-f_\epsilon})\Big(cu\beta'(u)+av_\epsilon+r_\epsilon-\psi_\epsilon v_\epsilon\Big).$$

□

Theorem 31. *There exists a unique renormalized solution $u\in C(0,T;L^0(\mathbb{R}^N))$ to Equation* (2.1) *under the conditions* (2.12) *(at page 12) and* (5.1) *(at page 33) with initial condition $u_0\in L^0$.*

Proof. From the above renormalization result, we have for all admissible function

$$\frac{\partial}{\partial t}\beta(u) - b\cdot\nabla_x\beta(u) = cu\beta'(u),$$

then considering $c=0$, we have

$$\frac{\partial}{\partial t}\beta(u) - b\cdot\nabla_x\beta(u) = 0$$

equation (2.1) with $c=0$. Now since $\beta(u^0)\in L^1\cap L^\infty(\mathbb{R}^N)$, from the result in Proposition(5) and Corollary(12.1), the above equation admits a unique solution corresponding to the initial condition $\beta(u_0)$. Hence there must exist a renormalized solution $u\in C(0,T;L^0(\mathbb{R}^N))$ corresponding to u^0. Now let us suppose u and v two renormalized solutions, and let fixed β. Then $\beta(u)$ and $\beta(v)$ solves the Equation (2.14), but since the solution is unique, we have no more choice than $\beta(u)=\beta(v)$. Now when we choice β be a map to map function out side the neighborhood of 0, on which be must vanish t, we see u must be equal to v. Hence the uniqueness. We then deduce the general case when for c not necessary zero since lemma (30) . Hence the result.

□

Theorem 32. *Let us consider equation* (2.1) *under the conditions* (2.12) *(at page 12) and* (5.1) *(at page 33). If $u\in L^\infty(0,T;L^p(\mathbb{R}^N))$ is renormalized solution of equation* (2.1)*, then u is a solution of* (2.1)*. If u is a weak solution of* (2.1) *with $b\in L^1(0,T;W^{1,q}_{\text{loc}}(\mathbb{R}^N))$, then u is a renormalized solution, a case which has been already proven in Corollary* (12.1)*.*

Proof. The second statement(if u is a renormalized solution of equation (2.1) with $b \in L^1(0,T;W^{1,q}_{\text{loc}}(\mathbb{R}^N))$, then u is a solution of (2.1)) is already showed in the proof of theorem 12. Now, let us proved the first statement. Let us assume that $u \in L^\infty(0,T;L^p(\mathbb{R}^N))$ is a renormalized solution of (2.1). Then for any admissible function β, equation (2.14) holds. Now we just need do consider a sequence (β_n) of admissible functions that converges to t $(\beta_n(t) \to t)$ and β' converges to some function g. Then $\beta'_n(t) = g(t) = 1$. Hence the corresponding equation (2.14) to those β_n is

$$\frac{\partial}{\partial t}\beta_n(u) - b \cdot \nabla_x \beta_n(u) = cu,$$

with initial condition $\beta_n(u^0)$. Letting n goes to infinity(of course in the weak sense as we dit in the prove of existence 6), we have

$$\frac{\partial}{\partial t}u - b \cdot \nabla_x u = cu,$$

□

Theorem 33 (Stability). *Let $u^0 \in L^0$, and let us assume that u^0_n converges to u^0 in L^0. Let the function u_n be a renormalized solution to the equation* (2.1) *with (b,c) replaced by (b_n, c_n), corresponding to the initial condition u^0_n, where $b_n,\ c_n \in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$ such that:*

1. $\text{div } b_n \in L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^N))$ *and*
2. $b_n,\ c_n,\ \text{div } b_n$ *converges as n goes to ∞ respectively to $b,\ 0,\ \text{div } b$, where b verifies the Conditions* (2.12) *(at page 12) and* (5.1) *(at page 33).*

   Then, u_n converges as n goes to ∞ in $C(0,T;L^0(\mathbb{R}^N))$. Furthermore, if we assume that the convergence of u^0_n to u_0 holds in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $p \in [1,\infty)$, and b_n, c_n $\text{div } b_n$ bounded in $L^1(0,T;L^\infty_{\text{loc}}(\mathbb{R}^N))$ and u^n is bounded in $L^\infty(0,T;L^p_{\text{loc}}(\mathbb{R}^N))$. Then the convergence of u_n to u holds in $C(0,T;L^p_{\text{loc}}(\mathbb{R}^N))$.

Remark: This above stability result is actually extended to global convergence, for which the proof is analogous to the prove of the local case, but here we will not do the proof and we then refer to the document of Di Perna and Lions.

5.2 Serrin's Theorem about Leray solution and its proof using the uniqueness result of Levy

Theorem 34. *Let $N \geq 3$ be an integer. Let ν be a positive real number. Let $2 \leq p < \infty$ and $N \leq q \leq \infty$ be real numbers satisfying $\frac{2}{p} + \frac{N}{q} = 1$. Let v be a fixed divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N))$. Let w be a fixed vector field in $L^2\Big(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N)\Big) \cap L^p(\mathbb{R}_+, L^q(\mathbb{R}^N))$. There exists a solution φ to the following Cauchy problem:*

$$(c'_{NS}) \begin{cases} \partial_t \varphi - \nabla \cdot (\varphi \otimes v) - \nu \Delta \varphi = -{}^t\nabla\varphi \cdot w \\ \varphi(0) = \varphi_0 \in \mathcal{D}(\mathbb{R}^N) \end{cases}$$

satisfying in addition, for almost every $t > 0$,

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^N)} exp\Big[\frac{C^p}{p\nu^{p-1}} \int_0^t \|w(s)\|_{L^q(\mathbb{R}^N)}\, \mathrm{d}s,\Big]$$

where C denotes a constant depending only on the dimension N.

We also remind that when the viscosity coefficient is small, the estimate degenerate. This also apply to the Navier-stokes equations with frozen coefficients.

On the right-hand side of the main equation, the quantity $-{}^t\nabla\varphi \cdot a$ is the shorthand for $-\nabla(\varphi \cdot a) + {}^t\nabla a \cdot \varphi$, where the gradient of a vector field is $N \times N$ dimensional matrix defined as:

$$(\nabla\varphi)_{ij} = \partial_j \varphi_i$$

for all $0 \leq i \leq N$. It is enough clear that we have

$$\left({}^t\nabla\varphi \cdot a\right)_i = \sum_{j=1}^{N} \partial_i \varphi_j a_j$$

and similarly

$$\left({}^t\nabla a \cdot \varphi\right)_i = \sum_{j=1}^{N} \partial_i a_j \varphi_j.$$

We defined a tensor product between two vector field let say a and v, an by:

$$(a \otimes v)_{ij} = a_i v_j$$

Let us assume an N dimensional squared matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \cdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}$$

We defined the divergence of A the operation $\nabla\cdot$ by:

$$(\nabla \cdot A)_{ij} = \sum_{j=1}^{N} \partial_j A_{ij}.$$

Using this equation, we can that whenever we have two vector field a and v, we can elaborate the formula

$$\nabla \cdot (a \otimes v) = a\text{div } v + v \cdot \nabla a, \tag{5.4}$$

which is clear that if v is divergence free, $\nabla \cdot (a \otimes v) = v \cdot \nabla a$,

$$v \cdot \nabla a := {}^t(v \cdot \nabla a_1, \cdots, v \cdot \nabla a_N),$$

the dot inside the parenthesize is the usual dot product between two vectors fields, i.e

$$(v \cdot \nabla a)_i = \sum_{j=1}^{N} v_j \partial_j a_i$$

Now let us prove the theorem 34.

Proof. As usual we consider the radial mollifying Kernel $\rho = \rho(x)$ and Let $\rho_\delta = \delta^{-N}\rho\left(\frac{x}{\delta}\right)$. Let $w_\delta = \rho * w$ and $v_\delta = \rho_\delta * v$. We now define $\left(C'_{NS,\delta}\right)$ to be the Cauchy problem (C'_{NS}) with w and v replaced respectively by w_δ and v_δ. The existence of smooth solution to the problem $\left(C'_{NS,\delta}\right)$. Now taking the scalar product of the equation in $\left(C'_{NS,\delta}\right)$ by φ^δ, we have

$$\varphi^\delta \cdot \partial_t \varphi^\delta - \varphi^\delta \cdot \nabla \cdot (\varphi^\delta \otimes v_\delta) - \nu\varphi^\delta \cdot \nabla\varphi^\delta = -\varphi^\delta \cdot \left({}^t\nabla\varphi^\delta \cdot w_\delta\right)$$

using formula (5.4) and the free divergenceness of the field v, and the fact that

$$\nu\varphi^\delta\cdot\Delta\varphi^\delta=\nu\varphi^\delta\cdot\nabla\cdot(\nabla\varphi^\delta)=\frac{\nu}{2}\nabla\cdot(\nabla|\varphi^\delta|^2)-\nu\nabla\varphi^\delta:\nabla\varphi^\delta,$$

which is obtain explicitly by taking the summation $\sum_{i,j}$ of

$$\varphi_i\partial_j\partial_j\varphi_i=\frac{1}{2}(\varphi_i\varphi_i)-(\partial_j\varphi_i)(\partial_j\varphi_i).$$

the above formula gives

$$\varphi^\delta\cdot\partial_t\varphi^\delta-\big(v_\delta\cdot\nabla\varphi^\delta\big)\cdot\varphi^\delta-\frac{\nu}{2}\nabla\cdot(\nabla|\varphi^\delta|^2)+\nu\nabla\varphi^\delta:\nabla\varphi^\delta=-\varphi^\delta\cdot\big({}^t\nabla\varphi^\delta\cdot w_\delta\big),$$

this is equivalent to

$$\frac{1}{2}\partial_t|\varphi^\delta|^2-\frac{1}{2}v_\delta\cdot\nabla|\varphi^\delta|^2-\frac{\nu}{2}\Delta|\varphi^\delta|^2+\nu\big|\nabla\varphi^\delta\big|^2=-\varphi^\delta\cdot\big({}^t\nabla\varphi^\delta\cdot w_\delta\big).$$

For notation convenience, we will denote $\psi^\delta=|\varphi^\delta|^2=\sum_j\big(\varphi^\delta_j\big)^2$.

multiplying this equation by $|\varphi^\delta|^{r-2}$ for some $r\geq 2$, and integrating in space and time taking use of the fact that $\nabla\cdot v^\delta=0$ and the decreasing at ∞ of φ^δ, we get

$$\frac{1}{r}\big\|\varphi^\delta(t)\big\|^r_{L^r(\mathbb{R}^N)}+\frac{r-2}{4}\nu\int_0^t\Big\|(\psi^\delta)^{\frac{r-2}{4}}\nabla\psi^\delta\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s+\nu\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s=\frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^N)}$$
$$-\int_0^t\int_{\mathbb{R}^N}|\varphi^\delta|^{r-2}(s,x)\varphi^\delta(s,x)\cdot({}^t\nabla\varphi^\delta\cdot w_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s,\quad(5.5)$$

we get this by doing the same tactics we have been using.

Now let denote

$$I(t):=\int_0^t\int_{\mathbb{R}^N}|\varphi^\delta|^{r-2}(s,x)\varphi^\delta(s,x)\cdot({}^t\nabla\varphi^\delta(s,x)\cdot w_\delta(s,x))\,\mathrm{d}x\,\mathrm{d}s,$$

We can rewrite the estimate I by

$$I(t)=\int_0^t\int_{\mathbb{R}^N}(\psi^\delta)^{\frac{r}{4}}(s,x)\big|\varphi^\delta\big|^{\frac{r-4}{2}}(s,x)\varphi^\delta\cdot({}^t\nabla\varphi^\delta\cdot w_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s,$$

using Hölder inequality, we get

$$|I(t)|\leq\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|_{L^2(\mathbb{R}^N)}\big\|(\psi^\delta)^{\frac{r}{4}}(s)\big\|_{L^{\tilde{q}}(\mathbb{R}^N)}\|w_\delta(s)\|_{L^q(\mathbb{R}^N)}\,\mathrm{d}s$$

where $\tilde{q}$ is such that $\frac{1}{2}+\frac{1}{q}+\frac{1}{\tilde{q}}=1$, where we deduce from the condition on p and q that $1-\frac{2}{p}=\frac{N}{q}=N(\frac{1}{2}-\frac{1}{\tilde{q}})$, then the sobolev space $\dot{H}^{1-\frac{2}{p}}$ embeds in $L^{\tilde{q}}(\mathbb{R}^N)$. Hence there exists a constant C depending only on p and such that

$$\big\|(\psi^\delta)^{\frac{r}{4}}(s)\big\|_{L^{\tilde{q}}(\mathbb{R}^N)}\leq C\big\|(\psi^\delta)^{\frac{r}{4}}(s)\big\|_{\dot{H}^{1-\frac{2}{p}}(\mathbb{R}^N)},$$

since $N\geq 3$ and $0\leq 1-\frac{2}{p}<1$, the sobolev exponents lies in a compact interval at a positive distance from the critical value $\frac{d}{2}$, near which the constant would blow up. Hence, for a fixed dimension N, we may choose C uniformly in p. Interpolate $\dot{H}^{1-\frac{2}{p}}$ with L^2 and $\dot{H}^1$, we have

$$\begin{aligned}\big\|(\varphi)^\delta(s)\big\|_{\dot{H}^{1-\frac{2}{p}}(\mathbb{R}^N)}\leq&\big\|(\psi^\delta)^{\frac{r}{4}}(s)\big\|^{\frac{2}{p}}_{L^2(\mathbb{R}^N)}\big\|\nabla(\psi^\delta)^{\frac{r}{4}}(s)\big\|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^N)}\\=&\big\|\varphi^\delta(s)\big\|^{\frac{r}{p}}_{L^2(\mathbb{R}^N)}\big\|\nabla(\psi^\delta)^{\frac{r}{4}}(s)\big\|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^N)}\end{aligned}$$

As $(\psi^\delta)^{\frac{r}{4}} = \frac{r}{4}(\psi^\delta)^{\frac{r-4}{4}}\nabla\psi^\delta$, we may now bound $|I(t)|$ from above by

$$C\int_0^t \Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|_{L^2(\mathbb{R}^N)}\left(\frac{r}{4}\Big\|(\psi^\delta)^{\frac{r-4}{4}}(s)\nabla\psi^\delta(s)\Big\|_{L^2(\mathbb{R}^N)}\right)^{1-\frac{2}{p}}\|\varphi^\delta(s)\|^{\frac{r}{p}}_{L^2(\mathbb{R}^N)}\|w_\delta(s)\|_{L^q(\mathbb{R}^N)}\,\mathrm{d}s,$$

using young's inequality for real number with the coefficient $\frac{1}{2}+\frac{1}{p}+\frac{1}{\tilde{p}}=1$, we get

$$|I(t)| \leq \frac{\nu}{2}\int_0^t \Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s + \frac{r\nu}{4\tilde{p}}\int_0^t\Big\|(\psi^\delta)^{\frac{r-4}{4}}(s)\nabla\psi^\delta(s)\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s$$
$$+\frac{C^p}{p\nu^{p-1}}\int_0^t\|\varphi^\delta(s)\|^r_{L^2(\mathbb{R}^N)}\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s.$$

We then have using the bound of $|I(t)|$ and the equation (4.1). we deduce that

$$\frac{1}{r}\|\varphi^\delta(t)\|^r_{L^r(\mathbb{R}^N)} + \frac{r-2}{4}\nu\int_0^t\Big\|(\psi^\delta)^{\frac{r-4}{4}}\nabla\psi^\delta\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s + \nu\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s$$
$$\leq \frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^N)} + \frac{\nu}{2}\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s + \frac{r\nu}{4\tilde{p}}\int_0^t\Big\|(\psi^\delta)^{\frac{r-4}{4}}(s)\nabla\psi^\delta(s)\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s$$
$$+\frac{C^p}{p\nu^{p-1}}\int_0^t\|\varphi^\delta(s)\|^r_{L^2(\mathbb{R}^N)}\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s$$

Arranging and simplifying this, we get

$$\frac{1}{r}\|\varphi^\delta(t)\|^r_{L^r(\mathbb{R}^N)} + \left(\frac{r-2}{4}\nu - \frac{r\nu}{4\tilde{p}}\right)\int_0^t\Big\|(\psi^\delta)^{\frac{r-4}{4}}\nabla\psi^\delta\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s + \frac{\nu}{2}\int_0^t\Big\|\nabla\varphi^\delta(s)|\varphi^\delta(s)|^{\frac{r-2}{2}}\Big\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}s \leq$$
$$\frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^N)} + \frac{C^p}{p\nu^{p-1}}\int_0^t\|\varphi^\delta(s)\|^r_{L^2(\mathbb{R}^N)}\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s.$$

For r sufficiently large,

$$\left(\frac{r-2}{4}\nu - \frac{r\nu}{4\tilde{p}}\right) > o,$$

we can the absorb the second two terms in the previous inequality to get

$$\frac{1}{r}\|\varphi^\delta(t)\|^r_{L^r(\mathbb{R}^N)} \leq \frac{1}{r}\|\varphi_0\|^r_{L^r(\mathbb{R}^N)} + \frac{C^p}{p\nu^{p-1}}\int_0^t\|\varphi^\delta(s)\|^r_{L^2(\mathbb{R}^N)}\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s.$$

Applying Grönwall to this, we get

$$\|\varphi^\delta(t)\|_{L^r(\mathbb{R}^N)} \leq \|\varphi_0\|_{L^r(\mathbb{R}^N)}\exp\bigg(\frac{C^p}{p\nu^{p-1}}\int_0^t\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s\bigg).$$

Using the fact that $\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)} \leq \|w(s)\|^p_{L^q(\mathbb{R}^N)}$, and letting r goes to infinity, we obtain

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi_0\|^\infty_{L^r(\mathbb{R}^N)}\exp\bigg(\frac{C^p}{p\nu^{p-1}}\int_0^t\|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)}\,\mathrm{d}s\bigg).$$

Hence, φ^δ is uniformly bounded in $L^\infty(\mathbb{R}_+\times\mathbb{R}^N)$, up to an extraction, there exists φ in $L^\infty(\mathbb{R}_+\times\mathbb{R}^N)$ such that

$$\varphi^\delta \rightharpoonup^* \varphi \quad \text{in } L^\infty(\mathbb{R}_+\times\mathbb{R}^N),$$

as $\delta \to 0$.

By Fatou's Lemma,

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi_0\|^\infty_{L^r(\mathbb{R}^N)} \exp\left(\frac{C^p}{p\nu^{p-1}} \int_0^t \|w_\delta(s)\|^p_{L^q(\mathbb{R}^N)} \, \mathrm{d}s\right)$$

follows.

By construction,

$$v_\delta, \ w_\delta \to v, \ w \quad \text{strongly in } L^2(\mathbb{R}_+, \dot{H}(\mathbb{R}^N)),$$

as $\delta \to 0$. Hence taking the limit as $\delta \to 0$, of φ^δ, in the equation in (C'_{NS}), we see that indeed φ satisfies the adjoin equation.

$\square$

Theorem 35. *Let $N \geq 3$ be an integer. Let ν be a positive real number. Let $2 \leq p < \infty$ and $N \leq q \leq \infty$ be real numbers satisfying $\frac{2}{p} + \frac{N}{q} = 1$. Let v be a fixed divergence free vector field in $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N))$. Let w be a fixed vector field in $L^2\Big(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N)\Big) \cap L^p(\mathbb{R}_+, L^q(\mathbb{R}^N))$. Let a be $L^2(\mathbb{R}_+ \times \mathbb{R}^N)$. Assume that a is a distributional solution of the Cauchy problem*

$$(c_{NS}) \begin{cases} \partial_t a + \nabla \cdot (a \otimes v) - \nu \Delta a = \nabla \cdot (w \otimes a) \\ \\ a(0) = 0 \end{cases}$$

with the initial condition understood in the sense of $C^0([0,T], \mathcal{D}'(\mathbb{R}^N))$. Then a is identically null on $\mathbb{R}_+ \times \mathbb{R}^N$. This is also True in the limit case $(p,q) = (\infty, d)$, provided that w satisfies the smallness condition

$$\|w\|_{L^\infty(\mathbb{R}_+, L^N(\mathbb{R}^N))} \leq \frac{2\nu}{C},$$

where C is the soblev constant corresponding to the embedding $\dot{H}^1 \hookrightarrow L^{\frac{2N}{N-2}}$ according to [7] (We do not verify that here). Recall that here a is also a vector field.

Proof. Let $\rho = \rho(x)$ be a radial mollifying kernel and $\rho_\epsilon(x) = \epsilon^{-N}\rho\big(\frac{\cdot}{\epsilon}\big)$. Convolving the equation (c_{NS}) by ρ_ϵ, we have

$$\rho_\epsilon * \partial_t a + \rho_\epsilon * \nabla \cdot (a \otimes v) - \nu \rho_\epsilon * \Delta a = \rho_\epsilon * \nabla \cdot (w \otimes a),$$

adding $\nabla \cdot (w \otimes a_\epsilon)$ and subtracting it back in the second member, also adding $\nabla \cdot (a_\epsilon \otimes v)$ and subtracting it back in the first member, where $a_\epsilon = \rho_\epsilon * a$, we have the equation

$$(c_\epsilon): \ \partial_t a_\epsilon + (\rho_\epsilon * \nabla \cdot (a \otimes v) - \nabla \cdot (a_\epsilon \otimes v)) + \nabla \cdot (a_\epsilon \otimes v) - \nu \Delta a_\epsilon = \rho_\epsilon * \nabla \cdot (w \otimes a) - \nabla \cdot (w \otimes a_\epsilon) \\ + \nabla \cdot (w \times a_\epsilon)$$

arranging this, we get

$$\partial_t a_\epsilon - C^\epsilon + \nabla \cdot (a_\epsilon \otimes v) - \nu \Delta a_\epsilon = Q^\epsilon + \nabla \cdot (w \otimes a_\epsilon),$$

where $Q^\epsilon := \rho_\epsilon * \nabla \cdot (a * w) - \nabla \cdot (w \otimes a_\epsilon)$, and C^ϵ the commutator we defined above but here we deal with tensor product and not the the ordinary product, and we recall that a_ϵ is also a vector field since a is, and we consider $a_\epsilon = (a_{\epsilon 1}, \cdots, a_{\epsilon N})$.

This is equivalent to

$$(c_\epsilon): \quad \partial_t a_\epsilon + \nabla \cdot (a_\epsilon \otimes v) - \nu \Delta a_\epsilon = \nabla \cdot (w \otimes a_\epsilon) + C^\epsilon + Q^\epsilon.$$

We recall that taking into consideration the fact that $a \in L^2(\mathbb{R}_+ \times \mathbb{R}^N)$ and $w \in L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^N))$, as the same way we proved in the lemma 22

$$\|Q^\epsilon\|_{L^1(\mathbb{R}_+\times\mathbb{R}^N)} \to 0, \quad \text{as } \epsilon \to 0.$$

From the equation (c_ϵ), we can see that without any smoothing in time, a_ϵ and $\partial_t a_\epsilon$ lie respectively in $L^\infty(\mathbb{R}_+, C^\infty(\mathbb{R}^N))$ and $L^1(\mathbb{R}_+, C^\infty(\mathbb{R}^N))$, a fact which make the upcoming computation rigorous (definition od derivatives of distribution). Now as we did before in theorem 27, Let φ^δ be a solution of the cauchy problem $(-C'_{NS,\delta})$. Multiplying the equation (c_ϵ) by φ^δ and integrating in space and times we get

$$\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\partial_t a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\nabla\cdot(a_\epsilon\otimes v)(s,x)\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}^N}\nu\Delta a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s=$$
$$\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\nabla\cdot(w\otimes a_\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)(C^\epsilon+Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s. \quad (5.6)$$

But when we integrate by part the first term in the previous equation, we have

$$\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\partial_t a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s=<a_\epsilon(T),\varphi_0>_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)}-\int_0^t\int_{\mathbb{R}^N}a_\epsilon(s,x)\cdot\partial_t\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

,

replacing this in equation (5.6), and arranging, we obtain

$$<a_\epsilon(T),\varphi_0>_{\mathcal{D}(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)}=\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)(C^\epsilon+Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}a_\epsilon(s,x)\cdot\partial_t\varphi^\delta(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+\int_0^t\int_{\mathbb{R}^N}\nu\Delta a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\nabla\cdot(a_\epsilon\otimes v)(s,x)\,\mathrm{d}x\,\mathrm{d}s. \quad (5.7)$$

But since the vector field v is divergence free, we have $\nabla\cdot(a_\epsilon\otimes v)=v\cdot\nabla a_\epsilon=(v\cdot\nabla a_{\epsilon 1},\cdots,v\cdot a_{\epsilon N})$, using the same argument of the vanisheness of integral of divergence of a multiplication of divergence free vector field and a smooth compactly supported fuction, we can easily showed that

$$\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot\nabla\cdot(a_\epsilon\otimes v)(s,x)\,\mathrm{d}x\,\mathrm{d}s=-\int_0^T\int_{\mathbb{R}^N}a_\epsilon(s,x)\cdot\nabla\cdot(\varphi^\delta\otimes v)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Also we have $\nabla\cdot(w\otimes a_\epsilon)=w\text{div }a_\epsilon+a_\epsilon\cdot\nabla w$, which gives by multiplying by φ^δ, we obtain

$$\begin{aligned}\varphi^\delta\cdot(\nabla\cdot(w\otimes a_\epsilon))&=\varphi^\delta\cdot w\text{div }a_\epsilon+\varphi^\delta\cdot(a_{\epsilon N}\cdot\nabla w)\\&=\sum_{j=1}^N\varphi_j^\delta w_j\text{div }a_\epsilon+\sum_{j=1}^N\varphi_j^\delta a_\epsilon.\nabla w_j,\end{aligned}$$

taking the integral of this and applying the same tactics of integration of divergence we mention above, we obtain

$$\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot(\nabla(w\otimes a_\epsilon))(s,x)\,\mathrm{d}x\,\mathrm{d}s=\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot w(s,x)\text{div}a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x$$
$$-\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot w(s,x)\text{div}a_\epsilon(s,x)\,\mathrm{d}s\,\mathrm{d}x-\int_0^T\int_{\mathbb{R}^N}a_\epsilon(s,x)\cdot{}^t\,\nabla\varphi^\delta(s,x)\cdot w(s,x)\,\mathrm{d}x,$$

simplifying this we got

$$\int_0^T\int_{\mathbb{R}^N}\varphi^\delta(s,x)\cdot(\nabla(w\otimes a_\epsilon))(s,x)\,\mathrm{d}x\,\mathrm{d}s=-\int_0^T\int_{\mathbb{R}^N}a_\epsilon(s,x)\cdot({}^t\nabla\varphi^\delta\cdot w)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Hence replacing all the previous equality in equation (5.7), we obtain by putting together some terms

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x)(C^\epsilon + Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x)\big(\partial_t \varphi^\delta + \nu\Delta\varphi^\delta + \nabla\cdot(\varphi^\delta \otimes v) -^t \nabla\varphi^\delta \cdot w\big)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

From the adjoin equation on φ^δ in the Cauchy problem $(-c'_{NS,\delta})$, we have

$$-\partial_t\varphi^\delta - \nabla\cdot(\varphi^\delta \otimes v_\delta) - \nu\Delta\varphi^\delta = -^t\nabla\varphi^\delta\cdot w_\delta,$$

taking in subject $\partial_t\varphi^\delta - \nu\Delta\varphi^\delta$, in the previous equation, and adding both sides $-\nabla\cdot(\varphi^\delta\otimes v) +^t \nabla\varphi^\delta\cdot w$, we obtain

$$\begin{aligned}-\partial_t\varphi^\delta - \nabla\cdot(\varphi^\delta\otimes v) - \nu\Delta\varphi^\delta +^t \nabla\varphi^\delta\cdot w &= \nabla\cdot(\varphi^\delta\otimes v_\delta) - \nabla\cdot(\varphi^\delta\otimes v) -^t \nabla\varphi^\delta\cdot w_\delta \\ &= \nabla\cdot\big(\varphi^\delta\otimes(v_\delta - v)\big) +^t \nabla\varphi^\delta\cdot(w - w_\delta).\end{aligned}$$

We then have

$$\int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x)\big(\partial_t\varphi^\delta + \nu\Delta\varphi^\delta + \nabla\cdot(\varphi^\delta\otimes v) -^t \nabla\varphi^\delta\cdot w\big)(s,x)\,\mathrm{d}x\,\mathrm{d}s =$$
$$-\int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x)\cdot\big(\nabla\cdot\big(\varphi^\delta\otimes(v_\delta - v)\big) +^t \nabla\varphi^\delta\cdot(w - w_\delta)\big)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

By integration by part, we have

$$-\int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x)\cdot\nabla\cdot\big(\varphi^\delta\otimes(v_\delta - v)\big)(s,x)\,\mathrm{d}x\,\mathrm{d}s = \int_0^T \int_{\mathbb{R}^N} \varphi^\delta\otimes(v_\delta - v)(s,x) : \nabla a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s$$

and
by a similar argument as we did above, we can easily prove that

$$-\int_0^T \int_{\mathbb{R}^N} a_\epsilon(s,x)\cdot^t \nabla\varphi^\delta\cdot(w - w_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s = \int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x)\cdot\nabla\cdot((w - w_\delta)\otimes a_\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Hence

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x)(C^\epsilon + Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^T \int_{\mathbb{R}^N} \varphi^\delta\otimes(v_\delta - v)(s,x) : \nabla a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x)\cdot\nabla\cdot((w - w_\delta)\otimes a_\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s. \quad (5.8)$$

Since $a \in L^2(\mathbb{R}_+ \times \mathbb{R}^N)$, by young's convolution inequality, we have $\nabla a_\epsilon \in L^2(\mathbb{R}_+ \times \mathbb{R}^N)$ as well. Also, by Hölder inequality

$$\big\|\varphi_i^\delta(v_\delta - v)_i\big\|_{L^2(\mathbb{R}_+\times\mathbb{R}^N)} \le \|\varphi_i^\delta\|_{L^\infty(\mathbb{R}^N)}\|(v_\delta - v)_i\|_{L^2(\mathbb{R}_+\times\mathbb{R}^N)},$$

for every $0 \le i \le N$. Hence we can write $\varphi^\delta\otimes(v_\delta - v) \in \big(L^2(\mathbb{R}_+\times\mathbb{R}^N)\big)^{\mathbb{N}\times N}$, and we can also see that

$$\varphi^\delta\otimes(v_\delta - v) \to 0 \quad \text{in } L^2(\mathbb{R}_+\times\mathbb{R}^N),$$

as $\delta \to 0$.

we recall that for a fixed epsilon $a_\epsilon \in L^2(\mathbb{R}_+ \times \mathbb{R}^N)$, we then have by weak convergence that

$$\int_0^T \int_{\mathbb{R}^N} \varphi^\delta \otimes (v_\delta - v)(s,x) : \nabla a_\epsilon(s,x)\,\mathrm{d}x\,\mathrm{d}s \to 0,$$

as $\delta \to 0$.
Also we can easily see that

$$\|\nabla \cdot ((w - w_\delta) \otimes a_\epsilon)\|_{L^1(\mathbb{R}_+ \times \mathbb{R}^N)} \longrightarrow 0,$$

as $\delta \to 0$, for a fixed ϵ. Using the notion of weak* convergence, we deduce from the previous computation that

$$\int_0^T \int_{\mathbb{R}^N} \varphi^\delta(s,x) \cdot \nabla \cdot ((w - w_\delta) \otimes a_\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s \longrightarrow 0,$$

as $\epsilon \to 0$.
Hence we deduce that

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} \varphi(s,x)(C^\epsilon + Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Taking the limit ϵ and using lemma (22) we obtain

$$< a_\epsilon(T), \varphi_0 >_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = \int_0^T \int_{\mathbb{R}^N} \varphi(s,x)(C^\epsilon + Q^\epsilon)(s,x)\,\mathrm{d}x\,\mathrm{d}s \longrightarrow 0.$$

Hence

$$< a(T), \varphi_0 >_{\mathcal{D}'(\mathbb{R}^N),\mathcal{D}(\mathbb{R}^N)} = 0,$$

for any test function φ_0.
Then $a \equiv 0$, this is null vector field.

□

Now we take use of this theorem 35 to prove another version of prove of the known Serrin' s theorem.

Theorem 36. *Let $u = u(t,x)$ be a Leray solution of the Navier-Stokes equations*

$$(NS)\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ \\ \text{div } u = 0, \\ \\ u(0) = u_0 \in L^2(\mathbb{T}^3) \quad \textit{on } \mathbb{R}_+ \times \mathbb{T}^3 \end{cases}$$

Assume the existance of times $T_2 > T_1 > 0$ and exponents $2 \leq p < \infty$, $3 \leq q \leq \infty$ such that u belongs to $L^p\big(]T_1, T_2[, L^q(\mathbb{T}^3)\big)$, then it also belongs to $C^\infty\big(]T_1, T_2[\times \mathbb{T}^3\big)$.

Proof. Let us consider $\omega := \nabla \wedge u$ and $\omega_0 := \nabla \wedge u_0$ where we remind that $\nabla \wedge \cdot$ is the rotational operator. ω so define is the vorticity. Now let us take the rotational of the entire

$$\nabla \wedge \partial_t u + \nabla \wedge (\nabla \cdot (u \otimes u)) - \nabla \wedge \Delta u = \nabla \wedge (\nabla p) \tag{5.9}$$

But since the rotational of the gradient is always zero, we have $\nabla \wedge (\nabla p) = 0$.
We also recall that we have for any vector field A the following formula

$$\nabla \wedge (\nabla \wedge A) = \nabla(\nabla \cdot A) - \Delta A$$

,

applying this formula when taking itself rotational we prove that

$$\Delta(\nabla \wedge A) = \nabla \wedge \Delta A.$$

Hence we have

$$\nabla \wedge \Delta u = \Delta(\nabla \wedge u) = \Delta \omega,$$

also

$$\nabla \wedge \partial_t u = \partial_t(\nabla \wedge u) = \partial_t \omega.$$

We also have

$$\nabla \wedge (A \cdot \nabla A) = A \cdot \nabla(\nabla \wedge A) - (\nabla \wedge A) \cdot \nabla A.$$

We now compute $\nabla \wedge \nabla \cdot (u \otimes u)$, which gives

$$\begin{aligned}
\nabla \wedge \cdot (u \otimes u) &= \nabla \wedge (u \text{div } u + u \cdot \nabla u) \\
&= \nabla \wedge (u \cdot \nabla u) \quad \text{since div } u = 0 \\
&= u \cdot \nabla(\nabla \wedge u) - (\nabla \wedge u) \cdot \nabla u \\
&= u \cdot \nabla \omega - \omega \cdot \nabla u \\
&= \nabla \cdot (\omega \otimes u) - \nabla(u \otimes \omega) + u \text{div } \omega \quad \text{since divergence of a rotationel is zerom, div } \omega = 0 \\
&= \nabla \cdot (\omega \otimes u) - \nabla \cdot (u \otimes \omega).
\end{aligned}$$

Substituting all this we have done in equation (5.9), we get the Cauchy problem

$$(TNS) \begin{cases} \partial_t \omega + \nabla \cdot (\omega \otimes u) - \Delta \omega = \nabla \cdot (u \otimes \omega) \\ u(0) = \nabla \wedge u_0 \in L^2(\mathbb{T}^3) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^3 \end{cases}$$

Now let χ be a smooth cut off in time, supported inside $]T_1, T_2[$. Let φ be another cut off such that Supp $\chi \subset \{\varphi \equiv 1\}$. Let us denote $\omega' = \chi\omega$ and $u' = \varphi u$. We have

$$\partial_t \omega' = \chi \partial_t \omega + \omega \partial_t \chi$$
$$\nabla \cdot (\omega \otimes u) = \chi(t) \nabla \cdot (\omega' \otimes u)$$
$$\Delta \omega' = \chi(t) \Delta \omega.$$

We can deduce the following system

$$\begin{cases} \partial_t \omega' + \nabla \cdot (\omega' \otimes u) - \Delta \omega' = \nabla \cdot (u' \otimes \omega') + \omega \partial_t \chi \\ \omega'(0) = 0 \quad \text{since } \chi(0) = 0, \text{ for } \chi \text{ supported in }]T_1, T_2[\end{cases}$$

Now let us consider the problem

$$\begin{cases} \partial_t \omega'' + \nabla \cdot (\omega'' \otimes u) - \Delta \omega'' = \nabla \cdot (u' \otimes \omega'') + \omega \partial_t \chi \\ \omega''(0) = 0 \end{cases}$$

Following the same steps as those in theorem 34 we sketch a way to build a solution ω'' of the above Cauchy problem, belonging to $L^\infty\big(\mathbb{R}_+, L^2(\pi^3)\big) \cap L^2(\mathbb{R}_+, \dot{H}^1(\pi^3))$.

Let u_δ, u'_δ and ω_δ be smooth mollification of u, u' and ω respectively. by Friedrichs method and hest kernel estimates, there exists a smooth solution ω''_δ of

$$\begin{cases} \partial_t \omega''_\delta + \nabla \cdot (\omega''_\delta \otimes u_\delta) - \Delta \omega''_\delta = \nabla \cdot (u'_\delta \otimes \omega''_\delta) + \omega_\delta \partial_t \chi \\ \omega''_\delta(0) = 0 \end{cases} \tag{5.10}$$

Multiplying this equation by ω''_δ, we get

$$\frac{1}{2}\partial_t|\omega''_\delta|^2 + \omega''_\delta \cdot \nabla \cdot (\omega''_\delta \otimes u_\delta) - \omega''_\delta \cdot \nabla\omega''_\delta = \omega''_\delta \cdot \nabla \cdot (u'_\delta \otimes \omega'_\delta) + \omega''_\delta \cdot \omega_\delta \partial_t \chi$$

integrating over time and space, we have

$$\frac{1}{2}\|\omega''_\delta(t)\|_{L^2(\mathbb{T}^3)} + \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \nabla \cdot (\omega''_\delta \otimes u_\delta)(s,x)\,\mathrm{d}s\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \Delta\omega''_\delta(s,x)\,\mathrm{d}x\,\mathrm{d}x$$
$$= \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \nabla \cdot (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}x$$

doing integration by part of the first integral of the second hand side, we get the above equation equals

$$\frac{1}{2}\|\omega''_\delta(t)\|_{L^2(\pi^3)} + \int_0^t \int_{\mathbb{T}^3} \omega''_\delta \cdot \nabla \cdot (\omega''_\delta \otimes u_\delta)\,\mathrm{d}s\,\mathrm{d}x + \int_0^t \|\nabla\omega''_\delta(s)\|^2_{L^2(\pi)}$$
$$= \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \nabla \cdot (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}x,$$

we can now write the following inequality

$$\frac{1}{2}\|\omega''_\delta(t)\|_{L^2(\pi^3)} + \int_0^t \|\nabla\omega''_\delta(s)\|^2_{L^2(\pi)}\,\mathrm{d}s \leq \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \nabla \cdot (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}x,$$

by integration by part we have

$$\int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \nabla \cdot (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s = -\int_0^t \int_{\mathbb{T}^3} \nabla\omega''_\delta(s,x) : (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s,$$

hence we have replacing this in the above inequality

$$\frac{1}{2}\|\omega''_\delta(t)\|_{L^2(\pi^3)} + \int_0^t \|\nabla\omega''_\delta(s)\|^2_{L^2(\mathbb{T}^3)}\,\mathrm{d}s \leq \int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x) \cdot \omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}x$$
$$- \int_0^t \int_{\mathbb{T}^3} \nabla\omega''_\delta(s,x) : (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s. \quad (5.11)$$

Let us consider $\tilde{q}$ such that $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{1}{q} \Leftrightarrow \frac{1}{2} + \frac{1}{\tilde{q}} + \frac{1}{q} = 1$. using Hölder inequality we have

$$-\int_0^t \int_{\mathbb{T}^3} \nabla\omega''_\delta(s,x) : (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s \leq \int_0^t \|\nabla\omega''_\delta(s)\|_{L^2(\pi^3)}\|u'_\delta(s)\|_{L^q(\pi^3)}\|\omega''_\delta(s)\|_{L^{\tilde{q}}(\pi^3)}\,\mathrm{d}s.$$

Further, since $\frac{3}{q} = \frac{3}{2} - \frac{3}{\tilde{q}}$, we have the sobolev inequality

$$\dot{H}^{\frac{3}{q}}(\pi^3) \hookrightarrow L^{\frac{3\times 3}{3-b2\times\frac{3}{q}}=\tilde{q}}(\pi^3) = L^{\tilde{q}}(\pi^3),$$

which means that

$$\|\omega''_\delta(s)\|_{L^{\tilde{q}}(\pi^3)} \lesssim \|\omega''_\delta\|_{\dot{H}^{\frac{3}{q}}(\pi^3)}$$

By the assumption, $\frac{1}{p} + \frac{3}{q} = 1$, now using the sobolev interpolation, we have

$$\|\omega''_\delta\|_{\dot H^{\frac{3}{q}}(\pi^3)} \le \|\omega''_\delta\|_{\dot H^1}^{\frac{3}{q}} \|\omega''_\delta\|_{\dot H^0(\pi^3)=L^2(\pi^3)}^{\frac{2}{p}} \Leftrightarrow \|\omega''_\delta\|_{\dot H^{\frac{3}{q}}(\pi^3)} \le \|\nabla\omega''_\delta\|_{L^2(\pi^3)}^{\frac{3}{q}} \|\omega''_\delta\|_{\dot H^0(\pi^3)=L^2(\pi^3)}^{\frac{2}{p}}.$$

Hence

$$-\int_0^t \int_{\mathbb{T}^3} \nabla\omega''_\delta(s,x) : (u'_\delta \otimes \omega''_\delta)(s,x)\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \|\nabla\omega''_\delta(s)\|_{L^2(\pi^3)}^{1+\frac{3}{q}} \|u'_\delta(s)\|_{L^q(\pi^3)} \|\omega''_\delta(s)\|_{L^{\bar q}(\pi^3)}^{\frac{2}{p}}\,\mathrm{d}s.$$

But since $\frac{2}{p}+\frac{3}{q}=1, \Rightarrow \frac{2}{p}+1+\frac{3}{q}=2 \Rightarrow \frac{1}{p}+\frac{1}{2}(1+\frac{3}{q})=1$,
using young's inequality we have

$$\|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}^{1+\frac{3}{q}} \|u'_\delta(s)\|_{L^q(\mathbb{T}^3)} \|\omega''_\delta(s)\|_{L^{\bar q}(\mathbb{T}^3)}^{\frac{2}{p}} \le \frac{C}{2}\|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)} + \frac{1}{p}\|u'_\delta(s)\|_{L^q(\mathbb{T}^3)}^p \|\omega''_\delta(s)\|_{L^{\bar q}(\mathbb{T}^3)}^2,$$

this is also true,

$$\|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}^{1+\frac{3}{q}} \|u'_\delta(s)\|_{L^q(\mathbb{T}^3)} \|\omega''_\delta(s)\|_{L^{\bar q}(\mathbb{T}^3)}^{\frac{2}{p}} \le \frac{C}{2}\|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)} + \frac{1}{2}\|u'_\delta(s)\|_{L^q(\mathbb{T}^3)}^p \|\omega''_\delta(s)\|_{L^{\bar q}(\mathbb{T}^3)}^2.$$

Also by Cauchy Schwartz and Young's inequality we have

$$\int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x)\cdot\omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}s \le \int_0^t \|\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)} \|\omega_\delta(s)\|_{L^2(\mathbb{T}^3)} |\partial_t\chi(s)|\,\mathrm{d}s,$$

using Young's inequality, again, to the second hand side of the above inequality we finally get

$$\int_0^t \int_{\mathbb{T}^3} \omega''_\delta(s,x)\cdot\omega_\delta(s,x)\partial_t\chi(s)\,\mathrm{d}x\,\mathrm{d}s \le \frac{1}{2}\int_0^t \|\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}\,\mathrm{d}s + \frac{1}{2}\|\partial_t\chi\|_{L^\infty(\pi^3)}^2 \int_0^t \|\omega_\delta(s)\|_{L^2(\pi^3)}\,\mathrm{d}s.$$

Now putting together all those estimate, we finally get form inequality (5.11) that

$$\|\omega''_\delta(t)\|_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}^2\,\mathrm{d}s \le \int_0^t \Big(1 + C\|u'_\delta(s)\|_{L^p(\mathbb{T}^3)}^p\Big)\|\omega''_\delta(s)\|_{L(\mathbb{T}^3)}^2\,\mathrm{d}s$$
$$+ \|\partial_t\chi\|_{L^\infty(\mathbb{T}^3)}^2 \|\omega_\delta\|_{L^2(\mathbb{R}_+\times\mathbb{T}^3)}. \quad (5.12)$$

Since u'_δ and ω_δ are mollification of u' and ω respectively for $t\in\mathbb{R}_+$ and $\delta>0$, then
$\|u'_\delta(s)\|_{L^q(\mathbb{T}^3)} \le \|u'(s)\|_{L^q(\mathbb{T}^3)}$ and $\|\omega_\delta(s)\|_{L^2(\mathbb{T}^3)} \le \|\omega(s)\|_{L^2(\mathbb{T}^3)}$.
Using this and inequality (5.12) we have

$$\|\omega''_\delta(t)\|_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}^2\,\mathrm{d}s \le \int_0^t \Big(1 + C\|u'(s)\|_{L^p(\mathbb{T}^3)}^p\Big)\|\omega''_\delta(s)\|_{L(\mathbb{T}^3)}^2\,\mathrm{d}s$$
$$+ \|\partial_t\chi\|_{L^\infty(\mathbb{T}^3)}^2 \|\omega\|_{L^2(\mathbb{R}_+\times\mathbb{T}^3)}.$$

Using Grönwall inequality, we obtain

$$\|\omega''_\delta(t)\|_{L^2(\mathbb{T}^3)} + \int_0^t \|\nabla\omega''_\delta(s)\|_{L^2(\mathbb{T}^3)}^2\,\mathrm{d}s \le \|\partial_t\chi\|_{L^\infty(\mathbb{T}^3)}^2 \|\omega\|_{L^2(\mathbb{R}_+\times\mathbb{T}^3)} \exp\Big(t + C\int_0^t \|u'(s)\|_{L^q(\mathbb{T}^3)}^p\,\mathrm{d}s\Big).$$

We hence deduce that the sequence $(\omega''_\delta)_\delta$ is uniformly bounded in $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)\cap L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$. Up to an extraction there must exist $\omega \in L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)\cap L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$ such that

$$\omega''_\delta \rightharpoonup \omega'' \quad \text{in } L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big)\cap L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$$

as $\delta \to 0$.

Recall also that $u, u' \in L^2(\mathbb{R}_+, \dot H^1(\mathbb{T}^3))$ since it is Leray solution. Then

$$u_\delta,\ u'_\delta \longrightarrow u,\ u' \quad \text{in } L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3)).$$

We then deduce that ω'' belongs to $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$ and solves Problem (5.10).

Now let us pose $\tilde{\omega} := \omega' - \omega''$, we show by combining the Equation in ω' and $-\omega$ that $\tilde{\omega}$ solves the problem

$$\begin{cases} \partial_t \tilde{\omega} + \nabla \cdot (\tilde{\omega} \otimes u) - \Delta \tilde{\omega} = \nabla \cdot (u \otimes \tilde{\omega}) \\ \tilde{\omega}(0) = 0. \end{cases}$$

Recall that u and u' belongs to $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$ (because it is a Leray solution) and by assumption that $L^p(\mathbb{R}_+, L^q(\mathbb{T}^3))$. By construction of $\tilde{\omega}$, we entail that, it belongs to $L^2(\mathbb{R}_+ \times \mathbb{T}^3)$. We can then invoke the theorem 35, that $\tilde{\omega} = 0$, hence $\omega' = \omega''$. Thus $\omega' \in L^\infty\big(\mathbb{R}_+, L^2(\mathbb{T}^3)\big) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{T}^3))$. That is to say that

$$\omega \in L^\infty_{\text{loc}}\big(]T_1, T_2[, L^2(\mathbb{T}^3)\big) \cap L^2_{\text{loc}}(]T_1, T_2[, \dot{H}^1(\mathbb{T}^3)).$$

Since the rotational of u is of ω, the above regularity entails the regularity on u i.e

$$u \in L^\infty_{\text{loc}}\Big(]T_1, T_2[, \dot{H}^1(\mathbb{T}^3)\Big) \cap L^2_{\text{loc}}(]T_1, T_2[, \dot{H}^2(\mathbb{T}^3)).$$

We now improve on the regularity of u, an argument that relies on induction procedure. We want to show that

$$\omega \in L^\infty_{\text{loc}}\Big(]T_1, T_2[, \dot{H}^s(\mathbb{T}^3)\Big) \cap L^2_{\text{loc}}(]T_1, T_2[, \dot{H}^{s+1}(\mathbb{T}^3))$$

or

$$u \in L^\infty_{\text{loc}}\Big(]T_1, T_2[, \dot{H}^{s+1}(\mathbb{T}^3)\Big) \cap L^2_{\text{loc}}(]T_1, T_2[, \dot{H}^{s+2}(\mathbb{T}^3)).$$

As we have said earlier, The reasoning is going to be induction where the case $s = 0$, has been done already. Now let us suppose that the statement is True for some positive integer s. To prove the case $s+1$, we need to compute the derivatives of order $s+1$ in space of the entire Equation on ω', doing that we get somethings like this

$$\partial_t \partial_{s+1} \omega' + \nabla \cdot (\omega' \otimes \partial_{s+1} u) - \Delta \partial_{s+1} \omega' = \nabla \cdot (\partial_{s+1} \omega' \otimes u) + L.O.T(\omega'),$$

where $L.O.T(\omega')$ is the lower order term of the $\partial_{s+1}\omega'$. Performing the same step as above, one rove that

$$\partial_{s+1}\omega' \in L^\infty_{\text{loc}}\big(]T_1, T_2[, L^2(\mathbb{T}^3)\big) \cap L^2_{\text{loc}}(]T_1, T_2[, \dot{H}^1(\mathbb{T}^3)).$$

Hence result. We then conclude that $u \in C^\infty(\mathbb{R}_+ \times \mathbb{T}^3)$.

□

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Appendix A

Classical facts

In this section, we recall some important and well known theorem specially Cauchy-Lipschitz theorem and some relation connection between ODE and PDE.

A.1 Cauchy-Lipschitz theorem

Theorem 37 (Cauchy-Lipschitz). *Let us consider the Chaucy problem*

$$\begin{cases} x'(t) & = f(t, x(t)) \\ x(0) & = x_0 \end{cases} \tag{A.1}$$

A solution of the above problem x is called integral curve of f trough x_0.

1. *(Local version) Let $U \subset \mathbb{R}^N$ be an open subset and let $f : [0,T] \times U \to \mathbb{R}^N$ be continuous and satisfies Lipschitz condition*

   $$|f(t, x_1) - f(t, x_2)| \leq C|x_1 - x_2|$$

   for any $t \in [0,T]$, x_1 and x_2 in $U \subset \mathbb{R}^N$, where C is a given real constant. Then if $x_0 \in U$, for some $\delta > 0$, there exists a unique solution $x : [0, \delta] \to U$ of the Cauchy problem (A.1)*.*

   If the interval $[0,T]$ is replaced by the intervals $[-T, 0]$ or $[-T, T]$, the same statement holds, except the existence interval will be respectively $[-\delta, 0]$ and $[-\delta, \delta]$. The existence is limited to a small interval because the curve integral x might leave the domain U when the interval is big. In particular, either δ might be taken equals to T or there exists a maximum time interval $[0, T_0[\subset [0, T]$ of existence, characterized by the property that $x(t)$ approaches the boundary of U when $t \longrightarrow T_0$.

2. *(Global version) Let us maintain all the assumption in part one, but we mow take $U = \mathbb{R}^N$. Then for all $x_0 \in \mathbb{R}^N$, there exists a unique solution $x : [0,T] \to U$ of the Cauchy problem* (A.1)*. The statement holds when we replaced the interval $[0,T]$ by the unbounded half lines $]-\infty, 0]$, $[0, \infty[$ or the entire real line $\mathbb{R}$.*

Let $b \in C_b^0([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ such that $b_t \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ for every t, and :

$$\|b\|_{L^1\left(0,T;C_b^1\right)} = \int_0^T \left(\|b_t\|_\infty + \|Db_t\|_\infty\right) \mathrm{d}t < \infty, \tag{A.2}$$

and denotes

$$M = \int_0^T \|Db_t\|_\infty \, \mathrm{d}t < \infty. \tag{A.3}$$

Applying Theorem 37 with the conditions (A.2) and (A.3), there exists, for each $y_0 \in \mathbb{R}^d$, a unique solution $x \in C^1([0,T];\mathbb{R}^d)$ solution of the Cauchy problem

$$\begin{cases} x'(t) & = b_t(x(t)), \\ x(0) & = y_0 \end{cases} \qquad \text{or, equivalently,} \qquad x(t) = y_0 + \int_0^T b_s(x(s))\,\mathrm{d}s.$$

We denotes $X_t(y_0)$ this solution, and we have from the previous

$$X_t(y_0) = y_0 + \int_0^T b_s(X_s(y_0))\,\mathrm{d}s. \tag{A.4}$$

Further more, by using Gronwall inequality that we will proved later, we get

$$\big\|X_t(y_1) - X_t(y_0)\big\|_\infty \le \|y_1 - y_0\|_\infty \exp \int_0^t \|Db_s\|_\infty\,\mathrm{d}s.$$

Also, since b_t is C^1 for each t, as well X_t is C^1 and even is a C^1-diffeomorphism.

Now let Y_t the reciprocal of X_t, that is, for all t in $[0,T]$ and x and y in $\mathbb{R}^d$: $X_t(y) = x$ if and only if $y = Y_t(x)$. Now let us see the relation between the ODE

$$\partial_t X_t = b_t(X_t) \tag{A.5}$$

and the PDE

$$\partial_t u + b \cdot \nabla u = cu + f. \tag{A.6}$$

Lemma 38 (Gronwall Inequality)**.** *If $\psi \ge 0$ and ϕ are continuous functions that verify: $\forall\ t \ge t_0$*

$$\phi(t) \le K + \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s,$$

where K is a positive constant real number,then

$$\phi(t) \le K \exp\bigg(\int_0^t \psi(s)\,\mathrm{d}s\bigg).$$

Proof. Let

$$f(t) = \frac{K + \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s}{\exp\Big(\int_{t_0}^t \psi(s)\,\mathrm{d}s\Big)},$$

then

$$f'(t) = \psi(t)\frac{\phi(t) - K - \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s}{\exp\Big(\int_{t_0}^t \psi(s)\,\mathrm{d}s\Big)} \le 0,$$

thus $\forall\ \ t \ge t_0, \quad f(t) \le f(t_0) = K$.

We then deduce that

$$\phi(t) \le K + \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s \le K \exp\bigg(\int_{t_0}^t \psi(s)\,\mathrm{d}s\bigg).$$

□

A.2 Characteristics method of solving PDE)

Let u and v in $C^1([0,T]\times\mathbb{R}^d)$ linked by $v(t,y)=u(t,X_t(y))$, or, equivalently, $v(t,Y_t(x))=u(t,x)$. From de Leibniz rule, we obtain

$$\partial_t[v(t,y)] = \Big(\partial_t u\Big)(t,X_t(y)) + \Big(D_x u\Big)(t,X_t(y))\cdot\Big(\partial_t X\Big)_t(y) \tag{A.7}$$

$$= \Big(\partial_t u\Big)(t,X_t(y)) + \Big(D_x u\Big)(t,X_t(y))\cdot b_t(X_t(y)) \tag{A.8}$$

$$= \Big(\partial_t u + b\cdot\nabla u\Big)(t,X_t(y)). \tag{A.9}$$

From this, we see that, if u is a $C^1([0,T]\times\mathbb{R}^d)$ solution of PDE (A.6), then v is a $C^1([0,T]\times\mathbb{R}^d)$ solution of the ODE

$$\partial_t[v(t,y)] = c(t,X_t(y))v(t,y) + f(t,X_t(y)). \tag{A.10}$$

Technically, the characteristics method consists of finding solution of (A.6), that is relying on the Characteristics curves $(t,X_t(y))$, which makes sense, since X_t is C^1- diffeomorphism. And that lead us to an ODE (A.10), which gives a solution from which we can build the solution of the PDE (A.6).

We solve Equation (A.10) by constant variation method. We do that in two steps. First of all, we find a solution that we denote v_h, of the homogeneous equation corresponding to Equation (A.10),

$$\partial_t[v_h(t,y)] = c(t,X_t(y))v_h(t,y).$$

Solving this equation, we find that

$$v_h(t,y) = v(0,y)\exp\bigg(\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg).$$

Now we find a particular solution of Equation (A.10) by constant variation method that consist to claim that

$$v_p(t,y) = k(t)\exp\bigg(\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg),$$

where $k(t)$ is a real valued function of t, is a solution of Equation (A.10), and we solve for k. When we substitute v_p in Equation (A.10), we have

$$\partial_t\bigg(k(t)\exp\bigg(\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg)\bigg) = c(t,X_t(y))k(t)\exp\bigg(\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg).$$

This gives us when we develop

$$k'(t) = f(t,X_t(y))\exp\bigg(-\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg).$$

We thus infer that, choosing arbitrarily $k(0)=0$

$$k(t) = \int_0^t \bigg(f(r,X_r(y))\exp\bigg(-\int_0^r c(s,X_s(y))\,\mathrm{d}s\bigg)\bigg)\,\mathrm{d}r.$$

Hence

$$v_p(t,y) = \int_0^t \bigg(f(r,X_r(y))\exp\bigg(\int_r^t c(s,X_s(y))\,\mathrm{d}s\bigg)\bigg)\,\mathrm{d}r.$$

The general solution of Equation (A.10) can now be the sum of the homogeneous solution v_h and the particular solution v_p, that is

$$v(t,y) = v(0,y)\exp\bigg(\int_0^t c(s,X_s(y))\,\mathrm{d}s\bigg) + \int_0^t \bigg(f(r,X_r(y))\exp\bigg(\int_r^t c(s,X_s(y))\,\mathrm{d}s\bigg)\bigg)\,\mathrm{d}r.$$

Hence we have

$$u(t,x) \quad = \quad u(0,Y_t(x))\exp\left(\int_0^t c(s,X_s(Y_t(x)))\,\mathrm{d}s\right) \;+\; \int_0^t \left(f(r,X_r(Y_t(x)))\exp\left(\int_r^t c(s,X_s(Y_t(x)))\,\mathrm{d}s\right)\right)\mathrm{d}r.$$

This gives a formula for u being a $C^1([0,T]\times\mathbb{R}^d)$ solution when u_0 is $C^1(\mathbb{R}^d)$ and b is $C^0([0,T];C^1(\mathbb{R}^d)^d)$.

A.3 Others useful Theorem

Theorem 39 (Derivability under integral)**.** *Let $f :\mapsto f(t,x)$ a function from $I \times E$ to $\mathbb{C}$, where E is any space. Suppose that:*

1. *(Existence of F) For all $t \in I$, the function $x \mapsto f(t,x)$ is integrable;*
2. *(Differentiability) For almost every $x \in E$, $t \mapsto f(t,x)$ is differentiable on I, of derivative $\frac{\partial f}{\partial t}$;*
3. *(Domination of the derivative) There exists s function $\psi : E \mapsto \mathbb{R}_+$ measurable such that $\int f\,\mathrm{d}\mu < \infty$ and for all $t \in I$, almost every $x \in E$,*

$$\left|\frac{\partial f}{\partial t}(t,x)\right| \leq \psi(x).$$

Then the function

$$F : t \mapsto F(t) = \int f(t,x)\,\mathrm{d}\mu(x)$$

is differentiable on I and, for all $t \in I$,

$$F'(t) = \int \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}\mu(x).$$

Theorem 40 (Banach fixed point Theorem)**.** *Let E be a complete metric space, and f a contracting application from E to E. Then there exists in E an unique fixed point E, that is $x \in E$ such that $f(x) = x$. Further more, if we pose $X_{n+1} = f(X_n)$ for all integer n, the recurrent sequence $(X_n)_n$ is Cauchy convergent to an unique fixed point X^* of f.*