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Lagrangian Cobordisms And Surgery

Master Thesis

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Abstract

This document consists of three chapters. The first chapter introduce basic concepts and definitions in symplectic geometry and contact geometry. In the second chapter we define Lagrangian cobordism between two legenedrian submanifolds of a contact manifold, and prove that legendrian isotopy is realized by such cobordism. We also study gf-compatible Lagrangian cobordisms and prove a gf version of such realization. In the third chapter we give two descriptions of Lagrangian surgery, the second of which applies to exact symplectic manifolds and exact Lagrangians and is realized by Lagrangian cobordism from the lift of the Lagrangian submanifold before surgery to the lifting of Lagrangian submanifold after surgery.

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Contents

Chapter 1

Symplectic and Contact Geometry

In this chapter we introduce basic concepts of symplectic and contact geometries.

1.1 Symplectic Manifolds

1.1.1 Skew Symmetric Bilinear forms

In this section we will review symplectic linear algebra

Definition 1.1.1. Let V be an m-dimensional vector space over $\mathbb{R}$. A Bilinear form $\Omega : V \times V \to \mathbb{R}$ is skew symmetric if $\Omega(x, y) = -\Omega(y, x)$ for all $x, y \in V$.

We have standard form of such forms.

Proposition 1.1.1. *Let Ω be a skew-symmetric form on V. Then there is a basis $u_1, \cdots, u_k, e_1, \cdots, e_n, f_1, \cdots, f_n$ such that*

$$\Omega(u_i, \cdot) = 0 \ , \ \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0 \ , \ \Omega(e_i, f_j) = \delta_{ij}$$

Proof. Let $U = \ker \Omega = \{u \in V | \Omega(u, \cdot) = 0$. Choose a basis $u_1, \cdots, u_k$ of U, and let W be complementary space of U, that's $V = U \oplus W$. Let $e_1 \in W$, then there is f_1 such that $\Omega(e_1, f_1) = 1$. We let

$$W_1 = \text{span}\{e_1, f_1\}\ ,\ W_1^{\Omega} = \{w \in W | \Omega(w, v) = 0 \ \text{ for all } v \in W_1\}$$

We have

- $W_1 \cap W_1^{\Omega} = \{0\}$: For let $v = ae_1 + bf_1 \in W_1 \cap W_1^{\Omega}$, then $0 = \Omega(v, e_1) = -b$ and $0 = \Omega(v, f_1) = a$. Hence $v - 0$.
- $W = W_1 \oplus W_1^{\Omega}$: For let $v \in W$, and suppose that $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then
$$v = (-cf_1 + de_1) + (v + cf_1 - de_1),$$
where we note that $-cf_1 + de_1 \in W_1$ and $v + cf_1 - de_1 \in W_1^{\Omega}$

We continue the process by picking up $0 \neq e_2 \in W_1^{\Omega}$, then there is $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) = 1$. We let

$$W_2 = \text{span}\{e_2, f_2\}\ ,\ W_2^{\Omega|_{W_1^{\Omega}}} = \{w \in W_1^{\Omega} | \Omega(w, b) = 0 \ \text{ for all } \ v \in W_2\}$$

We get again $W_1^{\Omega} = W_2 \oplus W_2^{\Omega}$, continuing in this way we get $W_n^{\Omega|_{W_{n-1}}} = 0$ for some n (as $\dim V < \infty$, and hence

$$V = U \oplus W_1 \oplus \cdots \oplus W_n,$$

where $W_i = \text{span}\{e_i, f_i\}$ and all summands are orthogonal with respect to Ω. $\square$

We say Ω is non-degenerate(or symplectic) if the map $\tilde{\Omega} : V \to V^*$ defined by $\tilde{\Omega}(u) = \Omega(u, \cdot)$ is isomorphism. The kernel of this map is U constructed in the proof above, so Ω is non-degenerate if and only if $U = \{0\}$. In this case we say (V, Ω) a *symplectic vector space*. Note that by the above proposition V is even dimensional. A basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ as in the proposition is called a symplectic basis.

Corollary 1.1.2. *Let V be a vector space of dimension n, Ω is non-degenerate if and only if $\Omega^n \neq 0$. (Here $\Omega^n = \Omega \wedge \cdots \Omega$)*

The following proposition follows easily from definitions

Proposition 1.1.3. *Let (V, Ω) be a symplectic vector space. Let $W \subset V$ be a subspace, then $\dim Y + \dim Y^{\Omega} = \dim V$, where $Y^{\Omega} = \{v \in V | \Omega(v, u) = 0$ for all $u \in W\}$*

We say $W \subset (V, \Omega)$ an *isotropic* subspace if $W \subset W^{\Omega}$. It follows from above proposition that $\dim W \leq \frac{1}{2} \dim V$. If $W == W^{\Omega}$ we say W is *Lagrangian*, it follows in this case that $\dim W = \dim W^{\Omega} = \frac{1}{2} \dim V$.

1.1.2 Basic concepts for symplectic manifolds

We begin with the definition of symplectic manifolds.

Definition 1.1.2. Let M be a manifold. A 2-form ω which is closed and $\omega_p : T_p \times T_p M \to \mathbb{R}$ is symplectic for all p is called *symplectic form.* The pair (M, ω) is then called *symplectic manifold.* If $\omega = d\theta$, then M is called exact symplectic manifold.

A submanifold $L \subset M$ is called lagrangian if $T_x L$ is lagrangian subspace of $T_x M$ for each $x \in L$, so $\dim L = \frac{1}{2} \dim M$, and immersion $i : L \to M$ is lagrangian if $di(x)(T_x L)$ is lagrangian subspace of $T_x M$ for each $x \in L$ or equivalently $i^* \omega = 0$. Now we give the definiton of symplectomorphisms

Definition 1.1.3. Let (M_1, ω_1) and (M_2, ω_2) be $2n$- dimensional symplectic manifolds and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is *symplectomorphism* if $\varphi^* \omega_2 = \omega_1$.

We give the important examples

Example 1.1.1. 1. Let $M = \mathbb{R}^{2n}$ with coordinates $q_1, \cdots, q_n, p_1, \cdots, p_n$. Then the form

$$\omega_{std} = \sum_{i=1}^{n} dq_i \wedge dp_i$$

is symplectic. So $(\mathbb{R}^{2n}, \omega_{std})$ is symplectic manifold.

2. (Important example) Let X be a manifold, and $M = T^*X$ be its cotangent bundle, we have a canonical symplectic form ω_{can} defined on M defined as follows: We first define canonical 1-form λ_{can}. Denote by $\pi : M \to X$ the projection and let $p = (x, \alpha) \in T_xX$, define $\lambda_{can}|_{(x,\alpha)}(v) = \alpha(d\pi(x, \alpha)(v))$. To check smoothness, we compute λ_{can} in coordinates. Let $(U, q_1, \cdots, q_n)$ be coordinate chart of M, and $(T^*U, q_1, \cdots, q_n, p_1, \cdots, p_n)$ be the associated coordinate chart of T^*M (which means that for $p = (x, \alpha) \in T^*U$, we have $\alpha = \sum_i p_i(p)(dq_i)_x$). It is easily seen that

$$\lambda_{can} = \sum_{i=1}^{n} p_i dq_i$$

which is smooth. Define

$$\omega_{can} = -d\lambda_{can},$$

this is trivially closed, and in local coordinates

$$\omega_{can} = \sum_i dq_i \wedge dp_i.$$

So ω_{can} is symplectic, we call it the canonical symplectic form.For future reference, it can be checked that considering the natural identification $T_{(x,0)}T^*X \equiv T_xX \oplus T_x^*X$, we have

$$\omega_{can}|_{(x,0)}(v, w) = w_1^*(v_0) - v_1^*(w_0)$$

for $v = (v_0, v_1^*), w = (w_0, w_1) \in T_{(x,0)}T^*X$

Given a diffeomorphism $f : M_1 \to M_2$, we can lift it to a symplectomorphism $f_\# : (T^*M_1, \omega_1) \to (T^*M_2, \omega_2)$, where ω_1 and ω_2 are the corresponding canonical symplectic forms. Define

$$f_\#(x_1, \alpha_1) = (f(x), ((df_x)^*)^{-1}\alpha_1),$$

In fact $f_\#$ turns out to be exact symplectomorphism in the sense of the following claim

Claim. $f_\#^* \lambda_2 = \lambda_1$, where λ_i is the canonical 1-form of T^*M_i so that $\omega_i = -d\lambda_i$.

Proof. let $p_1 = (x_1, \alpha_1) \in T^*M_1$ and $p_2 = (x_2, \alpha_2) = f\#(p_1)$, then we have to show that

$$(df_\#)^*_{p_1}(\lambda_2)_{p_2} = (\lambda_1)_{p_1} \tag{1.1}$$

We have the following facts

- $(df_{x_1})^*\alpha_2 = \alpha_1$ and $x_2 = f(x_1)$ (by definition of $f_\#$)
- $(\lambda_1)_{p_1} = (d\pi_1)^*\alpha_1$ and $(\lambda_2)_{p_2} = (d\pi_2)^*\alpha_2$ where π_j is the projection $T^*M_j \to M_j$.
- $\pi_2 \circ f_\# = f \circ \pi_1$

The proof of 1.1 is

$$\begin{aligned}(df_\#)^*_{p_1}(\lambda_2)_{p_2} &= (df_\#)^*_{p_1}(d\pi_2)^*\alpha_2 = (d(f_\# \circ \pi_2))^*_{p_1}\alpha_2 \\ &= (d(f \circ \pi_1))^*_{p_1}\alpha_2 = (d\pi_1)^*_{p_1}(df)^*_{x_1}\alpha_2 \\ &= (d\pi_1)^*_{p_1}\alpha_1 = (\lambda_1)_{p_1}\end{aligned}$$

□

An important class of symplectomorphisms is the class of hamiltonian differemorphisms. To define it we have to define the Hamiltonian vector field. Fix a sympelctic manifold (M, ω). We say that a vector field $X : M \to TM$ is *Hamiltonian vector field* if there is a function (called *Hamiltonian function*) such that

$$\iota_X \omega = dH$$

We denote X by X_H. A diffeomrophism $\phi : M \to M$ is hamiltonian if there is isotopy

$$[0,1] \times M \to M \ , \ (t,x) \mapsto \psi_t(x)$$

such that ψ_t is generated by time dependent Hamitonian vector field X_{H_t} where H_t is smooth family of Hamiltonians and $\phi = \psi_1$. We can see that ϕ is symplectomorphism, for

$$\frac{d}{dt}\psi_t^* \omega = \psi_t^*(\mathcal{L}_{X_{H_t}}) = \psi_t^*(d\iota_{X_{H_t}}\omega) + \iota_{X_{H_t}} d\omega = \psi_t^*(ddH_t) = 0,$$

so $\psi_t^* \omega = \psi_0^* \omega = \omega$.

1.1.3 Moser-type theorems

One of the fundamental techniques in symplectic geometry is Moser's argument. Given smooth family of symplectic forms ω_t on M, with the property

$$\frac{d}{dt}\omega_t = d\sigma_t \tag{1.2}$$

The goal of Moser argument is to construct smooth family of diffeomorphisms ψ_t such that

$$\psi_t^* \omega_t = \omega \tag{1.3}$$

The idea is to construct ψ_t as flows of (to be determined) time dependent vector field X_t. That's

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \tag{1.4}$$

If this is the case then taking the derivative of 1.3 with respect to t, we get

$$\phi_t^*(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t) = 0$$

By 1.7 and Cartan formula,

$$0 = d\sigma_t + d(\iota_{X_t}\omega_t) + \iota_{X_t}(d\omega_t) = d\sigma_t + d(\iota_{X_t}\omega_t)$$

This equation is satisfied if

$$\sigma_t = \iota_{X_t}\omega_t$$

But by non-degeneracy of ω_t, we get a unique time dependent vector field X_t. It follows that ψ_t is determined by 1.4. Using this argument, we prove the following:

Theorem 1.1.4. *[Moser isotopy theorem] Let (M^{2n}, ω) be symplectic manifold, and $S \subset M$ be a submanifold (not necessarily compact). Suppose ω_0, ω_1 be symplectic forms such that for all $x \in X$, $\omega_0|_x$ and $\omega_1|_x$ are equal. Then there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of S in M and diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*

$$\psi|_S = \text{id} \quad , \quad \psi^*\omega_1 = \omega_0$$

Proof. We use Moser argument above. We find a neighborhood $\mathcal{U}_0$ of S such that there is $\sigma \in \Omega^1(\mathcal{U}_0)$ satisfying

$$\sigma_x = 0 \ \text{ for all } x \in S \quad , \quad d\sigma = \omega_1 - \omega_0$$

We endow M with riemmannian metric. We know by *(Tubular neighborhood theorem)* that there exists function $\epsilon : S \to \mathbb{R}_+$ such that the restriction of $\exp : TS^\perp \to M$ to

$$U_\epsilon = \{(x, v) | x \in S, v \in TS^\perp, \ |v| < \epsilon(x)\}$$

is embedding, we denote the image by $\mathcal{U}_0$. For $0 \le t \le 1$, define $\phi_t : \mathcal{U}_0 \to \mathcal{U}_1$ by

$$\phi_t(\exp(p, v)) = \exp(p, tv)$$

Clearly $\phi_0(\mathcal{U}_0) \subset S$ and ϕ_t is embedding whenever $t > 0$, with $\phi_t|_S = \text{id}$. Letting $\tau = \omega_1 - \omega_0$, we get

$$\phi_0^*(\tau) = 0 \ , \ \phi_1^* \tau = \tau$$

Define

$$Y_t = (\frac{d}{dt}\phi_t) \circ \phi_t^{-1}$$

This vector field may be singular at $t = 0$. However, we have

$$\frac{d}{dt}\phi_t^*\tau = \phi_t^*(\mathcal{L}_{Y_t}\tau) = \phi_t^*(d(\iota_{Y_t}\tau) + \iota_{Y_t}d\tau) = d(\phi_t^*(\iota_{Y_t}\tau)),$$

where σ_t is smooth family of 1-forms $\phi_t^*(\iota_{Y_t}\tau)$. Note that for $v \in T_x\mathcal{U}_0$ we have

$$\sigma_t|_x(v) = (\phi_t^*(\iota_{Y_t}\tau)(v))_x = (\iota_{Y_t}\tau)_{\phi_t(x)}(d\phi_t(x)(v) = \tau_{\phi_t(x)}(Y_t(\phi(x)), d\phi_t(x)(v))$$

which is smooth at $t = 0$, and it vanishes on S. Putting $\sigma = \int_0^1 \sigma_t \ dt$ (note that $\sigma_x = \int_0^1 \sigma_t|_x \ dt = 0$ for $x \in X$, we get

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}(\phi_t^*\tau) \ dt = \int_0^1 d\sigma_t \ dt = d\sigma$$

Now we start Moser argument, let $\omega_t = (1-t)\omega_0 + t\omega$, since $\omega_t|_x = \omega_0|_x$ for all $x \in S$, then by a compactness argument it follows that by shrinking $\mathcal{U}_0$, ω_t is nondegenerate on $\mathcal{U}_0$. By nondegenracy we find time dependent vector field X_t on $\mathcal{U}_0$ by

$$\iota_{X_t}\omega + \sigma = 0,$$

Because $\sigma_x = 0$ for all $x \in S$, it follows that $X_t = 0$ on S. Now we have to show that by shrinking $\mathcal{U}_0$ if necessary, the family of maps ψ_t defined as solution to the initial value problem

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \ , \ \psi_0 = \text{id},$$

is defined on $\mathcal{U}_0$ for all $t \in [0,1]$. It suffices to show that for every $x \in S$, there is a neighborhood $\mathcal{V}_x$ of x, such that for the integral curve $\theta(t)$ starting at $y \in \mathcal{V}_x$ is defined for $t \in [0,1]$, where then we can replace $\mathcal{U}_0$ by $\mathcal{U}_0 \cap \bigcup_{x \in S} \mathcal{V}_x$. Translating the claim to local neighborhood we find that we need to show

Claim. Let $F : [0,1] \times B(0,\epsilon) \to \mathbb{R}^n$ be continuous and lipschitz in the second argument uniformaly with respect to the first argument. Suppose that $F(t,0) = 0$, then there exist $\delta < \epsilon$ such that for any $x_0 \in B(0,\delta)$, the solution θ of initial value problem

$$\theta'(t) = F(t, \theta(t)) \ , \ \theta(0) = x_0,$$

is defined on $[0,1]$

To prove this claim, we find an apriori estimate on the solution. Because of lipschitz condition in the claim, there is constant C such that

$$|F(t,x)| = |F(t,x) - F(t,0)| \leq C|x|$$

Suppose θ is solution of the initial value problem in the claim, then

$$|\theta'(t)| = |F(t,\theta(t))| \leq C|\theta(t)|,$$

hence by Gronwall inequality, we get

$$|\theta(t)| \leq |x_0| e^{Ct} \tag{1.5}$$

We let $\delta = \frac{1}{2} e^{-C} \epsilon < \epsilon$, suppose $x_0 \in B(0,\delta)$. If θ (the solution of the initial value problem) has maximal interval $[0,\eta)$, then by ODE theory $|\theta(t)| \to \epsilon$ as $t \to \eta^-$. But from 1.5 we get

$$|\theta(t)| < \delta e^{C\eta} < \frac{1}{2} e^{C\eta - C} \epsilon < \frac{1}{2}\epsilon$$

which is a contradiction. So θ is defined on $[0,1]$ for any $x_0 \in B(0,\delta)$. This proves the claim.

Finally let $\mathcal{U}_1 = \psi_1(\mathcal{U}_0)$, then $\psi_1 : \mathcal{U}_0 \to \mathcal{U}_1$ is the desired diffeomorphism. Note that $\psi|_S = \mathrm{id}$ because $X_t = 0$ on S. □

We have a corollary

Corollary 1.1.5 (Darboux theorem)**.** *Every symplectic form is locally diffeomorphic to ω_{std} on $\mathbb{R}^{2n}$.*

Proof. Follows from 1.1.4 and 1.1.1 by setting $L = \{pt\}$ and □

A coordinate chart $(\mathcal{U}, (q_1, \cdots, q_n, p_1, \cdot, o_n))$ of symplectic manifold (M, ω) is called Darboux coordinate chart, if

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

The corollary states that there is Darboux coordinates around any point of M.

Theorem 1.1.6 (Weinstein-Darboux Theorem)**.** *Let (M^{2n}, ω) be symplectic manifold, and $i : L \hookrightarrow M$ be Lagrangian submanifold, then there exist a neighborhood $\mathcal{U}_0$ of the zero section L_0 in T^*L and neighborhood $\mathcal{U}$ of L and diffeomorhphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$i = \psi \circ i_0 \quad , \quad \psi^*\omega = \omega_{can}$$

Proof. Since L is Lagrangian, then for $x \in L$ the map

$$\beta : T_xM \to T_x^*L \ ; \ u \mapsto \Omega(u, \cdot)$$

descends to map

$$\tilde{\beta}_x : N_xL = T_xM/T_xL \to T_x^*L \ ; \ [u] \mapsto \Omega(u, \cdot),$$

in fact $\tilde{\beta}_x$'s give isomorphism $\tilde{\beta} : NL \to T^*L$. We have the following easy to prove claim

Claim. If J is a compatible almost complex structure on (M, ω), then JTL_x is Lagrangian subspace of T_xM and in fact with respect to the riemannian metric g_J, the bundle JTL is orthogonal to TL.

Fix an almost complex structure J, by the claim we can identify JTL with NL by the isomorphism $v \mapsto [v]$, so we can consider $\tilde{\beta} : JTL \to T^*L$. Fix the riemannian metric g_J and recall that there is a neighborhood $\mathcal{V}_0$ of 0-section of JTL and neighborhood $\mathcal{U}'$ of L in M such that the map $\mathcal{V}_0 \to \mathcal{U}'$ defined by $(x, v) \mapsto \exp_x(-v)$ is difeomorphism. Composing the later map with $\tilde{\beta}^{-1}$, we get a diffeomorhpism

$$\phi : \mathcal{U}_0' \to \mathcal{U}' \ ; \ (x, \alpha) \mapsto \exp_x(-\tilde{\beta}^{-1}\alpha)$$

where $\mathcal{U}_0'$ is a neighborhood of the zero section. We have to check that $\phi^*\omega$ and ω_{can} agree on the zero section. Indeed, for $(v_0, v_0^*) \in T_{(x,0)}T^*L = T_xL \oplus T_x^*L$, we have

$$d\phi(v_0, v_0^*) = v_0 - \tilde{\beta}^{-1}(v_0^*)$$

hence for $v = (v_0, v_0^*)$ and $w = (w_0, w_0^*)$ lying in $T^*_{(x,0)}L$

$$\begin{aligned}
\phi^*\omega_{(x,0)}(v, w) &= \omega_q(d\phi_{(x,0)}(v), d\phi_{(x,0)}(w)) \\
&= \omega_q(d\phi_{(x,0)}(v), d\phi_{(x,0)}(w)) \\
&= \omega_q(v_0 - \tilde{\beta}^{-1}(v_0^*), w_0 - \tilde{\beta}^{-1}(w_0^*)) \\
&= \omega_q(\beta^{-1}(w_0^*), v_0) - \omega_q(\beta^{-1}(v_0^*), w_0) \\
&= w_0^*(v_0) - v_0^*(w_0) \\
&= \omega_{can}|_{(q,0)}(v, w)
\end{aligned}$$

Now by Moser stability theorem, we can find neighborhoods $\mathcal{U}_0, \mathcal{U}_0'' \subset \mathcal{U}_0'$ of the zero section of T^*L and diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_0''$ which is identity on the zero-section and satisfies $\varphi^*(\phi^*\omega) = \omega_{can}$. It follows that $\psi = \phi \circ \varphi : \mathcal{U}_0 \to \phi(\mathcal{U}_0) =: \mathcal{U}$

is the required map. ∎

We have the following easy corollary

Corollary 1.1.7. *Let (M^{2n},ω) be a symplectic manifold and let L be a Lagrangian submanifold of M. Then around each point of L, there is Darboux coordinate chart $(\mathcal{U},q_1,\cdots,q_n,p_1,\cdots,p_n)$ such that*

$$L\cap\mathcal{U}=\{x|p_1(x)=\cdots=p_n(x)=0\}$$

Proof. Pick a point $x_0\in L$. By Weinstein-Darboux theorem, we can find neighborhoods $\mathcal{U}_1,\mathcal{U}_0$ of L and the zero section of T^*L respectively, such that there is symplectomorphism $\psi:\mathcal{U}_0\to\mathcal{U}_1$ satisfying

$$\psi^*(\omega)=\omega_{can}\ ,\ \psi\circ i_0=i$$

Let $(V,q_1,q_2,\cdots,q_n)$ be coordinate chart in L centered at x_0, let $(\pi^{-1}(V),\varphi=(q_1,\cdots,q_n,p_1,\cdots,p_n))$ be the associated coorindates on T^*L_1, then by composing, $\phi=\varphi\circ\psi^{-1}:\mathcal{U}\to\mathbb{R}^{2n}$ gives coordinate chart of M centered at x_0, we abuse notation and denote again $\phi=(q_1,\cdots,q_n,p_1,\cdots,p_n)$. We get clearly,

$$\omega=\sum_i dq_i\wedge dp_i\ ,\ L\cap\mathcal{U}=\{p|p_1(x)=\cdots=p_n(x)=0\}$$

∎

Theorem 1.1.8. *Let (M^{2n},ω) be a symplectic manifold. Let L_1,L_2 be Lagrangian submanifolds intersecting transversely at x_0, then there is Darboux coordinates $(\mathcal{U},\tilde{q}_1,\cdots,\tilde{q}_n,\tilde{p}_1,\cdots,\tilde{p}_n)$ around x, such that*

$$L_1\cap\mathcal{U}=\{x\in M|\tilde{p}_1(x)=\cdots=\tilde{p}_n(x)=0\}\ ,\ L_2\cap\mathcal{U}=\{y\in M|\tilde{q}_1(x)=\cdots=\tilde{q}_n(x)=0\}$$

Proof. (Sketch) The problem is local, so we can assume that L_1,L_2 are Lagrangian submanifolds of $(\mathbb{R}^{2n},\omega_{std})$ interescting transversally at the origin. By the above corollary, we can further assume that $L_1=\mathbb{R}^n\times\{0\}$. It follows by transversality that near the origin, L_2 coincides with a graph of function $f : \mathbb{R}^n \to \mathbb{R}^n$, that's L_2 coincides with $\{(f(p), p) | p \in \mathbb{R}^n\}$ near the origin. In other words, L_2 is defined by equations $q_i = f_i(p_1, \cdots, p_n)$ near 0, where f_i is the i-th component of f_i. Because L_2 is Lagrangian, we should have

$$\sum_i df_i \wedge dp_i = 0 \tag{1.6}$$

Now we set

$$\tilde{q}_i = q_i - f_i(p_1, \cdots, p_n) \ , \ \tilde{p}_i = p_i$$

and restrict to a small neighborhood around zero. It follows that

- L_2 is the set of all points satisfying $\tilde{q}_1 = \cdots = \tilde{q}_n$
- $\tilde{q}_1, \cdots, \tilde{q}_n, \tilde{p}_1, \cdots, \tilde{p}_n$ are Darboux coordinates:

$$\begin{aligned}
\omega_{std} &= \sum_i dq_i \wedge dp_i \\
&= \sum_i d(\tilde{q}_i + f_i) \wedge dp_i \\
&= \sum_i d\tilde{q}_i \wedge d\tilde{p}_i + df_i \wedge dp_i \\
&= \sum_i d\tilde{q}_i \wedge d\tilde{p}_i
\end{aligned}$$

  where the last equality follows by equation

□

1.2 Contact Manifolds

Throughout M is a manifold.

1.2.1 Basic concepts for contact manifolds

Definition 1.2.1. A k-dimensional Distribution on M is a choice of k-dimensional linear subspace $\xi_p \subset T_pM$ for each $p \in M$. This distribution is smooth if $\xi := \bigcup_{p\in M} \xi_p \subset TM$ is smooth sub-bundle. In this case we say ξ is a smooth distribution on M.

We have the following "local frame" criterion of smooth distributions

Proposition 1.2.1. *Let $\xi_p \subset T_pM$ constitute a k-dimensional distribution on M. Then $\xi = \bigcup_{p\in M} \xi_p \subset TM$ is smooth distribution on M if and only if every point $p \in M$ has a neighborhood U on which there are smooth vector fields $X_1, \cdots, X_k \in \Gamma(U, TM)$ such that $X_1|_q, \cdots, X_k|_q$ form a basis of ξ_q for every $q \in U$.*

We also have the the following 1-form criterion of smooth distributions

Proposition 1.2.2. *Let $\xi_p \subset T_pM$ constitute a k-dimensional distribution on M. Then $\xi = \bigcup_{p\in M} \xi_p \subset TM$ is smooth distribution on M if and only if every point $p \in M$ has a neighborhood U on which there are smooth $1-$forms $\alpha^1, \cdots, \alpha^{n-k}$ such that*

$$\xi_q = \ker \alpha_q^1 \cap \cdots \cap \ker \alpha_q^{n-k} \tag{1.7}$$

Proof. Suppose $\alpha^1, \cdots, \alpha^{n-k}$ are 1-forms satisfying equation (1.7) on neighborhood of p. We can extend them on possibly smaller neighborhood to smooth coframe $(\alpha^1, \cdots, \alpha^n)$. Let $(E_1, \cdots, E_n)$ be the dual frame. Then by (1.7) it follows that ξ is spanned by $E_{n-k+1}, \cdots, E_n$ on neighborhood of p. It follows by 1.2.1 that ξ is smooth distribution.
Conversely, suppose ξ is smooth. Then by 1.2.1 there is a neighborhood of any $p \in M$ on which there are smooth vector field $X_1, \cdots, X_k$ spanning ξ. On possibly smaller neighborhood of p, these can be extended to smooth frame $(X_1, \cdots, X_n)$. Let $(\epsilon^1, \cdots, \epsilon^n)$ be the dual coframe, then it is easy to see

$$\xi_q = \ker \epsilon_q^{k+1} \cap \cdots \cap \ker \epsilon_q^n$$

□

A particular case of the above proposition is the case of codimension 1 smooth distributions, for which at every point $p \in M$ there exist 1-form α on neighborhoood U of p such that $\ker \alpha = \xi$ on U. We say α is *local defining form* of ξ near p. One observes by basic linear that if α and α' are two local defining forms of ξ on U and U' respectively, then $\alpha' = f\alpha$ on $U \cap U'$ for some smooth non-vanishing function $f : U \cap U' \to \mathbb{R}$.

Definition 1.2.2. A *contact structure* on M is a smooth distribution ξ of codimension 1 on M such that for every p there is a local defining form α near p such that $d\alpha|_\xi$ is non-degenrate (i.e symplectic). The pair (M, ξ) is called *contact manifold*, and any local defining form α with the above property is called *(local) contact form.* If α is defined on all of M, we say α is a global contact form.

Remark 1.2.1. In fact it follows that *any* local defining form of contact structure ξ is contact form. Indeed, let α' be local defining form on U, pick $q \in U$ and let α be contact form on neighborhood U' of q. Then on $U \cap U'$ we have $\alpha' = f\alpha$. Taking differentials,

$$d\alpha' = df \wedge \alpha + f d\alpha$$

By restricting to ξ (noting that $\alpha|_\xi = 0$), we get $d\alpha'|_\xi = f d\alpha|_\xi$. Hence $d\alpha'|_\xi$ is non-degenerate on $U \cap U'$.

Remark 1.2.2. It follows from non-degeneracy that $\text{rank}\xi = 2n$, and hence M has odd dimension $2n + 1$

From now on (M, ξ) is a contact manifold. We have the following obvious proposition of symplectic linear algebra.

Proposition 1.2.3. *Let V be vector space and $\Omega : V \times V \to \mathbb{R}$ be skew symmetric bilinear map, such that $\Omega|_{W \times W}$ is non-degenerate where W is subspace of V of codimension 1, then $V = W \oplus \ker \Omega$*

Accordingly, given local contact form α on contact manifold (M, ξ). we get

$$T_pM = \ker \alpha_p \oplus \ker d\alpha_p$$

We have the following characterization of contact structures

Proposition 1.2.4. *ξ is contact structure on M if and only if $\alpha \wedge (d\alpha)^n \neq 0$ for any local defining 1-form.*

Proof. Take a basis $\{e_1, f_1, \cdots, e_n, f_n, r\}$ of T_pM such that $\text{span}\{e_1. \cdots, f_n\} = \xi_p = \ker \alpha_p$ and $\text{span}\{r\} = \ker d\alpha_p$, then

$$\alpha_p \wedge (d\alpha_p)^n(r, e_1, f_1, \cdots, e_n, f_n) = \alpha_p(r)(d\alpha_p)^n(e_1, f_1, \cdots, e_n, f_n)$$

Here we used the formula

$$\omega \wedge \eta(X_1, \cdots, X_n) = \sum_{(k,l) \text{ shuffles } \sigma} \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)})$$

and the observation $(d\alpha_p)^n(\cdots, r, \cdots) = 0$. As $\alpha_p(r) \neq 0$, we conclude $\alpha_p \wedge (d\alpha_p)^n \neq 0$ if and only if $(d\alpha_p)^n(e_1, f_1, \cdots, e_n, f_n) \neq 0$ if and only if $(d\alpha_p)^n|_{\xi_p}$ if and only if $(d\alpha_p)|_{\xi_p}$ is non-degenerate by corollary 1.1.2 □

Perhaps a natural question is when we can find a global contact, we have the following proposition.

Proposition 1.2.5. *A contact manifold (M, ξ) has a global contact form if and only if ξ is coorientable (that's TM/ξ is orientable).*

If a contact manifold (M, ξ) is given global contact form, we say (M, α) is contact manifold. We have the following definition.

Definition/Proposition 1.2.6. *Given a contact manifold (M, α), there is a* *unique vector field R_α satisfying*

$$\alpha(R_\alpha) = 1 \tag{1.8}$$

$$\iota_{R_\alpha} d\alpha = 0 \tag{1.9}$$

This vector field is called Reeb vector field determined by α. The contact form is invariant under the flow of R_α

Proof. $\ker d\alpha \simeq TM/\xi$ is a trivial bundle line bundle, hence there is a nowhere vanishing vector field $\tilde{R}$ such that $\tilde{R}_p \in \ker d\alpha_p$, so $\alpha(\tilde{R})$. It follows that $R_\alpha = \frac{\tilde{R}}{\alpha(\tilde{R})}$ satisfies 1.8 1.9. Uniqueness is clear. The last statement follows directly from Cartan formula of lie derivatives. □

Definition 1.2.3. A submanifold L of M is called *isotropic* if $T_xL \subset \xi_x$ for all $x \in L$

We have dimensional constraint on L

Definition/Proposition 1.2.7. *Suppose $L \hookrightarrow (M^{2n+1}, \xi)$ is isotropic, then* $\dim L \leq n$*. If* $\dim L = n$*, then L is called Legendrian submanifold.*

Proof. If L is isotropic, then $\alpha|_L = 0$. This implies $(d\alpha)|_L = d(\alpha|_L)$ if $T_x \subset (\xi_x, d\alpha_x)$ is isotropic, hence $\dim(T_xL) \leq \frac{1}{2}\dim(\xi_x) = n$ □

Now we define *contactomorphism* of contact manifolds

Definition 1.2.4. Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic if there is diffeomorphism $f : M_1 \to M_2$ such tha $Tf(\xi_1) = \xi_2$. If $\xi_i = \ker\ \alpha_i$, then this is equivalent to $f^*\alpha_2 = \lambda\alpha$, where $\lambda : M_1 \to \mathbb{R} \setminus \{0\}$

Let $(M, \xi = \ker\alpha)$ be a contact manifold. A *contact isotopy* is a smooth family ψ_t of contactomorphisms with $\psi_0 = \mathrm{id}$, then $\psi_t^*\alpha = \lambda_t\alpha$. Because $\lambda_0 = 1 > 0$, then

λ_t. Suppose that X_t is the generating time-dependent vector field. That's

$$\frac{d}{dt}\psi_t = X_t \circ \varphi_t$$

Then

$$\frac{d}{dt}\psi_t^* \alpha = \dot{\lambda}_t \alpha = \psi(\mu_t \alpha),$$

where $\mu_t = \dot{\lambda}_t \circ \psi_t^{-1}$. Because $\psi_t^* \mathcal{L}_{X_t} \alpha = \frac{d}{dt}\psi_t^* \alpha$, it follows that

$$\mathcal{L}_{X_t} \alpha = \mu_t \alpha$$

Conversely, given X_t satisfying this condition, we get

$$\frac{d}{dt}\psi_t^* \alpha = (\mu_t \circ \psi_t)\psi_t^* \alpha,$$

hence $\psi_t \alpha = e^{\int_0^t \mu_s \psi_s \; ds} \alpha$. A vector field X on M is called contact vector field. The discussion above illustrates that the lie algebra of group of contactomorphisms is the space of contact vector fields

Proposition 1.2.8. *Let $(M, \xi = \ker \alpha)$ be a contact manifold. Then there is a one-to-one correspondance between contact vector fields and functions $H : M \to \mathbb{R}$ given by*

- $X \to H_X = \alpha(X)$
- $X \to X_H$*, defined uniquely by*

$$\alpha(X_H) = H \ ; \ \iota_{X_H} d\alpha = dH(R_\alpha)\alpha - dH$$

Example 1.2.1.

(1) On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n, z)$, define $\alpha_1 = dz - \sum_{i=0}^n y_i dx_i$. Let's compute compute $\alpha_1 \wedge (d\alpha_1)^n$:

$$d\alpha_1 = \sum_i dx_i \wedge dy_i \implies (d\alpha_1)^n = n! dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

Hence

$$\alpha_1 \wedge d\alpha_1 = n! dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \neq 0.$$

α_1 is sometimes denoted by α_{std}. The reeb vector field is $R_{\alpha_{std}} = \frac{\partial}{\partial z}$

(2) Another contact structure on $\mathbb{R}^{2n+1}$ is defined by the form

$$\alpha_2 = dz + \sum_{j=1}^{n}(x_j dy_j - y_j dx_j) = dz + \sum_{j}^{n} r_j^2 d\theta_j,$$

where (r_j, θ_j) is the polar coordinates in $(x_j, y_j)-$plane $j = 1, \cdots, n$. It follows that $(\mathbb{R}^{2n+1}, \alpha_1)$ is strictly contactomorphic to $(\mathbb{R}^{2n+1}, \alpha_2$. Indeed, let

$$f(x_1, y_1, \cdots, x_n, y_n, z) = (\frac{y_1 - x_1}{2}, \frac{-x_1 - y_1}{2}, \cdots, \frac{y_n - x_n}{2}, \frac{-x_n - y_n}{2}, z - \frac{1}{2}\sum_j x_j y_j)$$

Then f is diffeomorphism and

$$\begin{aligned} f^*\alpha_2 &= d(z - \frac{1}{2}\sum_j x_j y_j) + \frac{1}{4}\sum_j (y_j - x_j) d(-x_j - y_j) + (x_j + y_j) d(y_j - x_j) \\ &= dz - \frac{1}{2}\sum_j (x_j dx_j + y_j dy_j) + \frac{1}{4}\sum_j x_j dx_j + x_j dy_j - y_j dx_j - y_j dy_j + x_j dy_j - x_j dy_j + y_j dy_j - \\ &= dz - \sum_j y_j dx_j = \alpha_1 \end{aligned}$$

(3) Regard S^{2n+1} as the set of unit vectors

$$\{(x_1, y_1, \cdots, x_{n+1}, y_{n+1}) | r = \sum_{i}^{n+1}(x_i^2 + y_i^2) = 1\}$$

We let $i : S^{2n+1} \hookrightarrow \mathbb{R}^{2n+1}$ be the inclusion. Let $\sigma = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$. We have

$$
\begin{aligned}
(d(\sigma))^{n-1} &= \left(\sum_i dx_i \wedge dy_k - dy_k \wedge dx_k\right)^{n-1} \\
&= 2^n \left(\sum_k dx_k \wedge dy_k\right) \\
&= 2^n (n-1)! \sum_k dx_1 \wedge dy_n \wedge \cdots \wedge d\widehat{x_k \wedge d}y_k \wedge \cdots \wedge dx_k \wedge dy_k
\end{aligned}
$$

and

$$
\begin{aligned}
rdr \wedge \sigma &= \frac{1}{2}\left(\sum_i (x_i dx_i - y_i dy_i)\right) \wedge \left(\sum_j (x_j dy_j - y_j dx_j)\right) \\
&= \frac{1}{2}\sum_i \sum_j x_i x_j dx_i \wedge dy_j + y_i x_j dy_i \wedge dy_j - x_i y_j dx_i \wedge dx_j - y_i y_j dy_i \wedge dx_j
\end{aligned}
$$

As can be easily seen, the wedge of each term on the right side of this equation with $i \neq j$ against $(d\sigma)^{n-1}$ is zero, hence $rdr \wedge \sigma \wedge (d\sigma)^{n-1}$ is

$$
2^{n-1}(n-1)! \left(\sum_i (x_i^2 + y_i^2) dx_i \wedge dy_i\right) \wedge \left(\sum_k dx_1 \wedge dy_1 \wedge \cdots \wedge dx_k \wedge dy_k \wedge \cdots \wedge dx_{n+1} \wedge dy_{n+1}\right)
$$

which is clearly equal to

$$
2^{n-1} n! r^2 dx_1 \wedge dy_1 \wedge \cdots dx_{n+1} \wedge dy_{n+1}
$$

which is not equal to 0 whenever $r \neq 0$. As S^{2n+1} is level surface of S^{2n+1} of r, then $\alpha = i^*\sigma$ is defines contact structure on S^{2n+1}.

(4) Let X be a manifold of dimension n, then the jet space $J^1(X) := T^*X \times \mathbb{R}$ is contact manifold with contact form $\alpha = dz - \lambda_{can}$, where z is the real coordinate.

Indeed,

$$\alpha \wedge (d\alpha)^n = (dz - \lambda_{can}) \wedge (-d\lambda_{can})^n = (-1))^n dz \wedge d\lambda_{can}$$

The reeb vector field is $R_\alpha = \frac{\partial}{\partial z}$, and for any function $f : X \to \mathbb{R}$. The submanifold

$$\{(q, df(q), f(q)|q \in X\} \subset J^1M$$

is Legendrian.

Now we describe how contact manifolds arise from symplectic manifolds, but first let's give a definition

Definition 1.2.5. Let (W^{2n+2}, ω) be symplectic manifold. A vector field X on W is called *Liouvelle vector field* if $\mathcal{L}_X\omega = \omega$.

Remark 1.2.3. If (W, ω) has liouvelle vector field, then it is exact. Indeed by Cartan formula $\omega = \mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega = d(\iota_X\omega)$

We have the required proposition

Proposition 1.2.9. *Let X be a liouvelle vector field on (W, ω). Let $M \hookrightarrow W$ be a hypersurface (i.e submanifold of codimension 1). If X is transverst to M (i.e $X_p \notin T_pM$ for all $p \in M$), then $(M, \iota_X\omega|_M)$ is contact manifold.*

Proof. let $\tilde{\alpha} = \iota_X\omega$, we have

$$\tilde{\alpha} \wedge (d\tilde{\alpha})^n = (\iota_X\omega) \wedge (d\iota_X\omega)^n = (\iota_X\omega) \wedge \omega^n = \frac{1}{n+1}\iota_X(\omega^{n+1})$$

where in the last equality we used the antiderivation property of interior multiplication. Because ω^{n+1} is volume form on W, it follows that $\tilde{\alpha} \wedge (d\tilde{\alpha})^n$ restricted to M is volume form, for if $v_1, \cdots, v_{2n+1}$ span T_pM, then $v_1, \cdots, v_{2n+1}, X_p$ span T_pW, Hence

$$\tilde{\alpha}_p \wedge (d\tilde{\alpha}_p)^n(v_1, \cdots, v_{2n+1}) = \omega_p^{n+1}(X_p, v_1, \cdots, v_{2n+1}) \neq 0$$

□

In fact every contact manifold arises in this way, as we will see after defining symplectization.

1.2.2 Symplectization and Contactisation

Symplectization

Definition/Proposition 1.2.10. *Let (M^{2n+1}, ξ) and (N^{2n+1}, ξ) be contact manifolds. Then*

(1) The following set

$$W = \{(x, \beta) | x \in M,\ \beta \in T^*_p M\ s.t\ \ker(\beta) = \xi_p\}$$

is symplectic submanifold of (T^*M, ω_{can}) and is called the symplectization of (M, ξ).

(2) If ξ is cooriented and α is global form giving the coorientation, then the map

$$\Phi_\alpha : \mathbb{R}^* \times M \to W,\ ;\ (t, x) \mapsto t\alpha_x$$

is symplectomorphism from $(\mathbb{R}^* \times M, d(t\pi_2^*\alpha))$ to (W, ω_{can}). In this case we call the component $(W_0 = \Phi_\alpha(\mathbb{R}_+ \times M), \omega_{can}|_{W_0})$ the intrinsic symplectization of M (It doesn't depend on choice of α). Given α, we call $(\mathbb{R} \times M, d(e^s\alpha))$ the extrinsic symplectization. It is symplectomorphic to W_0 via composition of $(s, x) \mapsto (e^s, x)$ and Ψ_α.

(3) let $f : M \to N$ be a diffeomorphism. Then, denoting by W and X the symplectizations of M and N respectively, f is contactmorphism if and only if $f_\#(W) = X$ in which case $f_\#|_W : W \to X$ is symplectomorphism. If M, N are cooriented by contact forms α, β, then the same assertion holds with W replaced with W_0 given that f preserves coorientations. If f is strict contactomorphism,

then under identification above of intrinsic symplectizations $\mathbb{R} \times M$ and $\mathbb{R} \times N$ with W_0 and X_0, $f_\#$ is given by $(s, x) \mapsto (s, f(x))$.

Proof.

(1) We begin by showing W is submanifold of dimension $2n+2$. We let α be local defining form of ξ near p, we extend to local coframe $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_n$ defined on $V \ni p$. Let $(U, x_1, \cdots, x_n)$ be a coordinate chart around p. We let for each $\gamma \in T_x^*U$, we let $y^i(\gamma)$ be the i-th component of γ, with respect to the above frame, that's $\gamma = \sum_i y^i(\gamma)\alpha_i|_x$. Defining $\overline{x_i} = x_i \circ \pi : T^*U \to \mathbb{R}$, we get coordinate chart of $(T^*U, \overline{x_1}, \cdots, \overline{x_n}, y_1, \cdots, y_n)$ of T_*M. Note that $W \cap T^*U$ is the intersection of $T^*M \setminus \{0\}$ and the set $\{y_2 = \cdots = y_n = 0\}$, which is submanifold of T^*U. It follows that W is submanifold of T^*M. Now we check it is symplectic, that is $d\lambda_{can}|_W$ is symplectic. Let α be local defining form defined on U, and let $\phi : \mathbb{R}^* \times U \to W$ be defined by $\phi(t, x) = t\alpha_x$. We observe that $\phi^*(\lambda_{can}|_W) = t\alpha$. But as $\dim W = 2n+1$ and

$$(d(t\alpha))^{n+1} = (dt \wedge \alpha + td\alpha)^{n+1} = (n+1)t^n dt \wedge \alpha \wedge (d\alpha)n$$

But the latter is non-zero, so $\phi^*(\omega_{can}|_W) = \phi^*(d\lambda_{can}|_W)$ is non-degenerate. This proves (1)

(2) follows from last part of the previous argument.

(3) Clear from the definitions. □

Now we make an important observation. Keeping the notation of the above proposition. If $\psi_t : M \to M$ is contact isotopy, then the the lift $\tilde{\psi}_t : W \to W$ Hamiltonian isotopy. Indeed, let $\tilde{X}_t$ be the vector field generating $\tilde{\psi}$ and observe that

$$\psi_t^*(\lambda_{can}|_W) = \lambda_{can}|_W =: \theta$$

Differentiating this equation with respect to t, we get

$$0 = \mathcal{L}_{\tilde{X}_t}\theta = d(\theta(\tilde{X}_t)) + \iota_{\tilde{X}_t} d\theta$$

which proves $\tilde{X}_t$ is Hamiltonian.

Contactisation

Let $(M, d\theta)$ be exact symplectic manifold. We define its contactisation to be (W, β) where $W = M \times \mathbb{R}$ with coordinate z on $\mathbb{R}$ and $\beta = dz + \theta$. It is obvious that β is indeed a contact form, for

$$\beta \wedge (d\beta)^n = (dz + \theta) \wedge (dz + \theta)^n = dz \wedge (d\theta)^n$$

which is non-degenerate. The Reeb vector field is $R = \frac{\partial}{\partial z}$.
Now we say (L, ι) is a nice exact Lagrangian immersed submanifold in $(M, d\theta)$ if it is a generic immersed submanifold (in the sense that all self intersections are double, transversal and isolated) and $\iota^*\theta = df$ such that for every self-intersection point p the values of the potential f at the two preimages of p are distinct.
Any nice exact Lagrangian submanifold (L, ι) of M with potential f defines Lagrangian embedding

$$\iota^+ : L \to W \ , \ x \to (\iota(x), -f(x))$$

The image of ι^+ is called Legendrian lift of (L, ι), it is indeed legendrian $(\iota^+)^*(dz + \theta) = -df + \iota^*\theta = 0$. On the other hand any Legendrian submanifold of W projects to Lagrangian immersed sumbanifold of M via the projection $(x, z) \mapsto x$. Points of self intersection of L corresponds to Reeb chords of L^+.

1.2.3 Moser type theorems

First of all we have the following theorem of Gray, which roughly states that there is not non-trivial deformations of contact structure.

Theorem 1.2.11. *If* $\{\xi_t\}_{t\in[0,1]}$ *is smooth family of contact structures on compact* M*. Then there is isotopy* $\{\phi_t : M \to M\}_{t\in[0,1]}$ *such that* $(\phi_t)_*(\xi_0) = \xi_t$

Proof. Let α_t be smooth family of 1-forms such that $\ker \alpha_t = \xi_t$. We want to find ϕ_t, such that

$$\phi_t^* \alpha_t = f_t \alpha_0 \tag{1.10}$$

for some non-vanishing functions $f_t : M \to \mathbb{R} - \{0\}$. We find ψ_t as flow of time dependent vector X_t, that's

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t \ , \ \psi_0 = \mathrm{id}$$

Differentiating 1.10, we get

$$\psi_t^*(\frac{d}{dt}\psi_t + \mathcal{L}_{X_t}\alpha_t) = \frac{1}{f_t}\frac{df_t}{dt}\psi_t^*\alpha_t = \psi_t^*(g_t\alpha_t),$$

where $g_t = (\frac{1}{f_t}\frac{df_t}{dt}) \circ \psi_t$. If we choose X_t to lie in $\ker \alpha_t = \xi_t$, then the equation is satisfied if

$$\psi_t^*(\frac{d}{dt}\alpha_t + \iota_{X_t} d\alpha_t) = g_t \alpha_t \tag{1.11}$$

Plugging the Reeb vector field R_t in 1.11, we get

$$\frac{d}{dt}\alpha_t(R_t) = g_t$$

We use this to define g_t. By the non-degenracy of $d\alpha_t|_{\xi_t}$, and the fact $R_t \in \ker(g_t\alpha_t - \frac{d}{dt}\alpha_t)$, there is a unique vector field X_t satisfying 1.11.

□

Theorem 1.2.12. *Let* (M^{2n+1}, α) *be a contact manifold. For any* $p \in M$*, there is* *a coordinate* $(x_1, y_1, \cdots, x_n, y_n, z)$ *around* p *in which* α *is standard form, that's*

$$\alpha = dz - \sum_{i=1}^{n} y_i dx_i$$

Proof. Let $(U, (u_1, \cdots, u_{2n}))$ be coordinate cube, such that $R = R_\alpha = \frac{\partial}{\partial u_1}$. Let S be the the submanifold defined by $u_1 = 0$. Since S is trasversal to R, then $d\theta|_S$ is symplectic, hence after shrinking U and S if necessary, we can find Darbouc coordinates $(x_1, y_1, \cdots, x_n, y_n)$ for S and extend it to U by being constant on integral curves of R. Let θ be the 1-form $\sum_i y_i dx_i$ on U, so $d\alpha|_S = -d\theta|S$. But as $\iota_R d\alpha = \iota_R d\theta$, it follows that $d\theta + d\alpha = 0$ at points of S. Then $\mathcal{L}_R\theta = \mathcal{L}_R\alpha = 0$ which implies $d(\theta+\alpha)$ is invariant under the flow of R, hence $d(\theta+\alpha) = 0$ on U. By Poincare lemma, there is smooth function $z : U \to \mathbb{R}$ such that $dz = \theta + \alpha$, we can assume that $z(p) = 0$. Since $dz_p(R_p) = 1$, we have $\{dx_i|_p, dy_i|_p, dz_p\}$ are linearly independent, so there is neighborhood around p on which $(x_1, y_1, \cdots, x_n, y_n, z)$ are coordinates. □

Theorem 1.2.13. *Let* $L \subset (M, \xi)$ *be Legendrian submanifold, then there exists neighborhood* $\mathcal{U}$ *of* L *and neighborhood* $\mathcal{U}_0$ *of the zero section of* $J^1(L)$ *such that there exist contactomorphism* $\psi : \mathcal{U}_0 \to \mathcal{U}$ *such that* $\psi(L_0) = L$*, where* L_0 *is the zero section of* L *in* $J^1(L)$ *(in otherwords* $L_0 = L \times \{0\}$

1.2.4 Generating families

Let M be a manifold, and let $f : M \times \mathbb{R}^N \to \mathbb{R}^N$ be a smooth map, where $M \times \mathbb{R}^N$ has coordinates (x, ξ). Suppose that 0 is regular value of $\partial_\xi f : M \times \mathbb{R}^N \to \mathbb{R}^N$. The submanifold $(\partial_\eta f)^{-1}(0)$ is called the *fiber critical set* and denote it by Σ_f. Define

$$i_f : \Sigma_f \to T^*M \quad ; \quad (x, \eta) \mapsto (x, \partial_x f(x, \eta)),$$

and

$$j_f : \Sigma_f \to J^1M \quad ; \quad (x, \eta) \mapsto (x, \partial_x f(x, \eta), f(x, \eta)),$$

it can be checked that i_f and j_f are immersions. The image L of i_f is immersed Lagrangian submanifold; the image λ of j_f is immersed Legendrian submanifold. We say that f generates L and Λ or that f is generating family. It is important to note that the Reeb chords of Λ are in bijective correspondence with critical points of $\delta(x, \xi, \xi') = f(x, \xi) - f(x, \xi')$ with positive critical value.

We say a generating family f is linear at infinity, if there is a linear function $A : \mathbb{R}^N \to \mathbb{R}$ such that $f(x, \xi) = A(\xi)$ outside a compact set.

Chapter 2

Lagrangian cobordism

2.0.1 Definition and compatible generating functions

Definition 2.0.1. Let Λ^- and Λ^+ be two legendrian submanifolds of $(M, \xi = \ker(\alpha))$, then a lagrangian cobordism between Λ_- and λ_+ is a lagrangian submanfiold of the symplectisation $(\mathbb{R} \times M, d(e^t\alpha))$ such that there exists T with

$$\overline{L} \cap ((-\infty, -T] \times X) = (-\infty, T] \times \Lambda_-$$

$$\overline{L} \cap ([T, \infty) \times X) = [T, \infty) \times \Lambda_+.$$

That cobordism is denoted by $\Lambda_- \prec \Lambda_+$.

We will be mainly concerned with lagrangian cobordisms of legendrian submanifolds in $(J^1M, dz - \lambda)$. To use generating families to study them, we need to identify its symplectisation with cotangent bundle.

Proposition 2.0.1. *Let M be a manifold, then then the symplectisation $(\mathbb{R} \times J^1M, d(e^t\alpha))$ of $(J^1M, \alpha = dz - \lambda)$ is symplectomorphic to $(T^*(\mathbb{R}_+ \times M), \omega_{can})$.*

In fact the symplectomorphism is given by

$$\theta : \mathbb{R} \times J^1 M \to T^*(\mathbb{R}_+ \times M) \quad ; \quad (s, q, p, z) \mapsto (e^s, q, z, e^s p)$$

Proof. Suppose that a point in $T^*(\mathbb{R}_+ \times M)$ has coordinates (t, x, z, y), then

$$\begin{aligned} \theta^*(\omega_{can}) &= \theta^*(-d(zdt + ydx)) \\ &= -d(zd(e^s) + e^s pdq) \\ &= -d(-e^s dz + e^s pdq) = d(e^s \alpha) \end{aligned}$$

□

So the lagrangian cobordism $\overline{L}$ can be viewed as Lagrangian submanifold of $T^*(\mathbb{R}_+ \times M)$, and hence may be constructed by generating function. We will be interested in the case, we have a generating function compatible with those of given ones for Λ_- and Λ_+

Definition 2.0.2. Let $f_\pm : M \times \mathbb{R}^N$ and $F : (\mathbb{R}_+ \times M) \times \mathbb{R}^N$ be functions. We say (F, f_-, f_+) is compatible if for some $S > 1$, get

$$f(t, x, \xi) = \begin{cases} tf_-(x, \xi) & t \leq 1/S \\ tf_+(x, \xi) & t \geq S \end{cases}$$

A *gf-compatible* lagrangian cobordism consists of a lagrangian cobordism $\Lambda_- \prec_{\overline{L}} \Lambda_+$ and a compatible triple of generating functions (F, f_-, f_+) of $\theta(\overline{L}) \subset T^*(\mathbb{R}_+ \times M)$, $\Lambda_-, \Lambda_+ \subset J^1 M$ respectively. A gf-compatible lagrangian cobordism is denoted by

$$(\Lambda_-, f_-) \prec_{(\overline{L}, F)} (\Lambda_+, f_+)$$

We should note that if we are given compatible triple (F, f_-, f_+) with 0 being a regular value of $\partial_\xi f_-, \partial_\xi f_+, \partial_\xi F$, then F determines an *immersed* lagrangian cobordism from λ_- to Λ_+. We call the resulting immersed cobordism together with the triple a *gf-compatible immersed lagrangian cobordism.*

2.0.2 Isotopy is realised by lagrangian cobordism

The aim of this section is to prove the following theorem which is due to Chantraine [3]

Theorem 2.0.2. *[legendrian isotopy gives "cylinderical" cobordism] Let* $(M, \xi = \ker\alpha)$ *be a compact contact manifold, and* $j_t : \Lambda \to M$ *be isotopy of legendrian embeddings into* M*. Then there is a lagrangian cobordism from* $\Lambda_0 = j_t(\Lambda)$ *to* $\Lambda_1 = j_1(\Lambda)$

We need to prove isotopy extension theorem, which says that an isotopy of isotropic submanifolds can be realized by ambient contact isotopy. First we need the following lemma

Lemma 2.0.3. *Let* M *be a manifold, and* S *be a submanifold of* M*. Suppose that* $f : S \to \mathbb{R}$ *is smooth function, and* λ *is one form in* N *along* S *(that's* $\lambda \in \Gamma(S, T^*M|_S)$*). If*

$$df_p(v) = \lambda_p(v) \quad \textit{for } v \in T_pS,$$

then there is $F : M \to \mathbb{R}$ *such that*

- $F|_S = f$
- $dF_p = \lambda_p$ *for all* $p \in S$

If S *is compact, then we can choose* F *to be compactly supported.*

We illustrate the lemma by proving the case $M = \mathbb{R}^m \times \mathbb{R}^n$, $S = \mathbb{R}^m \hookrightarrow M$ is submaninfold by inclusion of the first coordinate. So we have as given $f : \mathbb{R}^m \to \mathbb{R}$ and smooth family of linear maps $\alpha_x : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ parametrized by $x \in S$, such that $\alpha_x((v, 0)) = df_x(v)$. We construct F by the formula

$$F(x, y) = f(x) + \alpha_x((0, y))$$

We see that

- $F(x, 0) = f(x)$
- $dF_{(x,0)}(v_1, v_2) = df_x(v_1) + \alpha_x((0, v_2) = \alpha_x((v_1, v_2))$

This construction can be adapted to the general case by use of tubular neighborhood and cuf off function.

Theorem 2.0.4 (isotopy extension theorem)**.** *Let $j_t : L \to (M, \xi = \ker(\alpha))$ be an isotopy of isotropic embeddings of a closed manifold L in a contact manifold. Then there exist compactly supported contact isotopy ψ_t of (M, ξ)*

Proof. Define time-dependent vector field X_t along $j_t(L)$ by

$$X_t \circ j_t = \frac{d}{dt} j_t$$

Assume L is submanifold of M and j_0 the inclusion $L \subset M$. We want to find compactly supported smooth function $\tilde{H}_t : M \to \mathbb{R}$ whoose hamiltonian vector field $\tilde{X}_t$ equals X_t along $j_t(L)$, this will prove the theorem. $\tilde{X}_t$ is defined in terms of $\tilde{H}_t$ by

$$\alpha(\tilde{X}_t) = \tilde{H}_t \quad , \quad \iota_{X_t} d\alpha = d\tilde{H}_t(R_\alpha)\alpha - d\tilde{H}_t$$

We need

$$\alpha(X_t) = \tilde{H}_t \quad , \quad \iota_{X_t} d\alpha = d\tilde{H}_t(R_\alpha)\alpha - d\tilde{H}_t \quad \text{along } j_t(L) \tag{2.1}$$

We use the lemma above to construct $\tilde{H}_t$ with 2.1 satisfied. Define $H_t : j_t(L) \to \mathbb{R}$ by $H_t = \alpha(X_t)$, and let λ_t be the one form along $j_t(L)$ defined by

$$\lambda_t = -\iota_{X_t} d\alpha$$

In particular $\lambda(R_\alpha) = 0$. So, what we need is $\tilde{H}_t$ such that

- $\tilde{H}_t = H_t$ along $j_t(L)$

- $d\tilde{H}_t = \lambda$ along $j_t(L)$

By lemma, we need only to show that

$$dH_t(v) = \lambda_t(v) \quad \text{for } v \in T(j_t(L))$$

That's

$$d(\iota_{X_t}\alpha)(v) = -\iota_{X_t} d\alpha(v) \quad \text{for } v \in T(j_t(L))$$

This is equivalent to

$$j_t^*(\iota_{X_t} d\alpha) + j_t^* d(\iota_{X_t}\alpha) \equiv 0$$

which is in turn equivalent to

$$0 = j_t^*(\mathcal{L}_{X_t}) = \frac{d}{dt}(j_t^*\alpha)$$

but this is automatically true as j_t is isotropic embedding. □

Now we sketch the proof of theorem 2.0.2. We let j_t be such isotopy, then we extend j_t to compactly supported contact isotopy ψ_t. This in turn lift to $\tilde{\psi}_t : \tilde{M} \to \tilde{M}$. As we have already seen in subsection 1.2.2, $\tilde{\psi}_t$ is generated by Hamiltonian vector field X_{H_t} for function $H_t : \tilde{M} \to \mathbb{R}$. Pick $S > 0$ We define Hamiltonian $H_t' : \tilde{M} \to \mathbb{R}$ by

$$H_t'(s,x) = \begin{cases} H_t(s,x) & s > S \\ 0 & s < -S \end{cases},$$

and let ϕ_t be its Hamiltonian flow, that's the one generated by $X_{H_t'}$. The following properties of ϕ_t are clear

- $\phi_t(s,x) = \tilde{\psi}_t(s,x) = (s, \psi_t(x))$ for $s > S$
- $\phi_t(s,x) = (s,x)$ for $s < -S$.

Let T be large enough so that $\phi_t([-S,S]\times M)\subset(-T,T)\times M$. Denote by $\overline{L}$ the lagrangian submanifold $\phi_1(\mathbb{R}\times\Lambda_-)$ (note that ϕ_1 is hamiltonian diffeomorphism). Then by the properties above we get

- $\overline{L}\cap([T,\infty)\times M)=[T,\infty)\times\Lambda_+$: By choice of T and properties of ϕ_t above, we note that $\phi_1(s,x)$ lies in $[T,\infty)\times M$ if and only if $s\geq T$. So the intersection on the left hand side is $\phi_1([T,\infty)\times\Lambda_-)=(\mathrm{id}\times\psi_1)([T,\infty)\times\Lambda_-)=[T,\infty)\times\Lambda_+$ as required.
- $\overline{L}\cap((-\infty,-T]\times M)=(-\infty,T]\times\Lambda_-$: Similar.

2.0.3 Some results on gf-compatible lagrangian cobordisms

Theorem 2.0.5. *Let $\Lambda\subset J^1M$ be a legendrian submanifold with linear-at-infinity genergating function f, then there exist an immersed gf-compatible lagrangian cobordism $\emptyset\prec_{(\overline{L},F)}(\Lambda,f)$*

Proof. The idea is to deform the f to linear function. First let f agree with liner function A outside a compact subset of $M\times\mathbb{R}^N$. We have to construct a function $F:\mathbb{R}_+\times M\times\mathbb{R}^N\to\mathbb{R}$ satisfying

1. for all t, $F(t,x,\xi)=B_t(\xi)$ outside a compact set of $\{t\}\times M\times\mathbb{R}^N$ for some non-zero linear function B_t
2. There exists $T>0$ such that $F(t,x,\xi)$ equals $B_t(\xi)$ for $t<1/T$ and $tf(x,\xi)$ for all $t>T$.
3. 0 is regular value of $\partial_\xi F$

The construction of a function $G:\mathbb{R}_+\times M\times\mathbb{R}^N\to\mathbb{R}$ satisfying (1) and (2) is simple. Just let $\rho:\mathbb{R}_+\to\mathbb{R}$ which is 0 on $(0,1]$ and 1 on $[2,)$. Define G by

$$G(t,x,\xi)=t(\sigma(t)f(x,\xi)+(1-\sigma(t))A(\xi)).$$

For any $T > 2$ to see that (1) and (2) are satisfied. Moreover there for $t < 1/T$, $G(t, x, \xi)$ agrees whith $tA(\xi)$ outside a compact set of $\{t\ \} \times M \times \mathbb{R}^N$. Now we need to modify G to F to get the final property (3) satisfied. The modification is quite standard, we perturbate be adding $\epsilon \cdot \xi$ for well chosen ϵ. Note that as f is linear-at-infinity and the set of critical p, there exist a open and oints convex ball U around 0 which consists entirley of regular values of ∂_ξ. There exists $\epsilon \in U$ which is regular value of $\partial_\xi G$ and such that $\epsilon \cdot \xi \neq = tA(\xi)$ for all t. Choose a smooth path $\sigma : \mathbb{R}_+ \to U$ such that $\sigma(t) = \epsilon$ for $t < 2$ and $\sigma(t) \cdot \xi \neq tA(\xi)$ for $t \in [2, T]$ and $\sigma(t) = 0$ for $t \in [T,)$. We define F to be

$$F(t, x, \xi) = G(t, x, \xi) - \epsilon(t) \cdot \xi = \begin{cases} G(x, t, \xi) - \epsilon \cdot \xi & t \leq 2 \\ tf(x, \xi) - \sigma(t) \cdot \xi & t \in [2, T] \\ tf(x, \xi) & t \geq Tx \end{cases}$$

The properties (1) and (2) are clearly satisfied for G. Now we prove (3). One one hand ϵ is regular value of $\partial_\xi G$, so 0 is regular value of $\partial_\xi F = \partial_\xi G - \epsilon$ for $t \leq 2$. On the other hand we have for $t \geq 2$, $\partial_\xi f(t, x, \xi) = 0$ if and only if $\partial f(x, \xi) = \sigma(t)/t$, but by convexity of U, $\sigma(t)/t$ is regular value of $\partial f(x, \xi)$. It follows immediately that 0 is regular value of $\partial_\xi F$ as $d(\partial_\xi F)(t, x, \xi) : \mathbb{R} \oplus (T_x M \oplus \mathbb{R}^N) \to \mathbb{R}^N$ is

$$\begin{pmatrix} \partial_\xi f(t, \xi) - \sigma'(t) & t \cdot d(\partial_\xi f)(x, \xi) \end{pmatrix}$$

□

We need a lemma

Lemma 2.0.6. *Let $(f_t : M \times \mathbb{R}^N \to \mathbb{R})_{t \in \mathbb{R}_+}$ be a smooth one parameter family of generating functions. Let $F : \mathbb{R}_+ \times M \times \mathbb{R}^N \to \mathbb{R}^N$ be defined*

$$F(t, x, \xi) = tf_t(x, \xi)$$

Suppose that

- 0 *is regular value of* F
- *for all* (x, ξ, ξ') *in the fiber critical set of* $\delta_t(x, \xi, \xi') = f_t(x, \xi) - f_t(x, \xi')$, *we have*

$$\partial_x \delta_t(x, \xi, \xi') \implies \delta_t(x, \xi, \xi') \neq -t\partial_t \delta_t(x, \xi, \xi'),$$

then F *generates an embedded submanifold in* $T^*(\mathbb{R}_+ \times M)$

Proof. Since 0 is regular value of $\partial_\xi F$, then F generates an immersed lagrangian

$$\overline{L} = \{(t, x, f_t(x, \xi) + t\partial_t f_t(x, \xi), t\partial_x f(x, \xi)) | \partial_\xi f(x, \xi) = 0\}.$$

We need to find double points... So, double points are in bijective correspondence with points (t, x, ξ, ξ') with $\xi \neq \xi'$ satisfying

- (x, ξ, ξ') is in the fiber critical set of λ_t
- $\partial_x \delta_t(x, \xi, \xi') = 0$
- $\delta_t f(x, \xi, \xi') = -t\delta_t f(x, \xi, \xi')$

The lemma clearly follows. □

We have the following version of 2.0.2 with gf-compatible cobordisms.

Theorem 2.0.7. *Suppose that* Λ_- *is legendrian submanifold of* $J^1 M$ *with linear-at-infinity generating function* f_-, *and that* Λ_- *is legendrian isotopic to* Λ_+ *with linear-at-infinity generating function* f_+. *Then there is a gf-compatible cobordism* $(\Lambda_-, f_-) \prec_{(\overline{L}, F)} (\Lambda_+, f_+)$

Proof. (Sketch.) Let $(j_t : \Lambda \to M)_{t \in \mathbb{R}_+}$ be the legendrian isotopy. Denote $\Lambda_t = j_t(\Lambda)$ such that $\Lambda_t = \Lambda_-$ for $t \leq 1/T$ and $\Lambda_t = \Lambda_+$ for $t \geq T$. There is a one-parameter family of generating functions linear-at-infinity f_t that generates Λ_t.

We will reparametrize the one-parameter family f_t, so that $F(t,x,\xi) = tf_t(x,\xi)$ satisfies lemma 2.0.6. Since 0 is regular value of $\partial_\xi f_t$ for all t, then 0 is regular value of $\partial_\xi F$. Observe that $\partial_x \delta_t(x,\xi,\xi') = 0$ if (x,ξ) and (x,ξ') are endpoints of reeb chord, where then the length of the reeb chord is $|\delta_t(x,\xi,\xi')|$. Let $h > 0$ be the minimum length of reeb chords of all of the legendrians Λ_t of the legnedrian isotopy. It suffices to show that for every (x,ξ,ξ') in the fiber critical set of δ_t

$$|\partial_t \delta_t(x,\xi,\xi')| < \frac{h}{t}.$$

Since the fiber critical set of δ_t is compact for each t, and $\partial_t\delta_t$ vanishes outside a compact interval, then $\partial_t\delta_t$ is bounded as function of (t,x,ξ,ξ') where (x,ξ,ξ') lies in the fiber critical set of δ_t. Choose $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$0 < \rho'(t) < \frac{h}{t \max |\partial_t \delta_t|},$$

then letting $\tilde{f}_t = f_{\rho(t)}$, $\tilde{\delta}_t(x,\xi,\xi') = \tilde{f}_t(x,\xi) - \tilde{f}_t(x,\xi')$ we get

$$|\partial_t \tilde{\delta}_t(x,\xi,\xi')|_{t=t_0}| = \rho'(t_0)|\partial_t\delta_t(x,\xi,\xi')|_{t=\rho(t_0)}| < \frac{h}{t_0 \max|\partial_t\delta_t|} \cdot |\partial_t \delta_t(x,\xi,\xi')|_{t=\rho(t_0)}| < \frac{h}{t_0}$$

□

Chapter 3

Lagrangian Surgery

Recall that an immersed submanifold of M is a pair (L, ι) of manifold L and immersion $\iota : L \to M$. In this chapter all immersed submanifolds are considered to be generic. That's all points of self intersections are double,transversal and isolated. So if x is a point of self intersection, then the pre-image of ι consists of two points $\{x_1, x_2\}$ and there is open neighborhood $U \subset M$ of x containing no other self intersection such that $\iota^{-1}(U)$ is union of disjoint open neighborhoods U_1, U_2 of x_1, x_2 respectively and $\iota|_{U_i}$ is embedding, in particular $U \cap \iota(L) = \iota(U_1) \cup \iota(U_2)$ and $\iota(U_1)$ intersect $\iota(U_2)$ transversally at x. An equipment of (L, ι) at x, is an order of tangent spaces $T_x\iota(U_1)$ and $T_x(\iota(U_2))$. We say (L, ι) oriented if L is oriented.

Throughout $P^n = S^{n-1} \times S^1$ and Q^n the manifold obtained from $S^{n-1} \times I$ by identifying $(x, 1)$ with $(\tau(x), 0)$ where $\tau : S^{n-1} \to S^{n-1}$ is the standard orientation reversing involution (given by reflection).

All symplectic manifolds are oriented by its symplectic structure, that's (M, ω) is oriented by ω^n.

3.1 First Description

Lagrangian handles

Let $(\mathbb{R}^{2n}, \omega_{std})$ be the standard symplectic manifold, suppose it is oriented by the form ω^n. Denote by L_x and L_y the Lagrangian submanifolds $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ respectively. We have the following definition

Definition 3.1.1. Let $\epsilon > 0$ and $\sigma : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{2n}$ be a Lagrangian embedding satisfying:

- $\sigma(S^{n-1} \times (-\epsilon, \epsilon)) \subset B(0, \epsilon)$
- $\sigma(z, t) = -tz$ for $(z, t) \in S^{n-1} \times (-\infty, -\epsilon]$
- $\sigma(z, t) = tz$ for $(z, t) \in S^{n-1} \times [\epsilon, \infty)$.

So in particular, denoting the two $n-$dimensional discs of radius ϵ containing 0 by $B_x \subset L_x$ and $B_y \subset L_y$ respectively, we get

$$\sigma(S^{n-1} \times (-\infty, -\epsilon]) = L_x \setminus B_x \ ; \ \sigma(S^{n-1} \times [\epsilon, \infty)) = L_y \setminus B_y$$

σ or its image is called an *ϵ-Lagrangian handle.* If Γ denotes this image and L_x and L_y are oriented, then we say the Lagrangian handle is *positive* (denoted $\operatorname{sgn} \Gamma = 1$) if $L_x \setminus B_x \hookrightarrow \Gamma$ and $L_y \setminus B_y \hookrightarrow \Gamma$ induce the same orientations on Γ and *negative* (denoted $\operatorname{sgn} \Gamma = -1$) if these orientations are different.

Lagrangian handles exist by the following theorem

Theorem 3.1.1. *Suppose that L_x and L_y are oriented and let $\epsilon > 0$. Then there exists an ϵ-Lagrangian handle Γ with* $\operatorname{sgn} \Gamma = (-1)^{\frac{n(n-1)}{2}+1} L_x \cdot L_y$

Proof. Let f and g be functions $\mathbb{R} \to \mathbb{R}_+$ such that

- $f(t) = t$ for $t \geq \epsilon$ and $f(t) = 0$ for $t \leq -\epsilon$
- $g(t) = -t$ for $t \leq -\epsilon$ and $g(t) = 0$ for $t \geq \epsilon$
- $t \mapsto (f(t), g(t))$ is embedding $\mathbb{R} \to \mathbb{R}^2$
- $\sqrt{f(t)^2 + g(t)^2} \leq \epsilon$ for $t \in (-\epsilon, \epsilon)$

Define $\sigma : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{2n}$ by

$$\sigma(z, t) = (f(t)z, g(t)z).$$

We have

1. $\sigma^* \omega_{std} = 0$: Let $\iota : S^{n-1} \times \mathbb{R} \hookrightarrow \mathbb{R}^n \times \mathbb{R}$ be the inclusion, and define $\tilde{\sigma} : \mathbb{R}^n \times \mathbb{R}$ by $\tilde{\sigma}(z, t) = (f(t)z, g(t)z)$, with coordinates of first factor $\mathbb{R}^n$ be $z_1, \cdots z_n$. We have $\sigma = \tilde{\sigma} \circ \iota$, and

$$\begin{aligned} \tilde{\sigma}^* \omega_{std} &= \sum_i (f'(t) z_i dt + f(t) dz_i) \wedge (g'(t) z_i dt + g(t) dz_i) \\ &= \sum_i (f(t) g'(t) - g'(t) f(t)) z_i dz_i \wedge dt \end{aligned}$$

and it follows that

$$\sigma^* \omega_{std} = \sum_i (f(t) g'(t) - g'(t) f(t)) (z_i \circ \iota) d(z_i \circ \iota) \wedge dt = 0,$$

because $\sum_i (z_i \circ \iota)^2 = 1 \implies \sum_i (z_i \circ \iota) d(z_i \circ \iota) = 0$.

2. σ is immersion: We compute the differential, let $(v, s) \in T_{(t,z)}(\mathbb{S}^{n-1} \times \mathbb{R}) = T_z S^{n-1} \times T_t \mathbb{R} = T_z S^{n-1} \mathbb{R}$, we have

$$d\sigma(z, t)(v, s) = (f'(t) s z + f(t) v, g'(t) s z + g(t) v)$$

As $\{z, v\} = 0$, then z and v are linearly independent unless $v = 0$. Hence for $(v, s) \neq (0, 0)$

$$d\sigma(z, t)(v, s) = 0 \implies f'(t) = g'(t) \text{ or } f(t) = g(t)$$

But this is impossible by choice of f and g

3. σ is injective: This follows by positivity of f and g and the last property of f and g.

4. σ is proper: clear from the construction.

It follows that σ is Lagrangian embedding. It is clear that conditions in definition of handle are satisfied, by choice of f and g. So σ is ϵ-lagrangian handle. The statement about sign is straightforward. □

Construction of Lagrangian surgery

Next we turn to the construction of surgery of an immersed Lagrangian submanifold. Let (M, ω) be a symplectic manifold and (L, ι) be an immersed Lagrangian submanifold, and let x be a point of self-intersection of L, write $\iota^{-1}(x) = \{x_1, x_2\}$. Suppose that L is equipped at x by equipment (l_1, l_2). Pick a neighborhood $U \subset M$ of x small enough so that

- There is open neighborhood $U \subset M$ of x containing no other self intersection such that $\iota^{-1}(U)$ is union of disjoint open neighborhoods U_1, U_2 of x_1, x_2 respectively and $\iota|_{U_i}$ is embedding, in particular $U \cap \iota(L) = \iota(U_1) \cup \iota(U_2)$. Denoting $\iota(U_i) = L_i$, we get L_1 intersecting L_2 transversally at x. We assume $T_x L_1 = l_1$ and $T_x L_2 = l_2$

- There is symplectomorphism $\varphi : (U, x) \to (B(0, \epsilon), 0)$ such that $\varphi^{-1}(L_x) = L_1$ and $\varphi^{-1}(L_y) = L_2$.

This possible by theorem 1.1.8. Now we let $\sigma : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{2n}$ be a Lagrangian $\epsilon/3$-handle with image Γ. Let $V = \varphi^{-1}(\overline{B(0, 2\epsilon/3)})$

$$\tilde{L} = (L \setminus \iota^{-1}(V)) \cup (S^{n-1} \times (-\epsilon, \epsilon))/\sim$$

where $\sim$ is the equivalence class generated by $(z,t) \sim \iota^{-1}\varphi^{-1}\sigma(z,t)$ for $t \in (-\epsilon, -2\epsilon/3) \cup (2\epsilon/3, \epsilon)$. There is a unique manifold structure on $\tilde{L}$ such that the inclusions

$$i_1 : L \setminus \iota^{-1}(V) \hookrightarrow \tilde{L} \ , \ i_2 : S^{n-1} \times (-\epsilon, \epsilon) \hookrightarrow \tilde{L}$$

are open embeddings. We define

$$\tilde{\iota} : \tilde{L} \to M$$

by

$$\tilde{\iota}(p) = \begin{cases} \iota(x) & \text{if } p = [x] \ , \ x \in L \setminus \iota^{-1}(V) \\ \varphi^{-1}\sigma(z,t) & \text{if } p = [z,t] \ , \ (z,t) \in S^{n-1} \times (-\epsilon, \epsilon) \end{cases}$$

This is clearly well defined and is smooth Lagrangian immersion because the composition with i_1 and i_2 are smooth Lagrangian immersions. It is seen that the image of this immersion coincides with $\iota(L)$ outside V and has one self intersection removed. The image is exactly $(L \setminus U) \cup \varphi^{-1}(\Gamma)$. We call $(\tilde{L}, \tilde{\iota})$ a result of Lagrangian surgery at point x. It is a generic immersed submanifolds, because self intersections happen outside U, where the new immersion coincides with the old one. If L is oriented, then L_1 and L_2 inherits orientation and we say the surgery is positive(negative) if the handle Γ is positive(negative) with respect to orientations on L_x and L_y induced by those of L_1 and L_2 respectively. So the surgery sign depends on the equipment in fact the sign is exactly $(-1)^{n(n-1)/2+1} l_1 \cdot l_2$. By changing eqquipment, we can control the sign as in following proposition which is clear

Proposition 3.1.2. *Let (L, ι) be an oriented immersed Lagrangian submanifold with point of self intersection x*

1. *If n is odd, then there exists a Lagrangian surgery of any sign at point x.*

2. *If n is even, then there exists Lagrangian surgery of sign $(-1)^{n(n-1)/2+1}\mathrm{ind}(x)$.*

Remark 3.1.1. If (L, ι) is oriented and the surgery is positive, then $(\tilde{L}, \iota)$ inherits natural orientation which agrees with orientation of L under embdedding $i_1 : L \setminus \iota^{-1}(V) \hookrightarrow \tilde{L}$.

The following two propositions will follow directly from constructions:

Proposition 3.1.3. *Let (L, ι) be immersed Lagrangian with L connected and let x be a point of self intersection. Let $(\tilde{L}, \tilde{\iota})$ be a result of Lagrangian surgery at x.*

1. *If L is oriented, Then $\tilde{L} \cong L\#P$ if the surgery is positive and $\tilde{L} \cong L\#Q$ if the surgery is negative.*

2. *If L is non-orientable, then $\tilde{L} \cong L\#Q \cong L\#P$.*

On the other hand if $(L = L_1 \sqcup L_2, \iota)$ is immersed submanifold where L_1 and L_2 are connected and x is a point of self-intersection, then denoting by $(\tilde{L}, \tilde{\iota})$ a result of Lagrangian surgery at x, we have $\tilde{L} \cong L_1\#L_2$

Proposition 3.1.4. *Let (L, ι) be a closed immersed Lagrangian submanifold of (M^{2n}, ω), and let $(\tilde{L}, \tilde{\iota})$ be a result of Lagrangian surgery.*

1. *$\iota_*([L])$ and $\tilde{\iota}_*([\tilde{L}])$ are homologous (mod 2)*

2. *if L is oriented and the surgery is positive, then $\iota_*([L])$ and $\tilde{\iota}_*([\tilde{\iota}])$ are homologous, where $\tilde{L}$ is given the natural orientation of remark.*

Now we give an application to the surgery construction on embedding problems:

Theorem 3.1.5. *1. Let L_1 and L_2 be closed connected manifolds with Lagrangian embedding into $\mathbb{R}^{2n}$. Then there is a Lagrangian embedding of $L_1\#L_2\#Q$ into $\mathbb{R}^{2n}$. If n is odd, then $L_1, \#L_2\#P$ admits Lagrangian embedding into $\mathbb{R}^{2n}$.*

2. *Let L be a closed connected manifold having Lagrangian immersion into $\mathbb{R}^{2n}$. Then there is a Lagrangian embedding of $L\#kQ$ into $\mathbb{R}^{2n}$ for some k. If n is odd, then there is Lagrangian embedding of $L\#kP$.*

Proof. To prove (1) Let $i_j : L_j \hookleftarrow \mathbb{R}^{2n}$ denote the Lagrangiann embeddings. By linear map and perturbation we can assume that $i_1(L_1)$ and $i_2(L_2)$ intersect transversaly at exactly two points x_1 and x_2. Let $(\tilde{L} = L_1 \sqcup L_2, \tilde{\imath})$ be defined by $\imath(x) = i_j(x)$ if $x \in L_j$. Then $(\tilde{L}, \tilde{\imath})$ is immersed Lagrangian submanifold with two points of self intersections x_1, x_2. We perform Lagrangian surgery at x, to get an immersed Lagrangian submanifold $(\tilde{L}^{(1)}, \tilde{\imath}^{(1)})$. By proposition 3.1.3, $\tilde{L}^{(1)} \cong L_1\#L_2$. If one of L_1 or L_2 is non-orientable, then $\tilde{L}^{(1)}$ is non-orientable, and by the same proposition after applying a surgery at x_2 we get a generic immersed submanifold $(\tilde{L}^2, \tilde{\imath}^{(2)})$ such that $\tilde{L}^2 \cong \tilde{L}^{(1)}\#Q \cong L_1\#L_2\#Q$. But this immersed submanifold has no self interections and generic, so we get embedding $L_1\#L_2\#Q \hookrightarrow \mathbb{R}^{2n}$. Now assume that both L_1 and L_2 are oriented, and fix a choice of orientations. We have two cases

- Let n be odd, then $\tilde{L}^{(2)}$ is $\tilde{L}^{(1)}\#Q$ or $\tilde{L}^{(2)}\#P$ according to the choice of sign of surgery, so we get embeddings $L_1\#L_2\#P \hookrightarrow \mathbb{R}^{2n}$ and $L_1\#L_2\#Q \hookrightarrow \mathbb{R}^{2n}$.
- Let n be even. Suppose that $\text{ind}(x_1) = (-1)^{\frac{n(n-1)}{2}+1}$. Then the first surgery is positive and we get the natural orientation on $(\tilde{L}^{(1)}, \tilde{\imath}^{(1)})$, because of compatibility of this orientation with that of $(\tilde{L}, \imath)$, we get the new index at x_2 the same as old index at the same point which is $-\text{ind}(x_1)$, so the second surgery is negative, and we get $\tilde{L}^{(2)} = \tilde{L}^{(1)}\#Q$. So we get embedding $\tilde{\imath}^{(2)} : \tilde{L}^{(2)} \cong L_1\#L_2\#Q \hookrightarrow \mathbb{R}^{2n}$

Proving (2) is similar by performing surgery k times. □

Another application is

Theorem 3.1.6. *Let (M^{2n}, ω) be a symplectic manifold. Suppose that $G = \mathbb{Z}$ when n is odd and $G = \mathbb{Z}_2$ when n is even. Let $\alpha \in H_n(M, G)$ be represented by*

Lagrangian immersion (that's there is a G-oriented Lagrangian immersed submanfold (L, ι) such that $\iota_*([L]) = \alpha$). Then α can be represented by a Lagrangian embedding.

Proof. Let α be represented by (L, ι), we know that ι is homotopicc to ι' which is generic. ι' also represents α. We can apply successive Lagrangian surgeries to get embedded submanifold $(\tilde{L}, \tilde{\iota}')$. If L is oriented and n is odd, then we can demand the surgeries to be positive. The theorem follows by proposition 3.1.4.

□

3.2 Second Discritption via cobordism

Basically I will reproduce section 8.2 of the paper [4] here. In exact symplectic manifold, we construct surgery of lagrangian submanifold obtained by first lifting the submanifold to the contactization, and then remove reeb chords, and then projecting back. There is lagrangian cobordism from the lift before surgery to that after surgery.

Local model of Lagrangian surgery

Let $\eta, \delta > 0$, and consider the open subset

$$V_{\eta,\delta} = \{(q, p, z) | |q| < \eta \ , \ |p| < 2\delta \ , \ z \in \mathbb{R}\}$$

let $\zeta > 0$, and denote by $\Lambda^+_{\eta,\delta,\zeta}$ the submanifold given by the two sheets

$$\{(q, \pm df_{\eta,\delta,\zeta}(|q|), \pm f_{\eta,\delta,\zeta}(|q|)) | |q| \leq \eta)\} \text{ where } f_{\eta,\delta,\zeta}(s) = \frac{\delta}{2\eta}s^2 + \frac{\zeta}{2}$$

which is legendrian as we have seen in Chapter 1. More explicitly,

$$\Lambda^+_{\eta,\delta,\zeta} = \{(q, \pm\frac{\delta}{\eta}q, \pm\frac{\delta}{2\eta}|q|^2 \pm \frac{\zeta}{2})||q| < \eta\}$$

It is seen that it has one Reeb chord of length ζ, namely the one joining $(0,0,-\frac{\zeta}{2})$ to $(0,0,\frac{\zeta}{2}$. We note that $\Lambda_{\eta,\delta,\zeta}$ can be given by generating function

$$F^+_{\eta,\delta,\zeta} : B(0,\eta) \times \mathbb{R} \to \mathbb{R} \ , (q,\xi) \mapsto \frac{\xi^3}{3} - g^+_{\eta,\delta,\zeta}(|q|)\xi,$$

where

$$g^+(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{3/2}$$

Now let $g^-_{\eta,\delta,\zeta} : \mathbb{R}_+ \to \mathbb{R}$ be a function such that

- $g^-_{\eta,\delta,\zeta}(s) = \left(\frac{3}{2}f_{\eta,\delta,\zeta}(s)\right)^{3/2}$ for $s > \frac{3\eta}{4}$
- $g^-_{\eta,\delta,\zeta}(s) < 0$ for $s < \frac{\eta}{2}$
- $0 < (g^-_{\eta,\delta,\zeta})'(s) < \frac{2\delta\eta}{\delta\eta+\zeta}$

The last condition can be acheived if $\zeta < \frac{7\delta\eta}{16}$. Let $\Lambda^-_{\eta,\delta,\zeta} \subset J^1B(0,\eta)$ be the legendrian submanifold generated by

$$F^-_{\eta,\delta,\zeta} : B(0,\eta) \times \mathbb{R} \to \mathbb{R} \ , (q,\xi) \mapsto \frac{\xi^3}{3} - g^-_{\eta,\delta,\zeta}(|q|)\xi$$

By the last condition $\Lambda^-_{\eta,\delta,\zeta}$ is subset of $V_{\eta,\delta,\zeta}$. Also we should note that $\Lambda^+_{\eta,\delta,\zeta}$ agrees with $\Lambda^+_{\eta,\delta,\zeta}$ in $\{(q,p,z)|3\eta/4 < |q| < \eta \ , \ |p| < 2\delta, z \in \mathbb{R}\}$.

Constuction of Surgery and cobordism

Fix an exact symplectic manifold (M,θ). Let L be a generic nice exact Lagrangian immersed submanifold with self intersection points $x_1, \cdots, x_n$ and let L^+ be the legendrian lift to the contactization (W, β). The self intersection point x_i corresponds to a Reeb chord which we denote by a_i.

Definition 3.2.1. With the above notation, the set of Reeb chords $\{a_1, \cdots, a_n\}$ is called contractible if there are disjoint neighborhoods U_i of a_i such that there is a strict contactomorphism $(U_i, U_i \cap \iota^+(L^+)) \cong (V_{\eta_i,\delta_i,\zeta_i}, \Lambda^+_{\eta_i,\delta_i,\zeta_i})$ for $\eta_i, \delta_i, \zeta_i > 0$ with $\zeta_i < \frac{7\delta_i\eta_i}{16}$.

We let $\{a_1, \cdots, a_n\}$ be a set of contractible Reeb chords. Denote by $\phi_i : (U_i, U_i \cap \iota^+(L^+)) \to (V_{\eta,\delta,\zeta}, \Lambda^+_{\eta,\delta,\zeta})$ a strict contactomorphism as in definition. Let $L^+(a_1, \cdots, a_n)$ be the legendrian submanifold $(L^+ \setminus \bigcup_i (U_i)) \cup \bigcup_i \Lambda^-_{\eta_i,\delta_i,\zeta_i}$. This is indeed a legendrian submanifold by the fact that $\Lambda^+_{\eta,\delta,\zeta}$ agrees with $\Lambda^+_{\eta,\delta,\zeta}$ in $\{(q,p,z) | 3\eta/4 < |q| < \eta \ , \ |p| < 2\delta, z \in \mathbb{R}\} \subset V_{\eta_i,\delta_i,\zeta_i}$, and that ϕ_i is strict contactomorphism.

Denote by $L(x_1, \cdots, x_n)$ the Lagrangian projection of $L^+(x_1, \cdots, x_n)$, then $L(a_1, \cdots, a_n)$ agrees with L outside neighborhoods of x_i and has k self intersections removed. The latter fact follows because ζ_i can be made arbitrarily small, so no Reeb chord is created when passing from $\Lambda_{+\eta_i,\delta_i,\zeta_i}$ to $\Lambda^-_{\eta_i,\delta_i,\zeta_i}$.

Next we construct Lagrangian cobordism from $L^+(a_1, \cdots, a_n)$ to L^+, but first we construct a Lagrangian cobordism from $\Lambda^-_{\eta,\delta,\zeta}$ to $\Lambda^+_{\eta,\delta,\zeta}$ in $\mathbb{R} \times J^1B(0,\eta)$. In fact we construct a gf-compatible cobordism. Define $G_{\eta,\delta,\zeta} : \mathbb{R}_+ \to \mathbb{R}$ to be a function satisfying for $S > 1$ the following properties

- $G_{\eta,\delta,\zeta}(t,s) = g^-_{\eta,\delta,\zeta}(s)$ for $s < 1/S$
- $G_{\eta,\delta,\zeta}(t,s) = g^+_{\eta,\delta,\zeta}(s)$ for $s > S$
- $\frac{\partial}{\partial t} G(t,0) > 0$
- $G_{\eta,\delta,\zeta}(t,s) = g^+_{\eta,\delta,\zeta}(s) = g^-_{\eta,\delta,\zeta}(s)$ for $s > 3\eta/4$

We define

$$F_{\eta,\delta,\zeta} : (\mathbb{R}_+ \times B(0,\eta)) \times \mathbb{R} \to \mathbb{R}\ ,\ (t, x, \xi) \mapsto t \cdot \left(\frac{\xi^3}{3} + G_{\eta,\delta,\zeta}(t, |q|) \right)$$

According to Chapter 2, thi gives immersed Lagrangian cobordism $\Sigma_{\eta,\delta,\zeta}$ in $\mathbb{R} \times J^1B(0,\eta)$ from $\Lambda^-_{\eta,\delta,\zeta}$ to $\Lambda^+_{\eta,\delta,\zeta}$ with points of self intersection possibly corresponding to critcal points of

$$\delta_{F_{\eta,\delta,\zeta}}(t, q, \xi_1, \xi_2) = F_{\eta,\delta,\zeta}(t, q, \xi_1) - F_{\eta,\delta,\zeta}(t, q, \xi_2)$$

with positive values, But the third property of G eliminates this possibility. Thus the cobordism is embedded.

In the trivial cobordism $\mathbb{R}\times L^+$, we replace $\mathbb{R}\times(U_i \cap L^+)$ with the $(\mathrm{id}\times\phi_i)^{-1}(\Sigma_{\eta_i,\delta_i,\zeta_i})$. This can be glued to yield smooth embedded submanifold $\mathbb{R} \times W$ because of the fourth property of G, and it is Lagrangian because $\mathrm{id}\times\phi_i : \mathbb{R}\times W \to \mathbb{R}\times J^1B(0,\eta_i)$ is symplectomorphism. So we have obtained cobordism from $L^+(a_1, \cdots, a_n)$ to L^+.

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