



THE YAU-TIAN-DONALDSON CONJECTURE FOR TORIC KÄHLER MANIFOLDS

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by

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ABSTRACT. This *memoir* is devoted to the study of the *existence* of Calabi's extremal Kähler metrics on toric varieties, seen as symplectic toric manifolds (M, ω, \mathbf{T}) . First we present the theory developed by DELZANT [D88] which *classifies* toric symplectic manifolds by their associated Delzant polytopes $(\Delta, \mathbf{L}, \Lambda)$. We then describe the *formalism of Abreu-Guillemin* [A98, G94], *i.e.* the differential geometry aspect of toric varieties. After that we investigate the Calabi problem specialized to toric manifolds, initiated by DONALDSON [D02]. In the last part of this *memoir*, following APOSTOLOV [A19], we explain some key aspects of the YTD conjecture in the toric Kähler situation.

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0. Introduction

« *toric varieties have provided a remarkable fertile testing ground for general theories* ».

W. FULTON, Introduction to Toric Varieties, [F93]

An important problem in current research in Kähler geometry is to find and study the "best" metric on certain complex manifolds. In this direction, an important result is the *uniformization theorem*.

Theorem 0.1 (Poincaré, Koebe). *Any compact Riemann surface admits a riemannian metric g of constant scalar curvature, unique up to the action of complex automorphisms.*

Motivated by this theorem, E. CALABI in [CF85], asked the problem of existence of certain Kähler metrics on compact Kähler manifolds. Namely, the *problem of Calabi* is to find on a compact Kähler manifold a Kähler metric in a given De Rham class, precisely:

Problem 0.2 (Calabi problem). Let (M, ω_0) be a compact Kähler manifold, does the manifold M admit a Kähler metric ω such that $[\omega] = [\omega_0] \in H^2(M, \mathbf{R})$ with the scalar curvature of ω constant ?

In [Y77], YAU answers positively to the *Calabi problem* for Kähler-Einstein (KE for short) metrics, in the special case when the first chern class vanishes:

- if $c_1(M) = 0$, all Kähler class contains a unique KE metric such that $\text{Ric}(g) = 0$.

However, on a Fano manifold, corresponding to the case $c_1(M) > 0$, the existence is, *in general*, not systematic and obstructions arise given by many algebraic invariants.

In [CF85], CALABI introduced on a compact Kähler manifold, the notion of *extremal metrics*, for which *constant scalar curvature* Kähler (cscK for short) metrics constitute important examples. One predicts that the *existence* of extremal metrics is equivalent to a certain algebro-geometric notion of *stability*. Among experts, some of them think that it would be easier to search for extremal metrics when the manifold admits *more symmetry*. In this direction, Donaldson initiated a program to solve the conjecture for *toric varieties*. This was done in a series of 3 articles, wherein he obtained the first results in this direction.

In this paper, our concern will be the existence of extremal metrics, in the sense of Calabi, restricted to the *toric* setting. Namely, the main goal of this *memoir*, is to explain some key aspects of the following theorem.

Theorem 0.3 (YTD for Toric Kähler manifolds). *A Kähler toric manifold (M, ω, \mathbf{T}) admits a \mathbf{T} -invariant extremal Kähler metric if and only if its associated Delzant polytope (Δ, \mathbf{L}) is b -uniformly K -stable.*

This result has been established recently as a corollary of the general study of cscK metrics by CHEN-CHENG [CC18] and an enhancement to extremal metrics due to HE [H18], combined with previous results by DONALDSON [D02], CHEN-LI-SHENG [CLS14] and ZHOU-ZHU [ZZ08] in the toric case.

The specialization of Kähler geometry to toric varieties takes an elegant form and things become more elementary. Hereafter is some important results in the work.

In [D88], DELZANT gave a correspondence between toric (symplectic) manifolds (M^{2m}, ω) and certain polytopes Δ in \mathbf{R}^m called Delzant polytopes. The first works on the Kähler geometry aspects and the calculation of the scalar curvature were studied by ABREU and GUILLEMIN and expressed by data on the Delzant polytope Δ . In [G94], GUILLEMIN proved that \mathbf{T} -invariant Kähler metrics correspond to convex functions (*symplectic potentials*) on Δ . ABREU in [A98] obtained a characterization of extremal Kähler metrics given by a non-linear PDE of order 4 on the symplectic potential, *i.e.* the existence of extremal metrics is reduced on to the resolution of this PDE. The idea behind *Abreu's equation* uses the correspondence between complex and symplectic coordinates *via* the *Legendre transform*.

The obstructions to the existence of cscK metrics are given by many algebraic invariants such as K -stability, Futaki invariant ; TIAN defined an analytic notion of stability, which was generalized, with an algebro-geometric point of view, by DONALDSON [D02]. In the *toric framework*, DONALDSON [D02], expresses all these invariants in terms of a single linear functional acting on the space of continuous convex functions on the corresponding Delzant polytope.

The first progress towards theorem 0.3, was obtained in [D09], where the author proved that b -uniform K -stability is equivalent to K -stability on toric surfaces and, moreover, he proved the following key result for cscK metrics on toric surfaces.

Theorem 0.4 (Donaldson [D09]). *Any polarized complex toric surface with zero Futaki invariant is K -stable if and only if it admits a cscK metric.*

This result gives a positive answer to a more general existence problem on polarized algebraic varieties (M, L) , called the YAU-TIAN-DONALDSON (YTD for short) conjecture. The principle is the following.

Conjecture 0.5 (YTD). *The polarized algebraic manifold (M, L) admits a cscK metric in the class $c_1(L)$ if and only if (M, L) is K -polystable.*

Furthermore, HISAMOTO proved that the b -uniform stability is equivalent to the equivariant uniform K -stability relative to a maximal torus. The latter notion appears in a problem called the YTD conjecture for extremal metrics. Letting (M, L) be an algebraic polarized variety, we have:

Conjecture 0.6 (YTD). *Existence of extremal Kähler metrics on M in $c_1(L)$ is equivalent to an algebro-geometric of equivariant uniform K -stability with respect to a maximal torus.*

This paper starts with reminders on the notion of Kähler metrics, Chern connections, and the Ricci curvature.

In section 2, we present the theory of Delzant. We define the notion of toric symplectic manifolds (M, ω, \mathbf{T}) and Delzant polytopes $(\Delta, \mathbf{L}, \Lambda)$ and shows that they are in a 1:1 correspondence by the so-called Delzant theorem. We show the Delzant construction, namely how to construct (M, ω, \mathbf{T}) from the data of $(\Delta, \mathbf{L}, \Lambda)$, for the particular case of \mathbf{CP}^m in section 2.4 and in the general case in section 2.5. We finish by the consequences of this construction.

In the next section (3), we present the theory of ABREU-GUILLEMIN, *i.e.* the differential geometry aspect of symplectic toric manifolds (M, ω, \mathbf{T}) via the corresponding Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$. We obtain the local expression of \mathbf{T} -invariant ω -compatible Kähler metrics and of the scalar curvature s_g .

We conclude this *memoir*, with the theory developed by DONALDSON in section 4. We define the Donaldson-Futaki invariant and the Mabuchi K -energy in order to define (uniform) K -stability. This *memoir* ends with an account of the proof of theorem 0.3, following the exposition of [A19].

1. Generalities in Kähler Geometry

We recall some elements of Kähler geometry.

1.1. Kähler manifolds. Let M be a smooth manifold of (real) even dimension $n = 2m$ equipped with a riemannian metric g . An almost complex structure J on M is a field of automorphisms of the tangent bundle TM which satisfy $J^2 = -\text{Id}_{TM}$. If the riemannian metric g satisfies the identity $g(JX, JY) = g(X, Y)$ for any two vector fields X, Y then it is called a *hermitian metric*. Under the previous condition, $M = (M, g, J)$ is called an *almost hermitian manifold*. With the hermitian metric, if J is assumed integrable, we can define a (skew-symmetric) 2-form ω on M by $\omega(X, Y) = g(JX, Y)$. We call it the *kähler form* of g . Thus $\omega(\cdot, J\cdot)$ appears as a riemannian metric.

Definition 1.1. A Kähler metric on M is a hermitian metric g such that the Kähler form is closed, i.e. $d\omega = 0$.

The Kähler metric is uniquely determined by its Kähler form ω . Since ω is a closed $(1, 1)$ -form, it determines a cohomology class $[\omega] \in H^{1,1}(M)_{\mathbf{C}} \cap H^2(M, \mathbf{R})$ called the *Kähler class* of ω . If M is compact, one should mention that the set of all this classes associate to any Kähler structure on M is called the *Kähler cone*

$$\mathcal{K}_M \subset H^{1,1}(M, \mathbf{C}) \cap H^2(M, \mathbf{R}).$$

Kähler manifolds are almost hermitian manifolds with (mutually) compatible *complex and symplectic structures*.

Definition 1.2. An almost hermitian manifold (M, g, J, ω) is Kähler if and only if J is integrable (i.e. (M, J) is a complex manifold) and ω is closed (i.e. (M, ω) is a symplectic manifold)

Let D be the Levi-Civita connection of a riemannian metric g , then the Kähler condition i.e. $d\omega = 0$ is equivalent to that J is invariant under parallel transformation i.e. $DJ = 0$.

Lemma 1.3. A Kähler manifold $M = (M, J, g, \omega)$ is an almost hermitian manifold such that $DJ = 0$.

The integrability condition on J means the cancellation of the *Nijenhuis tensor* N^J i.e. satisfies $N^J(X, Y) := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0$. From the *Newlander-Nirenberg theorem* this is equivalent to the existence of a holomorphic atlas on M compatible with the almost complex structure J , whence (M, J) is a complex manifold. Then this complex structure on M allows us to rewritten the Kähler form in complex (local) coordinates as

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Indeed, the complex structure provides us local complex coordinates $\{z_1, \dots, z_n\}$ such that $z_i = x_i + \sqrt{-1}y_i$. The field J is defined by $J\partial/\partial x_i = \partial/\partial y_i$ and $J\partial/\partial y_i = -\partial/\partial x_i$. The complex tangent bundle $T_{\mathbf{C}}M := TM \otimes \mathbf{C}$ is spanned by $\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - \sqrt{-1} \frac{\partial}{\partial y_i} \right)$ and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + \sqrt{-1} \frac{\partial}{\partial y_i} \right)$. When evaluated on those elements, g satisfies $g_{ij} = g_{\bar{j}\bar{i}} = 0$ and $g_{i\bar{j}} = \bar{g}_{\bar{i}j}$. The complexified tangent bundle splits as $T_{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M$; into the $\sqrt{-1}$ and the $-\sqrt{-1}$ eigenspaces of J . So the metric g extends naturally by \mathbf{C} -linearity to a hermitian metric \tilde{g} on the *holomorphic tangent bundle* $T^{1,0}M$ by

$$\tilde{g} = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j.$$

We retrieve the riemannian structure and the symplectic one *via* $g = \text{Re}(\tilde{g})$ and $\omega = \text{Im}(\tilde{g})$.

Remark 1.4. The (compact) Hopf surface $S := \mathbf{S}^3 \times \mathbf{S}^1$ admits a complex structure *via* $S \simeq \mathbf{C}^2 \setminus (0)/\Gamma$ where $\Gamma = \{2^k \text{Id} : k \in \mathbf{Z}\}$ by the equivalence relation $(z_1, z_2) \sim (2z_1, 2z_2)$. However H is not a symplectic manifold as the second De Rham cohomology group is zero, $H^2(S, \mathbf{C}) = \{0\}$. This is a general fact for compact manifold explained as follows. Let (M^{2m}, ω) be a compact symplectic manifold. Since ω is closed it defines a De Rham class $[\omega] \in H^2(M)$. By non-degeneracy of the symplectic form, $\omega^m := \omega \wedge \dots \wedge \omega$ (m times) is a volume form. Thus, $\int_M \omega^m \neq 0$, as M is compact. Then $[\omega^m] \in H^{2m}(M)$ is nonzero, whence $[\omega] \neq 0$ in $H^2(M)$. For example, the only sphere which admits a symplectic structure is the 2-sphere \mathbf{S}^2 , see example (2.10).

The converse fails as well, *i.e.* there exist manifold with symplectic but no complex structure. FERNÁNDEZ-GOTAY-GRAY in [FGG88] exhibited such an example. The authors showed that the tower of circle fibrations given by circle bundles over circle bundle over the a 2-dimensional torus, is a compact symplectic 4-manifold which do not admit *complex* structures. Hence by definition 1.2 shows the *peculiar* aspect of Kähler manifolds.

Example 1.5. - The quotient of \mathbf{C}^m by the lattice $\mathbf{Z}^{2m} \subset \mathbf{C}^m$ gives a complex tori $\mathbf{C}^m/\mathbf{Z}^{2m}$.

As the exterior derivative is invariant by translation by integers the Kähler form on the complex tori is $\omega = \sum_{i,j=1}^m dz^i \wedge d\bar{z}^j$, induced by the Kähler form on \mathbf{C}^m .

- On the complex projective space \mathbf{CP}^m the Kähler form is the Fubini-Study form, see Section 2.4.
- A Kähler manifold (M, ω) which admits a *global* proper Kähler potential ρ is called a *Stein manifold* *i.e.* $\omega = \frac{i}{2} \partial \bar{\partial} \rho$ where ρ is a smooth proper real-valued function on (M, J) .

1.2. Holomorphic vector bundles. The purpose of this section is to give an overview on Cauchy-Riemann operators on holomorphic vector bundles and their links with the Chern connection. The latter naturally extends the Levi-Civita connection on an almost hermitian manifold and coincides with it when the manifold is Kähler. In section 4.19, we will see how ABREU deduces the *Ricci form*, defined in terms of the Chern connection of the *anti-canonical line bundle*, and the scalar curvature for a toric symplectic manifold. This will be achieved with proposition 1.8. We give the framework for a complex vector bundle E .

If E is a complex vector bundle of rank r over an almost complex manifold (M, J) a Cauchy-Riemann operator $\bar{\partial}^E$ on E is defined as a first order \mathbf{C} -linear differential operator acting on sections of E with values in the complex tensor product $E \otimes \Lambda^{0,1} M$ satisfying

$$(1) \quad \bar{\partial}^E(fs) = s \otimes \bar{\partial}f + f \bar{\partial}^E s.$$

Here, $\bar{\partial} = \frac{1}{2}(d - id^c)$ denotes the usual Cauchy-Riemann operator acting on functions and d^c the twisted exterior differential (*cf. infra*). A \mathbf{C} -linear connection provides an important example of Cauchy-Riemann operator. Indeed, any \mathbf{C} -linear connection ∇ on E can be written in terms of its $(0, 1)$ and $(1, 0)$ -parts by $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where

$$\nabla_X^{1,0} s = \frac{1}{2} (\nabla_X s - i \nabla_{JX} s), \quad \nabla_X^{0,1} s = \frac{1}{2} (\nabla_X s + i \nabla_{JX} s),$$

and $\nabla^{0,1}$ clearly satisfies equation (1), hence is a Cauchy-Riemann operator. Conversely, any Cauchy-Riemann operator can be obtained uniquely in this fashion. Suppose (E, h) is a hermitian complex fiber bundle with inner product h . We say that ∇ is hermitian if it preserves h *i.e.* if $X \cdot h(s_1, s_2) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2)$, for any sections s_1, s_2 of E and vector field X .

Proposition 1.6. *Let E a complex vector bundle over a complex manifold (M, J) and h a given hermitian metric on E . Then, any Cauchy-Riemann operator $\bar{\partial}^E$ on E is the $(0, 1)$ -part*

of a \mathbf{C} -linear hermitian connection ∇ :

$$\bar{\partial}^E = \nabla^{0,1}.$$

When E is furthermore endowed with a holomorphic structure, defined below, there is a canonical Cauchy-Riemann operator $\bar{\partial}^E$ and leading *via* proposition (1.6) to the notion of canonical hermitian connection on the hermitian holomorphic vector bundle (E, h) . A *holomorphic vector bundle* E is a complex vector bundle (of rank r) on a complex manifold (M, J) such that E (viewed as a manifold) as a complex structure such that all fibers are complex submanifolds of E and the two operations coming from the vector space structure are holomorphic. On a given holomorphic vector bundle E over a complex manifold (M, J) there exists a *canonical Cauchy-Riemann operator* $\bar{\partial}^E$ defined by

$$\bar{\partial}^E s = \sum_{i=1}^r \bar{\partial} f_i \otimes s_i,$$

for any section s of E which is written in a holomorphic trivialization as $s = \sum_{i=1}^r f_i s_i$ with respect to a holomorphic local frame s_i of E , and where the twisted exterior differential d^c is defined by $d^c \psi = J d J^{-1} \psi$, for any p -forms ψ , where $J^{-1} = (-1)^p J$ is the inverse of J acting on p -forms.

Definition 1.7. *Let (E, h) be a hermitian holomorphic vector bundle. The Chern connection ∇ on E is the \mathbf{C} -linear hermitian connection associated to the canonical Cauchy-Riemann operator.*

As mention earlier we concentrate our attention on line bundle *i.e.* when $r = 1$. If L is a hermitian holomorphic line bundle and $\nabla = \nabla^L$ the Chern connection on L , the curvature R^∇ is then equal to $i \text{Id} \otimes \rho^\nabla$, where ρ^∇ is a real 2-form called the *curvature form* of ∇ .

Proposition 1.8. *Let (L, h) a holomorphic line bundle (endowed with a hermitian metric h) over a complex manifold (M, J) . For any non vanishing holomorphic section s of L , the Chern connection ∇ and the curvature form ρ^∇ have the following expressions:*

$$\begin{aligned} \nabla s &= \partial \log |s|_h^2 \otimes s \\ &= \frac{1}{2} \left(d \log |s|_h^2 + i d^c \log |s|_h^2 \right) \otimes s \end{aligned}$$

and,

$$\rho^\nabla = -\frac{1}{2} d d^c \log |s|_h^2.$$

Proof. Let X be a (real) holomorphic vector field. For simplicity, we denote by (\cdot, \cdot) the hermitian inner product h . Since ∇ is a *metric* connection, it is consistent with h , *i.e.* we have:

$$(2) \quad X \cdot |s|_h^2 = (\nabla_X s, s) + (s, \nabla_X s)$$

$$(3) \quad JX \cdot |s|_h^2 = (\nabla_{JX} s, s) + (s, \nabla_{JX} s).$$

Recall that the $(0, 1)$ -part $\nabla^{(0,1)}$ is equal to the Cauchy-Riemann operator, that determines the holomorphic structure of L and since s is a holomorphic section (viewed as a map from an open set \mathcal{U} of M to L), it satisfies $\nabla^{(0,1)} s = 0$ *i.e.* $\nabla_{JX} s = i \nabla_X s$. With this identity, equation (3) can be written as $(\nabla_{JX} s, s) = JX \cdot |s|_h^2 - i (s, \nabla_X s)$. Now, L is a holomorphic line bundle hence $\nabla_X s = \theta(X)s$, where θ is a complex 1-form on \mathcal{U} . By combining the latter two identities, and the semi-linearity of h on the second variable we have

$$i\theta(X)|s|_h^2 = d^c |s|_h^2(X) + i\bar{\theta}(X)|s|_h^2.$$

We infer that $\text{Im}(\theta) = \frac{1}{2|s|_h^2} d^c |s|_h^2 = \frac{1}{2} d^c \log |s|_h^2$. Considering $\nabla_X s = \theta(X)s$ and (2) instead of (3), similar arguments leads to $\text{Re}(\theta) = \frac{1}{2} d \log |s|_h^2$. We thus have $\theta = \frac{1}{2} (d \log |s|_h^2 + i d^c \log |s|_h^2) = \partial \log |s|_h^2$. This proves the first part. The curvature R^∇ of ∇ , satisfy by its very definition, $R^\nabla s = -d\theta \otimes s = -\frac{1}{2} dd^c \log |s|_h^2 \otimes s$. \square

2. Delzant Theory

For a given hamiltonian action of a compact Lie group G on a compact symplectic manifold (M, ω) , the image of the momentum mapping $\mu : M \rightarrow \mathfrak{g}^*$ (coming from the hamiltonian action) is hard to describe. GUILLEMIN and STENBERG conjectured that $\mu(M)$ intersected with any positive *Weyl chamber* is a *convex polytope* and a proof of it was given by KIRWAN [K84]. In the same vein, when G is abelian, the *abelian convexity theorem* of ATIYAH [A82], GUILLEMIN and STENBERG [GS82] offers much more quantitative informations such as an explicit description near a *vertex* p of the momentum polytope and the fact that p is the image of a *fixed point* in M . In particular, this theorem tells us that, the image of the moment map of the hamiltonian action of a *torus* on a compact symplectic manifold (M, ω) is a *convex polyhedron* $\Delta \subset \mathbf{R}^k$, called, the *moment polytope*.

If (M, ω, \mathbf{T}) is a symplectic toric manifold, *grosso modo*, a symplectic manifold with a hamiltonian action of a torus \mathbf{T} of dimension $\dim_{\mathbf{C}}(M)$, not all polytopes in \mathbf{R}^k determine completely (M, ω, \mathbf{T}) . In [D88], DELZANT showed that a symplectic toric manifold (M, ω, \mathbf{T}) is entirely determined by the data of the moment polytope $(\Delta, \mathbf{L}, \Lambda)$ satisfying certain *combinatorial conditions* given by (\mathbf{L}, Λ) and a lattice Λ . This class of polytope arising from toric symplectic manifold is called *Delzant polytope*. *Delzant's theorem* classifies symplectic toric manifolds (M, ω, \mathbf{T}) in terms of their Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$.

2.1. Hamiltonian actions. We give the definition of a *hamiltonian action*. A Lie group G acts on a smooth manifold M via a group homomorphism

$$\psi : G \rightarrow \text{Diff}(M)$$

with value in the *diffeomorphism* group of M . The action is *smooth* if the evaluation map $\text{ev} : G \times M \rightarrow M$, $\text{ev}(g, p) := \psi(g)(p)$ is a smooth map between manifolds. In this setting, if $M = (M, \omega)$ is a symplectic manifold, we have

Definition 2.1. We say that G acts symplectically on (M, ω) if $\psi(g)^*(\omega) = \omega$, for all $g \in G$.

Example 2.2. From the classical isomorphism $\mathbf{R}^{2m} \simeq \mathbf{C}^m$, writing $z_i = x_i + \sqrt{-1}y_i$ for $i = 1, \dots, m$; the *standard symplectic form* on \mathbf{R}^{2m} is

$$(4) \quad \omega_{std} = \sum_{i=1}^m dx_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum_{i=1}^m dz_i \wedge d\bar{z}_i.$$

Furthermore, if $\mathbf{T}^m = (e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m})$ is the m -dimensional torus, its action on $(\mathbf{R}^{2m}, \omega_{std})$ via

$$\rho(e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m})(z) := (e^{\sqrt{-1}t_1}z_1, \dots, e^{\sqrt{-1}t_m}z_m)$$

is symplectic.

Remark 2.3. The *Darboux theorem* states that *any* symplectic manifold (M^{2n}, ω) is locally symplectomorphic (*i.e.* isomorphic in the category of symplectic manifolds) to $(\mathbf{R}^{2m}, \omega_{std})$. Thus, ω can be expressed on a *Darboux chart* by means of an open set \mathcal{U} of M with local coordinates $\{(x_i), (y_i)\}_{i \in [1, m]}$ by $\omega|_{\mathcal{U}} = \sum_{i=1}^m dx_i \wedge dy_i$.

Example 2.4. Another important example of such action arises from *hamiltonian flows*. Consider (M, ω) a symplectic manifold and f a smooth function on it. By non-degeneracy of ω *i.e.* from the isomorphism $T_p M \simeq T_p^* M$ induced by ω at each point p of M , we define the so-called *hamiltonian vector field* of f (sometimes called the *symplectic gradient* of f), by

$$X_f := -\omega^{-1}(df).$$

Suppose X_f is complete (always true when M is compact) *i.e.* its flow φ_t is defined for all $t \in \mathbf{R}$. The action ρ of \mathbf{R} on (M, ω) defined by $\rho(t) := \varphi_t$ is symplectic. Indeed, at time 0,

$\varphi_0^*(\omega) = \omega$ and:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=s} (\varphi_t^* \omega) &= \varphi_s^* \left(\frac{d}{dt}\bigg|_{t=0} (\varphi_t^* \omega) \right) \\ &= \varphi_s^* (\mathcal{L}_{X_f} \omega) \\ &= \varphi_s^* ((d\iota_{X_f} + \iota_{X_f} d)(\omega)) \\ &= \varphi_s^* (d(-df)) = 0. \end{aligned}$$

Let G be a Lie group and $\mathfrak{g} = \text{Lie}(G)$ its associated Lie algebra defined as the vector space of all left-invariant vector fields on G . The dual vector space of \mathfrak{g} is denoted by \mathfrak{g}^* . Consider (M, ω) a symplectic manifold and $\psi : G \rightarrow \text{Diff}(M)$ an action of G on (M, ω) . For any $\xi \in \mathfrak{g}$, let X_ξ be the vector field on M generated by the one-parameter subgroup $\{\psi \circ \exp(t\xi) \mid t \in \mathbf{R}\} \subset \text{Diff}(M)$ i.e.

$$X_\xi(p) := \frac{d}{dt}\bigg|_{t=0} \left(\psi(\exp(t\xi))(p) \right).$$

This vector field is the *fundamental vector field* of $\xi \in \mathfrak{g}$.

Recall that the conjugation action of G on itself induces a linear action on \mathfrak{g} denoted $\text{Ad} : G \rightarrow \mathbf{GL}(\mathfrak{g})$. This induces a linear action, called the *co-adjoint action*, on the dual vector space \mathfrak{g}^* , denoted $\text{Ad}^* : G \rightarrow \mathbf{GL}(\mathfrak{g}^*)$ and defined by

$$\text{Ad}_g^* \circ \alpha := \alpha \circ \text{Ad}_{g^{-1}},$$

for all $g \in G$ and $\alpha \in \mathfrak{g}^*$.

Definition 2.5. An action $\psi : G \rightarrow \text{Diff}(M)$ is called *hamiltonian* if there exists a smooth map

$$\mu : M \rightarrow \mathfrak{g}^*$$

called a *moment map* which satisfies the following two properties:

(i) For any $\xi \in \mathfrak{g}$, the fundamental vector field X_ξ satisfies

$$\omega(X_\xi, \cdot) = -d\langle \mu, \xi \rangle$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between \mathfrak{g} and \mathfrak{g}^* .

(ii) The moment map μ is equivariant with respect to the action ψ of G on M and the coadjoint action Ad^* of G on \mathfrak{g}^* , i.e.

$$\mu \circ \psi_g = \text{Ad}_g^* \circ \mu,$$

for all $g \in G$.

Example 2.6. To illustrate hamiltonian actions, we expose an important property of coadjoint orbit called the Kirillov-Kostant-Souriau theorem. It states that each coadjoint orbit carries a natural symplectic structure with canonical symplectic 2-forms sometimes referred as the KKS form. To simplify the presentation a coadjoint orbit \mathcal{O}_α for $\alpha \in \mathfrak{g}^*$ is seen as the quotient $M := G/G_\alpha$ of G by the stabilizer of α for the coadjoint action. We assume that G is semi-simple (i.e. \mathfrak{g} is semisimple) and $\mathfrak{g} = \text{Lie}(G)$ is algebraically compact i.e. the Killing form of \mathfrak{g} is negative definite. Taking the derivative of Ad at the identity gives the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathbf{aut}(\mathfrak{g}) := \text{Lie}(\text{Aut}(\mathfrak{g}))$.

By semisimplicity of \mathfrak{g} , the inner product $\langle \cdot, \cdot \rangle$ is non-degenerate and thus gives the identification $\mathfrak{g} \simeq \mathfrak{g}^*$. This allows us to work with the adjoint action rather than the coadjoint action but then it is no longer canonical. For $\alpha, \beta \in \mathfrak{g}$, the Killing form of \mathfrak{g} is denoted by

$\langle \alpha, \beta \rangle := \text{tr}(\text{ad}_\alpha \circ \text{ad}_\beta)$. By compactness of \mathfrak{g} , the Killing form $\langle \cdot, \cdot \rangle$ is negative definite. So we define the Kirillov-Kostant-Souriau (KKS) 2-form by $\omega_y(\text{ad}_\alpha y, \text{ad}_\beta y) := -\langle y, [\alpha, \beta] \rangle$ for any $y \in M$ and $\alpha, \beta \in \mathfrak{g}$. To make it clearer, we should identify $\text{Im}(\text{ad}_y)$ with $T_y M$ so that ω becomes

$$\omega_y(u, v) := -\langle y, [u, v] \rangle,$$

for any $y \in M$ and $u, v \in T_y M \simeq \text{Im}(\text{ad}_y)$.

Proposition 2.7 (Kirillov-Kostant-Souriau). *The KKS 2-form ω defines on the coadjoint orbit $M = G/G_\alpha$ a symplectic structure. The adjoint action of G on (M, ω) is hamiltonian with moment map $\mu := \iota \circ \langle \cdot, \cdot \rangle : M \rightarrow \mathfrak{g}^*$ identified with the inclusion $\iota : M \subset \mathfrak{g}$ composed with the Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g} \rightarrow \mathfrak{g}^*$.*

2.2. Hamiltonian actions of \mathbf{T} . Now we describe hamiltonian actions of torus, *i.e.* we specialize when $G = \mathbf{T}^k$ is a k -dimensional torus (*i.e.* \mathbf{T}^k is a product of k circles $\mathbf{S}^1 \times \cdots \times \mathbf{S}^1$). In the case, things are easier: the coadjoint action is trivial on \mathbf{T}^k , thus condition (ii) reduces to ask the moment map $\mu : M \rightarrow \mathfrak{t}^*$ to be a \mathbf{T}^k -invariant map. Moreover, from general facts on connected abelian Lie group, Lie's theory tells us that $\text{Lie}(\mathbf{T}^k) =: \mathfrak{t} \simeq \mathbf{R}^k$ and thus $\mathfrak{t}^* \simeq \mathbf{R}^k$. In the tori setting, hamiltonian action can be easily described as follows.

Consider (M, ω) a symplectic manifold on which \mathbf{T}^k acts symplectically and suppose that each fundamental vector field X_ξ (with $\xi \in \mathbf{R}^k$) is hamiltonian with respect to a smooth function μ_ξ on M . Let $\{\xi_1 \dots \xi_k\}$ be the canonical basis of $\mathfrak{t} \simeq \mathbf{R}^k$ and let $\{X_i\}_{i=1, \dots, k}$ be the corresponding fundamental vector fields for this basis and $\{\mu_i\}_{i=1 \dots k}$ the associates hamiltonian functions. The induced fundamental vector field X_ξ has for hamiltonian function

$$\mu_\xi := \sum_{i=1}^k a_i \mu_i,$$

where $\xi = \sum_{i=1}^k a_i \xi_i$. This follows from the identity $\exp(t\xi) = \exp(ta_1 \xi_1) \circ \cdots \exp(ta_k \xi_k)$, since \mathbf{T}^k is abelian, and thus the induced fundamental vector field is $X_\xi = \sum_{i=1}^k a_i X_{\xi_i}$. Then, condition (i) of definition 2.8 is trivially satisfied. Let the moment map $\mu : M \rightarrow \mathbf{R}^k$ be defined by

$$\langle \mu(p), \xi \rangle := \mu_\xi.$$

It remains to see that condition (ii) holds, coming from standard arguments in Lie's theory, condition (ii) is equivalent to $\{\mu_\xi, \mu_\nu\}_\omega = -\mu_{[\xi, \nu]}$ where $\{\mu_\xi, \mu_\nu\}_\omega = \omega(X_\xi, X_\nu)$ is the so-called Poisson bracket. Thus we have to show that for any X_ξ, X_ν the function $\omega(X_\xi, X_\nu)$ vanishes identically on M . Note that, $\mathcal{L}_{X_\ell} \omega = 0$ for any fundamental vector field X_ℓ since a hamiltonian vector field is always symplectic (trivial using both *Cartan formula* and that ω is closed, cf. example 2.4). Since \mathbf{T} is abelian, $[X_\ell, X_\xi] = [X_\ell, X_\nu] = 0$ so $\mathcal{L}_{X_\ell}(\omega(X_\xi, X_\nu)) = 0$ thus the function $\omega(X_\xi, X_\nu)$ is constant on each orbit $\mathcal{O} \subset M$ for the \mathbf{T}^k -action. A standard fact is that \mathcal{O} is a homogeneous manifold of G *i.e.* \mathcal{O} is diffeomorphic to G/G_p , where G_p is the stabilizer of $p \in \mathcal{O}$. So, \mathcal{O} is compact as G is and thus, μ_ξ restricted to \mathcal{O} admits a critical point. At this point, $\omega(X_\xi, X_\nu) = -d\mu_\xi(X_\nu) = 0$, whence the function $\omega(X_\xi, X_\nu)$ vanishes identically on \mathcal{O} , thus on M . Thus, we obtain the following characterization of hamiltonian actions of tori.

Lemma 2.8. *Suppose that $G = \mathbf{T}^k$ acts symplectically on (M, ω) . Then the action is hamiltonian, if and only if, for any $\xi \in \mathfrak{g}$ there exists a smooth map μ_ξ on M which satisfies Hamilton's equation:*

$$\iota_{X_\xi} \omega = -d\mu_\xi.$$

Proof. If the action is hamiltonian just consider $\mu_\xi(p) := \langle \mu(p), \xi \rangle$ the projection of μ along ξ and the claim follows by the definition of the moment map. We already have shown the converse. \square

Example 2.9. Consider the \mathbf{T}^k -action on $(\mathbf{R}^{2k}, \omega_{\text{std}})$ given in example (2.2). Introduce polar coordinates $z_i = r_i e^{\sqrt{-1}\varphi_i}$ on $\mathbf{C}^k \setminus (0) \simeq \mathbf{R}^{2k} \setminus (0)$ for all $i = 1 \dots k$, then the symplectic 2-form ω_{std} becomes

$$\omega_{\text{std}} = \sum_{i=1}^k r_i dr_i \wedge d\varphi_i.$$

The fundamental vector fields associated to the canonical basis of \mathbf{R}^k are $\frac{\partial}{\partial \varphi_i}$. Thus $\iota_{\frac{\partial}{\partial \varphi_i}} \omega = -r_i dr_i = -d(\frac{1}{2}r_i^2)$ so the momentum mapping of the \mathbf{T}^k -action on $\mathbf{C}^k \setminus (0)$ (hence on \mathbf{C}^k by continuity) is defined (up to an additive constant) by

$$\mu(z) := \frac{1}{2}(|z_1|^2, \dots, |z_k|^2).$$

One notices that $\text{Im}(\mu) = \{(x_1, \dots, x_k) \in \mathbf{R}^k : x_i \geq 0\}$ is the nonnegative orthant of \mathbf{R}^k .

Example 2.10. The 2-sphere $\mathbf{S}^2 := \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is equipped with an atlas containing two charts (U_1, ϕ_1) and (U_{-1}, ϕ_{-1}) defined by the stereographic projections from the North pole $N = (0, 0, 1)$ and the South pole $S = (0, 0, -1)$ as:

$$(U_1, \phi_1(x, y, z)) = (\mathbf{S}^2 \setminus N, \frac{(x, y)}{1 - z})$$

and

$$(U_{-1}, \phi_{-1}(x, y, z)) = (\mathbf{S}^2 \setminus S, \frac{(x, y)}{1 + z}).$$

From the stereographic projection from N , one obtains that the coordinates on U_1 are $x = \frac{2u}{1+u^2+v^2}$, $y = \frac{2v}{1+u^2+v^2}$, $z = -\frac{1-u^2-v^2}{1+u^2+v^2}$ for $(u, v) \in \mathbf{R}^2$ and thus the 2-form

$$\omega_{\mathbf{S}^2} := \frac{4du \wedge dv}{(1 + u^2 + v^2)^2}$$

induces a symplectic structure on the 2-sphere \mathbf{S}^2 . The action of $\mathcal{S}^1 = \mathbf{T}^1$ on $(\mathbf{S}^2, \omega_{\mathbf{S}^2})$ around the z -axis of \mathbf{R}^3 is hamiltonian. Indeed, in the basis (du, dv) the 1-form $-dz$ is $\frac{(-4u, -4v)}{(1+u^2+v^2)^2}$ and the interior product of $\omega_{\mathbf{S}^2}$ by the induced fundamental vector field $-u\partial_u, v\partial_v$, for u, v respectively, gives $\frac{-4u}{(1+u^2+v^2)^2}$ and $\frac{-4v}{(1+u^2+v^2)^2}$ respectively. Thus, for $\xi = u, v$ one gets $\iota_{X_\xi} \omega_{\mathbf{S}^2} = -dz$ so the action is hamiltonian with momentum map the z -coordinate

$$\mu(x, y, z) = z.$$

In fact, lemma (2.8) works for an *abelian* Lie group G since we needed only the abelian condition on G . A moment map $\mu : M \rightarrow \mathbf{R}^k$ for a hamiltonian torus action is determined up to the addition of a vector in \mathbf{R}^k . Indeed, under the action of \mathbf{T}^k , any $\mu + c$ with $c \in \mathbf{R}^k$ is also a moment map for that action. Reciprocally, two moment maps for a given hamiltonian torus action differ by a constant cf. *infra*. A remarkable fact on *faithful* \mathbf{T} -action is the following general result about faithful actions of compact Lie group. We refer to ([GGK02], corollary B.48) for a proof.

Proposition 2.11. *Suppose the \mathbf{T}^k -action is effective on M . Then*

$$M^0 = \{p \in M : \text{the } \mathbf{T}\text{-action is free at } p\}$$

is an open dense subset of M .

Remark 2.12. If a compact Lie group G acts effectively on M and if \mathcal{O} denotes a \mathbf{T}^k -orbit for this action, then \mathcal{O} is a compact homogeneous manifold G/G_p of dimension $\leq k$, where G_p denotes the stabilizer of $p \in M$. We call *principal orbits* of G the subset of points of M^0 which orbits are of dimension $= \dim G$.

An important consequence is the following lemma, using the fact that the symplectic structure is identically zero on $T_p\mathcal{O}$, where \mathcal{O} is a principal orbit. In other words, $T_p\mathcal{O}$ is an isotropic submanifold of the symplectic vector space T_pM .

Lemma 2.13. *Suppose that (M, ω) admits an effective hamiltonian action of $G := \mathbf{T}^k$. Then $\dim M \geq 2k$.*

Proof. Let $p \in M$ a point where the action of G is free, i.e. G_p is trivial. Then the orbit $G \cdot p$ is diffeomorphic to $G/G_p = G$, so has dimension k . But we have seen that $T_p(G \cdot p)$ is a ω -isotropic submanifold of T_pM , in other words $T_p(G \cdot p) \subset T_p(G \cdot p)^\perp$ with respect to ω . Therefore, $k = \dim(G \cdot p) \leq \frac{1}{2}\dim M$. \square

Hereafter is the *central result* on the theory of hamiltonian action of torus.

Theorem 2.14 (Atiyah [A82], Guillemin-Stenberg [GS82]). *Suppose that a compact connected symplectic manifold (M, ω) admits a hamiltonian action of \mathbf{T}^k , with momentum map $\mu : M \rightarrow \mathbf{R}^k$. Then,*

- (i) *the image of μ is the convex hull $\Delta \subset \mathbf{R}^k$, of the images of the fixed points for the \mathbf{T}^k -action on M ;*
- (ii) *for all $p \in \Delta$, the variety $\mu^{-1}(p)$ is connected.*

2.3. Toric symplectic manifolds and Delzant theorem.

Definition 2.15. *A symplectic toric manifold is a connected compact symplectic manifold (M^{2k}, ω) of real dimension $2k$ endowed with an effective hamiltonian action ρ of a torus \mathbf{T} such that*

$$\dim \mathbf{T} = k = \frac{1}{2}\dim M.$$

We say that two toric symplectic manifolds $(M_1, \omega_1, \mathbf{T}_1, \rho_1)$ and $(M_2, \omega_2, \mathbf{T}_2, \rho_2)$ are *equivalent* if there exist an isomorphism of Lie group $\phi : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ and a diffeomorphism $\Phi : M_1 \rightarrow M_2$ with $\Phi^*\omega_2 = \omega_1$ so that,

$$\Phi(\rho_1(g)(p)) = \rho_2(\phi(g))(\Phi(p)),$$

for all $g \in \mathbf{T}_1, p \in M_1$. In this case, one has that $\mu_1(M_1)$ and $\mu_2(M_2)$ differ by a translation of \mathfrak{t}^* . Indeed, for any $\xi \in \mathfrak{t}$, we have $d\langle \Phi^*\mu_2, \xi \rangle = \Phi^*d\langle \mu_2, \xi \rangle = -\Phi^*\iota_{X_\xi}\omega_2 = -\iota_{X_\xi}\omega_1$. Hence $\Phi^*\mu_2$ is also a moment map for the action of \mathbf{T}_1 on M_1 , so there exist a constant $\gamma \in \mathfrak{t}^*$ such that $\Phi^*\mu_2 = \mu_1 + \gamma$. Thus

$$\mu_2(M_2) = \Phi^*\mu_2(M_1) = \mu_1(M_1) + \gamma.$$

Up to this equivalence, Delzant's theorem classifies symplectic toric manifolds (M, ω, \mathbf{T}) in terms of the corresponding Delzant polytope. One notices that the vector space \mathfrak{t} being the Lie algebra of a torus \mathbf{T} , it comes with a lattice $\Lambda \subset \mathfrak{t}$ satisfying $2\pi\Lambda = \exp^{-1}(\mathbf{1})$, where $\mathbf{1}$ is the identity element of \mathbf{T} . To put it another way, Λ is the lattice of \mathfrak{t} such that $\exp : \mathfrak{t}/2\pi\Lambda \simeq \mathbf{T}$.

Definition 2.16 (Delzant polytope). *Let V be a real vector space of dimension m and Λ a lattice in V . Consider a convex polytope Δ written as the minimal number of d linear inequalities*

$$\Delta = \{x \in V^* \mid L_j(x) := \langle u_j, x \rangle + \lambda_j \geq 0, \forall j = 1 \dots d\},$$

where $\lambda_j \in \mathbf{R}$ and $u_j \in V$ are called (labelled) normals of Δ . We shall refer the collection of affine function $\mathbf{L} := \{L_1, \dots, L_d\}$ as a labelling of Δ , and the couple (Δ, \mathbf{L}) as a labelled polytope in V . We define a Delzant polytope to be the triple $(\Delta, \mathbf{L}, \Lambda)$ satisfying:

- (i) Δ is compact ;
- (ii) Δ is simple, i.e. each vertex x_0 of Δ annihilates exactly m of the L_j 's and that the corresponding normals constitute a basis of V ;
- (iii) Δ is integral, i.e. for each vertex x_0 of Δ ,

$$\text{span}_{\mathbf{Z}}\{u_j \in V : L_j(x_0) = \langle u_j, x_0 \rangle + \lambda_j = 0\} = \Lambda.$$

Remark 2.17. One takes d as the number of facets (i.e. faces of codimension 1) of Δ . If all the λ_j are in \mathbf{Z}^m , or alternatively if all the vertices of Δ are in \mathbf{Z}^m , then from Δ we can construct a polarized toric manifold.

Example 2.18. From example (2.10), recall that the momentum map for the \mathcal{S}^1 -action on $(\mathcal{S}^2, \omega_{\mathcal{S}^2})$ by the rotation around the z -axis is $\mu(x, y, z) = z$ so the Delzant polytope is $[-1, 1]$. The Delzant polytope of $\mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1$ is the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

Theorem 2.19 (Delzant [D88]). *There exists a bijection between the equivalence classes of $2m$ -dimensional toric symplectic manifolds and the equivalence classes of Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$ in a vector space V^* of dimension m , up to the natural action of the affine group $\text{Aff}(V^*)$ on the triples $(\Delta, \mathbf{L}, \Lambda)$ i.e. we have a one-to-one correspondence:*

$$\{\text{symplectic toric manifold } (M^{2m}, \omega, \mathbf{T})\} \sim \xleftrightarrow{1:1} \{\text{Delzant polytope } (\Delta, \mathbf{L}, \Lambda)\} \sim.$$

The equivalence relation \sim of the lhs (resp. rhs) is understood to be for equivariant symplectomorphisms (resp. lattice isomorphisms). The statement, M is toric symplectic $\implies \mu(M)$ is Delzant, follows from the proof of theorem (2.14). We shall sketch the *Delzant construction* which associates to a toric symplectic manifold (M, ω, \mathbf{T}) a Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$.

2.3.1. *Symplectic reduction.* Without enter deeper in the construction, this *digression* on symplectic reduction will be of help when, in the next section, we will be concerned with the symplectic structure on $\mathbf{CP}^m = \mathbf{C}^{m+1} \setminus \{0\} / \mathbf{C}^*$.

If one considers a symplectic manifold X with an (effective, smooth, proper) action of the torus \mathbf{T}^1 (of real dimension 1). One can ask what is the symplectic structure of the smooth quotient manifold $\tilde{X} = X / \mathbf{T}$. Unfortunately, since $\dim \tilde{X} = \dim X - 1$ is odd, \tilde{X} cannot carry a symplectic structure. The *reduction theorem* allows us, at least, to retrieve an even dimensional manifold. The idea is the following ; assume that \mathbf{T} has a subgroup N acting in a hamiltonian fashion on X then one restricts the action of N on a level set of the momentum map μ_N (for the N -action). Thus, by taking the quotient of the level set by N , one can *lowered the dimension by 1* thus get (at least) a manifold with even dimension. *He who can do less can do more*, the reduction theorem is the fundamental result in symplectic reduction, it allows one to construct *symplectic quotient*. The context is explained hereafter.

Consider G a compact Lie group with a closed normal subgroup $N \triangleleft G$ acting in a hamiltonian fashion on a symplectic manifold (M, ω) . Let $\mu_G : M \rightarrow \mathfrak{g}^*$ the momentum map for the G -action. The momentum map for the N -action, $\mu_N : M \rightarrow \mathfrak{n}^*$ is obtained by composing μ_G with the natural projection $\iota^* : \mathfrak{g}^* \rightarrow \mathfrak{n}^*$.

Proposition 2.20 (Marsden-Weinstein [MW74], Meyer [M73]). *Suppose further that $C \in \mathfrak{g}^*$ is a fixed point for the coadjoint action of G on \mathfrak{g}^* and such that $\iota^*(C) = c$ is a regular value for μ_N . Assume that the action of N on $\mu_N^{-1}(c)$ is free. Then, the N -invariant symplectic 2-form ω , becomes a symplectic form on the manifold $M_{\text{red}} := (\mu_N^{-1}(c)) / N$, when restricted to*

$\mu_N^{-1}(c)$ i.e. if $\omega|_{\mu_N^{-1}(c)} =: \omega_{\text{red}}$ then

$$(M_{\text{red}}, \omega_{\text{red}})$$

is a symplectic manifold. Furthermore, the natural action of G/N on M_{red} is hamiltonian with momentum map (viewed as an N -invariant function on $\mu_N^{-1}(c)$) given by

$$\mu = \mu_G - C.$$

Remark 2.21. The *symplectic quotient* M_{red} is sometimes denoted by $M//N$ and is called the *reduction of (M, ω) at $c \in \mathfrak{n}^*$* with respect to G, μ . The symplectic form ω_{red} is called the *reduced symplectic form*. The dimension of the manifold $M//N$ is $\dim M - 2\dim N$. One may find this reduction theorem at the level $c = 0$. Following APOSTOLOV [A19] (to which we refer for a detailed proof), the construction can be summarize in the following diagram, where we let $S := \mu_N^{-1}(c)$,

$$\begin{array}{ccc} M & \xrightarrow{\mu_G} & \mathfrak{g}^* \\ \uparrow i & & \parallel \\ S & \xrightarrow{(\mu_G)|_S} & \mathfrak{g}^* \\ \downarrow \pi & & \uparrow \ell := j + C \\ M_{\text{red}} & \xrightarrow{\mu} & (\mathfrak{g}/\mathfrak{n})^* \end{array}$$

where $i : S \hookrightarrow M$ is the inclusion map and $\pi : S \rightarrow M_{\text{red}}$ the projection map. The natural inclusion of $(\mathfrak{g}/\mathfrak{n})^*$ in \mathfrak{g}^* is denoted by j , and the map $\ell : (\mathfrak{g}/\mathfrak{n})^* \rightarrow \mathfrak{g}^*$ is the *affine map* defined by $\ell(x) = j(x) + C$, for all $x \in (\mathfrak{g}/\mathfrak{n})^*$.

2.4. Example: \mathbf{CP}^m . To illustrate the theorem of Delzant, we shall see that the complex projective space \mathbf{CP}^m together with its Fubini-Study metric g_{FS} defines a symplectic toric manifold and that the Delzant polytope of $(\mathbf{CP}^m, g_{\text{FS}})$ corresponds to the usual m -simplex. Recall that the complex projective space \mathbf{CP}^m is the quotient of $\mathbf{C}^{m+1} \setminus (0)$ by the holomorphic action of \mathbf{C}^* given by $(\lambda, z) \mapsto \lambda z$ where $(\lambda, z) \in \mathbf{C}^* \times \mathbf{C}^{m+1} \setminus (0)$,

$$\mathbf{CP}^m = \mathbf{C}^{m+1} \setminus (0) / \mathbf{C}^*.$$

The homogeneous coordinates $[z_0 : \dots : z_m]$ are equivalent classes under the \mathbf{C}^* -action of elements $(z_0, \dots, z_m) \in \mathbf{C}^{m+1} \setminus (0)$. The affine charts $U_i := \{[z_0 : \dots : z_m] \in \mathbf{CP}^m : z_i \neq 0\} \simeq \mathbf{C}^m$ recover \mathbf{CP}^m , and we defined a complex atlas with the maps $\phi_i : \mathcal{U}_i \rightarrow \mathbf{C}^m$ defined by

$$\phi_i([z_0 : \dots : z_m]) = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_m}{z_i} \right)$$

for $i = 0, \dots, m$. Thus \mathbf{CP}^m carries a structure of a complex manifold of (complex) dimension m . In order to describe the Fubini-Study metric on \mathbf{CP}^m it is more convenient to work with the topological identification

$$\mathbf{CP}^m = \mathbf{S}^{2m+1} / \mathbf{S}^1$$

as the quotient of the unit hypersphere \mathbf{S}^{2m+1} of \mathbf{C}^{m+1} by the group of rotation \mathbf{S}^1 . The quotient is realized in the following fashion ; one takes the quotient of $\mathbf{C}^{m+1} \setminus (0)$ under the dilatation map $\mathbf{C}^{m+1} \setminus (0) \ni z \mapsto |z|/z \in \mathbf{S}^{2m+1}$ and retaking the quotient of \mathbf{S}^{2m+1} by the diagonal (i.e. componentwise) circle action \mathbf{S}^1 gives us the announced result. The standard flat metric of $\mathbf{C}^{m+1} \simeq \mathbf{R}^{2m+2}$

$$g_0 = \sum_{i=1}^m dx_i^2 + dy_i^2,$$

is compatible with the symplectic standard form ω_{std} and the almost complex structure J_0 is the usual one on \mathbf{C}^{m+1} (cf. § 1.1). As both g_0 and ω_{std} are preserved under the action of

\mathbf{T}^{m+1} , the flat Kähler manifold $(\mathbf{C}^{m+1}, J_0, g_0, \omega_{\text{std}})$ is \mathbf{T}^{m+1} -invariant. The unit hypersphere \mathbf{S}^{2m+1} carries the *canonical round metric* $g^{\mathbf{S}^{2m+1}}$ induces by the restriction of the euclidean metric g_0 on \mathbf{S}^{2m+1} . The Fubini-Study metric g_{FS} is defined as the unique metric such that the projection map (called the *Hopf fibration*):

$$\pi : \mathbf{S}^{2m+1} \longrightarrow \mathbf{CP}^m$$

is a *riemannian submersion*. In order to determine the *symplectic structure* on $\mathbf{CP}^m = \mathbf{S}^{2m+1}/\mathbf{S}^1$ one starts with the \mathbf{T}^{m+1} -hamiltonian action on the symplectic manifold $(\mathbf{C}^{m+1}, \omega_{\text{std}})$ (cf. example 2.2). We restrict this action to the diagonal action of the circle \mathbf{S}^1 on the sphere $\mathbf{S}^{2m+1} \subset \mathbf{C}^{m+1}$ given by $\rho(e^{\sqrt{-1}t})(z_0, \dots, z_m) = e^{\sqrt{-1}t}(z_0, \dots, z_m)$. In order to *highlight* proposition (2.20), we let $N := (e^{\sqrt{-1}t}, \dots, e^{\sqrt{-1}t}) \subset \mathbf{T}^{m+1}$, so N acts on \mathbf{S}^{2m+1} and induces a natural action of $\mathbf{T}^m = \mathbf{T}^{m+1}/N$ on $\mathbf{CP}^m = \mathbf{S}^{2m+1}/N$. At this point, one notice that the diagonal action of \mathbf{S}^1 on \mathbf{T}^{m+1} is the same as the natural action of $\mathbf{N} \subset \mathbf{T}^{m+1}$ on \mathbf{T}^{m+1} , so one has the identification $\mathbf{T}^{m+1}/\mathbf{S}^1 = \mathbf{T}^{m+1}/N$. By composing the moment map (for the \mathbf{T}^{m+1} -action) $\mu_{\mathbf{T}^{m+1}}(z) = \frac{1}{2}(|z_0|^2, \dots, |z_m|^2)$ with the projection map $\iota^* : \mathbf{R}^{m+1} \twoheadrightarrow \mathbf{R}$ (which is adjoint to the Lie algebras of the inclusion $N \subset \mathbf{T}^{m+1}$) thence one found

$$\mu_N(z) = \frac{1}{2} \sum_{i=0}^m |z_i|^2.$$

Then, as announced,

$$\mathbf{S}^{2m+1} = \mu_N^{-1} \left(\frac{1}{2} \right),$$

and since N acts freely on this level set, applying proposition (2.20), we can conclude that $(\mathbf{CP}^m, \omega_{\text{FS}}, \mathbf{T}^m)$ is a toric symplectic manifold for the action of the torus $\mathbf{T}^m = \mathbf{T}^{m+1}/N$ equipped with the so-called *Fubini-Study symplectic form* ω_{FS} . For example, for $m = 1$ the Fubini-Study form on $\mathcal{U}_0 = \{[z_0 : z_1] \in \mathbf{CP}^1 : z_0 \neq 0\}$ is given by

$$\omega_{\text{FS}} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2},$$

where $z_1/z_0 = z$ is the usual local coordinate on $\mathcal{U}_0 \subset \mathbf{CP}^1$, with $z = x + \sqrt{-1}y \in \mathbf{C}^1$ and one notice that $\omega_{\text{FS}} = \frac{1}{4}\omega_{\mathbf{S}^2}$.

The momentum map, induced by the action of \mathbf{T}^{m+1} , and restricted to the action of \mathbf{S}^1 gives

$$\mu : \mathbf{CP}^m \longrightarrow (\mathfrak{t}^{m+1}/\mathfrak{t}^1)^*.$$

Its image is thus identified with the intersection of the nonnegative orthant $\text{Im}(\mu_{\mathbf{T}^{m+1}}) = \{(x_0, \dots, x_m) \in \mathbf{R}^m : x_i \geq 0\}$ with the hyperplane $x_0 + \dots + x_m = \frac{1}{2}$. For example, for $m = 1$,

$$\mu([z_0 : z_1]) = \frac{1}{2} \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2},$$

and thus the moment polytope of \mathbf{CP}^1 is $[0, 1/2]$. Another way to see the *moment polytope* is to consider the natural action of the subtorus $\mathbf{T}^m = (1, e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m}) \subset \mathbf{T}^{m+1}$ on \mathbf{CP}^m defined by:

$$(5) \quad \rho(1, e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m})([z_0 : \dots : z_m]) = [z_0 : e^{\sqrt{-1}t_1}z_1 : \dots : e^{\sqrt{-1}t_m}z_m].$$

The subtorus \mathbf{T}^m fits in the short exact sequence of Lie groups,

$$\mathbf{1} \longrightarrow \mathbf{T}^m \longrightarrow \mathbf{T}^{m+1} \longrightarrow \mathbf{T}^{m+1}/\mathbf{S}^1 \longrightarrow \mathbf{1},$$

in particular the subtorus \mathbf{T}^m is isomorphic to $\mathbf{T}^{m+1}/\mathbf{S}^1 = \mathbf{T}^{m+1}/N$. From the induced projection map between the dual of Lie algebras $(\mathfrak{t}^{m+1})^* \twoheadrightarrow (\mathfrak{t}^m)^* \simeq \mathbf{R}^m$ one deduces that the

image of the momentum polytope is the m -simplex:

$$(6) \quad \Delta_m = \{(x_1, \dots, x_m) \mid \sum_{i=1}^m x_i \geq \frac{1}{2} \text{ and } x_i \geq 0, i = 1, \dots, m\}.$$

Proposition 2.22. *The complex projective manifold $(\mathbf{CP}^m, \omega_{\text{FS}}, \mathbf{T}^m)$ is a symplectic toric manifold endowed with the action of the torus \mathbf{T}^m (via equation 5). The Delzant polytope of \mathbf{CP}^m is the standard m -simplex $\Delta_m \subset \mathbf{R}^m$ (cf. equation 6) with the standard lattice \mathbf{Z}^m and labelled by*

$$\mathbf{L} = \{L_i(x) = x_i, i = 1, \dots, m \text{ and } L_{m+1} = \frac{1}{2} - \sum_{i=1}^m x_i\}.$$

Moreover, since by construction the metric g_{FS} is invariant by \mathbf{T}^m , we obtain $(\mathbf{CP}^m, \omega_{\text{FS}})$ as a \mathbf{T}^m -invariant Kähler structure. In particular, if $V \subset \mathbf{CP}^m$ is a projective complex variety, since the exterior derivative commutes with the pullback, one can restrict the Fubini-Study metric of \mathbf{CP}^m on V , thus V admits a Kähler structure. Note that the Kähler structure depends on the embedding $V \subset \mathbf{CP}^m$.

2.5. Delzant's construction. We give the recipe to get a symplectic toric manifold M_Δ from the data of a Delzant polytope Δ *i.e.* Delzant's construction. We begin with the ingredients. Let $(\Delta, \Lambda, \mathbf{L})$ be a Delzant polytope in the dual space V^* , where V is a m -dimensional vector space. By the *integral* condition of Delzant, the lattice Λ is defined entirely from the data of the labelling \mathbf{L} as the span of $dL_i =: u_i \in V^*$ over \mathbf{Z} . We let $\mathbf{T} := V/2\pi\Lambda$ be the corresponding torus. Under the action of a translation in $\text{Aff}(V^*)$, one can assume without loss of generality that $0 \in \Delta$ and thus $L_i(0) = \lambda_i \geq 0$, for all $i = 1, \dots, d$. Delzant's construction, *grosso modo*, constructs M_Δ as the symplectic quotient (cf. section 2.3.1) of \mathbf{C}^d by a $(d-m)$ -dimensional torus N which acts in a hamiltonian fashion on \mathbf{C}^d *i.e.*

$$M_\Delta := \mathbf{C}^d / N.$$

Step 1. *Construction of the $(d-m)$ -torus N .*

Let $\{e_1, \dots, e_d\}$ be the canonical basis of \mathbf{R}^d consider the linear map $\tau : \mathbf{R}^d \rightarrow V$ defined by

$$e_i \mapsto u_i.$$

Delzant's conditions, satisfied by Δ , show that τ is onto and sends the standard lattice \mathbf{Z}^d of \mathbf{R}^d in the lattice Λ of V . So we get a well defined homomorphism of tori

$$\tau : \mathbf{T}^d = \mathbf{R}^d / 2\pi\mathbf{Z}^d \rightarrow \mathbf{T} = V / 2\pi\Lambda$$

and we take N to be the kernel of the induced homomorphism τ , $N := \text{Ker } \tau$. Recall that V is of dimension m . By definition N is a connected subgroup of \mathbf{T}^d of dimension $(d-m)$. One note that the subtorus \mathbf{T}^d fits in the short exact sequence,

$$\mathbf{1} \longrightarrow N \xrightarrow{\iota} \mathbf{T}^d \xrightarrow{\tau} \mathbf{T} \longrightarrow \mathbf{1},$$

which induces a short exact sequence at the level of the corresponding Lie algebras

$$\mathbf{0} \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbf{R}^d \xrightarrow{\tau} \mathfrak{t} \longrightarrow \mathbf{0},$$

and dually we have

$$(7) \quad \mathbf{0} \longrightarrow \mathfrak{t}^* \xrightarrow{\tau^*} (\mathbf{R}^d)^* \xrightarrow{\iota^*} \mathfrak{n}^* \longrightarrow \mathbf{0}.$$

Step 2. *The hamiltonian N -action on \mathbf{C}^d .*

Recall that the \mathbf{T}^d -action hamiltonian action on \mathbf{C}^d has momentum map

$$\mu_{\mathbf{T}^d}(z) = \frac{1}{2}(|z_1|^2, \dots, |z_d|^2) + c.$$

We take the constant c to be $\lambda = (\lambda_1, \dots, \lambda_d) = (L_1(0), \dots, L_d(0))$ where the last identity comes from the assumption $0 \in \Delta$. Let $\mu_N := \iota^* \circ \mu_{\mathbf{T}^d} : \mathbf{C}^d \rightarrow \mathfrak{n}^*$ be the momentum map for the action of the subtorus $N \subset \mathbf{T}^d$.

Step 3. *The zero level set $S := \mu_N^{-1}(0)$ is a compact submanifold of \mathbf{C}^d .*

We consider $\Delta' := \tau^*(\Delta)$ the compact image of Δ under the inclusion τ^* . We claim that $\text{Im}(\tau^*) \cap \text{Im}(\mu_{\mathbf{T}^d}) = \Delta'$. According to this claim and using the s.e.s (7) we infer that

$$S = \mu_N^{-1}(0) = (\iota^* \circ \mu_{\mathbf{T}^d})^{-1}(0) = \mu_{\mathbf{T}^d}^{-1}(\text{Ker } \iota^*) = \mu_{\mathbf{T}^d}^{-1}(\text{Im } \tau^*) = \mu_{\mathbf{T}^d}^{-1}(\Delta').$$

Since $\mu_{\mathbf{T}^d} : \mathbf{C}^d \rightarrow (\mathbf{R}^d)^*$ is proper, S is compact. The set S is a closed manifold in \mathbf{C}^d . Indeed, since $\lambda_i > 0$, λ are in the interior of the nonnegative orthant of $(\mathbf{R}^d)^*$, which is the momentum image of \mathbf{C}^d . So λ is a regular value of $\mu_{\mathbf{T}^d}$ and it follows that $\iota^*(0)$ is as well a regular value of μ_N . It remains to prove the claim : $\text{Im}(\tau^*) \cap \text{Im}(\mu_{\mathbf{T}^d}) = \Delta'$.

Proof of the claim. Trivially, $\Delta' = \tau^*(\Delta) \subset \text{Im}(\tau^*)$. By the very definition of $\mu_{\mathbf{T}^d}$, $y \in \text{Im}(\mu_{\mathbf{T}^d})$ iff $\langle y, e_i \rangle \leq \lambda_i$. For any $x \in \Delta$, $\langle \tau^*(x), e_i \rangle = \langle x, u_i \rangle \leq \lambda_i$, so $\tau^*(\Delta) \subset \text{Im}(\mu_{\mathbf{T}^d})$ and thus $\text{Im}(\tau^*) \cap \text{Im}(\mu_{\mathbf{T}^d}) \supset \Delta'$. For the converse we suppose $y = \tau^*(z) = \mu_{\mathbf{T}^d}(w)$, it follows that $\langle z, u_i \rangle = \langle z, \tau(e_i) \rangle = \langle \tau^*(z), e_i \rangle = \langle y, e_i \rangle \leq \lambda_i$, i.e. $y \in \Delta$. Thus $\text{Im}(\tau^*) \cap \text{Im}(\mu_{\mathbf{T}^d}) \subset \Delta'$. We proved $\text{Im}(\tau^*) \cap \text{Im}(\mu_{\mathbf{T}^d}) = \Delta'$. □

Step 4. *N acts freely on S .*

Let $z \in S$ be a point mapping to a vertex of Δ ; we first determine the stabilizer \mathbf{T}_z of z under the action of \mathbf{T}^d . We claim that the stabilizer \mathbf{T}_z is the subtorus of dimension m :

$$\mathbf{T}_z = (e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m}, 1, \dots, 1) \subset \mathbf{T}^d.$$

Moreover, from the integral condition of Delzant, the map $\tau : \mathbf{T}_z \rightarrow \mathbf{T}$ is injective (it is thus an isomorphism). Thus it follows trivially,

$$N_z = N \cap \mathbf{T}_z = \text{Ker}(\tau) \cap N = \{\mathbf{1}\} \cap N = \{\mathbf{1}\}.$$

Since the stabilizer at any point $z' \in S$ is included in the stabilizer of a point $z \in S$ corresponding to a vertex, which is trivial, we conclude that N acts freely on S .

Proof of the claim. We first determine the stabilizer \mathbf{T}_z at points $z = (z_1, \dots, z_d) \in S$. It is a subtorus of \mathbf{T}^d of dimension d minus the cardinal of the set $I := \{i \mid z_i = 0\}$. The pattern is the following, z is in Δ^0 iff \mathbf{T}^d acts freely on z , z is in one facet iff \mathbf{T}^d acts with a 1-dimensional stabilizer, z is in the intersection of two facets iff \mathbf{T}^d acts with a 2-dimensional stabilizer, etc. Note that if $I \subset I'$ then the faces they determines, satisfy $F_{I'} \subset F_I$, which is maximal when the face F_I corresponds to a vertex. The corresponding largest set I is describe as follows.

According to the s.e.s (7) and the previous claim $\mu_{\mathbf{T}^d}(z) \in \Delta'$. By the definition of Δ' , there exists $x \in \Delta$ such that $\mu_{\mathbf{T}^d}(z) = \tau^*(x)$. Thus, $z_i = 0 \Leftrightarrow \langle \mu_{\mathbf{T}^d}(z), e_i \rangle = \lambda_i \Leftrightarrow \langle x, u_i \rangle = \lambda_i$ (which determines a face F_I) i.e. x is a point in the intersection of facets whose adjacent normal vectors are u_i . By the condition (ii) of the Delzant polytope Δ we thus have that $\dim(\mathbf{T}_z) = m$ and it is achieved precisely at (the images of) the vertices of Δ . Hence,

$$\mathbf{T}_z = (e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_m}, 1, \dots, 1) \subset \mathbf{T}^d.$$

□

Finally we savour the above construction. From the Delzant polytope Δ , we have constructed a compact symplectic toric manifold $(M_\Delta, \omega_\Delta)$ of real dimension $2d - 2(d - m) = 2m$, where ω_Δ is the reduced symplectic form and

$$M_\Delta := \mathbf{C}^d // N = S/N.$$

We explain how the momentum polytope of M_Δ is Δ . This comes mainly from the Delzant construction with help of the so-called reduction by stages. *Via* the isomorphism τ we get an embedding $\hat{\tau} : \mathbf{T}^m \hookrightarrow \mathbf{T}^d$ satisfying $\tau \circ \hat{\tau} = \text{Id}_{\mathbf{T}^m}$. We get a hamiltonian action of \mathbf{T}^m on \mathbf{C}^d with moment map $\hat{\tau} \circ \mu_{\mathbf{T}^d}$. This action commutes with the N -action thus by performing "reduction in stages" (see e.g. [MMOPR07]), we have that the following diagram

$$\begin{array}{ccccc} S & \xhookrightarrow{i} & \mathbf{C}^d & \xrightarrow{\mu_{\mathbf{T}^d}} & (\mathbf{R}^d)^* \\ \parallel & & & & \downarrow \hat{\tau}^* \\ S & \xrightarrow{\pi} & M_\Delta & \xrightarrow{\mu} & (\mathbf{R}^m)^* \end{array}$$

commutes. Where i is the inclusion $S \hookrightarrow \mathbf{C}^d$ and π is the natural projection on classes $S \twoheadrightarrow M_\Delta$. From these arguments we obtain an induced hamiltonian action of \mathbf{T}^m on M_Δ whose moment map μ makes the diagram commutative *i.e.* $\mu \circ \pi = \hat{\tau}^* \circ \mu_{\mathbf{T}^d} \circ i$. The \mathbf{T}^m -action is free as it is induced by the \mathbf{T}^d -action, which is free. Thus, from the *Delzant construction* we immediately infer

$$\begin{aligned} \mu(M_\Delta) &= \mu \circ \pi(S) = \hat{\tau}^* \circ \mu_{\mathbf{T}^d} \circ i(S) = \hat{\tau}^* \circ \mu_{\mathbf{T}^d} \left((\iota^* \circ \mu_{\mathbf{T}^d})^{-1}(0) \right) \\ &= \hat{\tau}^*(\text{Ker}(\iota^*)) = \hat{\tau}^*(\tau^*(\Delta)) = (\tau \circ \hat{\tau})^*(\Delta) \\ &= \Delta. \end{aligned}$$

2.6. Kähler structures on symplectic toric manifolds. As observed by DELZANT himself, from his construction, symplectic toric manifolds admit a *Kähler structure*. Later, following Delzant's construction LERMAN and TOLMAN, showed that these manifolds admits various structures and obtained a generalization of Delzant's theorem in the *toric orbifold* setting.

2.6.1. Kähler toric manifolds. Following Delzant's construction, since M_Δ is obtained *via* \mathbf{C}^d , which admits a (flat) Kähler structure, one can show that M_Δ is in fact a Kähler manifold. Hence the following result suggests that the \mathbf{T}^m -action on M_Δ preserves both symplectic and complex structures.

Proposition 2.23 (Delzant [D88]). *Any symplectic toric manifold (M, ω, \mathbf{T}) admits a ω -compatible \mathbf{T} -invariant Kähler structure (g_0, J_0) .*

2.6.2. Complex toric varieties from Delzant polytopes. For a symplectic toric manifold (M, ω, \mathbf{T}) , corresponding to a Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$ one can associate a complex manifold $M_\Delta^{\mathbf{C}}$ of dimension m . We give two constructions, one by patchwork using affine charts and the other by means of the Geometric Invariant Theory.

For the first construction, consider a vertex $v \in \Delta$ and at v , take a copy of $\mathbf{C}_v^m = \{(z_1^v, \dots, z_m^v)\}$. By *integral* condition of Delzant polytope Δ , the collection $\{u_{v_1}, \dots, u_{v_m}\}$ is a basis for the lattice Λ . Thus we have an identification $\mathbf{T}^m \simeq \mathbf{T}$, induced by the linear map $e_i \mapsto u_{v_i}$ where $\{e_i\}_i$ is the canonical basis of \mathbf{R}^m . To emphasis, we let $\mathbf{T}_v^m := \mathbf{T}^m$. The action of \mathbf{T}_v^m on the chart \mathbf{C}_v^m is the standard one (example 2.2).

Let $w \in \Delta$ be another vertex, it comes with a basis $\{u_{w_1}, \dots, u_{w_m}\}$ of Λ , corresponding to the normal of the facets adjacent to w . The change of coordinates between the basis $\{u_{v_1}, \dots, u_{v_m}\}$ and $\{u_{w_1}, \dots, u_{w_m}\}$ is given by the transition matrix $A = (a_{ij}) \in \mathbf{SL}_m(\mathbf{Z})$. It

allows us to identify the dense subset $(\mathbf{C}^*)^m_v \subset \mathbf{C}^m_v$ with the dense subset $(\mathbf{C}^*)^m_w \subset \mathbf{C}^m_w$ via the identification:

$$(8) \quad z_i^w = (z_1^v)^{a_{i1}} \dots (z_m^v)^{a_{im}}$$

for $i = 1, \dots, m$. This transition map is equivariant with respect to the map $\mathbf{C}^m_v \rightarrow \mathbf{C}^m_w$ where \mathbf{T}^m_v (resp. \mathbf{T}^m_w) acts on \mathbf{C}^m_v (resp. \mathbf{C}^m_w). To conclude, one obtains a complex manifold $M_\Delta^{\mathbf{C}}$ of dimension m , covered by an equivariant atlas of affine charts \mathbf{C}^m_v , parametrized by the vertices $v \in \Delta$. The intersection of charts $\mathbf{C}^m_v \cap \mathbf{C}^m_w$ is identified with $(\mathbf{C}^*)^m$ via equation (8). Moreover $M_\Delta^{\mathbf{C}}$ inherits an effective action of a complex algebraic torus $\mathbf{T}^{\mathbf{C}} := \{(z_1, \dots, z_m) \in \mathbf{C}^m : z_i \neq 0\} \simeq (\mathbf{C}^*)^m$ by complexify the effective action of \mathbf{T} on (M, J) .

One can explicitly construct $M_\Delta^{\mathbf{C}}$ in the category of algebraic varieties by means of the Geometric Invariant Theory (GIT), see KIRWAN [K84¹]. The idea is to complexifies the $(d - m)$ -torus N and to delete "unstable" points of \mathbf{C}^d for the action of $N^{\mathbf{C}}$. One defines

$$M_\Delta^{\mathbf{C}} := (\mathbf{C}^d)_{ss} / N^{\mathbf{C}}$$

as the *orbit space* of the holomorphic action of the complexified $(d - m)$ -dimensional torus $N^{\mathbf{C}} \simeq (\mathbf{C}^*)^{d-m}$ on the subset $(\mathbf{C}^d)_{ss} \subset \mathbf{C}^d$ of *semi-stable points* for $N^{\mathbf{C}}$ -action on \mathbf{C}^d , *i.e.* the points such that the closure of the $N^{\mathbf{C}}$ -orbit does not contain $0 \in \mathbf{C}^d$. The heart of the construction of $M_\Delta^{\mathbf{C}}$ is based on the data of the normals $u_i = dL_i$ in \mathfrak{t} . They are encoded by the so-called *fan* $\mathcal{F}(\Delta, \mathbf{L})$ associated to Δ .

Definition 2.24. *Let $(\Delta, \mathbf{L}, \Lambda)$ be a Delzant triple and $\mathcal{P} = \{F \subset \Delta\}$ the poset of closed facets of Δ , partially ordered by the inclusion. The fan $\mathcal{F}(\Delta, \mathbf{L})$ of (Δ, \mathbf{L}) is the union*

$$\bigcup_{F \in \mathcal{P}} \mathcal{C}_F$$

of polyhedral cones $\mathcal{C}_F = \{dL \mid L(x) \geq 0 \ \forall x \in \Delta, \text{ s.t. } L(x) = 0 \ \forall x \in F\}$ in V^ .*

Theorem 2.25 (Lerman-Tolman [LT97]). *Suppose J is \mathbf{T} -invariant ω -compatible complex structure on the toric manifold (M, ω, \mathbf{T}) . Then, (M, J) is \mathbf{T} -equivariantly biholomorphic to the complex manifold $M_\Delta^{\mathbf{C}}$ associated to the fan $\mathcal{F}(\Delta, \mathbf{L})$ of the corresponding Delzant triple $(\Delta, \mathbf{L}, \Lambda)$.*

Actually, through the Delzant construction, the complex manifold $M_\Delta^{\mathbf{C}}$ is a smooth toric projective variety. Under the identification by the above theorem we have the following result.

Proposition 2.26. *Every symplectic toric manifold (M, ω, \mathbf{T}) endowed with a \mathbf{T} -invariant ω -compatible Kähler structure J is a projective variety.*

Proof. Let $s \in H^{2,0}(M, \mathbf{C})$ be a holomorphic 2-form. In view of the GIT construction, recall that $M_\Delta^{\mathbf{C}}$ is the quotient of the (complexifies) torus $N^{\mathbf{C}} \simeq (\mathbf{C}^*)^{d-m}$ acting in a holomorphic fashion on $(\mathbf{C}^d)_{ss}$. Then, fix a basis $\{e_1, \dots, e_{d-m}\}$ of $\text{Lie}((\mathbf{C}^*)^{d-m}) = \mathbf{C}^{d-m}$ and denote by $K_j := X_{e_j}$ the fundamental vector fields on M induced by \mathbf{C}^{d-m} . As the torus \mathbf{C}^{d-m} acts in a holomorphic way, the K_j 's defined thus holomorphic vector fields, equivalently, the $(1, 0)$ -part $Z = K_j^{1,0}$ is holomorphic in the usual sense, *i.e.* can be written in (local) holomorphic coordinates z_j as $Z = \sum_{j=1}^m Z_j \partial_{z_j}$ where the component Z_j are holomorphic functions of the coordinates z_j . Thus the function $s(K_i^{1,0}, K_j^{1,0}) \in H^0(M, \mathcal{O})$ is constant as M is compact. Moreover, this function vanishes on the (reciprocal image by the momentum map μ of the) vertices of Δ . Indeed, let p be such a point, by ATIYAH-GUILLEMIN-STENBERG, p is a fixed point for the action and moreover, it is thus a zero of K_i for all i (see [T20]). Furthermore, $s(K_i^{1,0}, K_j^{1,0})$ has holomorphic components, hence by analytic continuation, one can extend this function to all M and thus this function vanishes identically on M . Then one conclude by Hodge theorem:

$$H^{0,2}(M, \mathbf{C}) = H^{2,0}(M, \mathbf{C}) = \{0\}.$$

The Hodge decomposition is then written $H^2(M, \mathbf{C}) = H^{1,1}(M, \mathbf{C})$ *i.e.* $H^2(M, \mathbf{C})$ is the group of closed $(1,1)$ -forms. Thus, one can assume the Kähler form $[\omega] \in H^{1,1}(M)$ to be with integer coefficients. This is equivalent to M being projective by the Kodaira embedding theorem.

Indeed, consider the exponential sequence on the complex manifold (M, J)

$$0 \longrightarrow \underline{\mathbf{Z}} \xrightarrow{\iota} \mathcal{O}_M \rightarrow \mathcal{O}_M^* \longrightarrow 0,$$

where $\underline{\mathbf{Z}}$ is the subsheaf of germs of holomorphic functions with value in \mathbf{Z} . It is the kernel of the surjective morphism of sheaves $\mathcal{O}_M \ni s \mapsto \exp(2i\pi s) \in \mathcal{O}_M^*$; the surjectivity is due to the existence of the logarithm. Hence the short exact sequence. It induces a long exact sequence in cohomology, in particular one has the exact sequence

$$(9) \quad H^1(M, \mathcal{O}_M^*) \xrightarrow{c_1} H^2(M, \underline{\mathbf{Z}}) \xrightarrow{\iota_*} H^2(M, \mathcal{O}_M).$$

Recall that $H^1(M, \mathcal{O}_M^*)$ is encoding isomorphism classes of line bundles. By the exact sequence (9), since $\iota_*([\omega]) = 0$, there exists a holomorphic line bundle $L \rightarrow M$ such that $c_1(L) = [\omega]$. Then, L is positive as ω is and therefore, by the *Kodaira embedding theorem* also ample. Hence M is projective. □

2.6.3. Polarized projective toric varieties. Smooth compact toric symplectic manifolds are related to the notion of polarized toric variety. Let's first recall the definition. A *smooth polarized (projective) complex variety* is a compact complex manifold M of complex dimension m endowed with an holomorphic very ample line bundle $L \rightarrow M$. By *very ample*, one means that the map

$$M \longrightarrow \mathbf{P}(H^0(M, L)^*) \simeq \mathbf{CP}^N$$

is an holomorphic embedding and thus L is the pullback, *via* this map, of the anti-tautologic line bundle $\mathcal{O}(1)$ restricted to the dual of the $(N+1)$ -dimensional complex vector space of holomorphic sections of L , $H^0(M, L)^*$. One denotes by \tilde{M} the embedded image of M in \mathbf{CP}^N and consider the identification $(M, L) \simeq \tilde{M}$, where the polarization on \tilde{M} is just the restriction of $\mathcal{O}(1)$ on \mathbf{CP}^N .

Definition 2.27. *A toric polarized projective variety $\tilde{M} \subset \mathbf{CP}^N$ is an m -dimensional complex submanifold of \mathbf{CP}^N which is the Zariski closure of a principal orbits for the action of an m -dimensional complex torus $\mathbf{T}^{\mathbf{C}}$ (viewed as a complex Lie subgroup of $\mathbf{SL}_{N+1}(\mathbf{C})$).*

For example, any smooth polarized toric complex variety is a symplectic toric manifold. Indeed, if \mathbf{T}^m is the real m -dimensional torus corresponding to $\mathbf{T}^{\mathbf{C}}$, let $\mathbf{T}^N \subset \mathbf{SL}_{N+1}(\mathbf{C})$ be a maximal real torus such that $\mathbf{T}^m \subset \mathbf{T}^N$. Thus, on \tilde{M} a symplectic form $\tilde{\omega}$ is given by the restriction (on \tilde{M}) of a \mathbf{T}^N -invariant (cf. *infra*) Fubini-Study Kähler metric ω_{FS} defined on \mathbf{CP}^N ; moreover $\tilde{\omega}$ belongs to $2\pi c_1(L)$. The polarized toric variety \tilde{M} admits thus a symplectic structure *via* $\tilde{\omega}$. Furthermore, this structure is compatible with the *toric structure*, defined as follows. The \mathbf{T}^m -action on \tilde{M} is hamiltonian with respect to the symplectic structure $\tilde{\omega}$ as $\mathbf{T}^m \subset \mathbf{T}^N$ and \mathbf{T}^N acts in a hamiltonian fashion on \mathbf{CP}^N .

From this fact, one can ask how this translates in terms of Delzant polytopes, *i.e.* how to construct a polarized toric variety from the data of a Delzant polytope. The corresponding class of Delzant polytopes is given by *lattice Delzant polytopes*.

Definition 2.28. *A lattice Delzant polytope is a Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$ such that all the vertices of Δ belongs to the dual lattice $\Lambda^* \subset V^*$.*

Let $(\Delta, \mathbf{L}, \Lambda)$ be such a polytope. One can take a basis $\{e_1, \dots, e_m\}$ of Λ . Then, one denotes by $A(\Delta)$, the set of all lattice points $\lambda^{(i)}$ (for $i = 0, \dots, N$) written in coordinates as $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_m^{(i)})$. Considering $\Delta \subset (\mathbf{R}^m)^*$ with the standard lattice $\Lambda^* = \mathbf{Z}^m$, one can view the vertices of Δ as elements of \mathbf{Z}^m . Now one considers the action ρ of $\mathbf{T}^{\mathbf{C}} \simeq (\mathbf{C}^*)^m$ on \mathbf{CP}^N via:

$$\rho(z_1, \dots, z_m)[s_0 : \dots : s_N] := [(s_1^{\lambda_1^{(0)}} \dots s_m^{\lambda_m^{(0)}})s_0 : \dots : (s_1^{\lambda_1^{(N)}} \dots s_m^{\lambda_m^{(N)}})s_N].$$

To this data, one associates the toric polarized variety $M_{A(\Delta)} \subset \mathbf{CP}^N$ as the Zariski closure (in \mathbf{CP}^N) of the $(\mathbf{C}^*)^m$ -orbit of the point $[1 : \dots : 1] \in \mathbf{CP}^N$ via ρ .

Theorem 2.29 (Section 6.6 in [C03]). *For any lattice Delzant polytope Δ , $M_{A(\Delta)}$ is a smooth polarized toric projective variety whose Delzant polytope is $\Delta \subset \mathbf{R}^m$ with dual lattice $\Lambda^* = (\mathbf{Z}^m)$. In particular, $M_{A(\Delta)}$ is biholomorphic to the complex manifold $M_{\Delta}^{\mathbf{C}}$.*

With this last point and proposition 2.26, one deduces a *correspondence* between smooth toric projective varieties and complex toric varieties.

Remark 2.30. Moreover, with the notion of lattice polytope we retrieve easily the projectiveness of toric symplectic manifolds (M, ω, \mathbf{T}) . Indeed, one notices that in the fan $\mathcal{F}(\Delta, \mathbf{L})$, the λ 's does not appears, thus one can take $\Lambda = \mathbf{Z}^m$ and $\lambda_i \in \mathbf{Q}$, then the vertices are solutions of systems with rational coefficients and thus are rational too. Then Δ is a lattice polytope. Thus $M_{\Delta}^{\mathbf{C}}$ is projective.

3. Abreu-Guillemin Theory

We describe in this section the differential geometry aspect of toric varieties. The original references for this section are [A98, G94]. Using Delzant theory, one considers (M, ω, \mathbf{T}) an m -dimensional symplectic toric manifold with momentum map $\mu : M \rightarrow \Delta$. We seek to describe locally and latter on, globally, toric Kähler metrics, *i.e.* \mathbf{T} -invariant ω -compatible Kähler metric on (M, ω, \mathbf{T}) . The local framework is due to GUILLEMIN and generalized by CALDERBANK-DAVID-GAUDUCHON. The compactification, *i.e.* the extends of the local metric to the whole M , is done via Abreu's boundary condition. Inspecting \mathbf{T} -invariant metrics, one considers the space of orbits

$$M_{\text{red}} := M/\mathbf{T}$$

for the \mathbf{T} -action. On it the moment map μ is well defined as it is \mathbf{T} -invariant and defined a continuous function $\hat{\mu}$ (when M_{red} is endowed with the quotient topology). Moreover, from Delzant's construction $\hat{\mu} : M_{\text{red}} \rightarrow \Delta$ is a bijection. In fact, one can equipped these "manifolds" with a differentiable structure of manifolds with corners, *i.e.* locally modelled on $[0, \infty)^k \times \mathbf{R}^{m-k}$. The map $\hat{\mu}$ will be a diffeomorphism in this category. The differential geometry of the polytope $\Delta \subset \mathfrak{t}^*$ is given by restricting smooth functions on \mathfrak{t}^* . The latter smooth structure is related to the one on M as follows.

Lemma 3.1 (Schwarz [S74]). *A \mathbf{T} -invariant function $f(p)$ on M is smooth if and only if $f(p) = \phi(\mu(p))$ for some smooth function $\phi(x)$ on \mathfrak{t}^* .*

In other words, the \mathbf{T} -invariant function f is the pullback of the moment map by a smooth function on \mathfrak{t}^* . In full generality, SCHWARZ proved it for any compact Lie group acting orthogonally on \mathbf{R}^n .

Recall that in Step 4 of Delzant's construction, the pre-image $p \in \mu^{-1}(y)$ of a point $y \in \Delta$ situated on an open face of codimension k has a stabilizer which is a torus of dimension k . As so, if Δ^0 denotes the interior of Δ , *i.e.* the set of points $x \in \Delta$ which admit an euclidean open ball B_x centred on x such that $B_x \subseteq \Delta$. Then the pre-image of $x \in \Delta^0$ is a principal orbit isomorphic to \mathbf{T} . One considers the dense open subset in M :

$$M^0 := \mu^{-1}(\Delta^0)$$

consisting of points having principal orbits. Thus, $\mu : M^0 \rightarrow \Delta^0$ is a principal \mathbf{T} -bundle over Δ^0 . For the local theory, we want the \mathbf{T} -action to be free ; this is precisely achieved when one restricts the \mathbf{T} -action to M^0 .

3.1. Toric Kähler metrics: local theory. One describe Kähler metrics on M^0 by considering a basis $\{e_1, \dots, e_m\}$ of \mathfrak{t} . For an element $x \in \mathfrak{t}^*$, we write $x = (x_1, \dots, x_m)$ its decomposition in the dual basis. From now on, to make things clearer we will identify the momentum function $\mu_i := \langle \mu, e_i \rangle$ with the coordinate function $x_i = \langle x, e_i \rangle$. We let $K_j := X_{e_j}$ be the induced fundamental vector field for each j . They are functionally independent on the dense open subset $M^0 := \mu^{-1}(\Delta^0)$ of M , meaning that $K_1 \wedge \dots \wedge K_m$ is nonvanishing on M^0 .

One let (g, J) be a \mathbf{T} -invariant ω -compatible Kähler structure on M and consider:

$$H_{ij} := g(K_i, K_j).$$

By Schwarz's lemma, one identify this smooth function on M with a smooth function H_{ij} on Δ . Furthermore, the corresponding symmetric matrix, will be denoted $\mathbf{H}_{ij} = (H_{ij}(x))$; it is smooth w.r.t $x \in \Delta$. Without taking into account the basis we fixed, in this more intrinsic fashion, \mathbf{H} is defined as a $S^2\mathfrak{t}^*$ -valued smooth function over Δ as:

$$\mathbf{H}_x(\xi_1, \xi_2) := g_p(X_{\xi_1}, X_{\xi_2}),$$

for any $\xi_1, \xi_2 \in \mathfrak{t}$ and any p in the fiber $\mu^{-1}(x)$.

On the boundary $\partial\Delta$ of Δ , the inverse of \mathbf{H} happens to be singular, for example, on the vertices of Δ , the smooth function \mathbf{H} vanishes. This follows since the vertices corresponds to fixed points of the action by ATIYAH, GUILLEMIN-STENBERG, and fixed points for the action corresponds to the vanishing of all (fundamental) vector fields on this point, by a classical result (e.g. [T20]). Thus, we restricts our attention on the interior polytope Δ^0 ; on Δ^0 the matrix \mathbf{H} is positive definite; so let $\mathbf{G} := \mathbf{H}^{-1}$. Consider the vector field JK_j and one notice that, with the identification $\mu_i \longleftrightarrow x_i$, one has:

$$(10) \quad dx_i(JK_j) = -\omega(K_i, JK_j) = -g(JK_i, JK_j) = -g(K_i, K_j) = -H_{ij}(x).$$

With this observation, one considers the family (K_1, \dots, K_m) that span an m -dimensional space together with the family (JK_1, \dots, JK_m) that span its orthogonal w.r.t g . Since the complex structure J is preserves by the \mathbf{T} -action, i.e. $\mathcal{L}_{K_i}J = 0$, for all i , and by $(\mathcal{L}_{K_i}J)(K_j) = \mathcal{L}_{K_i}(JK_j) + J\mathcal{L}_{K_i}(K_j)$, one infer $\mathcal{L}_{K_i}(JK_j) = 0$ has the other term (written as a Lie bracket) in the r.h.s vanishes because $\mathfrak{t} = \text{Lie } \mathbf{T}$ is abelian. Recall that the integrability condition on J means that the *Nijenhuis tensor* vanishes, thus $\mathcal{L}_{JK_i}JK_j = 0$. In terms of Lie bracket we have thus:

$$[K_i, K_j] = [K_i, JK_j] = [JK_i, JK_j] = 0.$$

One considers the family of vector fields $\{K_1, \dots, K_m, JK_1, \dots, JK_m\}$ which hence forms a basis of TM^0 orthogonal pairwise for the Lie bracket. To this basis correspond the dual basis of T^*M^0 :

$$\{\theta_1, \dots, \theta_m, J\theta_1, \dots, J\theta_m\}.$$

For any vector field X one set $J\theta(X) = -\theta(JX)$ for any 1-form θ . The pairwise orthogonality for the Lie bracket is equivalent to $0 = d\theta_i = d(J\theta_i)$, for $i = 1, \dots, m$. Each 1 form $J\theta_i$ is *basis* w.r.t the fibration $\mu : M^0 \rightarrow \Delta^0$, i.e. $J\theta_i = \mu^*(\alpha_i)$ for $\alpha_i \in H^1(\Delta^0, \mathbf{R})$; because (in fact it is equivalent) they satisfy $\iota_{K_j}J\theta_i = 0$ and $\mathcal{L}_{K_j}J\theta_i = 0$. As $\Delta^0 \subset \mathfrak{t}^*$ is contractible, the fundamental group $\pi_1(\Delta^0)$ is trivial, then $H^1(\Delta^0, \mathbf{R})$ is trivial thus, $\alpha_i = dy_i$ for some smooth function $y_i(x)$ defined on Δ^0 up to an additive constant. Futhermore, with the convention above on the 1-form θ (and omitting the pull-back by μ as usual) we infer that:

$$(11) \quad J\theta_i = -dy_i.$$

One finally obtain, with 10, the two identities

$$(12) \quad -J\theta_i = dy_i = \sum_{j=1}^m G_{ij}(x) dx_j$$

$$(13) \quad Jdx_i = \sum_{j=1}^m H_{ij}(x) \theta_j$$

or equivalently, $-J\theta_i = \mathbf{G}dx$ and $Jdx_i = \mathbf{H}\theta$, when one see \mathbf{H}, \mathbf{G} as matrices. One defines the \mathfrak{t} -valued 1-form $\boldsymbol{\theta}$, as

$$\boldsymbol{\theta} = \sum_{i=1}^m \theta_i \otimes e_i.$$

This 1-form, seen alternatively as a matrix of 1-forms, determine entirely the connection locally with zero curvature. Hence $\boldsymbol{\theta}$ defines a flat connection 1-form.

With this framework, the symplectic 2-forms ω on M^0 becomes :

$$(14) \quad \omega = \sum_{i=1}^m dx_i \wedge \theta_i,$$

or in a more concise way $\omega = \langle d\mu \wedge \theta \rangle$. One way to think of this system coordinates is as follows. The universal cover of $M^0 := \mu^{-1}(\Delta^0)$ is identified with $\Delta^0 \times \mathfrak{t}^* \simeq \Delta^0 \times \mathfrak{t}$. On it we have the system of coordinates (x_i, θ_i) , for $i = 1, \dots, m$, which one also think of it as a coordinate system on M^0 . Lets prove 4.9.

We evaluate ω on the basis $\{K_1, \dots, K_m, JK_1, \dots, JK_m\}$. *Primo*, for any i, j , we have that $\omega(K_i, K_j) = 0$ vanishes on M . Indeed, as \mathbf{T} acts symplectically on M , each fundamental vector field K_ℓ preserves ω i.e. $\mathcal{L}_{K_\ell} \omega = 0$ and since \mathbf{T} is abelian, $\mathcal{L}_{K_\ell}(\omega(K_i, K_j)) = \omega(\nabla_{K_\ell} K_i, K_j) + \omega(K_i, \nabla_{K_\ell} K_j) = 0$. If \mathcal{O} denote a \mathbf{T} -orbit, it follows that the function $\omega(K_j, K_i)$ is constant on the submanifold $\mathcal{O} \subset M$. (An equivalent fashion to see this is by considering $d\omega(K_i, K_j) = -d(\iota_{K_i} \iota_{K_j} \omega)$ and applying twice *Cartan formula*). Furthermore, \mathcal{O} is a compact manifold as \mathbf{T} is compact (image of a compact by continuous function) and since μ is a moment map, $\omega(K_i, K_j) = -d\mu_i(K_j)$. Therefore, μ_i admits a critical point when restricted to \mathcal{O} . At this point $\omega(K_i, K_j) = -d\mu_i(K_j) = 0$, hence $\omega(K_i, K_j)$ is identically zero on \mathcal{O} , so on M . *Secundo*, as ω is J -invariant and from the last point we immediately infer that $\omega(JK_i, JK_j)$ vanishes on M . *Tercio*, and at last, one has $\omega(K_i, JK_j) = -dx_i(JK_j) = H_{ij}(x)$ by 10. Thus, one conclude that:

$$\begin{aligned}
 \omega &= \sum_{i,j=1}^m \omega(K_i, JK_j) \theta_i \wedge J\theta_j \\
 (15) \quad &= - \sum_{i,j=1}^m H_{ij}(x) J\theta_j \wedge \theta_i \\
 &= \sum_{i=1}^m dx_i \wedge \theta_i
 \end{aligned}$$

where the last line comes from (12) ; so $-\mathbf{H}J\theta_j = \mathbf{H}\mathbf{G}dx_i$ and this term is simply dx_i as $\mathbf{H} = \mathbf{G}^{-1}$ on M^0 . By definition of \mathbf{H}_{ij} , and from the calculus above, the Kähler metric is:

$$\begin{aligned}
 g &= \sum_{i,j=1}^m H_{ij} (\theta_i \otimes \theta_j + J\theta_i \otimes J\theta_j) \\
 (16) \quad &= \sum_{i,j=1}^m (G_{ij} dx_i \otimes dx_j + H_{ij} \theta_i \otimes \theta_j).
 \end{aligned}$$

We used successively that $J\theta_i = -dy_i$ with the \mathbf{R} -bilinearity of $\cdot \otimes \cdot$, and the latter argument as for ω . In a more concise way, $g = \langle d\mu, \mathbf{G}, d\mu \rangle + \langle \theta, \mathbf{H}, \theta \rangle$.

Lemma 3.2 (Guillemin [G94]). *Let (M, w, \mathbf{T}) be a symplectic toric manifold with Delzant polytope Δ and (g, J) an ω -compatible \mathbf{T} -invariant Kähler structure. Then on M^0 , the Kähler metric and Kähler form (g, ω) are of the form 15-16, where $G_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ for a smooth strictly convex function $u(x)$ on Δ^0 .*

Conversely, for any strictly smooth function u on Δ^0 , the riemannian metric on M^0 defined by 16 with $\mathbf{G} = \text{Hess}(u)$, $\mathbf{G} = \mathbf{H}^{-1}$ and the flat connection 1-form θ defines an ω -compatible \mathbf{T} -invariant Kähler structure on M^0 .

Definition 3.3 (Symplectic potential). *The strictly smooth function, convex function u on Δ^0 from lemma above is called the symplectic potential of g .*

Proof. We will always denote, unless exception, the partial derivative $\frac{\partial f}{\partial x_i}$ by simply $f_{,i}$. The direction \Rightarrow was almost done in this section, thus one just have to check that G is the hessian of a symplectic potential u . For $\beta := \sum_{i=1}^m y_i dx_i$, by (12), one get

$$d\beta = \sum_{i=1}^m dy_i \wedge dx_i = \sum_{i,j=1}^m G_{ij} dx_i \wedge dx_j = 0,$$

so there exist a smooth function u on Δ^0 such that $\beta = du$ by Poincaré's lemma. Thus $y_i = u_{,i}$ and so

$$(17) \quad G_{ij} = y_{i,j} = u_{,ij}$$

which is the hessian of u . For the converse, one only have to check that the almost complex structure J on M^0 is integrable. Assume that equation (17) remains true and let $G_{ij} = g(K_i, K_j)$ thus the 1-forms

$$-J\theta_i = \sum_{j=1}^m G_{ij} dx_j = \sum_{j=1}^m u_{,ij} dx_j$$

are closed. Also, as θ is flat, the 1-forms θ_i are closed too so the 1-forms $-J\theta_i + \sqrt{-1}\theta_i$ forms a basis of $\Lambda^{1,0}M^0$, thus locally this basis is written $dy_i + \sqrt{-1}dt_i$. Hence, one has holomorphic coordinates $y_i + \sqrt{-1}t_i$ for J . This conclude. \square

3.2. The scalar curvature on toric varieties. If (M^{2m}, J, g, ω) is a Kähler manifold, the riemannian metric g lifts to a hermitian metric on the anti-canonical bundle $K_M^{-1} = \bigwedge^m(T^{1,0}M)$ whose canonical holomorphic structure is induces by the one on $T^{1,0}$ describe in the first chapter. The corresponding Chern connection is the one induced by the Chern connection of $T^{1,0}$, so the Chern curvature, denoted by $R^{K_M^{-1}}$, of the Chern connection of K_M^{-1} , is the trace of the Chern curvature of $T^{1,0}$ and by definition 1.2, this is simply the riemannian curvature $R = R_{X,Y}$. Then, $R^{K_M^{-1}}$ is a purely imaginary 2-form which can be written as

$$R_{X,Y}^{K_M^{-1}} = \sqrt{-1}\text{tr}(-J \circ R_{X,Y}) = \sqrt{-1}\rho_g(X, Y).$$

Thus, the Ricci form ρ_g is the curvature form of the Chern connection of the anti-canonical line bundle K_M^{-1} . The Ricci form is related to the Ricci tensor Ric_g as the symplectic form is to the riemannian metric g , *i.e.*

$$\rho_g(X, Y) = \text{Ric}_g(JX, Y).$$

The *scalar curvature* is by definition the trace of the Ricci tensor w.r.t the riemannian metric

$$s_g := \text{tr}_g(\text{Ric}_g),$$

or equivalently, $s_g = \text{tr}_\omega(\rho_g)$. Thus in view of its definition, one considers local complex coordinates (z_1, \dots, z_m) such that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^m dz^i \wedge d\bar{z}^i \quad \text{and} \quad \rho_g = \frac{\sqrt{-1}}{2} \sum_{i=1}^m \lambda_i dz^i \wedge d\bar{z}^i,$$

in this form, clearly $\lambda_1 + \dots + \lambda_m = \text{tr}_\omega(\rho_g)$. If one writes $\omega^{(m)} := dz^1 \wedge \bar{z}^1 \wedge \dots \wedge dz^m \wedge \bar{z}^m$, then

$$\begin{aligned} \omega^m &= \left(\frac{\sqrt{-1}}{2} \right)^m (2m)! \omega^{(m)} \\ \rho_g \wedge \omega^{m-1} &= \left(\frac{\sqrt{-1}}{2} \right)^m (2m-1)! (\lambda_1 + \dots + \lambda_m) \omega^{(m)} \end{aligned}$$

Thus

$$(18) \quad s_g = 2m(\rho_g \wedge \omega^{m-1})/\omega^m.$$

Consider a symplectic toric manifold (M, ω, \mathbf{T}) , any \mathbf{T} -invariant Kähler metric, ω -compatible on M is locally given on M^0 by (16), where $\mathbf{G} = \text{Hess}(u)$ and $u \in S(\Delta, \mathbf{L})$ is a symplectic potential (cf. infra). Recall that $H_{ij} = (\text{Hess}(u))_{ij}^{-1} = u^{ij}$.

Lemma 3.4 (Abreu [A98]). *The expression of the Ricci form of (g, J) is:*

$$\rho_g = -\frac{1}{2} \sum_{i,j,k=1}^m H_{ij,ik} dx_k \wedge \theta_j$$

whereas the scalar curvature is:

$$s_g = - \sum_{i,j=1}^m H_{ij,ij}.$$

Proof. As the fundamental vector fields K_i preserve J , the section

$$\sigma := (K_1 - \sqrt{-1}JK_1) \wedge \dots \wedge (K_m - \sqrt{-1}JK_m)$$

is a holomorphic section of the anti-canonical line bundle K_M^{-1} and furthermore, non-vanishing on M^0 . From proposition 1.8 the Ricci form is given on M^0 by

$$\rho_g = -\frac{1}{2} dd^c \log |\sigma|_g^2.$$

We readily compute $|\sigma|_g^2 = 2^m g(K_i, K_j) = \det \mathbf{H}$ and so using¹ $d \log \det \mathbf{H} = \text{tr}(\mathbf{H}^{-1} d\mathbf{H})$, and the identity $0 = (G_{ij}H_{ij})' = G_{ij,k}H_{ij} + G_{ij}H_{ij,k}$ one has:

$$\begin{aligned} d^c \log \det \mathbf{H} &= \text{tr}(\mathbf{H}^{-1} d^c \mathbf{H}) \\ &= \sum_{i,j,k} G_{ij} H_{ij,k} J dx_k \\ &= \sum_{i,j,k,l} G_{ij} H_{ij,k} H_{kl} \theta_l \\ &= - \sum_{i,j,k,l} G_{ij,k} H_{ij} H_{kl} \theta_l \\ &= - \sum_{i,j,k,l} G_{ik,j} H_{ij} H_{kl} \theta_l \\ &= \sum_{i,j,k,l} G_{ik} H_{ij} H_{kl,j} \theta_l \\ &= \sum_{j,l} H_{jl,l} \theta_l. \end{aligned}$$

It follows,

$$\begin{aligned} \rho_g &= -\frac{1}{2} dd^c \log \det \mathbf{H} \\ (19) \quad &= -\frac{1}{2} \sum_{i,j,k} H_{ij,ik} dx_k \wedge \theta_j. \end{aligned}$$

With this expression of ρ_g and recalling that, in this context of a \mathbf{T} -invariant ω -compatible Kähler structure on M , ω is written $\omega = \sum_{i=1}^m dx_i \wedge \theta_i$, and *via* (18), the trace of ρ_g with respect to ω is $-\sum_{i,j=1}^m H_{ij,ij}$. This conclude. \square

¹Let $t \in \mathbf{R}$, the scalar $\partial_t \log \det \mathbf{H}$ is by the chain rule:

$$\frac{\partial \log \det \mathbf{H}}{\partial \det \mathbf{H}} \frac{\partial \det \mathbf{H}}{\partial t}.$$

The first term is just $1/\det(\mathbf{H})$ and the second is the well-known Jacobi's formula: $\partial_t \det \mathbf{H} = \det(\mathbf{H}) \text{tr}(\mathbf{H}^{-1} \partial_t \mathbf{H})$.

3.3. Legendre transform. This coordinate transform identifies Kähler potentials over M^0 (as Legendre duals) to symplectic potentials on Δ^0 . Thus the spirit is to identify \mathbf{T} -invariant J -compatible symplectic structures within a fixed cohomology class to, \mathbf{T} -invariant ω -compatible complex structures on M^0 within a fixed diffeomorphism class. For a given \mathbf{T} -invariant complex structure J on (M, ω, \mathbf{T}) , recall that the action of \mathbf{T} on (M, J) complexifies to an effective holomorphic action of the torus $\mathbf{T}^{\mathbf{C}} \simeq (\mathbf{C}^*)^m$. Let $p \in M^0$ be a fixed point for this action, then M^0 is identified with the orbit $\mathbf{T}^{\mathbf{C}} \cdot p \simeq (\mathbf{C}^*)^m$. Hence the polar coordinates (r_i, t_i) (on each \mathbf{C}^*) gives us the *angular coordinates*

$$\mathbf{t} = (t_1, \dots, t_m) : M^0 \rightarrow \mathfrak{t}/2\pi\Lambda.$$

The functions $\{x_1, \dots, x_m; t_1, \dots, t_m\} \in \Delta^0 \times \mathbf{T}$ are called *momentum-angle coordinates* associated to the data (g, J) . One writes $\boldsymbol{\theta} = d\mathbf{t}$, i.e. $\theta_j = dt_j$. In particular the fundamental vector fields K_i 's, dual to θ_i 's, are written $K_i = \partial/\partial t_i$; then in symplectic coordinates i.e. momentum-angle coordinates:

$$\omega = \sum_{i=1}^m dx_i \wedge dt_i$$

where x_i are the momentum coordinates and t_i are the angular coordinates such that $K_i = \frac{\partial}{\partial t_i}$. Let $\{e_1, \dots, e_m\}$ be the basis of \mathfrak{t} .

Now, following the description of GUILLEMIN we describe how to pass from this symplectic coordinates to holomorphic coordinates on M^0 . We turn our attention on the function $y_j = u_{,j}$ on M^0 . One has globally defined coordinate on M^0 . Indeed, $J\theta_j + \sqrt{-1}\theta_j = dy_j + \sqrt{-1}dt_j$ and thus $y_j + \sqrt{-1}t_j$ defined holomorphic coordinate on (M, J) .

Definition 3.5 (Legendre transform). Let u be a strictly convex, smooth function on Δ^0 . We denote by $y_j(x) := u_{,j}(x)$ the first derivative of u . Consider the derivative of u at x :

$$y(x) := \sum_{i=1}^m y_i(x)e_i = (du)_x \in (T_x\Delta^0)^* \simeq (\mathfrak{t}^*)^* = \mathfrak{t}$$

viewed as a map $\Delta^0 \rightarrow \mathfrak{t}$. The Legendre transform of $u(x)$ is the function $\phi(y) = \phi(y_1, \dots, y_m)$ such that:

$$(20) \quad \phi(y(x)) + u(x) = \langle y(x), x \rangle$$

where in the basis of \mathfrak{t}^* , $x = (x_1, \dots, x_m)$ is viewed as a smooth function $\Delta^0 \rightarrow \mathfrak{t}^*$

The main result of [G94] is

Lemma 3.6 (Guillemin [G94]). Let (g, J) be an ω -compatible \mathfrak{t} -invariant Kähler structure on (M, ω, \mathbf{T}) with symplectic potential $u(x)$ on Δ^0 . Then, the Legendre transform $\phi(y)$ of $u(x)$ is a Kähler potential on M^0 of the symplectic form ω i.e.

$$\omega = dd^c\phi,$$

recall $d^c\phi = Jd\phi$.

The proof uses easy computations and equation (13). By the very definition of the Legendre transform, $\phi(y(x)) = \sum_{i=1}^m x_i u_{,i} - u(x)$ so $d\phi = \sum_{i,j=1}^m x_i G_{ij} dx_j$ and one compute successively $dd^c\phi = dJd\phi$ to be equal to ω .

3.4. The Guillemin metric. The symplectic toric manifold M_Δ canonically associated to the Delzant polytope Δ , has a canonical \mathbf{T} -invariant Kähler metric called the *Guillemin metric*, which we denoted by g_0 . The *key point* is that a Kähler metric on M_Δ determines and (modulo the choice of angular coordinates) is determined by a metric on Δ^0 called the *reduced metric*. Reduced metrics behave well with respect to symplectic reductions (i.e. symplectic quotients), in fact CALDEBANK-DAVID-GAUDUCHON in [CDG03] show that they are *functorials via* the

(pullback of the) affine map ℓ (cf remark 2.22). One has a map $\Delta \rightarrow \mathbf{C}^d$, where d is the number of facets of Δ , which descends by means of ℓ to the symplectic quotient M_Δ . Thus one compute the reduced metric and the symplectic potential for the flat Kähler structure on \mathbf{C}^d .

In polar coordinates (r_i, t_i) the flat metric h of \mathbf{C}^d is written $h = \sum_{i=1}^d dr_i^2 + r_i^2 dt_i^2$, which in momentum coordinates x_i (satisfying $x_i = r_i^2/2$, as we computed the momentum map $\mu_{\mathbf{C}^d}$) is written on the open cone $\{x_i > 0\}$:

$$h = \sum_{i=1}^d \frac{dx_i^2}{2x_i} + \sum_{i=1}^d 2x_i dt_i^2.$$

Here the first sum is the reduced metric for the flat Kähler structure on \mathbf{C}^d . The second is the metric on the d -dimensional torus fibres. By taking the derivatives of 2nd order of the symplectic potential, we recover the reduced metric. Thus the symplectic potential (for the flat structure on \mathbf{C}^d) is $1/2 \sum_{i=1}^d x_i \log x_i$. Now, by writing $x_i = L_i(x)$ one has:

Theorem 3.7 (Guillemin [G94]). *Equip M_Δ with the induced Kähler structure (g_0, J_0) via the Delzant construction. The reduced metric for the canonical Guillemin metric g_0 is*

$$g_{\text{red}}^0 = \frac{1}{2} \sum_{i=1}^d \frac{dL_i \otimes dL_i}{L_i}$$

and so the symplectic potential on Δ^0 is:

$$u_0(x) = \frac{1}{2} \sum_{i=1}^d L_i \log L_i.$$

In all generality, for a given \mathbf{T} -invariant ω -compatible Kähler structure (g, J) on a symplectic toric manifold, the *reduced metric* associated to the metric g (16) is, *by definition*,

$$g_{\text{red}} = \sum_{i,j=1}^m G_{ij} d\tilde{x}_i \otimes d\tilde{x}_j,$$

where \tilde{x}_i are momentum coordinates (previously denoted simply x_i in the section: *local theory*) identified with $\mu_i = \langle \mu, e_i \rangle$.

3.5. Toric Kähler metrics : global theory. ABREU in [A01] extends Kähler toric metrics on M^0 to a (global) Kähler metric on M . To extends to the whole compact symplectic toric manifold M the (general) toric Kähler metric g , defined as in (16), one will assume g_0 to be a globally \mathbf{T} -invariant ω -compatible Kähler metric on (M, ω, \mathbf{T}) . Without loss of generality, one can fix the metric g_0 to be the Guillemin metric. The hessian of the symplectic potential u_0 and the angular coordinates will be denoted $\mathbf{G}_0 = \text{Hess}(u_0)$ and $\boldsymbol{\theta}_0$ respectively. A key observation in order to extends the metric is that it suffices to show that it can be extends smoothly on M . If its the case, it will defined and almost complex structure J on M whose integrability condition is satisfy by continuity. The non-degeneracy of g will follows by continuity too. At the level of the polytope one want thus to extends a Kähler metric on the interior Δ^0 to the boundary, i.e. to *compactify* the metric. The sufficient conditions of compactification are given by *Abreu's boundary conditions*.

Lemma 3.8. *Let g be an invariant Kähler structure on $M^0 = \mu^{-1}(\Delta^0)$ written as in (16), where $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ is the angular coordinates of g_0 . Then g extends into a Kähler structure on $M = \mu^{-1}(\Delta)$ provided Abreu's boundary conditions:*

$$(21) \quad \mathbf{G} - \mathbf{G}_0 \text{ is smooth on } \Delta$$

$$(22) \quad \mathbf{G}_0^{-1} \mathbf{G} \text{ is smooth and nondegenerate on } \Delta.$$

Conversely, to show that these condition are also necessary, is difficult. This was done by a work in [ACGT04] of APOSTOLOV-CALDERBANK-GAUDUCHON-TØNNESSEN ; they established the following criterion for a metric on the open dense subset M^0 to extends on the "boundary" to M .

Proposition 3.9 ([ACGT04]). *Let \mathbf{H} be positive definite $S^2\mathfrak{t}^*$ -valued function on Δ^0 . Then, \mathbf{H} comes from a \mathbf{T} -invariant ω -compatible almost-Kähler structure on M if and only if \mathbf{H} satisfies the following conditions:*

- (i) [smoothness] \mathbf{H} is the restriction to Δ^0 of a smooth $S^2\mathfrak{t}^*$ -valued function on Δ ;
- (ii) [boundary condition] for any point y on the facet $F_j \subset \Delta$ with inward normal $u_j \in \mathfrak{t}$, we have

$$\mathbf{H}_y(u_j, \cdot) = 0 \text{ and } (d\mathbf{H})_y(u_j, u_j) = 2u_j,$$

where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$ -valued function on Δ ;

- (iii) [positivity] for any point y in the interior of a face $F \subset \Delta$, $\mathbf{H}_y(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$.

Definition 3.10 ($\mathcal{S}(\Delta, \mathbf{L})$). *For any compact labelled polytope (Δ, \mathbf{L}) , lets denote by $\mathcal{S}(\Delta, \mathbf{L})$ the space of smooth strictly convex functions u defined on Δ^0 , such that $\mathbf{H} = \text{Hess}(u)^{-1}$ satisfies proposition 3.9 or equivalently conditions (21)-(22).*

By a fundamental result of ABREU for Kähler metric on the toric setting, the space of Kähler structures is parametrized by convex functions, namely symplectic potential satisfying additional conditions.

Theorem 3.11 (Abreu [A98, A01]). *There exists a bijection between the space of \mathbf{T} -invariant ω -compatible Kähler structure (g, J) on (M, ω, \mathbf{T}) (modulo the action of the group of \mathbf{T} -equivariant symplectomorphisms) and the space $\mathcal{S}(\Delta, \mathbf{L})$ (modulo the additive action of the space of affine linear functions).*

By this result, the study of Kähler metrics on toric manifold is equivalent to study convex functions defined entirely from the data of the polytope. Thus, the study of Kähler metrics seems easier for toric manifolds. A refinement of this theorem was given by DONALDSON on the space $\mathcal{C}(\Delta)$, where \mathcal{C} stands for *convex* and *continuous*.

Proposition 3.12 (Donaldson). *The space $\mathcal{S}(\Delta, \mathbf{L})$ can be alternatively defined as the subspace of the space $\mathcal{C}(\Delta)$ of convex continuous functions on Δ , satisfying*

- (i) [convexity] the restriction of u to the interior of any face of Δ (including Δ^0) is a smooth strictly convex function;
- (ii) [asymptotic behaviour] $u - \frac{1}{2} \sum_{i=1}^d L_i \log L_i$ is smooth on Δ .

4. Extremal metrics & the YTD conjecture

4.1. Calabi's extremal Kähler metrics. In the field of complex geometry, given a Kähler manifold (M, ω_0) , an important problem is to find a "canonical" metric in the fixed Kähler class $[\omega_0]$. CALABI in the 80's introduced extremal metrics as good candidates for this problem. Assume (M, ω_0) is a compact Kähler manifold.

Definition 4.1 (Extremal metrics). *A Kähler metric on M in the class $[\omega_0]$, is called extremal if it is a critical point of the Calabi functional:*

$$\omega \mapsto \int_M s_\omega^2 \omega^m,$$

that sends a metric $\omega \in [\omega_0]$ to the L^2 -norm of the scalar curvature s_ω .

For example, Kähler metrics with *constant scalar curvature* (cscK) are trivially extremal metrics, as so they are said to be trivial extremal metrics. As *Kähler-Einstein* metrics have constant scalar curvature, they are in particular extremal metrics.

As noticed by CALABI there is a strong interplay between *extremal metrics* and *holomorphic vector fields*. Namely, the Euler-Lagrange equation of the Calabi functional is that the gradient of the scalar curvature s_g is a holomorphic vector field.

Proposition 4.2 (Calabi [CF85]). *A Kähler metric g is extremal if and only if the ω -hamiltonian vector field $X_g := \omega^{-1}(ds_g)$ is a holomorphic vector field, i.e. $\mathcal{L}_{X_g} J = 0$.*

The main question on these metrics is their *existence* and *unicity*. The latter problem is solved but the existence problem is still open. With this characterization (proposition 4.2) if the identity component $\text{Aut}_0(M, J)$ of the automorphism group of (M, J) is reduced to the identity $\{\text{Id}\}$, i.e., if M has no non-trivial J -holomorphic vector fields, then all extremal metrics have necessary constant scalar curvature. For example, using the Bochner formula for 1-forms, see e.g. [G17], a compact Kähler manifold whose Ricci tensor Ric_g is negative definite admits no non-zero holomorphic vector fields.

On the toric setting extremal metrics admit a more practical definition. In view of Abreu's correspondence, one has:

Lemma 4.3. *Let (g, J) be a ω -compatible toric Kähler metric on (M, ω, \mathbf{T}) and $u \in \mathcal{S}(\Delta, \mathbf{L})$ the corresponding symplectic potential. Then, g is an extremal metric if and only if s_g is the pullback by the moment map of an affine function $s(x) = \langle \xi, x \rangle + \lambda$ on Δ .*

Proof. As g is \mathbf{T} -invariant, the scalar curvature s_g defined a \mathbf{T} -invariant function on M . Then, by Schwarz's lemma, s_g is the pullback by the moment map of a smooth function $s(x)$ on Δ . Hence, by the very definition of ω , in the basis K_i one has, $X_g = \omega^{-1}(ds) = \sum_i s_{,i} K_i$. By hypothesis, $\mathcal{L}_{X_g} J = 0$, in other words:

$$0 = JK_j \cdot s_{,i} = (ds_{,i})(JK_j) = - \sum_{i,k} s_{,ik} J dx_k(K_j) = - \sum_{i,k,l} s_{,ik} H_{kl} \theta_l(K_j),$$

by duality the last term is computed as $-\sum_{i,k} s_{,ik} H_{kj}$. Then $s_{,ik} = 0$ by non-degeneracy of \mathbf{H} on Δ^0 . This means $s(x)$ is an affine-linear function on Δ^0 , hence on Δ .

For the converse, given such an affine linear function $s(x)$ on Δ , then $ds = \xi$, thus $\omega^{-1}(ds) = X_\xi$ and this vector field preserves J , i.e. is an holomorphic vector field. \square

The idea behind this lemma is that a \mathbf{T} -equivariant function f has a holomorphic gradient if and only if, it is an affine function in the symplectic coordinates. One introduces a measure on $\Delta \subset \mathfrak{t}^*$ induced by the Lebesgue measure $dv = dx_1 \wedge \dots \wedge dx_m$ on $\mathfrak{t}^* \simeq \mathbf{R}^m$. A measure $d\sigma$ on the boundary $\partial\Delta$ is induced by the labelled $\mathbf{L} = (L_j)$ by letting, on each facets $F_i \subset \partial\Delta$,

$$dL_i \wedge d\sigma = -dv.$$

In the toric situation, the necessary condition of extremality of a metric is encoded via the *extremal function*. This result was proved by DONALDSON for a more general class of Delzant polytopes.

Proposition 4.4 (Donaldson [D02]). *Assume (Δ, \mathbf{L}) is a simple compact convex labelled polytope in \mathfrak{t}^* . Then, there exists a unique affine-linear function $s_{(\Delta, \mathbf{L})}$ on \mathfrak{t}^* such that, for any affine-linear function f on Δ :*

$$2 \int_{\partial\Delta} f d\sigma - \int_{\Delta} s_{(\Delta, \mathbf{L})} f dv = 0.$$

The affine linear function $s_{(\Delta, \mathbf{L})}$ is called the extremal affine function of (Δ, \mathbf{L}) . Furthermore, if for $u \in \mathcal{S}(\Delta, \mathbf{L})$ the metric (16) is extremal i.e.

$$s_g = - \sum_{i,j=1}^m H_{ij,ij} = s(x) = \langle \xi, x \rangle + \lambda,$$

then the affine linear function $s(x)$ must be equal to $s_{(\Delta, \mathbf{L})}$.

Proof. One writes the affine linear function as $s_{(\Delta, \mathbf{L})} = a_0 + \sum_{j=1}^m a_j x_j$. The condition on the integrals in proposition 4.4, applied to $f = x_i$ for $i = 0, \dots, m$ gives rise to a linear system with positive definite matrix. Hence the coefficients $(a_i)_i$ determines uniquely $s_{(\Delta, \mathbf{L})}$. For the second part of the proposition, suppose $\mathbf{H} = (H_{ij})$ is a smooth $S^2\mathfrak{t}^*$ -valued function on Δ which satisfies the boundary condition of proposition 3.9 and also the extremal condition i.e.

$$s_g = s(x) = \langle \xi, x \rangle + \lambda.$$

Then necessarily, $s(x) = s_{(\Delta, \mathbf{L})}$ i.e. $s(x)$ satisfies the condition on the integrals in the proposition 4.4. Indeed by lemma 4.5, applied trivially to an affine linear function $\phi = f$. \square

In the proof we make use of the following technical lemma using integration by parts, for a proof we refer to [A19]. A generalization of this result is exposed in lemma 4.28.

Lemma 4.5. *Let \mathbf{H} be any smooth $S^2\mathfrak{t}^*$ -valued function on Δ which satisfies the boundary conditions of proposition 3.9 (but not necessarily the positivity condition). Then, for any smooth function ϕ on Δ :*

$$\int_{\Delta} \left(\sum_{i,j=1}^m H_{ij,ij} \right) \phi dv = \int_{\Delta} \left(\sum_{i,j=1}^m H_{ij} \phi_{,ij} \right) dv - 2 \int_{\partial\Delta} \phi d\sigma.$$

Definition 4.6 (Abreu equation). *The Abreu equation is the non linear PDE of order 4:*

$$s(u) := - \sum_{i,j=1}^m (u^{,ij})_{,ij} = s_{(\Delta, \mathbf{L})}$$

where $u^{,ij} := (\text{Hess}(u))_{ij}^{-1} = H_{ij}$.

Thus, the condition for toric Kähler metrics to be extremal reduced to solve the Abreu equation. Let $(\Delta, \mathbf{L}, \Lambda)$ be a Delzant polytope associated to a toric symplectic manifold (M, ω, \mathbf{T}) and consider a local toric Kähler metric g in the form (16). By proposition (4.4), if g is extremal then the corresponding symplectic potential $u \in \mathcal{S}(\Delta, \mathbf{L})$ satisfies the Abreu equation. Conversely, if the scalar curvature s_g is an affine-linear function on \mathfrak{t}^* then by proposition (4.3), g is therefore an extremal Kähler metric. Then one has the correspondence : if $(\Delta, \mathbf{L}, \Lambda)$ is a Delzant polytope, solutions of Abreu's equation corresponds to extremal \mathbf{T} -invariant, ω -compatible Kähler metrics on the symplectic manifold (M, ω, \mathbf{T}) .

To the end of this *memoir*, our concern will not be directly on this PDE but rather with an algebraic invariant given by the corresponding functional appearing in proposition (4.4), called the *Donaldson-Futaki invariant*.

4.2. Donaldson-Futaki invariant. The *Futaki invariant* is an obstruction to the existence of Kähler-Einstein metrics for Fano manifolds due to FUTAKI. A refinement of this definition was given by DONALDSON [D02], which is more algebro-geometric and uses the so-called *Donaldson-Futaki invariant*. This functional is computed by DONALDSON in the toric case and appears as an obstruction to the existence of solutions of Abreu's equation. In the toric case the *Donaldson-Futaki invariant* as the following definition.

Definition 4.7 (Relative Donaldson-Futaki invariant). *Given a simple compact convex labelled polytope (Δ, \mathbf{L}) in \mathbf{R}^m , the Donaldson-Futaki invariant associated to (Δ, \mathbf{L}) is the functional*

$$\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) = 2 \int_{\partial\Delta} \phi d\sigma - \int_{\Delta} s_{(\Delta, \mathbf{L})} \phi dv,$$

acting on the space of continuous functions on Δ .

Remark 4.8. In the particular case $s_{(\Delta, \mathbf{L})} = cste$, this functional is called the non-relative Donaldson-Futaki invariant and corresponds to the case of cscK metrics. By the definition of $s_{(\Delta, \mathbf{L})}$, whenever ϕ is affine linear one has $\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) = 0$.

Proposition 4.9 (Donaldson [D02]). *If (Δ, \mathbf{L}) admits a solution of Abreu's equation, then*

$$\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) > 0$$

for any smooth convex function ϕ on Δ which is not affine linear.

Proof. By lemma 4.5, one computes immediately

$$\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) = \int_{\Delta} \sum_{i,j=1}^m H_{ij} \phi_{,ij} dv \geq 0,$$

where the positivity comes from the convexity of ϕ . As \mathbf{H} is positive definite on the interior Δ^0 , this inequality is strict unless $\phi_{,ij}$ vanishes *i.e.* ϕ is affine linear. \square

4.3. Toric test configurations & K-stability. The existence of extremal metrics on a manifold M is conjectured to be related to a notion of *stability* on the manifold M itself. The origin of this notion comes from the Fano case (for Kähler-Einstein metrics) defined by TIAN. A similar definition is given by DONALDSON [D02], in the context of polarized algebraic variety *via* the Donaldson-Futaki invariant and is conjectured to characterize the existence of cscK metrics. Inspired by this notion, CHEN-CHENG proved that the existence of cscK metrics is characterize by the notion of *geodesic stability*, introduced in [CC18]. Historically, *K-stability* is an analogy of the stability in GIT, namely the Hilbert-Mumford criterion. In the toric setting,

Definition 4.10. *By, PL convex function ϕ on Δ , we mean a convex rational piecewise-linear (PL) function $\phi = \max(f_1, \dots, f_k)$, where f_i are affine-linear function with rational coefficient.*

Definition 4.11 (Toric K-stability). *We say that a labelled compact convex simple polytope (Δ, \mathbf{L}) in \mathbf{R}^m is*

- (i) *K-semistable, if $\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) \geq 0$ for all PL convex function ϕ ;*
- (ii) *K-stable, if it is K-semistable and $\mathcal{F}_{(\Delta, \mathbf{L})}(\phi) = 0$ only for affine linear functions ϕ ;*
- (iii) *K-unstable, if it is not K-semistable.*

Definition 4.12. Let $(\Delta, \mathbf{L}, \Lambda)$ be a Delzant polytope. The corresponding toric symplectic manifold (M, ω, \mathbf{T}) is said to be K -stable iff (Δ, \mathbf{L}) is.

From proposition 4.9, one has immediately,

Corollary 4.13 (Donaldson [D02]). If the Abreu equation admits a solution, then (Δ, \mathbf{L}) is K -semistable.

An enhancement of this was done by ZHOU-ZHU to the K -stable case.

Theorem 4.14 (Zhou-Zhu [ZZ08]). If the Abreu equation admits a solution, then (Δ, \mathbf{L}) is K -stable.

Motivated by this result, the YTD conjecture for extremal metrics on toric manifold is the following.

Conjecture 4.15 (Donaldson [D02]). The Abreu equation admits a solution in $\mathcal{S}(\Delta, \mathbf{L})$ iff (Δ, \mathbf{L}) is K -stable.

Toric test configurations. Now, we explain the notion of *toric test configuration* introduced by DONALDSON [D02], giving a geometrical meaning for *convex PL* functions. We consider a Delzant polytope $(\Delta, \mathbf{L}, \Lambda)$ in \mathbf{R}^m , with $\Lambda = (\mathbf{Z}^m)^*$, corresponding to the symplectic toric manifold (M, ω, \mathbf{T}) . Also, consider a PL convex function $f = \max(f_1, \dots, f_i)$ defined as the minimal set of affine linear function with rational coefficients on Δ defining f . By a suitable normalization, one can assume without loss of generality that the coefficients of the f_i 's are integers. Fix an integer $R > 0$ such that $R - f(p) > 0$. With this data one define the polytope $\mathcal{Q} \subset \mathbf{R}^{m+1} = \mathbf{R}^m \times \mathbf{R}$,

$$\mathcal{Q} = \{(p, t) \in \Delta \times \mathbf{R} : 0 \leq t \leq R - f(p)\}.$$

One can assume that \mathcal{Q} is a *rational* Delzant polytope, i.e. for each $L_j \in \mathbf{L}$, the normal $u_j \in \Lambda$, with respect to the labelling of \mathcal{Q} given by:

$$\{L_j(p) \geq 0, (R - t - f_k(p)) \geq 0, t \geq 0, j = 1, \dots, d, k = 1, \dots, i\}$$

where $\mathbf{L} = (L_j)_{j=1, \dots, d}$ are the labels of Δ . Thus \mathcal{Q} gives rise to a symplectic toric *orbifold* of *complex* dimension $(m + 1)$. For simplicity, we consider \mathcal{Q} to be Delzant, in other words \mathcal{Q} is integral. Furthermore, one notice that \mathcal{Q} is defined *via* integral equations, thus (§ 2.6.3) the vertices of \mathcal{Q} are located in \mathbf{Z}^{m+1} . Thus by the Delzant theorem, one get a smooth toric polarized variety $\mathcal{M} \subset \mathbf{CP}^N$ with a polarization given by the line bundle $\mathcal{L} \rightarrow \mathcal{M}$ where $\mathcal{L} = \mathcal{O}(1)|_{\mathcal{M}}$. The Kähler metric on \mathcal{M} is given by the restriction of a $(\mathbf{C}^*)^N$ -invariant Fubini-Study metric ω_{FS} on \mathbf{CP}^N . As $\mathbf{T} \simeq \mathbf{T}^m$ is the torus acting on M , the corresponding torus acting on \mathcal{M} is given by $\mathbf{T}^{m+1} := \mathbf{T}^m \times \mathbf{S}_{m+1}^1$ where, by \mathbf{S}_{m+1}^1 , we means the circle subgroup of \mathbf{T}^{m+1} which act on the $(m + 1)$ -th factor of \mathbf{T}^{m+1} .

One shall note that Δ appears as a copy of a facet of \mathcal{Q} by $\mathcal{Q} \cap \{t = 0\} = \Delta \times \{0\}$. Thus, (the preimage of) the face $\Delta \subset \mathcal{Q}$ corresponds to a smooth submanifold $\tilde{M} \subset \mathcal{M}$ which is a smooth polarized toric manifold.

Via the Delzant construction, the stabilizer of points in \tilde{M}^0 is identified with the circle group \mathbf{S}_{m+1}^1 . Thus, $(\tilde{M}, \omega_{\text{FS}}|_{\tilde{M}})$ is equivariantly isomorphic to (M, ω) with respect to the action of $\mathbf{T}^{m+1}/\mathbf{S}_{m+1}^1 \simeq \mathbf{T}^m$ (resp. \mathbf{T}) on \tilde{M} (resp. M). So without loss of generality one can take (M, ω, \mathbf{T}) to be a *smooth toric polarized manifold*.

Let fix a ω_{FS} -compatible \mathbf{T}^{m+1} -invariant complex structure \mathcal{J} on \mathcal{M} which induces a \mathbf{T}^m -invariant ω -compatible complex structure J on (M, ω) . Following DONALDSON ([D02], § 4.2), with respect to the \mathbf{C}^* -action $\rho : \mathbf{C}^* \rightarrow \text{Aut}(\mathcal{M})$ induced by the circle \mathbf{S}_{m+1}^1 , the complex $(m + 1)$ -dimensional $(\mathcal{M}, \omega_{\text{FS}}|_{\mathcal{M}}, \rho)$ is an example of a *Kähler test configuration* associated to

the Kähler manifold (M, ω) , in the sense that definition 4.16 is satisfied.

One considers a compact complex m -dimensional Kähler manifold M and ω a Kähler form with cohomology class $[\omega]$.

Definition 4.16 (Kähler test configuration). *With this data, a smooth Kähler test configuration associated to $(M, [\omega])$, is a Kähler manifold $(\mathcal{M}, \mathcal{A} = [\Omega])$ endowed with an holomorphic \mathbf{C}^* -action ρ such that:*

- *there is a surjective holomorphic map $\pi : \mathcal{M} \rightarrow \mathbf{CP}^1$ such that for $x \neq 0 \in \mathbf{CP}^1$, $(M_x := \pi^{-1}(x), \omega_x := \Omega|_{M_x})$ is \mathbf{T} -equivariant isomorphic to $(M, [\omega])$;*
- *π is equivariant with respect to the \mathbf{C}^* -action ρ on \mathcal{M} and the standard action on \mathbf{CP}^1 fixing 0 and ∞ ;*
- *there is a $\mathbf{C}^* \times \mathbf{T}^{\mathbf{C}}$ -equivariant biholomorphism*

$$\mathcal{M}/M_0 \simeq (M \times \mathbf{CP}^1 \setminus \{0\}).$$

When (M, ω) is a toric manifold, the situation is easier. In this context, the Kähler test configuration $(\mathcal{M}, \mathcal{A})$ is given by the data of the PL convex function defining \mathcal{Q} . The instructive example is given by figure 4.3. The Delzant polytope in the l.h.s of figure 4.3 is obtained by *chopping-off a corner* of the square Delzant polytope $[0, 1/2] \times [0, 1/2] \subset \mathbf{R}^2$ representing, via the Delzant theorem, the surface $\mathbf{CP}^1 \times \mathbf{CP}^1$. This operation correspond precisely to the *blow-up* of $\mathbf{CP}^1 \times \mathbf{CP}^1$ at one point ; the blow-down map is denoted by β . Thus the polytope in the l.h.s correspond to $(\mathbf{CP}^1 \times \mathbf{CP}^1) \# \overline{\mathbf{CP}}^2$ which is symplectomorphic to \mathbf{CP}^2 blow-up at 2 points i.e. $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$. We refer to e.g. [A19], chapt. 1.6, for more information on equivariant blow-up. In the l.h.s polytope, the Delzant polytope of \mathbf{CP}^1 (in red) correspond to the same segment in the r.h.s ; and the roof (in blue) corresponds to the *concave* piecewise affine-linear function $-f(p)$. Hence $M = \mathbf{CP}^1$ in the r.h.s is identified with the red segment and the other copy of \mathbf{CP}^1 is the projection of the roof that defines the map π .

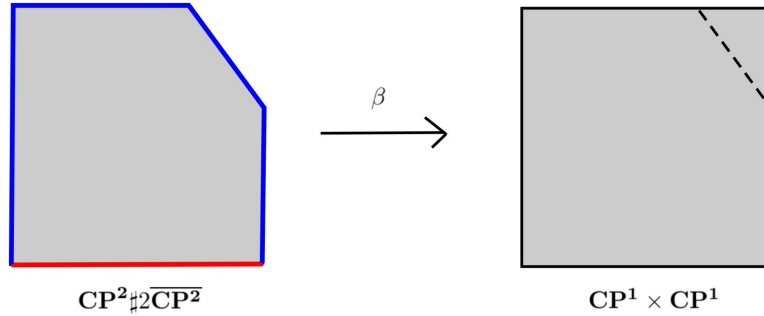


FIGURE 1. Toric test configuration for $M = \mathbf{CP}^1$.

Remark 4.17. TIAN proved that $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$ is one of the two exceptional cases (where the other is $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$) of complex surfaces with $c_1(S) > 0$ which *do not* admits a Kähler-Einstein metric.

Definition 4.18 (Toric test configuration). *Let (M, ω) be a toric Kähler manifold with labelled Delzant polytope (Δ, \mathbf{L}) in \mathbf{R}^m with respect to the lattice \mathbf{Z}^m . A Kähler test configuration $(\mathcal{M}, \mathcal{A}, \rho)$ associated to (M, ω) obtained from a rational PL convex function f as above is called a toric test configuration.*

By analogy with the stability in GIT, an invariant is attached to each toric test configuration, namely the Donaldson-Futaki invariant associated to the corresponding PL convex function. It can be computed in terms of differential-geometric quantity on \mathcal{M} . One denotes by $s(\Omega)$ the scalar curvature of the Kähler metric Ω on \mathcal{M} . The pullback on \mathcal{M} of the extremal affine-linear function $s_{(\Delta, \mathbf{L})}$ of (M, ω) will be, with a slightly abuse of notation, denoted $s_{(\Delta, \mathbf{L})}$.

Lemma 4.19. *With this data, the Donaldson-Futaki invariant 4.7 of the PL convex function f is given by*

$$(2\pi)^{m+1} \mathcal{F}_{(\Delta, \mathbf{L})}(f) = - \int_{\mathcal{M}} (s(\Omega) - s_{(\Delta, \mathbf{L})}) \Omega^{[m+1]} + 8\pi \int_M \omega^{[m]},$$

where $\psi^{[m]} := \psi^m/m!$.

Remark 4.20. When looking at a toric test configuration (given by a PL function f) associated to (M, ω) when ω is a cscK metric, ODAKA proved a co-homological formulae for the Donaldson-Futaki invariant $\mathcal{F}_{(\Delta, \mathbf{L})}(f)$. ODAKA first used this formula to study (possibly singular) polarized projective test configurations. We explain how to derive from the formula in prop. 4.19 in the context of cscK metrics. As $s_{(\Delta, \mathbf{L})}$ is constant, by the very definition of $\mathcal{F}_{(\Delta, \mathbf{L})}(\phi)$, one get

$$s_{(\Delta, \mathbf{L})} = 2 \frac{\int_{\Delta} d\sigma}{\int_{\Delta} dv}$$

with $\phi = 1$. If u is the symplectic potential of ω , from lemma 4.5 one get $\int_{\Delta} s(u) dv = 2 \int_{\partial \Delta} d\sigma$. Combining these two identities, one has

$$s_{(\Delta, \mathbf{L})} = \frac{\int_{\Delta} s(u) d\sigma}{\int_{\Delta} dv} = \frac{\int_M s(\Omega) \omega^m}{\int_M \omega^m} = 4\pi m \frac{c_1(M) \cdot [\omega]^{m-1} [M]}{[\omega]^m [M]}.$$

Following [A19], by substitution, one get the Donaldson-Futaki invariant for this special toric test configuration, denoted as $\mathcal{F}(\mathcal{M}, \Omega) := (2\pi)^m \mathcal{F}_{(\Delta, \mathbf{L})}(f)$ by the co-homological expression:

$$\mathcal{F}(\mathcal{M}, \Omega) = -2 \left[\left(c_1(\mathcal{M}) \cdot [\Omega]^{[m]} [\mathcal{M}] \right) - A \left(c_1(M) \cdot [\omega]^{[m-1]} [M] \right) \right] + 4 \text{Vol}(M, \omega),$$

where $A = \text{Vol}(\mathcal{M}, \Omega) / \text{Vol}(M, \omega)$. Notice that this formula depends only on the De Rham classes. This formula is used for any Kähler test configuration for a Kähler manifold (M, ω) which is equipped, *grosso modo*, with a maximal compact torus \mathbf{T} in its reduced group of complex automorphisms. This gives rise to the notion of \mathbf{T} -relative Donaldson-Futaki invariant $\mathcal{F}^{\mathbf{T}}(\mathcal{M}, [\Omega])$ of a compatible test configuration.

4.4. Mabuchi K-energy & unicity. Whilst on a Stein manifold, one can describe globally the Kähler form using a single function, namely the Kähler potential ; in general its not the case. However, provided that (M, ω) is *compact*, by the $\partial\bar{\partial}$ -lemma, one can consider the space

$$\mathcal{H} := \{ \phi \in C^\infty(M) : \omega_\phi = \omega + i\partial\bar{\partial}\phi > 0 \}$$

of smooth Kähler potentials relative to the metric ω . This space is an infinite dimensional Fréchet manifold endowed with a riemannian metric usually called the Mabuchi-Semmes-Donaldson metric given by $\langle \phi_1, \phi_2 \rangle_\phi := \int_M \phi_1 \phi_2 \omega_\phi^{[m]}$ for $\phi_1, \phi_2 \in T_\phi \mathcal{H} \simeq C^\infty(M, \mathbf{R})$. The fact that \mathcal{H} is contractible allows one to study all Kähler metrics in a fixed Kähler class simultaneously. This *variational approach* proposed by MABUCHI is very fruitful to the study of existence of extremal metrics in a fixed Kähler class. Letting $\omega_\phi^{[m]} := \omega_\phi^m/m!$, one defines the Mabuchi K -energy by its first variation.

Definition 4.21 (Variational Mabuchi). Let $\mathcal{M} : \mathcal{H}/\mathbf{R} \rightarrow \mathbf{R}$ be the Mabuchi K -energy defined by

$$(\delta\mathcal{M})(\phi) = - \int_M \left(s(\omega_\phi) - \underline{s} \right) \delta\phi \omega_\phi^{[m]},$$

where

$$\underline{s} := \frac{\int_M s(\omega_\phi) \omega^m}{\int_M \omega^m}.$$

Clearly, $s(\omega_\phi) = \underline{s}$ iff ϕ is a critical point of $\delta\mathcal{M}$. So, in the same vein as extremal metrics are defined via the Calabi functional, cscK metrics are precisely critical points of the Mabuchi functional. The derivative $\delta\mathcal{M}$ of the K -energy depends only on the variation of the metric (i.e. the Kähler metric) since $\delta\mathcal{M}(\phi + c) = \delta\mathcal{M}(\phi)$, for any constant $c \in \mathbf{R}$. One can see $\delta\mathcal{M}$ as a closed 1-form on \mathcal{H} , and as \mathcal{H} is contractible, it defines $\mathcal{M} : \mathcal{H} \rightarrow \mathbf{R}$ up to a constant ; and one can normalize \mathcal{M} so that $\mathcal{M}(\mathbf{0}) = 0$. Specialize to the case when M is a toric symplectic manifold, we will denote by $\mathcal{M}_{(\Delta, \mathbf{L})}$ the Mabuchi K -energy. Given a labelled compact convex simple polytope (Δ, \mathbf{L}) in \mathbf{R}^m , one has the following ramification.

Proposition 4.22 (Relative Mabuchi K -energy [D02]). With a suitable normalization by adding a constant, the Mabuchi K -energy $\mathcal{M} = \mathcal{M}(\omega)$ is given, up to a factor $(2\pi)^m$, by

$$\mathcal{M}_{(\Delta, \mathbf{L})}(u) = - \int_{\Delta} \left(\log \det \text{Hess}(u) \right) dv + \mathcal{F}_{(\Delta, \mathbf{L})}(u),$$

where the functional $\mathcal{F}_{(\Delta, \mathbf{L})}(u)$ is the Donaldson-Futaki invariant:

$$\mathcal{F}_{(\Delta, \mathbf{L})}(u) = 2 \int_{\partial\Delta} u d\sigma - \int_{\Delta} s_{(\Delta, \mathbf{L})} u dv.$$

Proof. One shows that this formula of \mathcal{M} coincide with the Mabuchi defines by its first variation. The functional $\mathcal{M}_{(\Delta, \mathbf{L})}$ is well-defined on \mathcal{C}_∞ , by a result of Donaldson [D02]. Using the formulae $d \log \det \mathbf{G} = \text{tr}(\mathbf{G}^{-1} d\mathbf{G})$, with the non-degenerate matrix $\mathbf{G} = \text{Hess}(u)$ and with the help of lemma 4.5, one computes the first variation of $\mathcal{M}_{(\Delta, \mathbf{L})}$ at a path $u = u(t) \in \mathcal{S}(\Delta, \mathbf{L})$ in the direction $\dot{u} = \dot{u}(t)$,

$$\begin{aligned} \left(d\mathcal{M}_{(\Delta, \mathbf{L})} \right)_u (\dot{u}) &= - \int_{\Delta} \sum_{i,j=1}^m H_{ij}^u \dot{u}_{,ij} dv + \mathcal{F}_{(\Delta, \mathbf{L})}(\dot{u}) \\ &= \int_{\Delta} \left[\left(- \sum_{i,j=1}^m H_{ij}^u \right) - s_{(\Delta, \mathbf{L})} \right] \dot{u} dv. \end{aligned}$$

In fact $u \in \mathcal{S}(\Delta, \mathbf{L})$ corresponds to a Kähler potential $\phi = \phi_u \in \mathcal{H}$ (with respect to ω_ϕ). Then for any path $u(t) \in \mathcal{S}(\Delta, \mathbf{L})$ with corresponding $\phi(t) := \phi_{u_t} \in \mathcal{H}$, taking the differentiate of (20) gives

$$(23) \quad \dot{u}(t) = -\dot{\phi}(t).$$

Recall that the scalar curvature of $\omega_\phi = \omega_{\phi_u}$ is exactly

$$s(\omega_\phi) = - \sum_{i,j=1}^m H_{ij}^u.$$

To see that the average scalar curvature \underline{s} coincide with the extremal affine function, one may refer to ([G17], 4.14). \square

Unicity of extremal toric metrics. Two solutions of the Abreu equation given by symplectic potentials $u_1, u_2 \in \mathcal{S}(\Delta, \mathbf{L})$ (corresponding to ω -compatible \mathbf{T} -invariant extremal Kähler metrics) are unique up to the action of the linear affine group on a symplectic toric manifold, or orbifold, (M, ω, \mathbf{T}) . In particular, on any compact symplectic toric manifold (resp. orbifold), there exists *at most one*, up to a \mathbf{T} -equivariant isometry, extremal Kähler metric (if they exist). GUAN proved that a geodesic in the space of \mathbf{T} -invariant extremal Kähler metrics corresponds, to a line in the space of symplectic potentials $\mathcal{S}(\Delta, \mathbf{L})$. Thus any two \mathbf{T} -invariant extremal Kähler metrics, are systematically linked via a geodesic ; the convexity of the Mabuchi K -energy conclude.

Theorem 4.23 (Guan [G99]). *Let $u_1, u_2 \in \mathcal{S}(\Delta, \mathbf{L})$ be two solutions of Abreu's equation. Then, $u_1 - u_2$ is an affine function. In particular, on any compact symplectic toric manifold (M, ω, \mathbf{T}) each Kähler class contains, at most one \mathbf{T} -invariant ω -compatible extremal Kähler metric (g, J) .*

Proof. By the computation of $(d\mathcal{M}_{(\Delta, \mathbf{L})})_u(\dot{u})$ on the previous result, immediately, critical points of the Mabuchi K -energy \mathcal{M} corresponds to solutions of Abreu's formula i.e. extremal \mathbf{T} -invariant ω -compatible metric. Furthermore, by the identity $d\mathbf{G}^{-1} = -\mathbf{G}^{-1}d\mathbf{G}\mathbf{G}^{-1}$, the second variation of \mathcal{M} is

$$\left(d^2\mathcal{M}_{(\Delta, \mathbf{L})}\right)_u(\dot{u}, \dot{v}) = \int_{\Delta} \text{tr} \left(\left(\text{Hess}(u) \right)^{-1} \text{Hess}(\dot{u}) \left(\text{Hess}(u) \right)^{-1} \text{Hess}(\dot{v}) \right) dv.$$

Since $\mathbf{H} = \mathbf{G}^{-1} > 0$, the second variation $d^2\mathcal{M}$ is ≥ 0 , \mathcal{M} is convex on $\mathcal{S}(\Delta, \mathbf{L})$. Moreover, as $\text{Hess}(\dot{u})$ is symmetric, the derivative \dot{u} is an affine function iff $\left(d^2\mathcal{M}_{(\Delta, \mathbf{L})}\right)_u(\dot{u}, \dot{u}) = 0$. From Abreu's boundary condition, for any $u_1, u_2 \in \mathcal{S}(\Delta, \mathbf{L})$, $u_t = tu_1 + (1-t)u_2$ for $t \in [0, 1]$ is a curve in $\mathcal{S}(\Delta, \mathbf{L})$ with $\dot{u} = u_1 - u_2$. In conclusion, if u_1, u_2 are two solutions of the Abreu equation, they are critical points of \mathcal{M} and it follows that the difference $u_1 - u_2$ must be affine by the convexity of \mathcal{M} . \square

From the proof, one infer that if $u_* \in \mathcal{S}(\Delta, \mathbf{L})$ is a solution of Abreu's equation, thus a critical point of $\mathcal{M}_{(\Delta, \mathbf{L})}$, then the convexity of $\mathcal{M}_{(\Delta, \mathbf{L})}$ show that u_* is a global minima of the Mabuchi K -energy $\mathcal{M}_{(\Delta, \mathbf{L})}$.

Remark 4.24. DERVAN-ROSS [DR16] proved that, if the Mabuchi K -energy \mathcal{M} (see the next §), defined on a Kähler manifold, is bounded below (resp. coercive), then this implies K -semistability (resp. uniform K -stability). On the toric situation the coercivity of $\mathcal{M}_{(\Delta, \mathbf{L})}$ is known to be equivalent to the uniform K -stability. This result is due to HISAMOTO [H16].

The Chen-Tian formulae. We conclude this section by introduces important functional which will be *crucial* in the proof of the YTD conjecture of toric manifolds. In the general context of a Kähler manifold, the Mabuchi functional is explicitly given by the Chen-Tian formula. Consider the *Aubin-Mabuchi functional* $\mathbb{I}_{\omega} : \mathcal{H} \rightarrow \mathbf{R}$ defined as

$$(24) \quad \mathbb{I}_{\omega}(\phi) = \int_M \phi \sum_{i=0}^m \omega_{\phi}^{[i]} \wedge \omega^{[m-i]}.$$

One writes simply $\mathbb{I}_{\omega} = \mathbb{I}$ when the dependence on ω is clear. For a real $(1, 1)$ -form θ , consider the *twisted Aubin-Mabuchi* \mathbb{I}^{θ} defined as

$$\mathbb{I}^{\theta}(\phi) = \int_M \sum_{i=0}^{m-1} \phi \theta \wedge \omega_{\omega}^{[i]} \wedge \omega^{[n-1-i]}.$$

Proposition 4.25 (Chen-Tian). *The Mabuchi K -energy reads,*

$$\mathcal{M}(\omega_{\phi}) = \int_M \log\left(\frac{\omega_{\phi}^m}{\omega^m}\right) \omega_{\phi}^{[m]} + \underline{s}\mathbb{I}(\phi) - 2\mathbb{I}^{\theta}(\phi),$$

where θ is the Ricci form $\rho_\omega =: \text{Ric}(\omega)$ of the metric ω .

The r.h.s term in this proposition is unchanged if ϕ is changed by an additive constant, thus \mathcal{M} is well-defined on the variation of the metric, whence the notation $\mathcal{M}(\omega_\phi) := \mathcal{M}(\phi)$.

4.5. Uniform K-Stability. In this section we expose a strengthened notion of K-stability, namely *uniform K-stability*, due to SZÉKELYHIDI [S06] and BOUCKSOM-HISAMOTO-JONSSON, inspired by [D02] in the toric situation, and which is conjectured to be the right one in order to characterize the *existence of extremal metrics* on a smooth polarized complex variety. We relate its relationship with the properness of the Mabuchi K -energy. In the toric setting, by a result of HISAMOTO [H16], these two are equivalent, namely the uniform stability is equivalent to the coercivity of the K -energy. In this section, the definition of uniform K-stability in the toric setting is given, relying on the coercivity of the Mabuchi K -energy \mathcal{M} .

Recall that $\mathcal{C}(\Delta)$ is the set of continuous *convex* function on Δ , notice that continuity follows from the convexity on Δ^0 . One introduces subset of $\mathcal{C}(\Delta)$; namely, the space $\mathcal{C}_\infty(\Delta)$ of those smooth functions smooth on Δ^0 . The pattern is,

$$\mathcal{S}(\Delta, \mathbf{L}) \subset \mathcal{C}_\infty(\Delta) \subset \mathcal{C}(\Delta).$$

Note that $\mathcal{S}(\Delta, \mathbf{L})$ is *unchanged* by adding an element of $\mathcal{C}_\infty(\Delta)$ which is additionally supposed smooth on Δ . Indeed, from proposition 3.12, it follows, for $u \in \mathcal{S}(\Delta, \mathbf{L})$ and $f \in \mathcal{C}_\infty(\Delta)$, the latter one is moreover assumed smooth on all Δ , then $u + f \in \mathcal{S}(\Delta, \mathbf{L})$. Conversely, the difference of any two functions in $\mathcal{S}(\Delta, \mathbf{L})$ is a function in $\mathcal{C}_\infty(\Delta)$, smooth on all Δ . Affine linear functions acts on $\mathcal{C}(\Delta)$ and $\mathcal{C}_\infty(\Delta)$ by translations. For this purpose one introduces a slice $\mathcal{C}^*(\Delta)$ for the action on $\mathcal{C}(\Delta)$ which is closed under positive linear combinations; the induced slice in $\mathcal{C}_\infty(\Delta)$ will be denoted by $\mathcal{C}_\infty^*(\Delta)$. *Grosso modo*, a slice is a (local) subspace which is transversal to the orbit. Then, any $f \in \mathcal{C}(\Delta)$ is *uniquely* written as

$$f = \pi(f) + g,$$

where g is an affine function and $\pi : \mathcal{C}(\Delta) \rightarrow \mathcal{C}^*(\Delta)$ is a linear projection, so $\pi(f) \in \mathcal{C}^*(\Delta)$.

Example 4.26. The two following slices will be of importance in the definition of *uniform K-stability*. DONALDSON [D02], gave for an interior point $x \in \Delta^0$ the slice:

$$(25) \quad \mathcal{C}^*(\Delta) := \{f \in \mathcal{C}(\Delta) : f(x) \geq f(x_0) = 0\}.$$

One can also take the following slice:

$$(26) \quad \mathcal{C}^*(\Delta) := \{f \in \mathcal{C}(\Delta) : \int_{\Delta} f(x)g(x)dv = 0, \text{ for any } g \text{ affine linear}\}.$$

This various examples of slice corresponds to different notions of stability. Each of them is closely related to the coercivity of the Mabuchi K -energy $\mathcal{M}_{(\Delta, \mathbf{L})}$; precisely, to the growth of $\mathcal{M}_{(\Delta, \mathbf{L})}$. To measure it we define a norm $\|\cdot\|$ on any slice $\mathcal{C}^*(\Delta)$. So let $\|\cdot\|$ be a semi-norm on $\mathcal{C}(\Delta)$ which induces a *tamed* norm on $\mathcal{C}^*(\Delta)$. By *tamed* norm, we means that there exists $C > 0$ such that

$$\frac{1}{C} \|\cdot\|_1 \leq \|\cdot\| \leq C \|\cdot\|_\infty,$$

where $\|\cdot\|_1 := \int_{\Delta} |\cdot| dv$ is the L^1 -norm and $\|\cdot\|_\infty$ is the \mathcal{C}^0 -norm on $\mathcal{C}(\Delta)$. For example the L^p -norm $\|\cdot\|_p := (\int_{\Delta} |\cdot|^p dv)^{1/p}$ defines a tamed norm on the two slices given in example 4.26. Following [D02], the *boundary norm*

$$(27) \quad \|f\|_b := \int_{\partial\Delta} f d\sigma$$

is a tamed norm with slice $\mathcal{C}^*(\Delta) := \{f \in \mathcal{C}(\Delta) : f(x) \geq f(x_0) = 0, x \in \Delta^0\}$. One claim that for any tamed norm $\|\cdot\|$, the space of PL convex functions and smooth convex functions on the whole Δ are both dense in $\mathcal{C}^*(\Delta)$. With respect to a tamed norm $\|\cdot\|$, the *Donaldson-Futaki invariant* (definition 4.7) $\mathcal{F}_{(\Delta, \mathbf{L})}$ is, well defined and continuous on $\mathcal{C}^*(\Delta)$.

Proposition 4.27 (Donaldson [D02], Zhou-Zhu [ZZ08]). *For any $\delta > 0$, the following assertions are equivalent:*

(1) for all $f \in \mathcal{C}(\Delta)$,

$$\mathcal{F}_{(\Delta, \mathbf{L})}(f) \geq \delta \|\pi(f)\|$$

(2) for all $0 \leq \varepsilon < \delta$ there exists C_ε such that

$$\mathcal{M}_{(\Delta, \mathbf{L})}(u) \geq \varepsilon \|\pi(u)\| + C_\varepsilon$$

for all $u \in \mathcal{S}(\Delta, \mathbf{L})$.

This result suggests that the positivity (for $f \neq 0$) of the Donaldson-Futaki invariant $\mathcal{F}_{(\Delta, \mathbf{L})}$ corresponds to the properness of the Mabuchi K -energy $\mathcal{M}_{(\Delta, \mathbf{L})}$. The direction (1) \rightarrow (2), is given by DONALDSON (be aware of the sign error) and ZHOU-ZHU [ZZ08], and makes use of the following technical lemma which extends lemma 4.5. The fact that the Mabuchi is well-defined on $\mathcal{C}_\infty(\Delta)$ *i.e.* that $\log \det(\text{Hess}(u))$ is integrable on Δ , hinge in the following result. For clarity, note that it is enough to show that $\log \det(u_{ij})$, with i, j fixed, is integrable on Δ , where $u_{ij} := (\text{Hess}(u))_{ij}$. This allows one to get rid of summation over the indices in the following.

Lemma 4.28 (Donaldson [D02]). *Let $u \in \mathcal{S}(\Delta, \mathbf{L})$. Then, for any $f \in \mathcal{C}_\infty(\Delta)$, $w^{ij}f_{,ij}$ is integrable on Δ and*

$$\int_{\Delta} w^{ij} f_{,ij} dv = \int_{\Delta} (w^{ij})_{,ij} f dv + \int_{\partial \Delta} f d\sigma.$$

Definition 4.29 (Toric Uniform K -stability). *A convex compact simple labelled polytope (Δ, \mathbf{L}) satisfying condition (1) of proposition 4.27 for some constant $\lambda > 0$ is called, uniformly K -stable, with respect to the chosen slice $\mathcal{C}_\infty^*(\Delta)$ with norm $\|\cdot\|$. One say that (Δ, \mathbf{L}) is*

- (i) L^p -uniformly K -stable, if it is uniformly K -stable w.r.t the slice 26 and the L^p -norm $\|\cdot\|_{L^p}$;
- (ii) b -uniformly K -stable, if it is uniformly K -stable w.r.t the slice 25 and the boundary norm $\|\cdot\|_b$.

This notion appears as a reinforcement of K -sability. Indeed, as PL convex functions are dense (for the norm $\|\cdot\|_\infty$) in $\mathcal{C}(\Delta)$, one define equivalently the assertion (1) with PL functions. Interested at toric surfaces, DONALDSON obtained that the K -stability corresponds to the b -uniform one, with the assumptions that $s_{(\Delta, \mathbf{L})}$ is positive. The two following theorems shows that K -stability is in fact uniform, with not so restrictive assumption.

Theorem 4.30 (Donaldson [D02]). *If (Δ, \mathbf{L}) is a compact convex labelled polytope in \mathbf{R}^2 such that $s_{(\Delta, \mathbf{L})} > 0$ on Δ . Then, (Δ, \mathbf{L}) is K -stable iff it is b -uniformly K -stable.*

Theorem 4.31 (Székelyhidi [S06]). *If (Δ, \mathbf{L}) is a compact convex labelled polygone in \mathbf{R}^2 such that $s_{(\Delta, \mathbf{L})}$ is constant. Then, (Δ, \mathbf{L}) is K -stable iff it is L^2 -uniformly K -stable.*

4.6. Existence results. In this section the proof of the YTD conjecture on the existence of extremal metrics on toric manifold is given *via* theorem 4.37 & 4.36. The notion of stability required for the characterization is the b -uniform K-stability.

We start with a review on the general theory and then we specialize to the toric situation. Toward this end, consider (M, g_0, J, ω_0) a complex compact m -dimensional manifold with complex structure J , Kähler metric g_0 and Kähler form ω_0 . The group $\text{Aut}(M)$ of complex automorphism of M , is a finite dimensional complex Lie group with Lie algebra given by the vector space of smooth (real) holomorphic vector field, i.e. the set of X such that $\mathcal{L}_X J = 0$. Recall that,

$$\mathcal{H} := \{\phi \in C^\infty(M) : \omega_\phi = \omega + i\partial\bar{\partial}\phi > 0\}$$

is the space of smooth Kähler potentials relative to the metric ω . Under the action of \mathbf{R} on \mathcal{H} by translation, the Kähler metric corresponding to the Kähler form ω_ϕ is preserved ; thus one want to work with a *normalization* \mathcal{H}^0 , for which, one has a bijection between elements $\phi^0 \in \mathcal{H}^0$ and Kähler metrics with Kähler forms in the fixed DeRham class $[\omega] \in H^2(M, \mathbf{R})$. Thus one can decompose $\phi \in \mathcal{H}$ as $\phi = \phi^0 + C^{te}$. Among all popular normalization, we consider the normalization

$$\mathcal{H}^0 := \mathcal{H} \cap \mathbb{I}^{-1}(\{0\}),$$

where \mathbb{I} is the *Aubin-Mabuchi functional*, introduced in 24. For any $\sigma \in \text{Aut}_0(M)$, we write $\sigma[\phi] \in \mathcal{H}^0(M)$ for the *normalized Kähler potential*, relative to ω , associated to the Kähler form $\sigma^*(\omega_\phi)$.

Let $K^{\mathbf{C}} \subset \text{Aut}_0(M)$ be the *reductive* group corresponding to the the complexification of a connected compact subgroup $K \subset \text{Aut}_0(M)$. In other words $K^{\mathbf{C}}$ is the smallest closed complex subgroup in $\text{Aut}(M)$ containing K . Such a group K always exists with satisfying moreover that the Kähler structure (g, J, ω) is K -invariant. So one considers the subspace $\mathcal{H}^K \subset \mathcal{H}$ of K -invariant Kähler potentials in \mathcal{H} . Finally, the space

$$\mathcal{H}^{0,K} := \mathcal{H}^0 \cap \mathcal{H}^K$$

parametrizes the K -invariant Kähler metrics on (M, J) whose Kähler form is in the DeRham class $[\omega]$. DARVAS proved that (\mathcal{H}, d_1) is a metric space for a certain metric d_1 , called the Finsler metric ; which is defined as follows. A characterization (e.g [G17], chap. 4) of *smooth segments* $\phi(t) \in \mathcal{H}^0$, for $t \in [0, T]$ starting at $\phi(0) = 0$ and endpoint $\phi(T)$, is as follows:

$$\phi(t) \in \mathcal{H}^0 \iff \int_M \dot{\phi}(t) \omega_{\phi(t)}^m = 0, \quad \forall t \in [0, T].$$

DARVAS defined the length of the smooth segment $\phi([0, 1]) \subset \mathcal{H}$ by

$$\int_0^1 \left(\int_M |\dot{\phi}(t)| \omega_{\phi(t)}^{[m]} \right) dt.$$

Thus the Finsler metric $d_1(\phi_1, \phi_2)$ is defined to be the infimum of the lengths of all segments with endpoints ϕ_1 and ϕ_2 . One considers

$$d_{1,K^{\mathbf{C}}}(\phi_1, \phi_2) := \inf_{\sigma \in K^{\mathbf{C}}} d_1(\phi_1, \sigma[\phi_2]),$$

for $\phi_1, \phi_2 \in \mathcal{H}$. In the next definition the functional \mathcal{M}^K is called the *relative Mabuchi energy* or sometimes K-relative energy (to K). The next notion of *analytic stability* is due to ZHU-ZHOU.

Definition 4.32 ($K^{\mathbf{C}}$ -relative properness). Let \mathcal{M}^K be a functional defined on the space \mathcal{H}^K . The functional \mathcal{M}^K is called $K^{\mathbf{C}}$ -proper with respect to d_1 if

- \mathcal{M}^K is bounded from below on \mathcal{H}^K ;
- for any sequence $\phi_n \in \mathcal{H}^{0,K}$, with $d_{1,K^{\mathbf{C}}}(\phi_0, \phi_n) \rightarrow \infty$, one has $\mathcal{M}^K(\phi_n) \rightarrow \infty$.

By a result of CALABI (see [G17], chap. 3), the invariant compact group $K \subset \text{Aut}_0(M)$, in the above definition, can be taken, without loss of generality, to be a *torus* \mathbf{T} in a subgroup of $\text{Aut}_0(M)$. Precisely, letting $\mathbf{T} \subset \text{Aut}_{\text{red}}(M)$ be a (real) maximal subtorus of the *reduced group* of automorphisms $\text{Aut}_{\text{red}}(M) \subset \text{Aut}_0(M)$. Recall that this group is closed and connected with Lie algebra given by holomorphic vector fields with non-empty zero set. Denote by $\mathbf{T}^{\mathbf{C}}$ its complexification which is a maximal subtorus of $\text{Aut}_{\text{red}}(M)$. The reason of considering this subclass of torus is the following observation by CALABI. A *key result* is that if it exists an extremal metric ω_ϕ (for some $\phi \in \mathcal{H}$), then there exists an isometric extremal metric $\omega_{\tilde{\phi}}$. Thus the *Calabi problem* is *reduced*, without loss of generality, on $\mathcal{H}^{0,\mathbf{T}}$.

Furhtermore, on this space, MABUCHI & GUAN [G99], introduced a functional $\mathcal{M}^K : \mathcal{H}^{0,K} \rightarrow \mathbf{R}$, called the Mabuchi K-energy relative to the group K . Actually, one can work without loss of generality with the Mabuchi K-energy $\mathcal{M}^{\mathbf{T}}$ relative to a torus \mathbf{T} , where $\mathbf{T} \subset K$ is any maximal torus in K .

The crucial fact ([G17], chap. 4) is that critical points of $\mathcal{M}^{\mathbf{T}}$ are precisely the Kähler potentials in $\mathcal{H}^{0,\mathbf{T}}$, corresponding to \mathbf{T} -invariant extremal metric in (the fixed class) $[\omega]$. The main result is that the properness of $\mathcal{M}^{\mathbf{T}}$ implies the existence of \mathbf{T} -invariant extremal Kähler metrics in $[\omega]$, precisely:

Theorem 4.33 (Chen-Cheng [CC18], He [H18]). *Suppose $\mathbf{T} \subset \text{Aut}_{\text{red}}(M)$ is a maximal real torus and $\mathbf{T}^{\mathbf{C}} \subset \text{Aut}(M)$ its complexification. If the relative K-energy $\mathcal{M}^{\mathbf{T}}$ acting on $\mathcal{H}^{\mathbf{T}}$ is $\mathbf{T}^{\mathbf{C}}$ -proper with respect to the distance d_1 and the normalization \mathcal{H}^0 . Then, there exists $\phi \in \mathcal{H}^{\mathbf{T}}$ such that ω_ϕ is an extremal Kähler metric.*

We specialize to the toric situation. Let (M, ω) be a toric symplectic manifold and $(\Delta, \mathbf{L}, \Lambda)$ its corresponding Delzant polytope. From our discussion above, the torus \mathbf{T} acting effectively, in a hamiltonian fashion on (M, ω) is a maximal torus in the reductive group $\text{Aut}_{\text{red}}(M, J)$, where J is any T -invariant compatible complex structure on M . Recall that from the *Abreu-Guillemin formalism*, one has J -holomorphic coordinates on M^0 , $y_j + \sqrt{-1}t_j$; sometimes called the *affine logarithmic system* of coordinate, because, exponentiating gives rise to $z_j := e^{y_j} e^{\sqrt{-1}t_j}$ which is another set of holomorphic coordinate on M^0 . Consider $u \in \mathcal{S}(\Delta, \mathbf{L})$ a symplectic potential; corresponding to a \mathbf{T} -invariant ω -compatible complex structure J . In the same spirit as the *orbit-cone correspondence* in toric geometry, one can identified \mathbf{C}^* -equivariantly, the complex manifold (M^0, J) with the orbit $(\mathbf{C}^*)^m \cdot p_u$. Precisely, $p_u \in M^0$ corresponds to the unique minima $x_u \in \Delta^0$ of the convex function u under the momentum map μ , with $t_j(p_u) = 0$. Hence, the action of the flows of $\{K_1, \dots, K_m, JK_1, \dots, JK_m\}$ around p_u is identified with the (local) system of coordinate (z_1, \dots, z_m) , whence

$$(\mathbf{C}^*)^m \simeq (M^0, J).$$

Now, the Legendre transform of $u \in \mathcal{S}(\Delta, \mathbf{L})$ is $\phi(y)$; with the coordinate y_j , simply given by $\log z_j$, since $z_j := e^{y_j} e^{\sqrt{-1}t_j}$ and the momentum coordinate t_j vanishes. So one has the corresponding function $F_u(z)$ of u , defined on $(\mathbf{C}^*)^m$ by

$$F_u(z) := \phi(y) = \phi(\log |z_1|, \dots, \log |z_m|).$$

Moreover, the key result of GUILLEMIN, proposition 3.6, tells us that \mathbf{T} -invariant function F_u introduces on $M^0 \simeq (\mathbf{C}^*)^m$ a Kähler form

$$\omega_u := dd^c F_u(z)$$

which is seen to extends as a smooth Kähler metric (g_u, ω_u, J_u) *via* on M via the identification $(\mathbf{C}^*)^m \simeq (M^0, J_u)$. Also, since the canonical symplectic potential u_0 corresponds to the Guillemin metric ω_0 , it gives birth to F_{u_0} *via*, $\omega_0 = dd^c F_{u_0}$. Thus one obtains two Kähler forms ω_0, ω_u related by

$$\omega_u = \omega_0 + dd^c(F_u(z) - F_{u_0}(z)) = \omega_0 + dd^c \phi_u,$$

where $\phi_u := F_u(z) - F_{u_0}(z)$ is a \mathbf{T}^m -invariant function defined on $(\mathbf{C}^*)^m$ such that ϕ_u extends smoothly on M . To resume, at this point ω_0 and ω_u define two different Kähler metrics on $M_\Delta^{\mathbf{C}}$; recall that the latter space is biholomorphic to (M, J) , § 2.6. Moreover, ϕ_u extends to a \mathbf{T} -invariant Kähler potential (w.r.t ω_0), in other words, $\phi_u \in \mathcal{H}^{\mathbf{T}}$; this is in fact a correspondence by taking the dual Legendre transform.

Now, fix a point $p_0 \in M^0$ corresponding to $x_0 \in \Delta^0$ and recall the slice of example 25 :

$$\mathcal{C}_\infty^*(\Delta) := \{f \in \mathcal{C}_\infty(\Delta) : f(x_0) = 0, d_{x_0}f = 0\}.$$

the corresponding slice in the space of symplectic potentials is $\mathcal{S}^*(\Delta, \mathbf{L}) := \mathcal{S}(\Delta, \mathbf{L}) \cap \mathcal{C}_\infty^*(\Delta)$. With this normalization, $p_u = p_0$. However, this slice of symplectic potentials not correspond to the slice of Kähler relative potentials $\mathcal{H}^0 := \mathcal{H} \cap \mathbb{I}^{-1}(\{0\})$. Thus, we consider the action of \mathbf{R} and $\mathbf{T}^{\mathbf{C}}$ on $\mathcal{H}^{0, \mathbf{T}}$. *Primo*, with the action of \mathbf{R} , for $u \in \mathcal{S}^*(\Delta, \mathbf{L})$ if $\tilde{u} = u + c$, then $\tilde{\phi} = \phi - c$ and the coordinates are unchanged i.e. $\tilde{y}_j = y_j$. *Secundo*, if $\gamma \in \mathbf{T}^{\mathbf{C}}$ acts on $\phi \in \mathcal{H}^{0, \mathbf{T}}$, the result is the same, i.e. differs by a coefficient y_j . Now, consider the slice on $\mathcal{C}_\infty(\Delta)$:

$$\mathcal{C}_\infty^0(\Delta) := \{f \in \mathcal{C}_\infty(\Delta) : d_{x_0}f = 0, \int_\Delta f dv = \int_\Delta u_0^* dv\}.$$

Recall that $u_0^* = \pi(u_0)$ is the $\mathcal{C}_\infty^*(\Delta)$ normalization of the canonical symplectic potential u_0 . As usual, the corresponding slice in the space of symplectic potentials is $\mathcal{S}^0(\Delta, \mathbf{L}) := \mathcal{S}(\Delta, \mathbf{L}) \cap \mathcal{C}_\infty^0(\Delta)$. In fact the latter is the one we need, in others words, we have the *wished correspondence*, between $\mathcal{H}^{0, \mathbf{T}}$ and symplectic potentials.

Lemma 4.34. *For any path $\tilde{u}(t)$ in $\mathcal{S}^0(\Delta, \mathbf{L})$, the corresponding ω_0 -relative Kähler potentials $\phi(t) = \phi_{\tilde{u}(t)}$ obtained via the Legendre transform is an element of $\mathcal{H}^{0, \mathbf{T}}$, and satisfy*

$$\frac{d}{dt}\tilde{u}(t) = -\frac{d}{dt}\phi(t).$$

Conversely, any path in $\mathcal{H}^{0, \mathbf{T}}$ comes from a path $\tilde{u}(t)$ in $\mathcal{S}^0(\Delta, \mathbf{L})$, up to the action of $\mathbf{T}^{\mathbf{C}}$.

Proof. The formula of first variation is obtained along the lines of the proof of proposition 4.22. This identity and the characterization of path in \mathcal{H}^0 , gives us $\mathbb{I}(\phi(t)) = cste$; then $cste = 0$ since $\mathcal{S}^0(\Delta, \mathbf{L})$ is convex and contains u_0^* . For the converse, the relative Kähler potential $\phi(t)$ can be pullback by an element $\gamma_t \in \mathbf{T}^{\mathbf{C}}$ so that the symplectic potential associated to $\gamma_t^*(\phi_t)$ satisfies the condition on the derivative at x_0 . \square

In [D02] and [ZZ08], the authors relate, the relative Mabuchi K -energy $\mathcal{M}^{\mathbf{T}}$ acting on $\mathcal{H}^{0, \mathbf{T}}$ (the Kähler case) to the Mabuchi K -energy $\mathcal{M}_{(\Delta, \mathbf{L})}$ acting on $\mathcal{S}^0(\Delta, \mathbf{L})$ (the Kähler toric case).

Proposition 4.35 (Apostolov [A19]). *Assume (Δ, \mathbf{L}) is a b -uniformly K -stable Delzant labelled polytope. Then the relative K -energy $\mathcal{M}^{\mathbf{T}}$ is $\mathbf{T}^{\mathbf{C}}$ -proper on $\mathcal{H}^{\mathbf{T}}$ with respect to the Finsler distance d_1 and the normalization $\mathcal{H}^0 := \{\phi \in \mathcal{H} : \mathbb{I}(\phi) = 0\}$, where \mathbb{I} is the Aubin-Mabuchi functional 24.*

Proof. Let $\phi_j, j = 1, \dots, \infty$ be a sequence in $\mathcal{H}^{0, \mathbf{T}}$ such that $d_{1, \mathbf{T}^{\mathbf{C}}}(0, \phi_j) \rightarrow \infty$. By lemma 4.34, from this sequence corresponds a sequence of normalized symplectic potentials $\tilde{u}_j \in \mathcal{S}^0(\Delta, \mathbf{L})$. The projection of \tilde{u}_j on the slice $\mathcal{S}^*(\Delta, \mathbf{L})$ is denoted $u_j^* := \pi(\tilde{u}_j)$, and thus $\tilde{u}_j = u_j^* + g$ where g is an affine linear function on \mathfrak{t}^* (for $j = 0$, we had the canonical symplectic potential u_0). The action of an element $\gamma_j \in (\mathbf{C}^*)^m$, let unchanged the symplectic potentials but g is translated by a nonzero factor, thus g is assumed to be constant; which is given as:

$$\tilde{u}_j = u_j^* + \frac{1}{\text{Vol}(\Delta)} \int_M (u_0^* - u_j^*) dv,$$

with $j = 0, 1, \dots, \infty$. Consider the path $u_j^*(t) := (1-t)u_0^* + tu_j^*$ in $\mathcal{S}^0(\Delta, \mathbf{L})$. By lemma 4.34, the length of this path, w.r.t d_1 , is $C \int_\Delta |\tilde{u}_j^* - u_j^*| dv$ where $C = (2\pi)^m$. By the , we have:

$$\begin{aligned}
d_{1,\mathbf{T}^{\mathbf{C}}}(0, \phi_j) &\leq d_1(0, \gamma_j[\phi_j]) \\
&\leq C \int_{\Delta} |\tilde{u}_j^* - u_j^*| dv \\
&= C \int_{\Delta} \left(|(u_j^* - u_0^*) + \frac{1}{\text{Vol}(\Delta)} \int_{\Delta} (u_0^* - u_j^*) dv| \right) dv \\
&\leq (C+1) \int_{\Delta} |u_j^* - u_0^*| dv.
\end{aligned}$$

We used, the very definition of $d_{1,\mathbf{T}^{\mathbf{C}}}$, in the first line ; in the third line the definition of \tilde{u}_j . As, for $j \rightarrow \infty$, $d_{1,\mathbf{T}^{\mathbf{C}}}(0, \phi_j) \rightarrow \infty$, the previous inequality gives us $\int_{\Delta} |u_j^*| dv \rightarrow \infty$. Finally the properness of $\mathcal{M}^{\mathbf{T}}$, comes from (2) proposition 4.27 (recall that uniform stability is defined with it) and by the assumption that (Δ, \mathbf{L}) is b -uniformly K-stable leads us to, $\mathcal{M}^{\mathbf{T}}(\omega_{\phi_j}) = \mathcal{M}_{(\Delta, \mathbf{L})}(\tilde{u}_j)$ is lowered by $\varepsilon \|\tilde{u}_j^*\|_b + C_{\varepsilon} \geq \varepsilon' \int_{\Delta} |u_j^*| dv + C_{\varepsilon}$ which goes to ∞ as $j \rightarrow \infty$. The last inequality comes from the fact $\|\cdot\|_b$ bounds the L^1 -norm. \square

As a corollary of this result, together with theorem 4.33 and the correspondence between ω_0 -relative Kähler potentials in \mathcal{H}^0 and symplectic potentials $\tilde{u} \in \mathcal{S}^0(\Delta, \mathbf{L})$, one get:

Theorem 4.36. *If (Δ, L) is a b -uniformly K-stable Delzant labelled polytope (corresponding to a toric Kähler manifold (M, ω_0, J)). Then, (M, J) admits a \mathbf{T} -invariant extremal Kähler metric whose Kähler form is in the De Rham class $[\omega_0]$.*

The converse is due to CHEN-LI-SHENG.

Theorem 4.37 (Chen-Li-Sheng [CLS14]). *If (Δ, \mathbf{L}) is a compact convex simple labelled polytope in \mathbf{R}^m such that the Abreu equation admits a solution in $\mathcal{S}(\Delta, \mathbf{L})$ then (Δ, \mathbf{L}) is b -uniformly K-stable.*

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