Massey product and A_∞ structure

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Abstract The aim of this Master thesis is to present some homological algebra operations which can be seen in geometry, namely Massey products. Then we introduce A_∞ categories which have a strong link with Massey products. In fact, A_∞ categories come with maps that are compositions, that work well with Massey product in homology. One of the main theorems about A_∞ categories is the Kadeishvili theorem, sometimes called minimal model theorem, which claims that one can push forward the A_∞ structure from an A_∞ category to a set of objects as long as you have the first composition on this set of objects and a pre-natural transformation on the A_∞ category.
In the first part we introduce Massey products through the example of the Borromean knot. We present a proof that this knot is knotted that generalize in higher dimension. Then we present A_∞ categories in a didactic way, going further and further in the properties and their implications. Finally, we present a proof of Kadeishvili minimal model theorem based on combinatorics with trees.

Keywords *homological algebra, Massey products, combinatorics, A_∞ category,*

Contents

1 Massey product

1.1 Cup product

The first thing to do is to introduce the definition of cup product and its properties. We will briefly give them since they are well known. The interesting point comes below. As the aim of this subject is to understand some operations in cohomology, we will give an example where the cup product distinguish two spaces who have the same homology and cohomology.
Even if we take for now cohomology in $\mathbb{Z}$ in this section, and in $\mathbb{Z}/2\mathbb{Z}$ later, these definition and properties are valid with cohomology in R any commutative ring with unity.

Definition 1.1. Cup product
Let X be a topological space and $C^p(X,\mathbb{Z}) = Hom(C_p(X),\mathbb{Z})$ denote the group of singular (or simplicial or cellular) p-cochains of X with coefficient in $\mathbb{Z}$. Then we define the cup product as a map

$$\smile : C^p(X,\mathbb{Z}) \times C^q(X,\mathbb{Z}) \longrightarrow C^{p+q}(X,\mathbb{Z})$$

as follows : if $\sigma : \Delta_{p+q} \longrightarrow X$ is a singular (p+q)-chain and $c^p \in C^p(X)$, $c^q \in C^q(X)$, then :

$$< c^p \smile c^q > (\sigma) =< c^p, \sigma \circ \iota(\epsilon_0,...,\epsilon_p) > . < c^q, \sigma \circ \iota(\epsilon_p,...,\epsilon_{p+q}) >$$

Remark 1.2. Recall ι is the simplex mapping the ϵ_i, $\sigma \circ \iota(\epsilon_0,...,\epsilon p)$ is the restriction of σ to the front p-face $\iota(\epsilon_0,...,\epsilon p)$ and and $\sigma \circ \iota(\epsilon_p,...,\epsilon_{p+q})$ is the restriction of σ to the back q-face $\iota(\epsilon_p,...,\epsilon p+q)$

Remark 1.3. Cup product is usually hard to compute, but it is a necessary and usefull operation in cohomology since it induces a ring structure on $H^1(X)$.

The followings properties are not proven. The proofs are presented in almost every general differential geometry, homological algebra or algebraic topology book.

Theorem 1.4. *(i) cup product is bilinear and associative*

(ii) the cochain z^0 whose value is 1 on each singular 0-cochain acts as a unity element

(iii) the following formula holds :

$$\partial(c^p \smile c^q = (\partial c^p) \smile c^q + (-1)^p c^p \smile (\partial c^q)$$

Theorem 1.5. *the cup product on cochains induces a cup product in cohomology :*

$$\smile : H^p(X,\mathbb{Z}) \times H^q(X,\mathbb{Z}) \longrightarrow H^{p+q}(X,\mathbb{Z})$$

which is still bilinear, associative and for which cohomology class $[z^0]$ is a unity.

Property 1.6. *If $f : X \longrightarrow Y$ is a continuous function. Let $f^* : H^*(X) \longrightarrow H^*(Y)$ be the induced homomorphism in cohomology. Then,*

$$\forall \alpha^p \in H^p(X), \forall \beta^q \in H^q(X) ; f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

Property 1.7. *The following formula holds $\forall \alpha^p \in H^p(X,\mathbb{Z})$, $\forall \beta^q \in H^q(X,\mathbb{Z})$:*

$$\alpha^p \smile \beta^q = (-1)^{pq} \beta^q \smile \alpha^p$$

Here comes the example where cup product help us to distinguish two spaces which have same homology and cohomology. It is taken from [4] and concerns the wedge product $S^1 \vee S^1 \vee S^2$ which if defined as follows : take two circles, one sphere and link them in one point p. Then we have :

Lemma 1.8. *The wedge product $S^1 \vee S^1 \vee S^2$ has the same homology and cohomology than the torus T.*

Proof. A cellular decomposition in CW complex of this space is a dimension 0 cell (the point which links the circles and sphere), two dimension one cell (the circles) and a dimension 2 cell (the sphere). Now, writting $[x, y, z]$ the simplex generated by the points x, y, z; we write the following cycles :

$$w_1 = [a, b] + [b, c] + [c, a]$$
$$z_1 = [a, d] + [d, e] + [e, a]$$
$$z_2 = \partial[a, f, g, h]$$

with the points a,b,c,d,e,f,g,h as in the picture1. These cycles represent fundamental classes of the circles and the sphere.

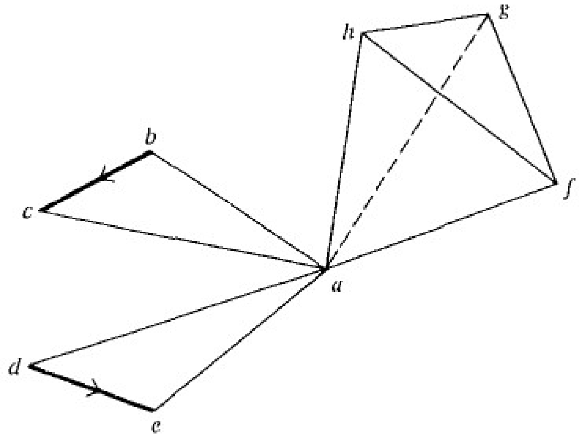


Figure 1: Representation of $S^1 \vee S^1 \vee S^2$ in [4]

Recall the boundary operator in a cellular chain complex X is the application $\partial_n : C_n(X_n, X_{n-1}) \longrightarrow C_{n-1}(X_{n-1}, X_{n-2})$, defined as

$$\partial_n(e_n^\alpha) = \sum_\beta deg(\text{attaching map of } e_n^\alpha \text{ on } e_{n-1}^\beta)e_{n-1}^\beta$$

hence ∂_i vanishes for every i. Then the wedge product has the same decomposition map and boundary operator as a torus T. This implies that they have same homology and cohomology □

Remark 1.9. Even if we proved that the torus T and $S^1 \vee S^1 \vee S^2$ have same cohomology, we also saw before that cup product raise a ring structure on cohomology. These ring will be different. This is where we see that cup product allows to distinguish two spaces with same homology and cohomology. We have to compute the cup product in the torus and in $S^1 \vee S^1 \vee S^2$.

Lemma 1.10. *The cohomology rings of the torus T and $S^1 \vee S^1 \vee S^2$ are different*

Proof. Let us show that the cup product of $X = S^1 \vee S^1 \vee S^2$ is null. We consider cocycles $w^1 = [b, c]^*$ and $z_1 = [d, e]^*$ which are a basis of $Hom(C_1(X), \mathbb{Z})$ as w_1 and z_1 give a basis of $C_1(X)$. Since no 2-simplex in X has vertice attached to w^1 nor z^1 then $w^1 \smile w^1 = z^1 \smile w^1 = z^1 \smile z^1 = 0$. Hence the cup product is null.

Now we compute the cup product in the cohomology ring if the torus. The fact it isn't null yields to the result.
We take cocycles a and b pictured bellow. They generate $H^1(T) \simeq \mathbb{Z}$. Since $[A, B, C]$ is a 2-simplex

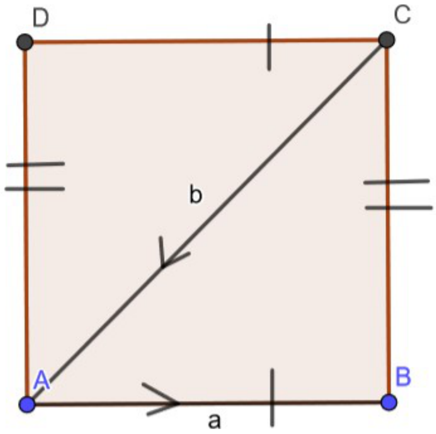


Figure 2: Representation of the torus and its cocycles

whose admit a and b as vertices. Then $a \smile b$ is non null.
(We can compute $< a \smile b, [A, B, C] >=< a, [A, B] > . < b, [A, C] >= 1.1 = 1$) □

In this part we saw that cup product could help us to distinguish two space with same homology and cohomology. But this raised a question: does it exists spaces which have same homology and cohomology, the same cup product but that are different. In this case, can we distinguish them ? We have to go a level further in cohomology and understand other operations : Massey products.

1.2 Borromean knot and higher order Massey product

In this section, we are going to see that even if cup product is null, we can build a new product to distinguish our space from an other. To do so we will introduce a classical link, namely the Borromean knot. We will show that this link is knotted using Massey products. Then we will extend this proof to a generalisation of the Borromean knot in higher dimension. In fact this is where this proof becomes usefull since it almost don't change and is still valid, where an easier proof that the Borromean knot is knoted is available but isn't valid in higher dimension.

Remark 1.11. Let X be a manifold, $p, q, r \in \mathbb{N}^*$ and $[a] \in H^p(X), [b] \in H^q(X)[c] \in H^r(X)$.
Suppose $a \smile b = b \smile c = 0$.
Then there exist $W \in \Omega^{p+q-1}(X), Z \in \Omega^{q+r-1}(X)$ such that :

$$a \smile b = d\,W$$
$$b \smile c = d\,Z$$

Let us call $T = W \smile c - (-1)^p a \smile Z \in \Omega^{p+q+r-1}$. Then $d\,T = 0$ hence the class of T in cohomology is well defined. This depends on the choices of W and Z, that we will remove with a quotient.

Definition 1.12. Triple Product
With the notations in 1.11, we define the triple product between $[a], [b], [c]$, denoted $\langle [a], [b], [c] \rangle$ to be the element of $H^{p+q+r-1}(X) / [a \smile H^{q+r-1}(X) + H^{p+q-1} \smile c]$ which is the cohomology classes of every such chains T.

Remark 1.13.
- The condition $a \smile b = b \smile c = 0$ makes the triple product non empty.
- We could have defined $\langle [a],[b],[c]\rangle$ as a subset of $H^{q+r-1}(X)$ whose elements are the same in $H^{p+q+r-1}(X) / [a \smile H^{q+r-1}(X) + H^{p+q-1} \smile c]$.
- Cup product can be seen as the order one Massey product, triple product as the order 3 Massey product, and recursively we can defined the order n Massey product with the same reasoning as in the remark 1.11 triple product definition.

Definition 1.14. Massey Product of order n
Let X be a manifold, $n \in \mathbb{N}, n \geqslant 2$ and suppose Massey product between $n-1$ elements defined. Let for $i \in [\![1,n]\!], a_{i,i} \in H^{p_i}(X)$. The we define the Massey product between the $a_{i,i}$, denoted $\langle a_{1,1},...,a_{n,n}\rangle$ to be the subset of elements of the form :

$$\overline{a_{i,i}} \smile a_{i,i} + \overline{a_{i,i}} \smile a_{i,i} + ... + \overline{a_{i,i}} \smile a_{i,i}$$

with $\overline{u} = (-1)^{deg\, u} u$ and $\forall (i,j), 1 \leqslant i \leqslant j \leqslant n, (i,j) \neq (1,n)$, $a_{i,j}$ is a solution of :

$$d a_{i,j} = \overline{a_{i,i}} \smile a_{i+1,j} + \overline{a_{i,i}} \smile a_{i+2,j} + ... + \overline{a_{i,j}} \smile a_{j,j}$$

Remark 1.15.
- $0 \in \langle a_{1,1},...,a_{n,n}\rangle$ if and only if these equations admits a solution;
- $\langle a_{1,1},...,a_{n,n}\rangle \neq \varnothing$ if and only if $\forall S \subsetneq [\![1;n]\!], S = s_1,...,s_k \langle a_{s_1,s_1},...,a_{s_k,s_k}\rangle = 0$;
- cohomology class of elements in $\langle a_{1,1},...,a_{n,n}\rangle$ is equal up to lower order Massey products between the $a_{i,j}$

There is still one thing wich is required to understand this proof that Borromean knot is knotted. That is Alexander duality and its application to interpret linking number. The following is presented in [5] where you can also find the proofs.

Theorem 1.16. *the Lefschetz isomorphism*
Let M^n be a compact, orientable manifold with boundary ∂M. We denote $[M]\ inH_n(M,\partial M)$ its fundamental class, and

$$\frown: H^{n-k}(M,L_1)\, x\, H_n(M,\partial M) \quad \rightarrow \quad H_k(M,L_2)$$

with $L_1 = \varnothing, L_2 = \partial M\, or L_1 = \partial M, L_2 = \varnothing$.
Then

$$\frown [M] : H^{n-k}(M) \quad \rightarrow \quad H_k(M,\partial M)$$

and

$$\frown [M] : H^{n-k}(M,\partial M) \quad \rightarrow \quad H_k(M)$$

are isomorphisms.

Remark 1.17.
- this isomorphism stand in homology and cohomology with coefficient in R an additive group of a ring.
- if M is nonorientable, this isomorphism stand with coefficient in $\mathbb{Z}/2\mathbb{Z}$

This theorem yields to Alexander duality wich will be the core of our proof.

Theorem 1.18. *Alexander duality*
Let $M \subsetneq S^n$ be a closed submanifold. Then $\forall 0 \leqslant k \leqslant n-1$, we have the following isomorphisms :

$$\tilde{H}^k(M) \simeq \tilde{H}_{n-k-1}(S^n \backslash M)$$
$$\tilde{H}_k(M) \simeq \tilde{H}^{n-k-1}(S^n \backslash M)$$

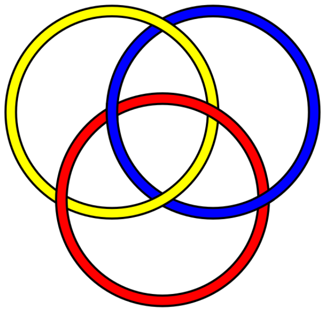


Figure 3: Borromean Ring, by AnonMoos https://commons.wikimedia.org/w/index.php?curid=927405

Remark 1.19. This theorem is valid for every k because of reduced homology. The group $H^0(X)$ is always a free abelian group of rank r. Reduced homology is the homology that is the same everywhere, except you replace $H^0(X)$ by the free abelian group of rank r-1. Reduced homology is denoted by $\tilde{H}_i(X)$ so we have $\tilde{H}_i(X) = H_i(X) \forall i > 1$ and for example $\tilde{H}_0(X) = \mathbb{Z}$ if $H_0(X) = \mathbb{Z}^2$. The intuition is that the homology of a point has to be 0.

Alexander duality provide an interpretation of linking number as a multiplication in cohomology. As a quick reminder, one can define the linking number of a link in S^3 as follows. This definition comes from [2] :

Definition 1.20. Let $K_1, K_2 \subset S^3$ be two knots. Suppose they are oriented and let $K = K_1 \sqcup K_2$ and C be the set of crossings between K_1 and K_2 -that is to say, considering a plan P, and the projection on this plan Π, a crossing is a point p such that $p \in \Pi(K_1) \cap \Pi(K_2)$. Then $\forall p \in C$ one can associate a sign following the right hand rule, written $\epsilon(p)$.
Then the linking number between K_1 and K_2 is defined as $lk(K_1, K_2) = \sum_{p \in C} \epsilon(p)$.

Remark 1.21. We will extend this definition to connected oriented manifolds embedded in S^n in following subsection.

Now with Alexander duality,since $\tilde{H}^{2-k}(K) \simeq \tilde{H}_k(S^3 \backslash K)$ hence there is a class in $\tilde{H}_1(S^3 \backslash M)$ that correspond to fundamental class of K_1, and an other that correspond to fundamental class of K_2. The cup product of these two is in $H^2(S^3 \backslash K) \simeq \tilde{H}_0(K) \simeq \mathbb{Z}$. The only thing that is left to be proved is that :

Lemma 1.22. *This integer is equal to linking number up to sign.*

The proof is in the next subsection, together with a generalized definition of linking number of two orientable connected submanifolds of S^n because the proof isn't clearer in S^3.
Now we can focus on the Borromean knot.

We can give explicit equation to realise this knot, for example :

$$x = 0, y^2 + \frac{z^2}{4} = 1 \text{ for circle } S_1 \tag{1}$$

$$y = 0, z^2 + \frac{x^2}{4} = 1 \text{ for circle } S_2 \tag{2}$$

$$z = 0, x^2 + \frac{y^2}{4} = 1 \text{ for circle } S_3 \tag{3}$$

Pose α, β, γ to be respectively fundamental class of S_1, S_2, S_3. One can compute with right hand rule than linking number between two circles if null. Then because of Alexander duality, we have :

$$\alpha \smile \beta = \alpha \smile \gamma = \beta \smile \gamma = 0$$

This implies that the triple product $< \alpha, \beta, \gamma >$ is well defined. Let us show that this triple product isn't null. Then we will show that if the Borromean knot wasn't knotted, this triple product would be null hence the conclusion.
Let us write $B = S_1 \sqcup S_2 \sqcup S_3$. We have $< \alpha, \beta, \gamma > \subset H^{1+1+1-1}(S^3 \backslash B) \simeq H_0(B)$.
Indeed, we have $H^2(S^3 \backslash B) \simeq H_0(B)$ and $\alpha \smile H^1(S^3 \backslash B) + H^1(S^3 \backslash B) \smile \gamma = \{0\}$ since $H^1(S^3 \backslash B)$ is generated by α, β, γ and any cup product between two of these vanishes (even $\alpha \smile \alpha = 0$ for example because of cup product property). Hence cup product is null.
Lets us call D_i the disc spanned by S_i, $i = 1, 2, 3$, and ω_i the corresponding chain. Since $H_2(S^3, B) \simeq H^1(S^3 \backslash B)$, this isomorphism make $[\omega_1]$ correspond to α, ω_2 to β and ω_3 to γ. Let c be half the disk D_2 and d be half the disk D_3. Then we have :

$$\omega_1 \cap \omega_2 = \partial c$$
$$\omega_2 \cap \omega_3 = \partial d$$

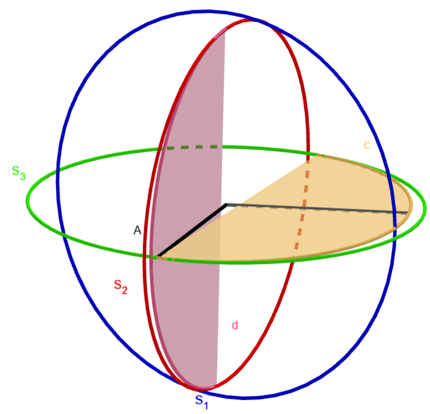


Figure 4: Illustration of c,d and A the cycle with non null contribution.

Then triple product $< \alpha, \beta, \gamma >$ is the set of cycles $c \cap \omega_3 + \omega_1 \cap d \in H_1(S^3, B)$ up to choice of c and d. But this cycle comes from S_1 and goes in S_3 so the isomorphism $\partial : H_1(S^3, B) \longrightarrow H_0(B)$ doesn't send it to zero. Hence the triple product isn't null.

Now suppose Borromean knot B is unknotted, then we can separate our three circles hence chains ω_i corresponding to spanned disks doesn't intersect anymore, then the triple product would be null.

Then we proved :

Property 1.23. *The Borromean knot is knotted*

There is an easier proof by colouring the circles that works well but cannot be generalised to higher order Borromean rings. The following part will extend definitions of Borromean rings and linking number in higher order, and with quite the same proof we will conclude that, again, this "Borromean ring in S^n will be knotted.

1.3 Generalisation of higher dimension Borromean ring

First we want to know if we can defined a linking number between submanifolds of S^n. In the following part, M_1^p, M_2^q will be to closed connected orientable submanifold of S^n with $n = p + q + 1$. To do so we need the notion of intersection number. Here goes some definitions and lemmas, proved in [5] in I)5)3).

1.3.1 The linking number in higher order

Let M^n be a closed orientable manifold. The followings definitions and properties are true on R a ring with identity, althought we enonciate them in $\mathbb{Z}$.

Lemma 1.24. *We can find two cellular decompositions of M such that, denoting σ_i cells of the first one and σ_j^* cells of the other :*

1. *there is bijection between k-cells of the first decomposition and the (n-k)-cells of the second one,*
2. $\sigma_i \cap \sigma_j^* = \emptyset, \forall i \neq j$,
3. σ_i *and* σ^* *intersect transversely,*
4. *cells are oriented such that if $e_1, ..., e_k$ is a positively oriented basis for a cell σ_i and $\epsilon_1, ..., \epsilon_{n-k}$ is for a cell σ_i^* then $e_1, ..., e_k, \epsilon_1, ..., \epsilon_{n-k}$ is a positively oriented basis for M^n.*

Remark 1.25. There is an algorithm to construct such decompositions. See the remark 1.29 the next lemma to have some clues about it.

Remark 1.26. This definition of linking number is equivalent to the other for $n = 3$. I gave the one with the right-hand-rule because it is clearer that it correspond to geometric considerations.

Definition 1.27. With the notations in 1.24, let $V = \sum a_i \sigma_i$ and $W = \sum b_i \sigma_i^*$. The intersection number between V and W is defined as $<< V, W >>= \sum a_i b_i$

Morevover here is lemma that will be usefull just after next definition :

Lemma 1.28. *If Δ_i and Δ_j are simplicies (in the first decomposition of M) of dimension k and k-1 respectively, then :*

$$<< \partial\Delta_i, \Delta_j^* >>= (-1)^k << \Delta_i, \partial\Delta_j^* >>$$

where Δ_j^* is the union of each simplicies of the second decomposition that intersect Δ_j transversaly in its barycenter. This is union of closed (n-k)-cells.

Remark 1.29. Actually given a decomposition of M into simplicies K and considering K' the barycentric subdivision of this decompostion,we can build the "dual" decomposition of the definition 1.27 by considering the set of the $\Delta_k^*$for $\Delta_k \in K'$

Let us go back to our closed connected orientable submanifolds $M_i \subset S^n$. Since $n \geqslant p, q \geqslant 0$; $H_p(S^n) = 0$. Moreover fundamental class $[M_1^p] \in H_p(S^n) = \{0\}$ and $[M_2^q] \in H_q(S^n) = \{0\}$ then $\exists W_1^{p+1} \in C_{p+1}(S^n), W_2^{q+1} \in C_{q+1}(S^n)$; $[M_1^p] = \partial W_1^{p+1}$and $[M_2^q] = \partial W_2^{q+1}$.

Definition 1.30. We define the linking number of M_1 and M_2 to be the intersection number between there fundamental class and the chains W_i:

$$lk(M_1, M_2) := | << W_1^{p+1}, [M_2^q] >> | = | << [M_1^p], W_2^{q+1} >> |$$

Remark 1.31. Because of the lemma 1.28, the second equality is clear. The first one is where the real definition stands.

Remark 1.32. This definition looks alike what we did in the proof that Borromean knot is knotted. In fact, this definition is valid as a definition of linking number in a knot. But I wanted a more computable definition of the linking number in order to make the proof to be easier at a first sight.

Now we want to interpret the linking number thanks to Alexander duality.

1.3.2 Alexander duality and linking number in higher order

Take $X = M_1^p \bigsqcup M_2^q$. Because of Alexander duality, $H^k(S^n \backslash X) \simeq H_{p+q-k}(X)$ Then :

$$[M_1^p] \in H_p(X) \simeq H^q(S^n \backslash X), \text{ then we choose } \alpha \in H^q(S^n \backslash X) \text{ corresponding to} [M_1^p]$$
$$[M_2^q] \in H_q(X) \simeq H^p(S^n \backslash X), \text{ then we choose } \beta \in H^p(S^n \backslash X) \text{ corresponding to} [M_2^q]$$

We have $\alpha \smile \beta \in H^{p+q}(S^n \backslash X) \simeq \tilde{H}_0(X) = \mathbb{Z}$. And just like before, we have :

Lemma 1.33.

$$\alpha \smile \beta = |lk(M_1^p, M_2^q|$$

Proof. In Alexander duality proof we have that following morphisms are isomorphisms:

$$H^{p+q}(S^n \backslash X) \longrightarrow H_1(S^n, X) \xrightarrow{\partial} \tilde{H}_0(X)$$

Then $\alpha \smile \beta \in H^{p+q}(S^n \backslash X)$ correspond to an element $W_1^{p+1} \pitchfork W_2^{q+1} \in H_1(S^n, X)$ beacause we also have isomorphisms :

$$\begin{aligned} H^q(S^n \backslash X) &\longrightarrow H_{p+1}(S^n, X) & &\xrightarrow{\partial} \tilde{H}_p(X) \\ \alpha &\mapsto W_1^{p+1} & &\mapsto [M_1^p] \end{aligned}$$

Then

$$\begin{aligned} \partial(W_1 \cap W_2) &= (\partial W_1 \cap W_2) \cup (W_1 \cap W_2) \\ &= ([M_1] \cap W_2) \cup (W_1 \cap [M_2]) \end{aligned}$$

whose are arcs.But know two more things. Every arc in $W_1 \cap W_2$ which starts and finishes in M_1 will have no contribution in homology because we work in reduced homology. Plus we only need to look at arcs starting in M_1, ending in M_2. So $\alpha \smile \beta$ correspond to $([M_1] \cap W_2)$ which is the intersection number between M_1 and M_2 □

Now that we know what linking number means in higher order, we just have to generalise Borromean ring, and to prove that again it is kind of knotted which will be easy because the same proof as before apply. The classical proof Borromean knot is knotted doesn't generalize in higher dimension.

1.3.3 Generalisation of the Borromean rings, and of the proof

We will generalize Borromean rings in Borromeans spheres with equations. Denoting $x = (x_1, ..., x_p,$ $y = (y_1, ..., y_q)$, $z = (z_1, ..., z_r$, $n = p + q + r$ and also :

$$S_1^{q+r-1} : x = 0, ||y||^2 + \frac{||z||^2}{4} = 1 \tag{4}$$

$$S_2^{p+r-1} : y = 0, ||z||^2 + \frac{||x||^2}{4} = 1 \tag{5}$$

$$S_3^{p+q-1} : z = 0, ||x||^2 + \frac{||y||^2}{4} = 1 \tag{6}$$

$$\tag{7}$$

and $B = S_1 \cup S_2 \cup S_3 \subset S^n$. Then B is our generalization of Borromean rings.

Lemma 1.34. *If $i, j \in \{1,2,3\}, i \neq j$ then $lk(S_i, S_j) = 0$*

Proof. We will embed a sphere in a compact disk whose intersection with the other is empty, which will prove the lemma. Let ($\mathcal{D}$ be the disk of equation $\frac{||x||^2}{3^2} + \frac{||y||^2}{(\frac{1}{2})^2} + \frac{||z||^2}{(\frac{3}{2})^2} \leqslant 1$

- If $(x, y, z) \in S_1, x = 0$ and $||y||^2 = 1 - \frac{||z||^2}{4}$.
  Then

$$\begin{aligned}\frac{||x||^2}{3^2} + \frac{||y||^2}{(\frac{1}{2})^2} + \frac{||z||^2}{(\frac{3}{2})^2} &= \frac{1 - \frac{||z||^2}{4}}{1/4} \\ &= 4 + (\frac{4}{9}||z||^2 - 1) \\ &= 4 - \frac{5}{9}||z||^2\end{aligned}$$

  Then since $0 \leqslant ||z||^2 \leqslant 4$ we have $1 < \frac{16}{9} \leqslant 4 - \frac{5}{9}||z||^2 \leqslant 4$
  So $S_1 \cap \mathcal{D} = \varnothing$

- if $(x, y, z) \in S_2, y = 0$ and $||z||^2 = 1 - \frac{||x||^2}{4}$
  Then

$$\begin{aligned}\frac{||x||^2}{3^2} + \frac{||y||^2}{(\frac{1}{2})^2} + \frac{||z||^2}{(\frac{3}{2})^2} &= \frac{||x||^2}{3} + \frac{1 - \frac{||x||^2}{4}}{9/4} \\ &= \frac{4}{9} \\ &< 1\end{aligned}$$

  So $S_2 \subset \mathcal{D}$

This ensure that $lk(S_1, S_2) = 0$ since chains corresponding to balls spanned in the spheres won't intersect.
We can find such a compact for the other couples of spheres. □

We deduce from the lemma that with α, β, γ fundamental classes of respectively S_1, S_2, S_3 we have :

$$\alpha \smile \beta = \alpha \smile \gamma = \beta \smile \gamma = 0$$

and we still have :

$$\alpha \smile \alpha = \beta \smile \beta = \gamma \smile \gamma = 0$$

because of cup product properties.
Then the triple product $< \alpha, \beta, \gamma > \in H^{n-1}(S^n \backslash B) \simeq \tilde{H}_0(B)$. Let us show it isn't null.
Let $\omega_1, \omega_2, \omega_3$ balls such that ∂w_i is a representative of the fundamental class S_i. We choose :

$$\omega_1 = \{(x, y, z) \in \mathbb{R}^3, x = 0, ||y||^2 + \frac{||z||^2}{4} \leqslant 1\} \tag{8}$$

$$\omega_2 = \{(x, y, z) \in \mathbb{R}^3, y = 0, ||z||^2 + \frac{||x||^2}{4} \leqslant 1\} \tag{9}$$

$$\omega_3 = \{(x, y, z) \in \mathbb{R}^3, z = 0, ||x||^2 + \frac{||y||^2}{4} \leqslant 1\} \tag{10}$$

$$\tag{11}$$

Then we have :

$$\omega_1 \cap \omega_2 = \{(x, y, z)\mathbb{R}^3, x = 0, y = 0, ||z|| \leqslant 1\}$$
$$\omega_2 \cap \omega_3 = \{(x, y, z)\mathbb{R}^3, y = 0, z = 0, ||x|| \leqslant 1\}$$

We introduce

$$c = \{(x, y, z)\mathbb{R}^3, x_1 \geqslant 0, x_2 = ... = x_p = 0, y = 0, ||z||^2 + \frac{x_1^2}{4} \leqslant 1\}$$

and

$$d = \{(x, y, z)\mathbb{R}^3, x = 0, y_1 \geqslant 0, y_2 = ... = y_q = 0, ||z||^2 + \frac{x_1^2}{4} \leqslant 1\}$$

such that we have $\omega_1 \cap \omega_2 = \partial c$ and $\omega_2 \cap \omega_3 = \partial d$ and triple product $< \alpha, \beta, \gamma >$ correspond to $c \cap \omega_3 + \omega_1 \cap d$.
We now compute $c \cap \omega_3$ and $\omega_1 \cap d$:

$$\omega_1 \cap d = \{(x, y, z)\mathbb{R}^3, x = 0, 0 \leqslant y_1 \leqslant 1, y_2 = ... = y_q = 0, z = 0\}$$
$$c \cap \omega_3 = \{(x, y, z)\mathbb{R}^3, 0 \leqslant x_1 \leqslant 1, x_2 = ... = x_p = 0, y = 0, z = 0\}$$

So the cycles $t \mapsto \begin{cases} (1-2t, 0, ..., 0), t \in [0, \frac{1}{2}] \\ (0, ..., y_1 = 2t-1, 0, ..., 0), t \in [\frac{1}{2}, 1] \end{cases}$ is in $c \cap \omega_3 + \omega_1 \cap d$, has starting point in S_1 and ending point in S_3 so it has a non null contricution in homology yields that triple product is not null.
We showed :

Property 1.35. *Higher order generalization of Borromean rings has non null triple product*

Corollary 1.36. *Higher order generalization of Borromean rings is still "knotted" that is to say, we can't find K_1, K_2, K_3 three compacts such that $S_i = \partial K_i$ and $K_i \cap K_j = \emptyset$ for $i \neq j$*

Now that we saw the Massey product geometrically, we want to link it with an algebraic structure to understand where does it stand in the comprehension of a geometric space.

Remark 1.37. We will denote $F_2(X)$ the configuration space of X ,and $L_{p,q)}$ the corresponding lens space. We saw example of spaces where all of the Massey products are null. There is an other example of such a space which is $F_2(L_{7,1})$ because it is homotopy equivalent to $\vee_6 S^2 \times S^3$. But since Massey products of $F_2(L_{7,2})$ are not trivial, then $F_2(L_{7,1})$ and $F_2(L_{7,2})$ are not homotopy equivalent. Since $L_{7,1}$ and $L_{7,2}$ are homotopy equivalent, and their configuration space are not, this induces that configuration space are not homotopy invariant. All of the work is done in [3].

2 A_∞ categories

In this section we introduce the A_∞ structure which have a strong link with Massey product as remarked in the first observation. The majority of the content here comes from [6]. The associative ring we work on in cohomology is $\mathbb{Z}/2\mathbb{Z}$. We could do the work on an other field but this would make the formulas more involving because of a lot of signs $(-1)^{\spadesuit}$ where $\spadesuit$ has a complicated formula. It is still possible to find those formulas and make the work on $\mathbb{R}$ for example. You can have a look at [6] for example.

2.1 A_∞ structure and homology

The first definition we need is the definition of a A_∞ category. This definition induce a lot of operations and properties as you're going to see right after. Just before we need a little bit of vocabulary that deals with categorical algebra. I used the definitions in [7], but there are many equivalent definitions of categories. Even if I consider definitions of categories and functors are known the following definition is here to present what I mean for a non-unital category.

2.1.1 Homological vocabulary

Definition 2.1. 1. A non-unital category $\mathscr{C}$ consists of a set $ob(\mathscr{C})$ of objects, a set $Hom_{\mathscr{C}}(A,B)$ of morphisms for every ordered pair (A,B) of $ob(\mathscr{C})$ and an associative composition between morphism, that is to say for any ordered triple (A,B,C) in $ob(\mathscr{C})$, there is a map

$$\begin{aligned} Hom_{\mathscr{C}}(A,B)\times Hom_{\mathscr{C}}(B,C) &\to Hom_{\mathscr{C}}(A,C)\\ (f,g) &\mapsto g\circ f \end{aligned}$$

which is associative

2. If moreover for every $A\in ob(\mathscr{C})$ there is an identitfy map $id_A\in hom_{\mathscr{C}}(A,A)$ such that $\forall A,B\in ob(\mathscr{C})$ and $\forall f\in hom(_{\mathscr{C}}(A,B)$,

$$id_B\circ f = f = f\circ id_A$$

then $\mathscr{C}$ is said to be a category.

3. A category is said to be linear over a field $\mathbb{K}$ (or at least a commutative ring) if moreover $\forall A,B\in ob(\mathscr{C})$, $hom_{\mathscr{C}}(A,B)$ is a $\mathbb{K}$-vector space and the composition operation is bilinear, that is to say that it is defined on $Hom_{\mathscr{C}}(A,B)\otimes Hom_{\mathscr{C}}(A,B)$

Remark 2.2. In fact, this is the definition of small categories. Replace "a set $ob(\mathscr{C})$ of objects" by "a class $ob(\mathscr{C})$ of objects" to have the proper definition of a category. Every category will be considered as small ones.

Remark 2.3. Note that most of the properties that comes from [6] concern non-unital categories because the goal of the book is to introduce Fukaya categories. Apparently, having an identity morphism in such categories causes tricky phenomenons.
Since this book is my main reference, in the followings, the categories are considered as non unital and small.

We need a final vocabulary, regarding vector spaces.

Definition 2.4. A vector space V is said to be graduaded over $\mathbb{Z}$ if it comes together if a decomposition $V=\bigoplus_{k\in\mathbb{Z}} V_k$ where V_k is a vector space.

Remark 2.5. For examples polynomials (in one or several variables) form a graded $\mathbb{N}$ vector space.

2.1.2 A_∞ category definition and first observations

In the following, X_i will be objects of the categories, and a_i will be morphisms of $hom(X_i, X_j)$, and [k] means shifting the grading of the vector space down by $k \in \mathbb{Z}$.

Definition 2.6. Let $\mathbb{K}$ be a field. A non-unital A_∞ category $\mathscr{A}$ is set of objects $ob(\mathscr{A}$ such that for every pair $(X_0, X_1) \in ob(\mathscr{A})$, a set of morphisms $hom_{\mathscr{A}}(X_0, X_1)$ which is a graded vector space ; that have multi-linear compositions maps of order $d \geqslant 1$:

$$\mu^d_{\mathscr{A}} : hom_{\mathscr{A}}(X_{d-1}, X_d) \otimes ... \otimes hom_{\mathscr{A}}(X_0, X_1) \longrightarrow hom_{\mathscr{A}}(X_0, X_d)[2-d]$$

that verify the A_∞-associativity equations :

$$\sum_{1 \leqslant m \leqslant d, 0 \leqslant n \leqslant d-m} (-1)^{\dagger} \mu^{d-m+1}_{\mathscr{A}}(a_d, ..., a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, ..., a_{n+1}), a_n, ..., a_1) = 0$$

with $\dagger = \sum |a_i| - n$, $|a_i$ the graduation of $a_i \in hom(a_i, a_{i+1})$.

Remark 2.7. We wanted to give the global definition, and this † is nice enough to do it. In the following, $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$ so that the $(-1)^{\dagger}$ vanishes, and $\mathscr{A}$ is a non-unital A_∞ category.

Now apply the formula for $d = 1$. We see that $\mu^1 \circ \mu^1 = 0$ hence

$$... \xrightarrow{\mu^1} hom_{\mathscr{A}}(X_0, X_1) \xrightarrow{\mu^1} hom_{\mathscr{A}}(X_0, X_1) \xrightarrow{\mu^1} ...$$

is a chain complex we will denoting C for now. So we can build its homology as usual.

Remark 2.8. Plus we may use ∂ instead of μ^1 when we will focus on an cohomological property. This is motivated by the fact that μ^1 is a differential of a chain complex.

Also we can apply the formula for $d = 2$. We see that

$$\mu^2(\mu^1(a_2), a_1) + \mu^2(a_2, \mu^1(a_1)) = \mu^1(\mu^2(a_2, a_1)$$

It is exactly (iii), then we can think μ^2 may be the cup product of some chain.

If a_1, a_2 are cycles, then $\mu^1(\mu^2(a_2, a_1)) = 0$ so we can define

$$\begin{aligned} \mu^2 : H.(C) \times H.(C) &\longrightarrow H.(C) \\ ([a_1], [a_2]) &\mapsto [\mu^2(a_1, a_2)] \end{aligned}$$

because we have

$$\begin{aligned} \mu^2(a_2 + \partial(b), a_1) &= \mu^2(a_2, a_1) + \mu^2(\partial(b), a_1) \\ &= \mu^2(a_2, a_1) + \mu^2(b, \partial(a_1)) + \partial(\mu^2(b, a_1)) \end{aligned}$$

taking a_1, a_2 cycles the second term is null, and the last term is in $Im(\partial)$. So in homology we can define it because we have $\mu^2(a_2 + \partial(b), a_1) = \mu^2(a_2, a_1)$. We will write this composition $[a_2] \circ [a_1] = [\mu^2(a_2, a_1)]$.

Remark 2.9. Now, we won't write the compositions anymore. That is to say that for example $[a_2] \circ [a_1]$ will be denoted $[a_2].[a_1]$ or $[a_2][a_1]$.

We continue grabbing some information on $H(\mathscr{A})$ by applying the formula for $d = 3$. We have :

$$\mu^3(\mu^1(a_3), a_2, a_1) + \mu^3(a_3, \mu^1(a_2), a_1) + \mu^3(a_3, a_2\mu^1(a_1)) + \mu^2(\mu^2(a_3, a_2), a_1) + \mu^2(a_3, \mu^2(a_2, a_1)) + \mu^1(\mu^3(a_3, a_2, a_1)) = 0$$

which induces in homology taking a_1, a_2, a_3 cycles :

$$[\mu^2(\mu^2(a_3, a_2), a_1)] = [\mu^2(a_3, \mu^2(a_2, a_1))]$$

that is to say that μ^2 is associative in $H(C)$: it is a real composition.

This implies that we can associate to $\mathscr{A}$ its cohomological category $H(\mathscr{A})$ which have the same objects as $\mathscr{A}$, which morphisms spaces are the cohomology groups denoted $[a_i] \in H(hom_{\mathscr{A}}(X_0, X_1), \mu^1_{\mathscr{A}})$ and which composition is $[a_2][a_1] = [\mu^2_{\mathscr{A}}(a_2, a_1)]$. Then it is a linear graded small category. We don't know if it is unital or non-unital.

We can't have more information with $d = 4$ because μ^3 doesn't have an induced application in homology : even if a_1, a_2, a_3 are cycles, we have

$$\partial\mu^3(a_1, a_2, a_3)) = \mu^2(\mu^2(a_3, a_2), a_1) + \mu^2(a_3, \mu^2(a_2, a_1))$$

which doesn't simplify. But we can try to compute the triple product in homology : Suppose now that $[a_3].[a_2] =_{2.1.2} [\mu^2(a_3, a_2)] = 0$ and $[a_2].[a_1] = [\mu^2(a_2, a_1)] = 0$. We also choose $h_2 \in hom_{\mathscr{A}}(X_1, X_3)$, $h_1 \in hom_{\mathscr{A}}(X_0, X_2)$ such that $\partial h_2 = \mu^2_{\mathscr{A}}(a_3, a_2)$ and $\partial h_1 = \mu^2_{\mathscr{A}}(a_2, a_1)$.

Then let $c = \mu^3_{\mathscr{A}}(a_3, a_2, a_1) + \mu^2_{\mathscr{A}}(h_2, a_1) + \mu^2_{\mathscr{A}}(a_3, h_1) \in hom_{\mathscr{A}}(X_0, X_3)$ is a cycle and we have :

$$< [a_3], [a_2], [a_1] >= [c] \in \frac{hom_{H(\mathscr{A})}(X_0, X_3)}{[a_3].hom_{H(\mathscr{A})}(X_0, X_2) + hom_{H(\mathscr{A})}(X_1, X_3).[a_1]}$$

Even if the compositions of order $\geqslant 3$ are not chain map, they encode Massey products. An other way to see it is that the Massey products of the chain correct the compositions so that there sum is a chain map.

Remark 2.10. Massey prodcuts came easily considering the composition of order > 2 which means that this notion I discussed about the geometrical aspect before have a really strong link with A_∞ that have for now nothing to do with geometry. The structure of A_∞ category will appear in a geometric (and also topological) counter example later on.

For now we will focus on what comes after the definition of an algebraic structure : how do we send a structure on an other.

2.2 Functors between A_∞ categories

Definition 2.11. Let $\mathscr{A}$ and $\mathscr{B}$ be two A_∞ categories with compositions respectively $\mu^d_{\mathscr{A}}$ and $\mu^d_{\mathscr{B}}$. An A_∞ functor between $\mathscr{A}$ and $\mathscr{B}$ is a map $\mathscr{F} : Ob(\mathscr{A}) \longrightarrow Ob(\mathscr{B})$ together with multilinears maps $\mathscr{F}^d$ of order $d \geqslant 1$

$$\mathscr{F}^d : hom_{\mathscr{A}}(X_{d-1}, X_d) \otimes ... \otimes hom_{\mathscr{A}}(X_0, X_1) \longrightarrow hom_{\mathscr{B}}(\mathscr{F}(X_0), \mathscr{F}(X_d))[1-d]$$

that verify the following equation :

$$\sum_{r\geqslant 1} \sum_{s_1+...+s_r=d, s_i\geqslant 1} \mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}(a_d, ..., a_{d-s_r+1}), ..., \mathscr{F}^{s_1}(a_{s_1}, ..., a_1) =$$
$$\sum_{1\leqslant m\leqslant d, 0\leqslant n\leqslant d-n} \mathscr{F}^{d-m+1}(a_d, ..., a_{n+n+1}, \mu^m_{\mathscr{A}}(a_{n+m}, ..., a_{n+1}), a_n, ..., a_1)$$

Remark 2.12. This formula may look complicated but fix d,for every r we take every partition of d in r integer and for each partition $s_r, ..., s_1$, you have to send a pack of s_i consecutive a_j in one element of $hom_{\mathscr{B}}(\mathscr{F}(X_0), \mathscr{F}(X_d))$ by $\mathscr{F}^{s_i}$. Then you can compose them with $\mu^r_{\mathscr{B}}$. Summing for every r, you want to have the global structure (as in the definition of A_∞ category) of $\mathscr{A}$ "sent" to $\mathscr{B}$. For example we have for d=1 and d=2 :

$$\mu^1_{\mathscr{B}}(\mathscr{F}^1(a)) = \mathscr{F}^1(\mu^1_{\mathscr{A}}(a)) \tag{12}$$

$$\mu^1_{\mathscr{B}}(\mathscr{F}^2(a_2,a_1)) + \mu^2_{\mathscr{B}}(\mathscr{F}^1(a_2),\mathscr{F}^1(a_1)) = \mathscr{F}^2(\mu^1_{\mathscr{A}}(a_2),a_1) + \mathscr{F}^2(a_2,\mu^1_{\mathscr{A}}(a_1)) + \mathscr{F}^1(\mu^2_{\mathscr{A}}(a_2,a_1)) \tag{13}$$

Remark 2.13. The first equation implies that $\mathscr{F}^1$ induces a map in homology. The second equation induces that

$$H(\mathscr{F}^1)([a_2]).H(\mathscr{F}^1)([a_1]) = [\mathscr{F}^1(a_2)].[\mathscr{F}^1(a_1)] = [\mathscr{F}^1(\mu^2_{\mathscr{A}}(a_2,a_1))]$$

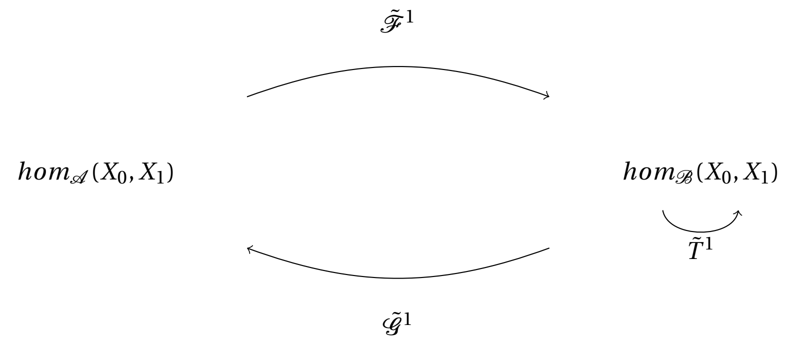
Remark 2.14. We can compose functors with the following formula :

$$(\mathscr{G}\circ\mathscr{F})^d(a_d,...,a_1) = \sum_{r\geqslant 1}\sum_{s_1+...+s_r=d, s_i\geqslant 1} \mathscr{G}^r(\mathscr{F}^{s_r}(a_d,...,a_{d-s_r+1}),...,\mathscr{F}^{s_1}(a_{s_1},...,a_1))$$

This will become usefull later on.

Here we write our main theorem, the Kadeishvili theorem. Then we will in next subsection explain the homotopy notion required in this theorem. We will prove it in the next section.

Theorem 2.15. *Let $\mathscr{B}$ be a non-unital A_∞ category. Suppose $\forall X_0, X_1 \in Ob(\mathscr{B})$ we have $(hom_{\mathscr{A}}(X_0,X_1),\mu^1_{\mathscr{A}})$ a vector space chain complex (for now, $\mathscr{A}$ isn't anything nor is $\mu^1_{\mathscr{A}}$ except a chain map). Plus suppose we have chain maps $\tilde{\mathscr{F}}^1$,$\tilde{\mathscr{G}}^1$, and also a linear map $\tilde{\mathscr{T}}^1$ of degree -1 such that we have :*



1.
2. $\mu^1_{\mathscr{B}}\tilde{T}^1 + \tilde{T}^1\mu^1_{\mathscr{B}} = \tilde{\mathscr{F}}^1\tilde{\mathscr{G}}^1 + id$

Then we can construct :

(i) An A_∞ category $\mathscr{A}$ with $Ob(\mathscr{A}) = Ob(\mathscr{B})$, with first order composition map $\mu^1_{\mathscr{A}}$,

(ii) non-unital functors $\mathscr{F}:\mathscr{A}\longrightarrow\mathscr{B}$ and $\mathscr{G}:\mathscr{B}\longrightarrow\mathscr{A}$ which are identity and for whose $\mathscr{F}^1 = \tilde{\mathscr{F}}^1$ and $\mathscr{G}^1 = \tilde{\mathscr{G}}^1$,

(iii) an homotopy T between $\mathscr{F}^1\mathscr{G}^1$ and $Id_{\mathscr{B}}$ for whose $T^1 = \tilde{T}^1$

Remark 2.16. Even if we chose to introduce $\tilde{\mathscr{F}}^1$,$\tilde{\mathscr{G}}^1$, and $\tilde{\mathscr{T}}^1$ with the tildes so we don't think they are at the beginning first order maps of functors, looking at the conclusion I will remove them for now. Still, remember they are not first order maps of functors, but just maps that verify the good conditions for it.

2.3 Natural transformation and homotopy between functors

As usual with categories, once you have the functors between elements of a category, you can look at the category of those morphism. This is the aim of next definitions where we will introduce this category and give the expression that verify there composition 2.17 and then finally explain what is an homotopy between two functors.

Definition 2.17. Let $\mathscr{A}$, $\mathscr{B}$ be two A_∞ categories. We define the category of nun-unital functors between $\mathscr{A}$ and $\mathscr{B}$, denoted $\mathscr{Q} = nu - fun(\mathscr{A}, \mathscr{B})$. Its objects are the functors, and an element of the chain space $T \in hom^g_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1)$ (g is the degree in the chain of T)is a sequence $(T^d)_{d\geqslant 0}$ of multilinears maps :

$$T^d : hom_{\mathscr{A}}(X_{d-1}, X_d) \otimes ... \otimes hom_{\mathscr{A}}(X_0, X_1) \longrightarrow hom_{\mathscr{B}}(\mathscr{F}(X_0), \mathscr{F}(X_d))[g-d]$$

Note that in particular, T^0 if a family (indexed by X) of elements in $hom^g_{\mathscr{Q}}(\mathscr{F}_0(X), \mathscr{F}_1(X))$. We call T a pre-natural transformation from $\mathscr{F}_0$ to $\mathscr{F}_1$.

The boundary operator of the chain is

$$\mu^1_{\mathscr{Q}}(T)^d(a_d, ..., a_1) = \sum_{r\geqslant 1, 1\leqslant i\leqslant r}\ \sum_{s_1+...+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}_1^{s_r}(\bullet), ..., \mathscr{F}_1^{s_{i+1}}(\bullet), T^{s_i}(\bullet), \mathscr{F}_0^{s_{i-1}}(\bullet), ..., \mathscr{F}_0^{s_1}(\bullet)) + \sum_{1\leqslant m\leqslant, 0\leqslant n\leqslant d-m} T^{d-m+1}(a_d, ..., a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, ..., a_{n+1}), a_n, ..., a_1)$$

where the $\bullet$ have to be replace by the a_j with the good indexes, for example $\mathscr{F}_1^{s_{i+1}}$ is evaluated in $(a_{s_1+...+s_{i-1}+1}, ..., a_{s_1+...+s_i+1})$.

Remark 2.18. Note that again, this boundary operator is the chain map of the chain composed of $hom_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1)$.

The higher order compositions are easier to formulate and follow the same pattern as the following. Take $T_1 \in hom_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1)$ and $T_2 \in hom_{\mathscr{Q}}(\mathscr{F}_1, \mathscr{F}_2)$. Then :

$$\mu^2_{\mathscr{Q}}(T_1, T_2)^d(a_d, ..., a_1) = \sum_{1\leqslant r\leqslant d}\ \sum_{1\leqslant i<j\leqslant r}\ \sum_{s_1+...+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}_2^{s_r}(\bullet), ..., \mathscr{F}_2^{s_{j+1}}(\bullet), T_2^{s_j}(\bullet), \mathscr{F}_1^{s_{j-1}}(\bullet), ..., \mathscr{F}_1^{s_{i+1}}(\bullet), T_1^{s_i}(\bullet), \mathscr{F}_0^{s_{i-1}}(\bullet), ..., \mathscr{F}_0^{s_1}(\bullet))$$

Remark 2.19. You can find in the appendix a python algorithm that compute all the terms involved in the second order composition map for pre-natural transformations.

Remark 2.20. Again the equations may be hard at a first sight. Note that T have to appear in every one of your terms for the first equations, and both T_1 and T_2 have to appear in the second one.

For the first one you place firstly some stacks of $\mathscr{F}_1$ on the left, then T, then the other stacks of $\mathscr{F}_0$ on the right. Finally, you add the "transformations of you compositions", with the same cyclic way as usual.

For the second one you place firstly some stacks of $\mathscr{F}_2$ on the left, then your T_2, you continue with your stacks of $\mathscr{F}_1$, then your T_1 and finally your stacks of $\mathscr{F}_0$.

Definition 2.21. Let T be a pre-natural transformation. Then if T is a cocycle ($\mu^1_{\mathscr{Q}}(T) = 0$), we will say that T is a natural transformation.

Remark 2.22. If T_1 and T_2 are two natural transformation, such that there exist there exists T_3 with $T^1 = T^2 + \mu^1_{\mathscr{Q}}(T_3)$, then T_1 and T_2 are chain homotopic (in the chain $(hom_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1), \mu^1_{\mathscr{Q}})$). This is a notion of chain homotopy between natural transformations.

Remark 2.23. Given a natural transformation T between two functors $\mathscr{F}_0, \mathscr{F}_1$, we see that the element $[T^0_X] \in hom_{H(\mathscr{B})}(\mathscr{F}_0(X), \mathscr{F}_1(X))$ satisfies $\forall [a] \in hom_{H(\mathscr{A})}(X^0, X^1)$:

$$[T^0_{X^1}].[\mathscr{F}^1_0(a)] = [\mathscr{F}^1_1(a)].[T^0_{X^0}]$$

Since this is the second order product of $H(\mathscr{A})$, $[T^d]$ verifies $\mu^1_{H(\mathscr{Q})}([T^d]) = 0$ (where $H(\mathscr{Q}) = nu-fun(H(\mathscr{A}), H(\mathscr{B}))$ hence [T] is a natural transformation between the functors induced in homology by $\mathscr{F}_0$ and $\mathscr{F}_1$.

Now let us define what is an homotopy between two functors of A_∞ category.

Let $\mathscr{F}_0, \mathscr{F}_1 \in Ob(\mathscr{Q})$; $\mathscr{Q} = nu-fun(\mathscr{A}, \mathscr{B})$ acting in the same way on objects. That is to say that $\mathscr{F}_0, \mathscr{F}_1 : \mathscr{A} \longrightarrow \mathscr{B}$ are two functors and $\forall X \in Ob(\mathscr{A})$; $\mathscr{F}_0(X) = \mathscr{F}_1(X)$. We introduce $D = \mathscr{F}_0 - \mathscr{F}_1 \in hom^1_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1)$ defined by $D^0 = 0$, $D^d = \mathscr{F}^d_0 - \mathscr{F}^d_1$, $\forall d \geqslant 1$.

Lemma 2.24. *D is a natural transformation.*

Proof. Every term in the first sum of 2.17 collapse with an other because of the multilinearity of $\mu^d_{\mathscr{B}}$. The second sum is null because $\mathscr{F}_0, \mathscr{F}_1$ act on the same way on objects. □

Definition 2.25. We say that $\mathscr{F}_0$ and $\mathscr{F}_1$ are homotopic if $\exists T \in hom^0_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1)$, $D = \mu^1_{\mathscr{Q}}$.

Lemma 2.26. *This homotopy is an equivalence relation.*

Proof. The proof is in [6]. Usually, this is not hard to proof but here computation are harder due to the "interpretation" of $T \in hom^0_{\mathscr{Q}}(\mathscr{F}_0, \mathscr{F}_1) = hom^0_{\mathscr{Q}}(\mathscr{F}_1, \mathscr{F}_2)$ because all this functors acts on the same way on objects. There is just one thing which is not clear in the book that we want to add. The very first step is to show the following formula, with $\{j,k,l\} = \{0,1,2\}$ and T_{jk} seen in $hom_{\mathscr{Q}}(\mathscr{F}_j, \mathscr{F}_k)$ and where • is the right collection of a_i:

$$\mu^2_{\mathscr{Q}}(\mathscr{F}_k - \mathscr{F}_l, T_{jk})^d(\bullet) = \mu^1_{\mathscr{Q}}(T_{jl})^d + \mu^1_{\mathscr{Q}}(T_{jk})^d$$

$$\begin{aligned}
\mu^2_{\mathscr{Q}}(\mathscr{F}_k - \mathscr{F}_l, T_{jk})^d(\bullet) = & \sum_{1\leqslant r\leqslant d}\sum_{1\leqslant i<j\leqslant r}\sum_{s_1+\ldots+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}_l(\bullet), \ldots, \mathscr{F}^{s_{j+1}}_l(\bullet), \\
& (\mathscr{F}_k - \mathscr{F}_l)^{s_j}(\bullet), \mathscr{F}^{s_{j-1}}_k(\bullet), \ldots, \mathscr{F}^{s_{i+1}}_k(\bullet), T^{s_i}_{jk}(\bullet), \mathscr{F}^{s_{i-1}}_j(\bullet), \ldots, \mathscr{F}^{s_1}_j(\bullet)) \\
\underset{\text{linearity}}{=} & \sum_{1\leqslant r\leqslant d}\sum_{1\leqslant i<j\leqslant r}\sum_{s_1+\ldots+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}_l(\bullet), \ldots, \mathscr{F}^{s_{j+1}}_l(\bullet), \\
& \mathscr{F}^{s_j}_k(\bullet), \mathscr{F}^{s_{j-1}}_k(\bullet), \ldots, \mathscr{F}^{s_{i+1}}_k(\bullet), T^{s_i}_{jk}(\bullet), \mathscr{F}^{s_{i-1}}_j(\bullet), \ldots, \mathscr{F}^{s_1}_j(\bullet)) \\
& + \sum_{1\leqslant r\leqslant d}\sum_{1\leqslant i<j\leqslant r}\sum_{s_1+\ldots+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}_l(\bullet), \ldots, \mathscr{F}^{s_{j+1}}_l(\bullet), \\
& \mathscr{F}^{s_j}_l(\bullet), \mathscr{F}^{s_{j-1}}_k(\bullet), \ldots, \mathscr{F}^{s_{i+1}}_k(\bullet), T^{s_i}_{jk}(\bullet), \mathscr{F}^{s_{i-1}}_j(\bullet), \ldots, \mathscr{F}^{s_1}_j(\bullet))
\end{aligned}$$

and now for every$s_1, \ldots, s_r$, each term of the first sum with both $\mathscr{F}_j, \mathscr{F}_k$and $\mathscr{F}_l$

$$\mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}_l(\bullet), \ldots, \mathscr{F}^{s_{j+1}}_l(\bullet), \mathscr{F}^{s_j}_k(\bullet), \mathscr{F}^{s_{j-1}}_k(\bullet), \ldots, \mathscr{F}^{s_{i+1}}_k(\bullet), T^{s_i}_{jk}(\bullet), \mathscr{F}^{s_{i-1}}_j(\bullet), \ldots, \mathscr{F}^{s_1}_j(\bullet))$$

collapse with following one in the second sum with both $\mathscr{F}_j, \mathscr{F}_k$and $\mathscr{F}_l$

$$\mu^r_{\mathscr{B}}(\mathscr{F}^{s_r}_l(\bullet), \ldots, \mathscr{F}^{s_j}_l(\bullet), \mathscr{F}^{s_{j-1}}_l(\bullet), \mathscr{F}^{s_{j-2}}_k(\bullet), \ldots, \mathscr{F}^{s_{i+1}}_k(\bullet), T^{s_i}_{jk}(\bullet), \mathscr{F}^{s_{i-1}}_j(\bullet), \ldots, \mathscr{F}^{s_1}_j(\bullet))$$

The only thing left is the terms with only $\mathscr{F}_l, \mathscr{F}_j$ or only $\mathscr{F}_k, \mathscr{F}_j$, that is to say :

$$\sum_{r\geqslant 1, 1\leqslant i\leqslant r} \sum_{s_1+\ldots+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}_l^{s_r}(\bullet), \ldots, \mathscr{F}_l^{s_{i+1}}(\bullet), T_{jl}^{s_i}(\bullet), \mathscr{F}_j^{s_{i-1}}(\bullet), \ldots, \mathscr{F}_j^{s_1}(\bullet)) =$$

$$= \mu^1_{\mathscr{Q}}(T_{jl})^d(\bullet) + \sum_{1\leqslant m\leqslant, 0\leqslant n\leqslant d-m} T_{jl}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

and

$$\sum_{r\geqslant 1, 1\leqslant i\leqslant r} \sum_{s_1+\ldots+s_r=d} \mu^r_{\mathscr{B}}(\mathscr{F}_k^{s_r}(\bullet), \ldots, \mathscr{F}_k^{s_{i+1}}(\bullet), T_{jk}^{s_i}(\bullet), \mathscr{F}_j^{s_{i-1}}(\bullet), \ldots, \mathscr{F}_j^{s_1}(\bullet)) =$$

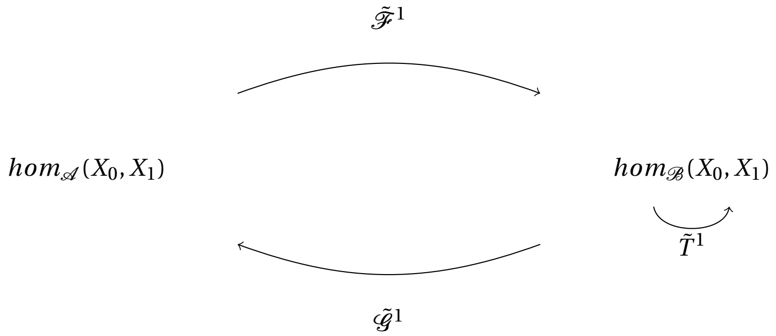
$$= \mu^1_{\mathscr{Q}}(T_{jk})^d(\bullet) + \sum_{1\leqslant m\leqslant, 0\leqslant n\leqslant d-m} T_{jk}^{d-m+1}(a_d, \ldots, a_{n+m+1}, \mu^m_{\mathscr{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1)$$

summing those to give the equality we wanted since $T_{jl} = T = T_{jk}$. □

3 Kadeishvili theorem and combinatorial proof

The aim of this section is to proove the Kaideishvilli theorem we already presented.

Theorem 3.1. *Let $\mathscr{B}$ be a non-unital A_∞ category. Suppose $\forall X_0, X_1 \in Ob(\mathscr{B})$ we have $(hom_{\mathscr{A}}(X_0, X_1), \mu^1_{\mathscr{A}})$ a vector space chain complex (for now, $\mathscr{A}$ isn't anything nor is $\mu^1_{\mathscr{A}}$ except a chain map). Plus suppose we have chain maps $\tilde{\mathscr{F}}^1$, $\tilde{\mathscr{G}}^1$, and also a linear map $\tilde{\mathscr{T}}^1$ of degree -1 such that we have :*

1. 

2. $\mu^1_{\mathscr{B}}\tilde{T}^1 + \tilde{T}^1\mu^1_{\mathscr{B}} = \tilde{\mathscr{F}}^1\tilde{\mathscr{G}}^1 + id$

   Then we can construct :

(i) *An A_∞ category $\mathscr{A}$ with $Ob(\mathscr{A}) = Ob(\mathscr{B})$, with first order composition map $\mu^1_{\mathscr{A}}$,*

(ii) *non-unital functors $\mathscr{F} : \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathscr{G} : \mathscr{B} \longrightarrow \mathscr{A}$ which are identity and for whose $\mathscr{F}^1 = \tilde{\mathscr{F}}^1$ and $\mathscr{G}^1 = \tilde{\mathscr{G}}^1$,*

(iii) *an homotopy T between $\mathscr{F}^1\mathscr{G}^1$ and $Id_{\mathscr{B}}$ for whose $T^1 = \tilde{T}^1$*

In a first part, we explain our notations, and the main results we will use. Then in a next section we present the first steps of the induction. Finally we present the combinatory in general.

We will build the functor $\mathscr{F}$ and the compositions maps by induction, with the following formulas :

$$\mathscr{F}^d(a_d,...,a_1) = \sum_{r\geqslant 2}\sum_{s_1+...+s_r=d} T^1(\mu^r_{\mathscr{B}}(\mathscr{F}_1^{s_r}(\bullet),...,\mathscr{F}_1^{s_{i+1}}(\bullet), T^{s_i}(\bullet), \mathscr{F}_0^{s_{i-1}}(\bullet),...,\mathscr{F}_0^{s_1}(\bullet)))$$

$$\mu^d_{\mathscr{A}}(a_d,...,a_1) = \sum_{r\geqslant 2}\sum_{s_1+...+s_r=d} \mathscr{G}^1(\mu^r_{\mathscr{B}}(\mathscr{F}_1^{s_r}(\bullet),...,\mathscr{F}_1^{s_{i+1}}(\bullet), T^{s_i}(\bullet), \mathscr{F}_0^{s_{i-1}}(\bullet),...,\mathscr{F}_0^{s_1}(\bullet)))$$

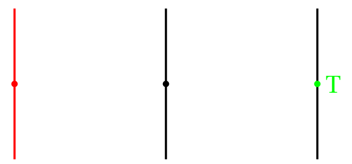
Then we will verify that for every integer r, $\mathscr{F}^r$ verifies the A_∞ functor definition 2.11 and that $\mu^d_{\mathscr{A}}$ verifies the A_∞ structure definition 2.6.

3.1 Rewriting the formulas with trees

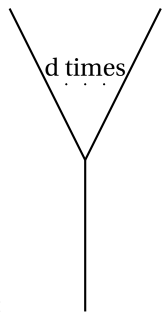
In order to make the computation easier, we will write the formulas with trees. Here are some definition to ensure we speak of the same thing. Every tree will be rooted, but we will speak of trees instead of rooted trees.

Definition 3.2. A rooted tree is a set of nodes and edges. Every edges has a unique parent node and a unique child node. A node can link several (or none) edges above it , but has a unique edge below it.
The root is a unique node without any parent.
A node which has no children is called a leaf.
An interior edge is an edge for which at least one children isn't a leaf, and which is not linked to the root.

Our tree will be colored in red if the element that is represented is in $\mathscr{A}$ and in black if it is in $\mathscr{B}$. The following figures are the representations of respectively $\mu^1_{\mathscr{A}}$, $\mu^1_{\mathscr{B}}$ and T^1 :

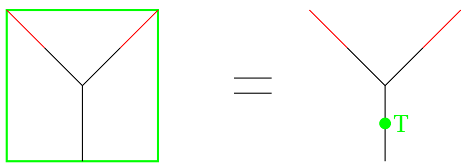


We will represent $\mu^d_{\mathscr{B}}$ as follows (same thing for $\mu^d_{\mathscr{A}}$ but in red):



Also, we will spanned the tree in a green rectangle to express it is $\mathscr{F}^d$ and a change of colours will indicate that we apply $\mathscr{F}_1$ for red to black, and $\mathscr{G}^1$ for black to red, read from the top to the bottom.

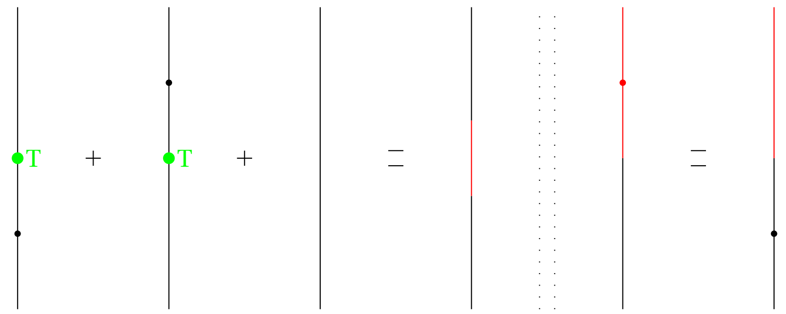
For example, we have $\mathscr{F}^2(a_2,a_1) = T^1(\mu^2_{\mathscr{B}}(\mathscr{F}^1(a_2),\mathscr{F}^1(a_1))$ which will be represented in trees as follows :



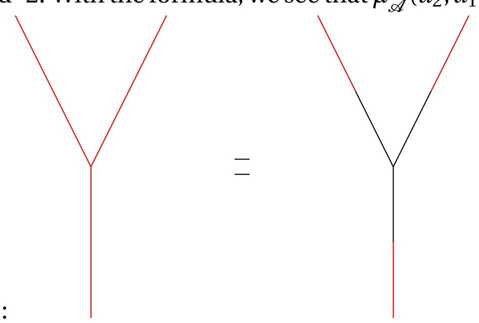
We will denote τ_d^k the set of all the trees with k interior edges and d leafs. Moreover we call colored a tree for which every interior edges carry a T^1, we will denote them $\tau_d^k\odot$; and we will denote $\tau_d^k\otimes$ the set of the trees which are not colored.

3.2 First steps and observation

First of all, we write the equation that verifies $\mathscr{F}^1$, $\mathscr{G}^1$ and T^1 on the left, and on 12 :

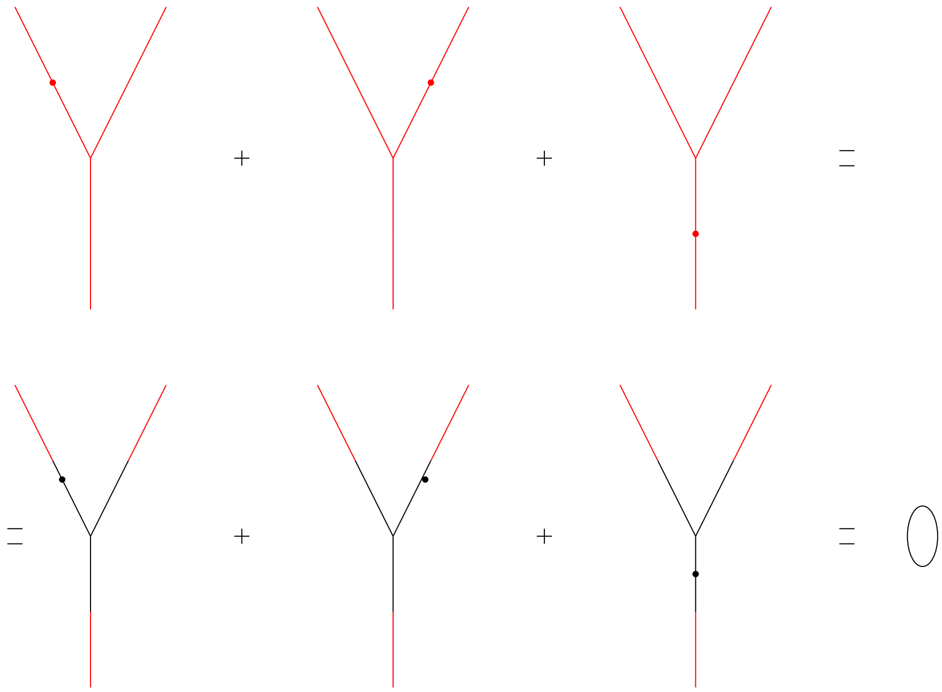


Let us do the computation for d=2. With the formula, we see that $\mu^2_{\mathscr{A}}(a_2, a_1) = \mathscr{G}^1(\mu^2_{\mathscr{B}}(\mathscr{F}_1(a_2), \mathscr{F}_1(a_1))$.

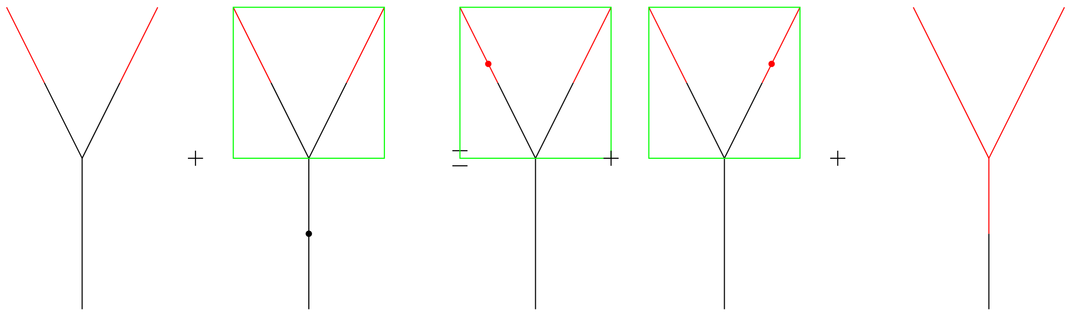


That is to say writting with trees that :

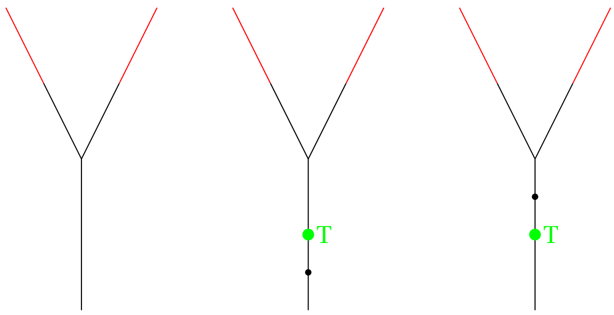
Then we can proove 2.6 for d = 2.



because we can use the equation verified by $\mu^d_{\mathscr{B}}$ since they are the composition of the A_∞ category $\mathscr{B}$. Now we can check that $\mathscr{F}_1, \mathscr{F}_2$ verify 13, that is to say :



For now and until the end, we won't right the "+" anymore in our trees equalities. We start from the right part of the equality. We use 3.2 and 3.2 to express the last tree as :



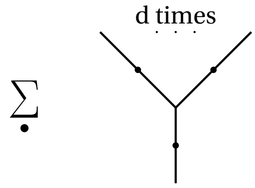
The last tree collapse with with the 2 others terms of the right part of the equality because of the equation verified by $\mu^2_{\mathscr{B}}$ and the right part in 3.2. The two terms that are left are exactly the terms on the left side of the equality, which yields to the result.

The combinatory is based on the way we wrote the last tree as three trees, and to recognize some of them. We will recursevely do this trick, which causes a combinatorial explosion of the terms.

3.3 Proof of the theorem

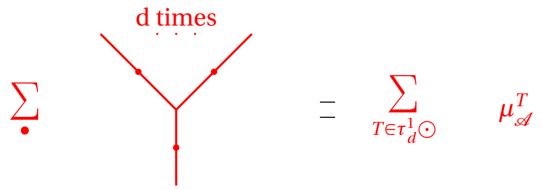
We suppose that $\mathscr{F}^k$ and $\mu^d_{\mathscr{A}}$ verify 2.11 and 2.6 until $d-1$.

We identity a tree T to the element it correspond to , writing μ^T for this element. For example

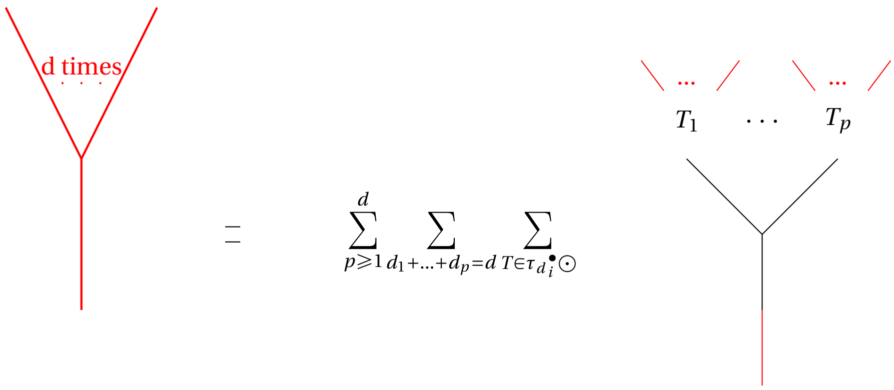


$\mu^2_{\mathscr{B}} = \mu^{\tau^0_2}_{\mathscr{B}}$ (there is only a unique tree in the set so we confound them). Also we write for the sum $\mu^d(\mu^1(.),..,.) + \mu^d(.,...,\mu^1(.)) + \mu^1(\mu^d(.,...,.))$.

The equation that have to verify $\mu_{\mathscr{A}}$ to be the compositions of a A_∞ category becomes :

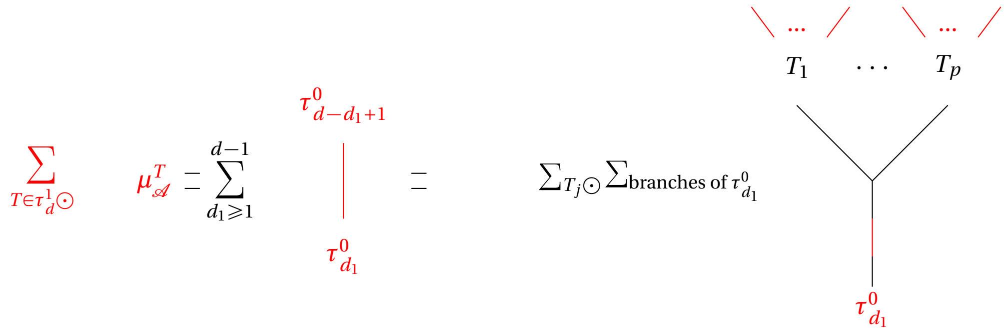


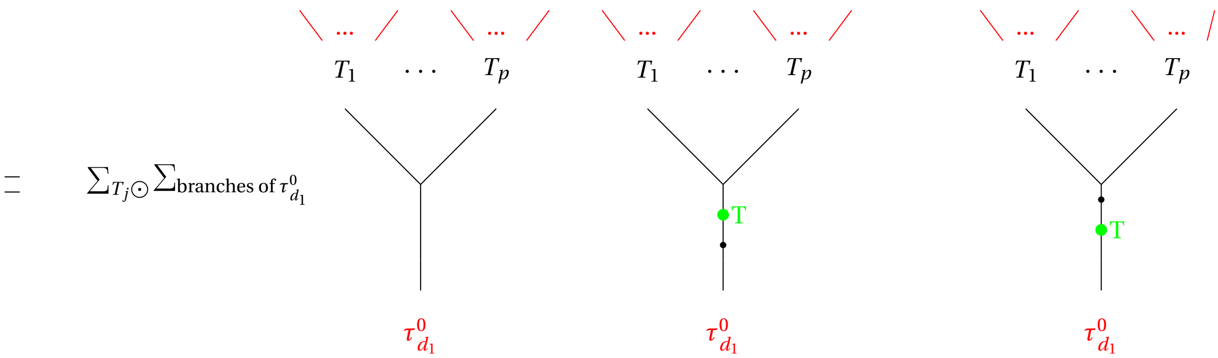
Also, note that recursively expressing F^k, we have the key of the combinatory :



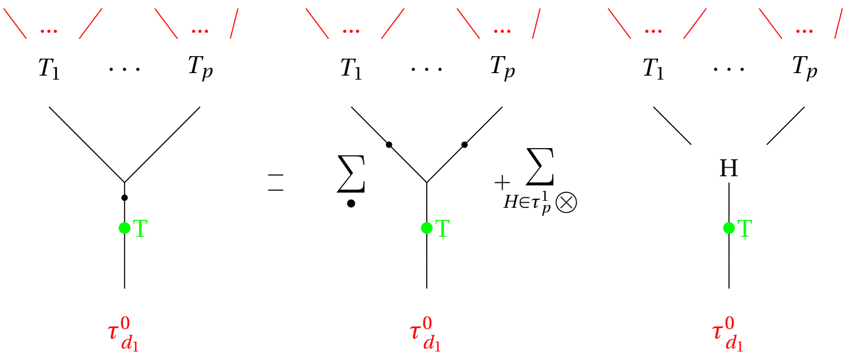
where $T \in \tau^{\bullet}_{d_i}\odot$ means $\sum_{k\geqslant 1}\sum_{T\in\tau^k_{d_i}\odot}$. Moreover in the following we will write this triple sum $\sum_{T_j\odot}$

Let's start with the right side. In a first time, we right the tree with a unique interior edge as 2 linked (by this interior edge) trees with no edges. Then since each leaf end and each root start with respectively, from bottom to top black-red and red-black; we colour this edge in an other way to use 3.2. In the same time, we use 3.3.In the following, $d_1 + d_2 = d, d_1 \geqslant 1, d_2 \geqslant 1$.



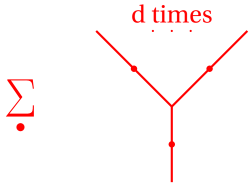


where the sum of the three trees are in the sum. Note that on the first tree on the right, the figured edge is the only one that doesn't carry a T (once the tree entirely developed). We call respectively A, B and C the three sums of trees. Lets split the sum into three one, fix A and the T_i; and figure out what does happen using 2.11 by induction.



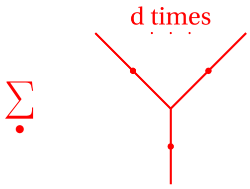
Remember in the final sum, we sum for all the possible T_j. Suppose the T_j linked to the edge that carry the dot isn't a leaf. Then T_j start with an edge that carries a T. This induces that this tree is already in the sum B because we recognize the pattern dot - T. On the other side, every term in the sum B is one of those trees, except for the case $d_1 = 1$. But in this case, we find the tree corresponding to $\mu^d_{\mathscr{A}}$ where the dot is carried by the root.

Now let's have a look at the trees where the edge that carries the dot linked to T_j is a leaf. Of course we can rise the (black) dot on the edge onto a (red) dot on the leaf. Summing for all others T_i and for all d_1, and using 3.3 (be careful, here T_i is just a leaf in the figure referenced); we find $\mu^d_{\mathscr{A}}$ with a dot on a leaf. This way we find :



Moreover looking at the trees of the second kind, the unique interior edge of H is the only interior edge of the tree which doesn't carry a T. Then we exactly find here all of the trees that are in A.

Then we effectively proved :



$$= \sum_{T\in\tau_d^1\odot} \mu_{\mathscr{A}}^T$$

Similar arguments apply to prove the formula 2.11 verified by $\mathscr{F}^d$. We proved the first part of the theorem.

The two other points of the theorem looks even more computational and we didn't find the right formulas. But we proved the most important thing : we can push forward the structure.

The Kadeishvili theorem is usefull to provide an A_∞ structure, especially on homology. For example, in [1], it is used to provide an A_∞ structure on "the linearized cohomology of the Legendrian contact homology". Then they use this structure to find "Legendrian knots that are not isotropic to their Legendrain mirror".

Appendix : Python script that return all of the terms in ?? for d=2, as a list of tuple, eacher tuple being what μ^k is evaluated on. Commentaries are in french

```
import numpy as np
def tableau_combin(d):
    " renvoie toutes les possibilites des indices dans le mu^d"
    #on note T l'element T, S l'element F - F , F les elements F
    #on renvoie en fait l'ensemble des listes avec l'interieur des mu^d
    #et aussi la place qu'occupent S et T
    if d < 0 : return None
    if d == 0 : return [[],[]]
    if d == 1 : return [[],[]]
    if d == 2 :
        return (([("S",0),("T",2)],[("S",1),("T",1)],[("S",2),("T",0)]),((0,1),(0,1),(0,1)))
    else :
        ens_liste , ens_indice = tableau_combin(d-1)

        L = []
        indice = []
        n = len(ens_liste)

        for i in range(n):
            L_i = ens_liste[i]
            temp_list = L_i.copy()
            taille_L_i = len(L_i)
            # on peut rajouter un Fk, un Fl, un Fj pour avoir les d elemnts
            (s,t) = ens_indice[i]
            temp_indice = (s,t)

            #on insert des Fk
            for k in range(s+1):

                temp_list.insert(k,("F",1))


                L.append(temp_list)
                indice.append((s+1,t+1))

                temp_list = L_i.copy()
                temp_indice = (s,t)

            #on insert des Fl
            for k in range(s+1,t+1):

                temp_list.insert(k,("F",1))

                L.append(temp_list)
                indice.append((s,t+1))

                temp_list = L_i.copy()
```

```
                temp_indice = (s,t)

            #on insert des Fj
            for k in range(t+1,taille_L_i+1):

                temp_list.insert(k,("F",1))

                L.append(temp_list)
                indice.append((s,t))

                temp_list = L_i.copy()
                temp_indice = (s,t)

        return (L,indice)

def est_pas_dedans(M,a):
    for x in M :
        if x == a : return False
    return True

def epuration(D):
    "D est un couple de deux listes, on veut éliminer les doublons de la premiere et"
    "le cas echeant"
    "enlever le couple d'indice correspondant dans la deuxieme"
    if len(D[0]) <= 1 : return D
    (L,indice) = D
    n = len(L)
    new_L, new_indice = [] , []

    for i in range(n):
        temp = L[i]
        #on eneleve les doublons , pas
        if est_pas_dedans(L[i+1:n],temp) :
            new_L.append(temp)
            new_indice.append(indice[i])

    return (new_L, new_indice)

def d_eme_terme_mu2(d):
    return epuration(tableau_combin(d))

################################################################################
# on note n_m les termes en mu^m dans le n eme morphisme de la collection      #
# tableau_combin(d) renvoie donc les termes en d_d et pours avoir les n_d      #
#il suffit d ajuster les puissances                                            #
# de sorte que la somme des puissances soit egales a n. Il faut donc repartir  #
#n-d puissance sur les d elements                                              #
################################################################################
```

```
def ajout_puissance_j_eme_pos(element,j):
    """ element est une liste de tuples ('F',x) et on veut mettre x+1 en jeme pos"""
    lettre, x = element[j]
    new_element = element.copy()
    new_element[j] = (lettre, x+1)
    return new_element

def distribution(L,a):
    """ renvoie l'ensemble des termes qu'on peut faire en distribuant a puissances sur
"les elemnts de L"""
    """ notons que L est un seul terme (une liste) qu'il faut doubler car le programme
"(recursif) renvoie une liste de listes ( de couples)"""
    if a < 0 or L == [[]]: return None
    if a == 0 : return L
    else :
        new_L = []
        n = len(L)
        d = len(L[0])
        element = []
        for i in range(n):

            element = L[i]
            new_element = []
            #on prend un terme, on ajoute une puissance a chacun de ses elemnts tour a tour,
            #et on ajoute ces exactements len(element) nvx termes
            # dans L. On a plus qu'a ajouter (a-1) puissances dans ce nouveau L
            for j in range(d):
                new_element = ajout_puissance_j_eme_pos(element,j)
                new_L.append(new_element)
        return distribution(new_L,a-1)




def termes_n_d(n,d):
    " renvoi tous les termes en mu^d dans mu^2_Q(T2,T1)^n"
    """ attention la liste indice contient 1 elements pour tous les choix de distribution
"qu'on a fait"""
    """ du coup c'est ce qu'on corrige dans le programme apres"""
    if d==1 : return "d doit etre >= 2"
    termes , indice = d_eme_terme_mu2(d)
    return distribution(termes,n-d)

def tous_les_termes(n):
    """ renvoie tous les termes de mu^2_Q(T2,T1)^n"""

    termes , indices = [], []
```

```
    temp = []
    A = []
    ind_A = 0
    for i in range(2,n):
        temp = termes_n_d(n,i)
        A = temp
        ind_A = len(A)
        for j in range(ind_A):
            termes.append(A[j])
    return termes


A = tous_les_termes(5)
print(A)
```

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