Spectrum of the Dirac operator on homogeneous spaces (according to C. Bär)

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Introduction

In the 1920's, physicists were trying to develop a relativistic version of quantum mechanics. The basis of quantum mechanics is the Schrödinger's Equation :

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

for the wave function Ψ where H is a differential operator determined by the classical energy in the system. For example, if we consider a free particle, its energy is $E = \frac{p^2}{2m}$ and the equation becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m}\Delta\Psi$$

This formulation of quantum mechanics is based on classical Newtonian mechanics. However, Eistein's Principle of Special Relativity stated that physical laws should be invariant under Lorentz transformations of space-time coordinates (our universe is seen as a 4-manifold with Lorentz metric), that is linear transformations of $\mathbb{R}^4$ which preserves the indefinite quadratic form :

$$(x, x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

where $x_0 = ct$ is the time-like coordinate and the constant c is the speed of light in vacuum.

Now the principle of Lorentz Invariance led Einstein to the conclusion that the energy of a free particle is given by

$$E = \sqrt{p^2c^2 + m^2c^4}$$

which is not what appeared in the previous example. Because of the square root, it is impossible to convert this expression into a classical differential operator. The first solution to this difficulty was to apply the Schrödinger's Equation twice. This gives a term corresponding to E^2 on the right-hand side, and yields the Klein-Gordon Equation

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} = -\hbar^2c^2\Delta\Psi + m^2c^4\Psi$$

which can be rewritten as

$$\left\{\Box + \left(\frac{mc}{\hbar}\right)^2\right\}\Psi = 0$$

where

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

In the late 1920's, British physicist Paul Dirac (1902-1984), was dissatisfied with the Klein-Gordon approach. According to Dirac, the principle of causality required the Schrödinger's Equation to be first-order in time. Lorentz invariance then implied it should be first-order in all variables. This was impossible to achieve for scalar wave functions, so Dirac decided to consider vector-valued wave functions

$$\Psi = (\psi_1, \ldots, \psi_N)$$

and to search for a first-order linear Lorentz-invariant operator $\mathcal{D}$ (later called a Dirac operator) with

$$\mathcal{D}^2 = \Box$$

Writing

$$\mathcal{D} = \sum_{\mu=0}^{3}\gamma_\mu\frac{\partial}{\partial x_\mu}$$

where the γ_μ are $N \times N$ matrices, we see that

$$\mathcal{D}^2 = \sum_{\mu,\nu=0}^{3} \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu} = \Box$$

implies the equations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \pm 2\delta_{\mu\nu}$$

These are easily solved for small values of N. For instance, for $N = 4$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\gamma_0 = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \ \gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}, \ \mu = 1, 2, 3$$

Using such matrices, Dirac wrote down in 1928 a relativistic Schrödinger equation for an electron in a magnetic field which eventually gave spectacular accord with experiment. However, there was a significant problem : the existence of negative energy solutions, that is eigenfunctions of the operator with negative eigenvalues.

Dirac, believing in the beauty of the equations, asserted that such particles should exist. He proposed the idea that the negative states are filled in general, but when a state is empty, we see it. Such empty states are called "antiparticles". The antielectron was called a positron, which was found in laboratory in 1932.

Underlying Dirac's discovery was an important mathematical fact : Lorentz transformations of space-time coordinates yield linear transformations of $\Psi = (\psi_1, \ldots, \psi_N)$ which were determined only up to a sign ± 1. The additional property determining the sign was thought of as an internal spin of the particle. The transformations of the spinor wave functions Ψ did not constitute a representation of the Lorentz group L but rather a representation of its simply-connected 2-fold covering group $\widetilde{L} \to L$.

Consider $\mathbb{R}^n$ and its standard norm $\|x\|^2 = \sum\limits_{i=1}^{n} x_i^2$, and denote by $M_{n\times n}$ the set of all $n \times n$ matrices with real coefficients. The orthonormal group ans its Lie algebra are then defined by

$$SO_n = \{g \in M_{n\times n} : g^t g = Id\}$$
$$\mathfrak{so}_n = \{g \in M_{n\times n} : g^t + g = 0\}$$

We know that for $n \geq 2$,

$$\pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$$

and there is a universal covering group which sits in the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Spin_n \to SO_n \to 1$$

The representations of $Spin_n$ (which correspond bijectively to representations of $\mathfrak{so}_n$) are generated, via direct sums and tensor products, by :

- The basic representations of $\mathbb{R}^n$;
- The spinor representations given by Clifford multiplication.

These latter representations are not representations of SO_n. To understand them systematically, we must first understand Clifford algebras.

But then, why are eigenvalues of the Dirac operator so interesting ? The physical interest stems from the observation that if $\mathcal{D}\Psi = \lambda\Psi$, then the time-dependant spinor field $\Phi(x,t) := e^{it\lambda}\Psi(x)$ satisfies the physical Dirac equation

$$\frac{\partial \Phi}{\partial t} = i\mathcal{D}\Phi$$

Hence, λ can be interpreted as the frequency or, equivalently, as the energy of the particle whose wave function is Φ. Computing the spectrum of the Dirac operator in particular spaces are the main goal of this report, especially in homogeneous spaces.

The first part will focus on basic definitions and results about Clifford algebras, spin groups, spin structures and the Dirac operator. This can be easily skipped by readers already familiar with these concepts, and one can come back at any time when needed to look at definitions or notations. An excellent reference for an introduction to Clifford algebra and spin group is [1]

Then, we will study the Dirac operator on homogeneous spaces, starting by giving some results on homogeneous spaces that will help our study. We will give an explicit formula for this operator, and then use it to computes the eigenvalues on the 3-sphere equipped with a Berger's metric, and on lens spaces. It is based on the article [2].

Finally, general results are given, and some of them proved, in the last part. It will show how the metric changes the eigenvalues, by computing again on the 3-sphere, but this time with the classical spherical metric. These, and more, are available in [3] and [4].

The appendices are here to help any reader who could have trouble with some of the notions or results used throughout this report.

1 Clifford algebras, spin structures and Dirac operators

To begin with, we start by giving some definitions and basic results about Clifford algebras[1] and spin groups. Although spin groups can be defined without Clifford algebras, the choice made here is to get the reader used to notations and properties of Clifford algebras widely used throughout this report. All of these will lead us then to the main concepts of spinor representations and Dirac operator.

1.1 Clifford algebras

Definition 1 (Clifford algebra)**.** Let V be an n-dimensional vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and q a bilinear symmetric form on V. The Clifford algebra $Cl(V, q)$ associated with (V, q) is the associative algebra with unit, defined by :

$$Cl(V, q) = {}^{V^{\otimes}}\!\diagup_{\mathcal{I}(V, q)} \tag{1}$$

where $V^{\otimes} = \bigoplus_{i \geq 0} V^{\otimes i}$ is the tensor algebra of V and $\mathcal{I}(V, q)$ the two-sided ideal generated by all elements of the form $x \otimes x + q(x, x)1$, for $x \in V$.

The product in the Clifford algebra will be denoted by the dot $\cdot$

Examples.

- If we consider the trivial quadratic form $q \equiv 0$ over a vector space V, then the Clifford algebra $Cl(V, q)$ is just the exterior algebra $\Lambda^*(V)$.
- If we take $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}$ and $q = -x^2$. Then $Cl(V, q) \simeq \mathbb{C}$.

Remark. There is a natural map $i : V \to Cl(V, q)$ obtained by considering the natural embedding $V \hookrightarrow V^{\otimes}$, followed by the projection $V^{\otimes} \to Cl(V, q)$.

Viewing V as a subset of $Cl(V, q)$ in that way, the algebra $Cl(V, q)$ is generated by V (and the unit 1), subject to the relation $v \cdot v = -q(v, v)1$.

Proposition 2 (Universal property)**.** Let $\mathcal{A}$ be an associative algebra with unit and $f : V \to \mathcal{A}$ a linear map such that :

$$f(v)^2 = -q(v, v)1_{\mathcal{A}} \tag{2}$$

Then there exists a unique $\mathbb{K}$-algebra homomorphism $\widetilde{f} : Cl(V, q) \to \mathcal{A}$ satisfying $\widetilde{f} \circ i = f$

Furthermore, if C is an associative $\mathbb{K}$-algebra with unit carrying a linear map $i' : V \to C$ satisfying $i'(v)^2 = -q(v, v)1_C$, with the property above, then C is isomorphic to $Cl(V, q)$.

Remark.

1. The Clifford algebra $Cl(V, q)$ can be abstractly defined as the algebra generated by $n + 1$ elements $\{v_0, \dots, v_n\}$ subject to the relations :

$$\begin{gathered} v_0 \cdot v_i = v_i \cdot v_0 = v_i \ , \ v_o^2 = v_0 \\ v_i^2 = -v_0 \ (i \geq 1) \ , \ v_i \cdot v_j = -v_j \cdot v_i \ (1 \leq i \neq j) \end{gathered}$$

   where v_0 corresponds to 1, and $\{v_1, \dots, v_n\}$ to a q-orthonormal basis of V.

2. If $\{e_1, \dots, e_n\}$ is a q-orthonormal basis of V, then the system

$$\{1, e_{i_1} \dots e_{i_k},\ 1 \leq i_1 < \dots < i_k \leq n,\ 1 \leq k \leq n\} \tag{3}$$

   spans $Cl(V, q)$ as vector space, thus $\dim Cl(V, q) \leq 2^n$ (actually this is an equality).

[1]Named after English mathematician William Kingdon Clifford (1845-1879), known for his work on graph theory and projective surfaces.

3. Clifford algebras are endowed with the following fundamental automorphisms.

   (a) Using the universal property, the injective morphism $V \hookrightarrow Cl(V,q)$ induced by the map $-Id : v \mapsto -v$ gives rise to the automorphism

$$\begin{aligned} \alpha : Cl(V,q) &\to Cl(V,q) \\ e_{i_1}\dots e_{i_k} &\mapsto (-1)^k e_{i_1}\dots e_{i_k} \end{aligned} \tag{4}$$

   As $\alpha^2 = Id$, we get the decomposition $Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$ where

$$Cl^i(V,q) = \{\varphi \in Cl(V,q),\ \alpha(\varphi) = (-1)^i\varphi\} \tag{5}$$

   Clearly, for $i,j \in {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$, we have $Cl^i(V,q)\cdot Cl^j(V,q) \subset Cl^{i+j}(V,q)$.
   Thus the Clifford algebra $Cl(V,q)$ is a ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$-graded algebra, that is a superalgebra. The subspace $Cl^0(V,q)$ (resp. $Cl^1(V,q)$) is called the even (resp. odd) part of $Cl(V,q)$.

   (b) Consider the $\mathbb{K}$-algebra anti-automorphism defined by :

$$\begin{aligned} {}^t : V^{\otimes} &\to V^{\otimes} \\ x_{i_1}\otimes\cdots\otimes x_{i_k} &\mapsto x_{i_k}\otimes\cdots\otimes x_{i_1} \end{aligned}$$

   Since ${}^t\,(\mathcal{I}(V,q)) \subset \mathcal{I}(V,q)$, there is an induced automorphism :

$$\begin{aligned} {}^t : Cl(V,q) &\to Cl(V,q) \\ x_{i_1}\cdots x_{i_k} &\mapsto x_{i_k}\cdots x_{i_1} \end{aligned}$$

An immediate application of the universal property (Proposition 2) shows that if there exists a $\mathbb{K}$-isomorphism f between two $\mathbb{K}$-vector spaces V and V', endowed respectively with two bilinear symmetric forms q and q', such that $f^*q' = q$, then f uniquely extends to a $\mathbb{K}$-algebra isomorphism between $Cl(V,q)$ and $Cl(V',q')$.

Lemma 3. Let $f : (V,q) \to (V',q')$ be an isometry. Then f uniquely extends to a $\mathbb{K}$-algebra isomorphism :

$$\widetilde{f} : Cl(V,q) \to Cl(V',q')$$

Proof. Let $g : (V,q) \to Cl(V',q')$ be the composition $i' \circ f$. Then g verifies :

$$g(v)^2 = g(v)\cdot g(v) = -q'(f(v),f(v))1_{Cl(V',q')} = -q(v,v)1_{Cl(V',q')}$$

since f is an isometry. So, by the universal property, g uniquely extends to $\widetilde{g} : Cl(V,q) \to Cl(V',q')$, and since $Cl(V',q')$ verifies the universal property, $\widetilde{g}$ is an isomorphism. The uniqueness comes from the fact that V and 1 generates $Cl(V,q)$. $\square$

Definition 4. The Clifford algebra $Cl_n := Cl(\mathbb{R}^n, q^{\mathbb{R}})$ (resp. $\mathbb{C}l_n := Cl(\mathbb{C}^n, q^{\mathbb{C}})$) associated with the canonical Euclidean scalar product $q^{\mathbb{R}}$ (resp. $q^{\mathbb{C}}$) defined by $q^{\mathbb{R}}(x,y) = \sum\limits_i x_i y_i$ (resp. $q^{\mathbb{C}}(z,w) = \sum\limits_i z_i w_i$) is called the n-dimensional real (resp. complex) Clifford algebra.

Examples. If $\{e_1, \cdots, e_n\}$ denotes the canonical orthonormal basis of $\mathbb{R}^n$, then the relations

$$e_i\cdot e_j + e_j\cdot e_i = 2\delta_{ij}$$

hold in Cl_n. Hence, we have the following facts :

1. A basis of Cl_1 is given by $\{1, e_1\}$. Since $e_1^2 = -1$, one has $Cl_1 \simeq \mathbb{C}$.

2. A basis of Cl_2 is given by $\{1, e_1, e_2, e_1 \cdot e_2\}$. Since the three vectors $e_1, e_2, e_1 \cdot e_2$ verify the same multiplication rules as the standard basis of imaginary quaternions, one has $Cl_2 \simeq \mathbb{H}$.

3. The volume element $\omega := e_1 \cdot e_2 \cdot e_3 \in Cl_3$ is central and $\omega \cdot \omega = 1$. Define

$$\Pi^{\pm} := \frac{1}{2}(1 \pm \omega)$$

Since

$$\Pi^+ + \Pi^- = 1 \ , \ \left(\Pi^{\pm}\right)^2 = \Pi^{\pm}$$
$$\Pi^- \cdot \Pi^+ = \Pi^+ \cdot \Pi^- = 0$$

one has the decomposition $Cl_3 = Cl_3^+ \oplus Cl_3^-$, where

$$Cl_3^{\pm} := \Pi^{\pm} \cdot Cl_3 = Cl_3 \cdot \Pi^{\pm}$$

A basis of $Cl_3^{\pm}$ is given by

$$\{\Pi^{\pm}, \Pi^{\pm} \cdot e_1, \Pi^{\pm} \cdot e_2, \Pi^{\pm} \cdot e_1 \cdot e_2\}$$

Hence $Cl_3^{\pm} \approx \mathbb{H}$ and $Cl_3 \simeq \mathbb{H} \oplus \mathbb{H}$.

Proposition 5. The n-dimensional real (resp. complex) Clifford algebras is isomorphic to the even part of the $(n+1)$-dimensional real (resp. complex) Clifford algebra.

$$Cl_n \simeq Cl^0_{n+1} \ \text{ and } \ \mathbb{C}l_n \simeq \mathbb{C}l^0_{n+1}$$

Proof. Denote by $\{e_1, \cdots, e_n\}$ and $\{e_1, \cdots, e_{n+1}\}$ the canonical basis of $\mathbb{K}^n$ and $\mathbb{K}^{n+1}$, $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$). This suggests to identify $\mathbb{K}^n$ with the image of the canonical injection in $\mathbb{K}^{n+1}$ as the subspace generated by the first n vectors. Define the linear map

$$\begin{aligned} f : \mathbb{K}^n &\to Cl^0_{n+1} \ (\text{resp } \mathbb{C}l^0_{n+1}) \\ e_i &\mapsto ei \cdot e_{n+1} \end{aligned}$$

By definition of f, we have $f(e_i)^2 = -1$. Thus, by the universal property, f extends to

$$\widetilde{f} : Cl_n \to Cl^0_{n+1} \ (\text{resp. } \mathbb{C}l_n \to \mathbb{C}l^0_{n+1})$$

Clearly, $\widetilde{f}$ is an injective linear map between vector spaces of same dimension ($\dim Cl_n = 2^n$ and $\dim Cl^0_{n+1} = \frac{1}{2} \dim Cl_{n+1} = \frac{1}{2} 2^{n+1} = 2^n$). Thus the map $\widetilde{f}$ is an isomorphism. □

1.2 Spin groups

We begin with the following remark : let v be any non zero vector of an Euclidean space (V, q). The symmetry σ_v with respect to the hyperplane orthogonal to v is defined by

$$\sigma_v(x) = x - 2\frac{q(x, v)}{q(v, v)} v$$

Viewing V as a subset of $Cl(V, q)$, we can write

$$\sigma_v(x) = x - (x \cdot v + v \cdot x) \cdot v^{-1} \cdot v = -v \cdot x \cdot v = -Ad(v)(x) \tag{6}$$

Since any element of $SO(V, q)$ is the product of an even number of symmetries of that type, it is the image under the map Ad of a certain subgroup of invertible elements in $Cl(V, q)$. We are thus led to having a closer look at the group units in $Cl(V, q)$.

We consider the map :

$$\begin{aligned} N : Cl(V,q) &\to Cl(V,q) \\ a &\mapsto \alpha({}^t a)\cdot a \end{aligned}$$

Note that $N(v) = |v|^2$, for all $v \in V$.

Denote by $Cl^*(V,q)$ the group of invertible elements of $Cl(V,q)$. It can be easily checked that $N(\alpha({}^t(a^{-1})))\cdot N(a) = 1$. Indeed, since α and ^{t} commutes, and are their own inverse, one has

$$N(\alpha({}^t(a^{-1}))) = \underbrace{\alpha({}^t\alpha({}^t a^{-1})))}_{=a^{-1}}\cdot\alpha({}^t a^{-1})$$

So

$$N(\alpha({}^t(a^{-1})))\cdot N(a) = a^{-1}\cdot\alpha({}^t a^{-1})\cdot\alpha({}^t a)\cdot a = 1$$

This implies $N\left(Cl^*(V,q)\right) \subset Cl^*(V,q)$. Since $v\cdot v = -|v|^2$, it is clear that any non zero vector $v \in V \subset Cl(V,q)$ lies in $Cl^*(V,q)$, and $v^{-1} = \frac{-v}{|v|^2}$. For each $a \in Cl^*(V,q)$, we consider the inner automorphism

$$\begin{aligned} Ad_a : Cl(V,q) &\to Cl(V,q) \\ b &\mapsto a\cdot b\cdot a^{-1} \end{aligned}$$

and the map

$$\begin{aligned} \widetilde{Ad_a} : Cl(V,q) &\to Cl(V,q) \\ b &\mapsto \alpha(a)\cdot b\cdot a^{-1} \end{aligned} \tag{7}$$

As we pointed out in (6), for any non zero vector v, since $\alpha(v) = -v$, the vector space $V \subset Cl(V,q)$ is stable under $\widetilde{Ad_v}$

$$\widetilde{Ad}_{v|V} = \sigma_v$$

For any $\mathbb{K}$-vector space V endowed with a symmetric bilinear form q, let $P(V,q)$ be the group defined by

$$P(V,q) := \{a \in Cl^*(V,q)\ ;\ \widetilde{Ad_a}(V) \subset V\}$$

As before, we will restrict to the case $(V,q) = (\mathbb{K}^n, q^{\mathbb{K}})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Lemma 6.
1. The kernel of the map $\widetilde{Ad} : P(V,q) \to GL_n(\mathbb{K})$ is $\mathbb{K}^*$.
2. $N(P(V,q)) \subset \mathbb{K}^*$
3. The map $N_{|P(V,q)} : P(V,q) \to \mathbb{K}^*$ is a group homomorphism.

Proof. 1. Consider $a \in Cl(V,q)$ such that $\widetilde{Ad_a} = Id$, that is, for all $v \in \mathbb{K}^n$,

$$\alpha(a)\cdot v\cdot a^{-1} = v \iff \alpha(a)\cdot v = v\cdot a \tag{8}$$

We assume this equality holds only for $v \in V$ since V generates $Cl(\mathbb{K}^n, q^{\mathbb{K}})$. As above, denote by $\{e_1,\dots,e_n\}$ the canonical basis of $\mathbb{K}^n$. For each fixed $1 \le i \le n$, any element $a \in P(V,q)$ can be uniquely written as $a = b + e_i\cdot c$, where b and c are linear combinations of elements in the basis $\{1, e_{i_1}\cdot\dots\cdot e_{i_k},\ 1 \le i_1 < \dots < i_k \le n,\ 1 \le k \le n\}$ that do not involve e_i.

Condition (8) written for $v = e_i$ is equivalent to $\alpha(b)\cdot e_i = e_i\cdot b$ and $\alpha(c)\cdot e_i = -e_i\cdot c$. Indeed,

$$\begin{aligned} \alpha(a)\cdot v = v\cdot a &\iff \alpha(b + e_i\cdot c)\cdot e_i = e_i\cdot(b + e_i\cdot c) \\ &\iff (\alpha(b) - e_i\cdot\alpha(c))\cdot e_i = (e_i\cdot b) - c \\ &\iff \alpha(b)\cdot e_i = e_i\cdot b \text{ and } \alpha(c)\cdot e_i = -e_i\cdot c \end{aligned}$$

Take a closer look at the last relation. If $d \in V$, such that $\alpha(d) \cdot e_i = -e_i \cdot d$, since $\alpha(d) = -d$, we have $-d \cdot e_i = -e_i \cdot d \iff e_i \cdot d = -e_i \cdot d$, so $d = 0$. Since V generates $Cl(V,q)$, and α is an automorphism of $Cl(V,q)$, then $c = 0$. So $a = b$, which does not involve e_i. The argument holds for each e_i, it follows that a is a scalar, and since $a \in Cl^*(V,q)$, a is a non zero scalar.

2. For all $v \in \mathbb{K}^n$ and $a \in P(V,q)$, we have $\widetilde{Ad}_a(v) \in \mathbb{K}^n$ (by definition of V and $P(V,q)$), hence $\widetilde{Ad}_a(v) = {}^t(\widetilde{Ad}_a(v))$.

   This implies :

$$\begin{aligned}\widetilde{Ad}_{N(a)}(v) &= \alpha(N(a)) \cdot v \cdot (N(a))^{-1} = \alpha(\alpha({}^t a)) \cdot v \cdot (\alpha({}^t a) \cdot a)^{-1} \\ &= \alpha(\alpha({}^t a)) \cdot \underbrace{\alpha(a) \cdot v \cdot a^{-1}}_{=\widetilde{Ad}_a(v) = {}^t\widetilde{Ad}_a(v)} \cdot \alpha({}^t a)^{-1} \\ &= {}^t a \cdot {}^t a^{-1} \cdot v \cdot {}^t\alpha(a) \cdot \alpha({}^t a)^{-1} \\ &= v\end{aligned}$$

   Hence from 1 of this proof, we conclude that $N(a) \in \mathbb{K}^*$. So $N(P(V,q)) \subset \mathbb{K}^*$.

3. By 2, for $a, b \in P(V,q)$,

$$\begin{aligned}N(a \cdot b) &= \alpha({}^t b \cdot {}^t a) \cdot a \cdot b = \alpha({}^t b) \cdot \alpha({}^t a) \cdot a \cdot b = \alpha({}^t b) \cdot N(a) \cdot b \\ &= N(a) \cdot \alpha({}^t b) \cdot b \text{ since } N(a) \in \mathbb{K}^n \\ &= N(a) \cdot N(b)\end{aligned}$$

   So $N_{|P(V,q)} : P(V,q) \to \mathbb{K}^*$ is a group homomorphism.

$\square$

Definition 7 (Spin group)**.** The spin group $Spin_n$ is the subgroup of $P(\mathbb{R}^n, q^{\mathbb{R}})$ generated by elements of the form $v_1 \dots v_{2k}$, with $k \geq 1$ and $v_i \in \mathbb{R}^n$, $\|v_i\| = 1$ for $1 \leq i \leq 2k$.

Proposition 8 (Covering of SO_n)**.** For $n \geq 2$, the homomorphism $\xi := \widetilde{Ad}_{|Spin_n}$ is a non trivial double covering of the special orthogonal group SO_n.

In particular, for $n \geq 3$, the group $Spin_n$ is the universal cover of SO_n.

Proof. We know that the image of a non zero vector by the map

$$\widetilde{Ad} : \mathbb{R}^n \backslash \{0\} \subset Cl_n^* \to GL_n$$

is the symmetry with respect to the hyperplane orthogonal to this vector. The image of $\widetilde{Ad}|_{Spin_n}$ is then the group of even product of such symmetries, which by the Cartan-Dieudonné theorem, is exactly the group SO_n.

Using 3 of the previous lemma, any element of $Spin_n$ satisfies $N(v) = 1$. Indeed,

$$N(v_1 \dots v_{2k}) = N(v_1) \dots N(v_{2k}) = 1 \text{ since } N(v_i) = \|v_i\| = 1$$

By 1 of the previous lemma, we conclude that :

$$\ker(\widetilde{Ad}) = \mathbb{K}^* \implies \ker(\widetilde{Ad}_{|Spin_n}) = \mathbb{K}^* \cap Spin_n$$

But

$$a \in (\mathbb{K}^* \cap Spin_n) \iff a \in \mathbb{K}^* \text{ and } \|a\| = 1 \iff a = \pm 1$$

So the kernel of $\widetilde{Ad} : Spin_n \to SO_n$ is $\{\pm 1\}$.

To show that the covering is non trivial, it is sufficient to check that 1 and -1 belong to the same connected component of $Spin_n$. To see this, choose unit orthogonal vectors $v, w \in \mathbb{R}^n$ $(n \geq 2)$ and note that the curve

$$\begin{aligned} c : \left[0, \frac{\pi}{2}\right] &\to Spin_n \\ t &\mapsto c(t) = (v\sin(t) + w\cos(t)) \cdot (v\sin(t) - w\cos(t)) \end{aligned}$$

satisfies $c(o) = 1$, $c\left(\frac{\pi}{2}\right) = -1$.

Finally, the last assertion follows from the fact that for $n \geq 2$, we have $\Pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$. □

Examples. 1. For $n = 1$: $Spin_1 \subset Cl_1^0 \simeq Cl_0 \simeq \mathbb{R}$. Thus $Spin_1 \simeq \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$. Moreover

$$\begin{aligned} \xi : Spin_1 \simeq \mathbb{Z}/2\mathbb{Z} &\to SO_1 \simeq \{1\} \\ t &\mapsto t^2 \end{aligned}$$

2. For $n = 2$, one has $Spin_2 \subset Cl_2^0 \simeq Cl_1 \simeq \mathbb{C}$. We verify that $S^1 \simeq U_1 \simeq Spin_2$ via the map

$$\begin{aligned} S^1 \simeq U_1 &\to Spin_2 \\ e^{i\theta} &\mapsto \left(\cos\left(\frac{\theta}{2}\right)e_1 + \sin\left(\frac{\theta}{2}\right)e_2\right) \cdot \left(-\cos\left(\frac{\theta}{2}\right)e_1 + \sin\left(\frac{\theta}{2}\right)e_2\right) = \cos(\theta) + \sin(\theta)e_1 \cdot e_2 \end{aligned}$$

- Let θ and $\varphi \in S^1$. We have

$$\cos(\theta) + \sin(\theta)e_1 \cdot e_2 = \cos(\varphi) + \sin(\varphi)e_1 \cdot e_2 \iff \begin{cases} \cos(\theta) = \cos(\varphi) \\ \sin(\theta)e_1 \cdot e_2 = \sin(\varphi)e_1 \cdot e_2 \end{cases}$$

$$\cos(\theta) + \sin(\theta)e_1 \cdot e_2 = \cos(\varphi) + \sin(\varphi)e_1 \cdot e_2 \iff \begin{cases} \cos(\theta) = \cos(\varphi) \\ \sin(\theta) = \sin(\varphi) \end{cases}$$

$$\cos(\theta) + \sin(\theta)e_1 \cdot e_2 = \cos(\varphi) + \sin(\varphi)e_1 \cdot e_2 \iff \theta \equiv \varphi \mod 2\pi$$

- This application is linear. Indeed :

$$\begin{aligned} e^{i(\theta+\varphi)} &\mapsto \cos(\theta+\varphi) + \sin(\theta+\varphi)e_1 \cdot e_2 \\ &= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) + \sin(\theta)\sin(\varphi)e_1 \cdot e_2 + \cos(\theta)\sin(\varphi)e_1 \cdot e_2 \end{aligned}$$

and

$$\begin{aligned} &(\cos(\theta) + \sin(\theta)e_1 \cdot e_2) \cdot (\cos(\varphi) + \sin(\varphi)e_1 \cdot e_2) \\ &= \cos(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi)e_1 \cdot e_2 \cdot e_1 \cdot e_2 + \cos(\theta)\sin(\varphi)e_1 \cdot e_2 + \sin(\theta)\cos(\varphi)e_1 \cdot e_2 \\ &= \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi) + \sin(\theta)\sin(\varphi)e_1 \cdot e_2 + \cos(\theta)\sin(\varphi)e_1 \cdot e_2 \end{aligned}$$

So if $w = v_1 \cdot v_2 \in Spin_2$, with $v_i \in \mathbb{R}^2$ and $\|v_i\| = 1$, then we can write $v_1 = \cos(\theta_1)e_1 + \sin(\theta_1)e_2$ and $v_2 = -\cos(\theta_2)e_1 + \sin(\theta_2)e_2$.
So we have

$$\begin{aligned} w = v_1 \cdot v_2 &= (\cos(\theta_1)e_1 + \sin(\theta_1)e_2) \cdot (-\cos(\theta_2)e_1 + \sin(\theta_2)e_2) \\ &= \cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)e_1 \cdot e_2 \end{aligned}$$

By considering $\theta = \theta_1 + \theta_2$, the image of $e^{i\theta}$ is exactly w.

The map $\xi : U_1 \simeq Spin_2 \to SO_2$ is then given by

$$e^{i\theta} \mapsto \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & cos(2\theta) \end{pmatrix}$$

Hence, under the identification $SO_2 \simeq U_1$, ξ is the square map

$$\begin{aligned} \xi : U_1 &\to U_1 \\ z &\mapsto z^2 \end{aligned}$$

3. The algebra Cl_3^0 is isomorphic to $\mathbb{H}$ by the isomorphism :

$$\Phi : \begin{cases} 1 & \mapsto 1 \\ e_1 \cdot e_2 & \mapsto i \\ e_2 \cdot e_3 & \mapsto j \\ e_3 \cdot e_1 & \mapsto k \end{cases}$$

By this isomorphism, $Spin_3 \subset Cl_3^0$ is identified with a subgroup of $\mathbb{H}^*$. If $a \in Cl_3^0$ and $q = \Phi(a)$, then

$$\begin{aligned} N(a) = 1 &\implies \alpha({}^t a) \cdot a = 1 \implies {}^t a \cdot a = 1 \text{ since } a \in Cl_3^0 \\ &\implies 1 = \Phi(1) = \Phi({}^t a \cdot a) = \Phi({}^t a) \cdot \Phi(a) = \overline{q} q \text{ since } {}^t(e_i \cdot e_j) = -e_i \cdot e_j \end{aligned}$$

So $\Phi(Spin_3) \subset Sp_1$ and since these two groups have the same dimension, we have $\Phi(Spin_3) = Sp_1$.

Under the identification, the covering map ξ is given by

$$\begin{aligned} \xi : Spin_3 \simeq Sp_1 &\to SO_3 \\ q &\mapsto \left(x \in \mathrm{Im}(\mathbb{H}) \mapsto qxq^{-1} = qx\overline{q}\right) \end{aligned}$$

where $\mathrm{Im}(\mathbb{H}) = \{bi + cj + dk \in \mathbb{H}\}$ denotes the imaginary quaternions (which is isomorphic to $\mathbb{R}^3$).

Furthermore, one can verify that the injective homomorphism

$$\begin{aligned} \mathbb{H} &\to \mathbb{C}(2) \\ q = \lambda + j\mu &\mapsto \begin{pmatrix} \lambda & -\overline{\mu} \\ \mu & \overline{\lambda} \end{pmatrix} \end{aligned}$$

induces an isomorphism between the groups Sp_1 and SU_2.

1.3 Representations of spin groups

We only consider complex finite dimensional representations of $Spin_n$

$$\rho : Spin_n \to GL(V)$$

where V is a finite dimensional complex vector space. Note that any representation of the group SO_n is obtained as a quotient of a representation ρ of the covering group $Spin_n$ verifying $\rho(\pm 1) = Id$.

We will show that by considering the standard complex representations of the Clifford algebras, the spin group $Spin_n$ inherits two irreducible complex representations if n is even and one if n is odd. As these representations do not descend to the group SO_n, they are called spinor representations.

We first consider complex representations of the Clifford algebras. Since every complex representation of Cl_n induces a complex representation of $\mathbb{C}l_n$ and conversely, we shall only consider complex Clifford algebras.

Theorem 9 (Representations of the Clifford algebra)**.** The complex Clifford algebra $\mathbb{C}l_n$ has a unique irreducible representation for n even :

$$\chi_n : \mathbb{C}l_n \to End(\Sigma_n)$$

and two inequivalent irreducible representation for n odd :

$$\chi_n^{\pm} : \mathbb{C}l_n \to End(\Sigma_n)$$

where Σ_n is a complex vector space of $\dim N = 2^{\lfloor \frac{n}{2} \rfloor}$, where $\lfloor . \rfloor$ denotes the floor function.

For a better understanding of the structure of the Clifford algebra, we introduce the complex volume element :

$$\omega^{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \dots e_n \in \mathbb{C}l_n$$

It verifies the relations :

$$\begin{aligned} \left(\omega^{\mathbb{C}}\right)^2 &= 1 \\ x \cdot \omega^{\mathbb{C}} &= (-1)^{n-1} \omega^{\mathbb{C}} \cdot x \text{ , for } x \in \mathbb{R}^n \subset \mathbb{C}l_n \end{aligned}$$

Proposition 10. For n odd,

$$\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$$

where $\mathbb{C}l_n^{\pm} := \Pi^{\pm} \cdot \mathbb{C}l_n = \mathbb{C}l_n \cdot \Pi^{\pm}$, and $\Pi^{\pm} = \frac{1}{2}(1 \pm \omega^{\mathbb{C}})$. Moreover, $\alpha(\mathbb{C}l_n^{\pm}) = \mathbb{C}l_n^{\mp}$.

Proof. Since $\left(\omega^{\mathbb{C}}\right)^2 = 1$, we have :

- $\Pi^+ + \Pi^- = 1$
- $\left(\Pi^{\pm}\right)^2 = \Pi^{\pm}$
- $\Pi^- \cdot \Pi^+ = \Pi^+ \cdot \Pi^- = 0$

Since n is odd, $\omega^{\mathbb{C}}$ and $\Pi^{\pm}$ are central in $\mathbb{C}l_n$. Indeed, remember that

$$x \cdot \omega^{\mathbb{C}} = (-1)^{n-1} \omega^{\mathbb{C}} \cdot x \text{ , } x \in \mathbb{R}^n \in \mathbb{C}l_n$$

. So since $n-1$ is even, and $\mathbb{C}^n$ generates $\mathbb{C}l_n$, $\omega^{\mathbb{C}}$ is central, and so is $\Pi^{\pm}$.

Then, $\mathbb{C}l_n^{\pm} = \Pi^{\pm} \cdot \mathbb{C}l_n$ are two ideals of $\mathbb{C}l_n$. Furthermore,

$$\forall v \in \mathbb{C}l_n \text{ , } v = \underbrace{\Pi^+ \cdot v}_{\in \mathbb{C}l_n^+} + \underbrace{\Pi^- \cdot v}_{\in \mathbb{C}l_n^-} \text{ since } \Pi^+ + \Pi^- = 1$$

and

$$\begin{aligned} v \in \left(\mathbb{C}l_n^+ \cap \mathbb{C}l_n^-\right) &\implies v = \Pi^+ \cdot v^+ = \Pi^- \cdot v^- \\ &\implies \Pi^+ \cdot v = \left(\Pi^+\right)^2 \cdot v^+ = \Pi^+ \cdot \Pi^- \cdot v^- = 0 \\ &\implies v = 0 \end{aligned}$$

So $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$.

Moreover, the volume element being odd, $\omega^{\mathbb{C}} \in \mathbb{C}l_n^1$, we have $\alpha(\Pi^{\pm}) = \Pi^{\mp}$, hence $\alpha(\mathbb{C}l_n^{\pm}) = \mathbb{C}l_n^{\mp}$. ∎

Proposition 11. For n odd, let χ_n be a complex irreducible representation of $\mathbb{C}l_n$. Then either $\chi_n(\omega^{\mathbb{C}}) = Id$, or $\chi_n(\omega^{\mathbb{C}}) = -Id$. The two possibilities occur and they are inequivalent.

Proof. Since $\left(\omega^{\mathbb{C}}\right)^2 = 1$, then $\chi(\omega^{\mathbb{C}})^2 = Id$, so $\chi(\omega^{\mathbb{C}})$ is diagonalizable, and the eigenvalues are ± 1. So the vector space Σ_n can be written as $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, associated with the ± 1 eigenvalue.

The volume element $\omega^{\mathbb{C}}$ being central, the eigenspaces $\Sigma_n^{\pm}$ are $\mathbb{C}l_n$-invariant. Indeed, if $v \in \Sigma_n^+$ then $v = \chi(\omega^{\mathbb{C}})(v)$ and for $u \in \mathbb{C}l_n$

$$\chi(u)(v) = \chi(u)\chi(\omega^{\mathbb{C}})(v) = \chi(u \cdot \omega^{\mathbb{C}})(v) = \chi(\omega^{\mathbb{C}})\chi(u)(v)$$

So $\chi(\omega^{\mathbb{C}})\chi(u)(v) = \chi(u)(v)$, which means $\chi(u)(v) \in \Sigma_n^+$. And the same holds for Σ_n^-

The representation being irreducible, we conclude that $\Sigma_n = \Sigma_n^+$ or $\Sigma_n = \Sigma_n^-$. It is clear that the two representations are inequivalent. By considering the action of $\mathbb{C}l_n$ on $\mathbb{C}l_n^+$ by left multiplication, we see that the two possibilities occur. $\square$

Definition 12 (Clifford multiplication)**.** The map

$$\begin{aligned} \mathbb{C}l_n \otimes \Sigma_n &\to \Sigma_n \\ \sigma \otimes \psi &\mapsto \sigma \cdot \psi \end{aligned}$$

where

$$\sigma \cdot \psi := \begin{cases} \chi_n(\sigma)(\psi) \text{ if } n \text{ is even} \\ \chi_n^+(\sigma)(\psi) \text{ if } n \text{ is odd} \end{cases}$$

is called the Clifford multiplication of σ with ψ.

Proposition 13.

1. For n even, the restriction of χ_n to $Spin_n$ (resp. $\mathbb{C}l_n^0$) splits into $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, where Σ_n^+ and Σ_n^- are complex inequivalent irreducible representations of $Spin_n$ (resp. $\mathbb{C}l_n^0$).
2. For n odd, the restriction of $\chi_n^{\pm}$ to $Spin_n$ (resp. $\mathbb{C}l_n^0$) are irreducible and equivalent.
3. For n even, for all $x \in \mathbb{R}^n \backslash \{0\}$, the linear map
$$\chi_n(x) : \Sigma_n^{\pm} \to \Sigma_n^{\mp}$$
are isomorphisms. Moreover, under the isomorphism $\mathbb{C}l_n^0 \simeq \mathbb{C}l_{n-1}$, the vector spaces $\Sigma^{\pm}$ corresponds to the two inequivalent irreducible representations of $\mathbb{C}l_{n-1}$.

Proof. Note the complex subalgebra generated by $Spin_n \in \mathbb{C}l_n$ is the even part $\mathbb{C}l_n^0$ of $\mathbb{C}l_n$. Hence, two representations of $\mathbb{C}l_n^0$ are irreducible or equivalent if and only if it is the case for their restriction to $Spin_n$. It is therefore sufficient to prove the assertion for $\mathbb{C}l_n^0$ instead of $Spin_n$.

1. Let $n = 2m$. Since $\omega^{\mathbb{C}}$ commutes with $\mathbb{C}l_n^0$, the restriction of χ_n to $\mathbb{C}l_n^0 \simeq \mathbb{C}l_{n-1}$ splits into $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, where
$$\Sigma_n^{\pm} = \chi_n\left(\frac{1 \pm \omega^{\mathbb{C}}}{2}\right)(\Sigma_n)$$
By Theorem 9, it is sufficient to prove that Σ_n^+ and Σ_n^- are non-trivial vector spaces. Since the linear map χ_n is an isomorphism, $\chi_n(\omega^{\mathbb{C}})$ cannot be equal to $\pm Id_{\Sigma_n} = \chi_n(\pm 1)$.

2. Again, we make use of the isomorphism $\mathbb{C}l_n^0 \simeq \mathbb{C}l_{n-1}$. For n odd, we know that $\alpha\left(\mathbb{C}l_n^{\pm}\right) = \mathbb{C}l_n^{\mp}$. Note that the even part sits diagonally in the decomposition $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$. In fact, since $\omega^{\mathbb{C}} \in \mathbb{C}l_n^1$ (since n is odd), we have

$$\omega^{\mathbb{C}} : \mathbb{C}l_n^0 \to \mathbb{C}l_n^1$$

hence $\mathbb{C}l_n^0 \cap \mathbb{C}l_n^{\pm} = \{0\}$.

More precisely, we have

$$\mathbb{C}l_n^0 = \{u^{\pm} + \alpha(u^{\pm}); u^{\pm} \in \mathbb{C}l_n^{\pm}\}$$

Since the two inequivalent irreducible representations of $\mathbb{C}l_n$ are distinguished by the isomorphism α, by restriction to $\mathbb{C}l_n^0$, they become equivalent.

3. Since n is even, we have $x \cdot \omega^{\mathbb{C}} = -\omega^{\mathbb{C}} \cdot x$, $\forall x \in \mathbb{R}^n$

Then

$$\begin{aligned}
\chi_n(x)\Sigma_n^{\pm} &= \chi_n(x)\chi_n\left(\frac{1 \pm \omega^{\mathbb{C}}}{2}\right)(\Sigma_n) \text{ (par 1)} \\
&= \chi_n\left(\frac{1 \mp \omega^{\mathbb{C}}}{2}\right)\chi_n(x)(\Sigma_n) \\
&\subset \Sigma_n^{\mp}
\end{aligned}$$

The linear map $\chi_n(x)$ is in fact an isomorphism since $\chi_n(x)^2 = -\|x\|^2 Id$. Indeed, let's write $x = \sum_{1 \le i \le n} x_i e_i$, where $\{e_1, \dots, e_n\}$ denotes an orthonormal basis of $\mathbb{R}^n$. Then

$$\begin{aligned}
\chi_n(x)^2(v) = \chi_n(x)(\chi_n(x)(v)) &= \chi_n(x)\left(\sum_{1 \le j \le n} x_j \chi_n(e_j)(v)\right) \\
&= \sum_{1 \le i \le n} x_i \chi_n(e_i)\left(\sum_{1 \le j \le n} x_j \chi_n(e_j)(v)\right) \\
&= \sum_{1 \le i,j \le n} x_i x_j \chi_n(e_i \cdot e_j)(v)
\end{aligned}$$

we have $e_i \cdot e_j = -e_j \cdot e_i$, since $\{e_1, \dots, e_n\}$ is an orthonormal basis. So the terms of the sum for which $i \neq j$ disappears. Also, $e_i \cdot e_i = -1$, so

$$\begin{aligned}
\chi_n(x)^2(v) &= \sum_{1 \le i \le n} x_i^2 \chi_n(-1)(v) \\
&= -\sum_{1 \le i \le n} x_i^2 v = -\|x\|^2 v
\end{aligned}$$

For the last statement, it is sufficient to note that the isomorphism $\mathbb{C}l_n \simeq \mathbb{C}l_{n+1}^0$ maps the volume element $\omega_{n-1}^{\mathbb{C}} := i^{\lfloor \frac{n}{2} \rfloor} e_1 \dots e_{n-1}$ of $\mathbb{C}l_{n-1}$ to the volume element $\omega_n^{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} e_1 \dots e_n$ of $\mathbb{C}l_n$.

$\square$

Definition 14. The representation

$$\rho_n := \begin{cases} \chi_{n|Spin_n} & \text{for n even} \\ \chi^+_{n|Spin_n} & \text{for n odd} \end{cases}$$

is called the canonical complex spin representation and is denoted by (ρ_n, Σ_n).

If n is odd, ρ_n is irreducible, whereas if n is even, it splits into two irreducible components $\rho_n^\pm$. Note that since $\rho(-1) = -Id$, the canonical complex spin representation does not descend to the group SO_n, and neither do its irreducible components $\rho_n^\pm$, if n is even.

1.4 Spin structure

We will now define what is a spin structure on a manifold M. The definitions given here are for a manifold with no other assumptions, and we will see later that on some specific spaces, the definition of a spin structure is much more easier, for instance on homogeneous spaces (see subsection 2.1).

Let M^n be an oriented n-dimensional manifold and let P be a principal bundle over M with group G. Recall that every representation

$$\rho : G \to Aut(V)$$

defines an associated vector bundle E, denoted by $E = P \times_\rho V$, defined as the quotient of $P \times V$ by the right G-action

$$g \cdot (u, X) := (ug, \rho(g^{-1})X)$$

The equivalence class of (u, X) is denoted by $[u, X]$ and the space of smooth sections of E is denoted by $\Gamma(E)$.

The principal bundle of positive linear frames of M will be denoted by

$$P_{GL_n^+} \to M$$

Let

$$\Xi : \widetilde{GL_n^+} \to GL_n^+$$

denotes the universal covering of GL_n^+.

Definition 15 (Spin structure)**.** A spin structure on an n-dimensional manifold M is given by a principal $\widetilde{GL_n^+}$-bundle $P_{\widetilde{GL_n^+}}M$ together with a projection :

$$\Theta : P_{\widetilde{GL_n^+}}M \to P_{GL_n^+}M$$

making the diagram :

$$\begin{tikzcd} \widetilde{GL_n^+} \arrow[r, "a \mapsto \tilde{u}a"] \arrow[d, "\Xi"'] & P_{\widetilde{GL_n^+}}M \arrow[d, "\Theta"] \arrow[dr] & \\ & & M \\ GL_n^+ \arrow[r, "A \mapsto \Theta(\tilde{u})A"'] & P_{GL_n^+}M \arrow[ur] & \end{tikzcd}$$

commutes for each $\tilde{u} \in P_{\widetilde{GL_n^+}}M$.

Equivalently, Θ is fibre preserving and satisfies $\Theta(\tilde{u}a) = \Theta(\tilde{u})\Xi(a)$, for all $\tilde{u} \in P_{\widetilde{GL_n^+}}M$ and $a \in \widetilde{GL_n^+}$.

Definition 16 (Spinorial metric). A spinorial metric in a Riemannian manifold (M, g) of dimension n is given by a principal $Spin_n$-bundle $P_{Spin_n}M$ together with a projection

$$\theta : P_{Spin_n}M \to P_{SO_n}M$$

making the diagram :

$$\begin{tikzcd}
Spin_n \arrow[r, "a\mapsto\tilde{u}a"] \arrow[dd, "\xi"'] & P_{Spin_n}M \arrow[dd, "\theta"] \arrow[dr] & \\
 & & M \\
SO_n \arrow[r, "A\mapsto\theta(\tilde{u})A"'] & P_{SO_n}M \arrow[ur] &
\end{tikzcd}$$

commutes for each $\widetilde{u} \in P_{Spin_n}M$

The spinorial metric is said to be subordinated to a spin structure P if $P_{Spin_n}M$ is a reduction to $Spin_n$ of the $\widetilde{GL_n^+}$-principal bundle $P_{\widetilde{GL_n^+}}M$.

Proposition 17. On a manifold endowed with a spin structure, each Riemannian metric gives rise to a spinorial metric.

Proof. The Riemannian metric g on M determines the subbundle $P_{SO_n}M$ inside $P_{GL_n^+}M$ (among all the bases of T_xM, we only keep the orthonormal ones for g). Since the group $Spin_n$ is the inverse image of SO_n in $\widetilde{GL_n^+}$ under Ξ, the inverse image of $P_{SO_n}M$ is a $Spin_n$-principal bundle, which determines the associated spinorial metric. $\square$

By Proposition 17, it is natural to give up the not so widely used term spinorial metric for the ore classical term, in the context of Riemannian geometry, of a spin manifold.

Definition 18 (Spinor bundle). To $Spin_n$, we associate a vector complex bundle $\Sigma M = P_{Spin_n} \times_{\rho_n} \Sigma_n$, called the spinor bundle.

Consider now a Riemannian metric g on M, and let $P_{SO_n}M$ be the principal SO_n-bundle of positive g-orthonormal frames over M. We denote by ι the representation of SO_n on $\mathbb{C}l_n$ obtained by extending every linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ of SO_n to an algebra morphism of $\mathbb{C}l_n$. We denote by $\mathbb{C}l(M)$ the vector bundle associated to $P_{SO_n}M$ for the representation ι, that is

$$\mathbb{C}l(M) = P_{SO_n}M \times_\iota \mathbb{C}l_n$$

Recall that $\mathbb{C}l(M)$ (which is called the Clifford bundle of (M, g)) is the set of equivalence classes $[u, a]$ of pairs $u \in P_{SO_n}M, a \in \mathbb{C}l_n$, with respect to the equivalence relation $[u, a] = [uA, \iota(A^{-1})(a)]$, for all $A \in SO_n$.

The equivariance of θ shows that there exists a representation of $\mathcal{C}^\infty(M)$-algebras

$$\mathbb{C}l(M) \to End(\Sigma M)$$

given by $[u, a]\,([\tilde{u}, \psi]) = [\tilde{u}, \rho_n(a)\psi]$ for each $\tilde{u} \in P_{Spin_n}M, u = \theta(\tilde{u}) \in P_{SO_n}M, a \in \mathbb{C}l_n$ and $\psi \in \Sigma_n$.

The action of $\mathbb{C}l(M)$ on ΣM is called the Clifford product and is denoted by

$$(\sigma, \psi) \mapsto \sigma \cdot \psi$$

Local sections of ΣM are called spinor fields.

1.5 The Dirac operator

Let ∇ be the Levi-Civita connection acting on section of ΣM. The morphism $\gamma_{|T^*M\otimes\Sigma M}$, denoted by the same letter γ, where

$$\begin{aligned}\gamma : \mathbb{C}l(M)\otimes\Sigma M &\to M\\ \sigma\otimes\psi &\mapsto \sigma\cdot\psi\end{aligned}$$

is the pointwise Clifford multiplication.

Definition 19 (Dirac operator)**.** The Dirac operator is the first-order differential operator acting on sections of the spinor bundles, given by

$$\mathcal{D} := \gamma\circ\nabla$$

Locally, on an open set $U\subset M$, we get :

$$\begin{array}{cccccc}\mathcal{D}: & \Gamma(\Sigma M) & \xrightarrow{\nabla} & \Gamma(T^*M\otimes\Sigma M) & \xrightarrow{\gamma} & \Gamma(\Sigma M)\\ & \psi & \longmapsto & \sum_{i=1}^{n} e_i^*\otimes\nabla_{e_i}\psi & \longmapsto & \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}\psi\end{array}$$

where $\{e_1,\dots,e_n\}\in\Gamma_U(P_{SO_n}M)$ is a local orthonormal frame of the tangent bundle and $\{e_1^*,\dots,e_n^*\}$ the dual frame.

Examples. 1. For $M=\mathbb{R}^n$, $\Sigma\mathbb{R}^n=\mathbb{R}^n\times\mathbb{C}^N$, where $N=2^{\lfloor\frac{n}{2}\rfloor}$.

Every spinor can be seen as a map $\Psi:\mathbb{R}^n\to\mathbb{C}^N$. Indeed , since $M=\mathbb{R}^n$, and Σ_n is a complex vector space of dimension N, so $\Sigma_n=\mathbb{C}^N$, and $P_{Spin_n}M=\mathbb{R}^n$, then $\Sigma M=P_{Spin_n}\times_\rho\Sigma_n=\mathbb{R}^n\times\mathbb{C}^N$, where ρ is the standard representation $\rho:\mathbb{R}^n\to GL(\mathbb{C}^N)=GL_N(\mathbb{C})$

So if $\Psi\in\Gamma(\Sigma M)$, then

$$\begin{aligned}\Psi:\mathbb{R}^n&\to\mathbb{R}^n\times\mathbb{C}^N\\ x&\mapsto[x,v_x]\end{aligned}$$

satisfying $[xy,v_x]=[x,\rho(y^{-1})v_x]$, and Ψ is entirely determined by $x\mapsto v_x$.

The Dirac operator is given by

$$\mathcal{D}=\sum_{i=1}^{n}e_i\cdot\partial_i$$

and acts on differentiable maps from $\mathbb{R}^n$ to $\mathbb{C}^N$, where $\partial_i = \nabla_{e_i}$. Then

$$
\begin{aligned}
\mathcal{D}^2 &= \left(\sum_{i=1}^{n} e_i \partial_i\right)\left(\sum_{j=1}^{n} e_j \partial_j\right) = \sum_{1\le i,j\le n} e_i \cdot e_j \partial_i \partial_j \\
&= -\sum_{i=1}^{n} \partial_i^2 + \sum_{1\le i<j\le n} e_i \cdot e_j \partial_i \partial_j + \sum_{1\le i>j\le n} e_i \cdot e_j \partial_i \partial_j \\
&= -\sum_{i=1}^{n} \partial_i^2 + \sum_{1\le i<j\le n} e_i \cdot e_j \partial_i \partial_j + \sum_{1\le i<j\le n} e_j \cdot e_i \partial_j \partial_i \\
&= -\sum_{i=1}^{n} \partial_i^2 + \sum_{1\le i<j\le n} e_i \cdot e_j \left(\partial_i \partial_j - \partial_j \partial_i\right) = -\sum_{i=1}^{n} \partial_i^2 \\
&= \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix}
\end{aligned}
$$

2. In the particular case where $M = \mathbb{R}^2$, we have the Clifford algebra $\mathbb{C}l_2 = \mathscr{M}_2(\mathbb{C})$, the complex volume element is $\omega^{\mathbb{C}} = ie_1 \cdot e_2$ and one can identify the spinor bundle $\Sigma_2 = \Sigma_2^+ \oplus \Sigma_2^- = \mathbb{C} \oplus \mathbb{C}$ with

   $$\Sigma_2^+ = \operatorname{span}_{\mathbb{C}}(e_1 + ie_2) \quad \text{and} \quad \Sigma_2^- = \operatorname{span}_{\mathbb{C}}(1 - ie_1 \cdot e_2)$$

   $\Big($check that $\chi(\omega^{\mathbb{C}})(e_1 + ie_2) = e_1 + ie_2$ and $\chi(\omega^{\mathbb{C}})(1 - ie_1 \cdot e_2) = -(1 - ie_1 \cdot e_2)\Big)$.

   Then each spinor field $\Psi \in \Gamma(\Sigma M)$ is given by two complex functions $f, g : \mathbb{R}^2 \to \mathbb{C}$, such that

   $$\Psi = (e_1 + ie_2)f + (1 - ie_1 \cdot e_2)g$$

   The Dirac operator acting on Ψ is then

   $$
   \begin{aligned}
   \mathcal{D}\Psi &= (e_1\partial_1 + e_2\partial_2)\left[(e_1 + ie_2)f + (1 - ie_1 \cdot e_2)g\right] \\
   &= e_1 \cdot e_1 \partial_1 f + ie_1 \cdot e_2 \partial_1 f + e_2 \cdot e_1 \partial_2 f + ie_2 \cdot e_2 \partial_2 f \\
   &\quad + e_1\partial_1 g + e_2 \partial_2 g - ie_1 \cdot e_1 \cdot e_2 \partial_1 g - ie_2 \cdot e_1 \cdot e_2 \partial_2 g \\
   &= -\partial_1 f + ie_1 \cdot e_2 \partial_1 f - e_1 \cdot e_2 \partial_2 f - i\partial_2 f + e_1 \partial_1 g + e_2 \partial_2 g + i_2 \partial_1 g - ie_1 \partial_2 g \\
   &= -\left(\partial_1 + i\partial_2\right)\Big((1 - ie_1 \cdot e_2)f\Big) + (\partial_1 - i\partial_2)\Big((e_1 + ie_2)g\Big)
   \end{aligned}
   $$

   Let $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\overline{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. Then

   $$\mathcal{D}\Psi = 2\left(-\partial_{\overline{z}}\Big((1 - ie_1 \cdot e_2)f\Big) + \partial_z\Big((e_1 + ie_2)g\Big)\right)$$

   That is, in the basis $\{e_1 + ie_2, 1 - ie_1 \cdot e_2\}$ of Σ_2

   $$\mathcal{D} = \begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\overline{z}} & 0 \end{pmatrix}$$

   Hence, the Dirac operator $\mathcal{D}$ can be considered as a generalization of the Cauchy-Riemann operator.

2 The Dirac operator on homogeneous spaces

2.1 Homogeneous spaces

Let us first give some result about homogeneous spaces, which will help us in our study of the Dirac operator, and more specifically about symmetric spaces. The main idea is to use the specific structure of homogeneous spaces and Frobenius reciprocity to narrow the computation of the eigenvalues on $\mathcal{D}$-invariant subspaces. Also, we will see, as noted before, that the spin structures and the bundle P_{Spin_n} are more easy to work with, since the latter can be identified to more practical spaces, and the former to lifts of the isotropy representation.

Theorem 20 (Cartan)**.** A Riemannian manifold (M, g_M) is a symmetric space if and only if there exists a triple (G, H, σ) satisfying the following conditions :

1. G is a connected Lie group, H is a compact subgroup of G, and σ is an involutive automorphism of G such that
$$G_e^\sigma \subset H \subset G^\sigma$$
where G^σ is the group $\{g \in G, \sigma(g) = g\}$ and G_e^σ is the connected component of the identity in G^σ.
2. There exists a G-invariant metric $g_{G/H}$ on G/H such that $\left(G/H, g_{G/H}\right)$ is isometric to (M, g_M).

In the following, any symmetric space (M, g_M) will be identified with the corresponding homogeneous space $\left(G/H, g_{G/H}\right)$ by means of the isometry

$$\iota : \left(G/H, g_{G/H}\right) \to (M; g_M)$$

The neutral element in G will be denoted e, the equivalence class in G/H of any $g \in G$ by $[g]$. The same notation L_g will be used to denote the left action of $g \in G$ of G and the induced action in the quotient G/H.

Now let's see what happens in the corresponding Lie algebras. We consider a compact and simply connected symmetric space (M, g_M), together with the corresponding triple (G, H, σ) satisfying the conditions of theorem 20.

Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of G (resp. H). The tangent map of ι at the point $[e]$ gives an isometry :

$$T_{[e]}G/H \simeq \left(\mathfrak{g}/\mathfrak{h}, g_{G/H,[e]}\right) \xrightarrow{T_{[e]}\iota} (T_pM, g_{M,p})$$

Now the structure of symmetric space of (M, g_M) provides an $Ad_G(H)$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ which complements $\mathfrak{h}$ in $\mathfrak{g}$. Indeed, let σ_* be the tangent map at e of the involutive automorphism σ. It is a Lie algebra automorphism of $\mathscr{G}$ such that $\sigma_*^2 = Id_{\mathfrak{g}}$. Since $H = G_e^\sigma$, one has :

$$\mathfrak{h} = \{X \in \mathfrak{g}; \sigma_*(X) = X\}$$

Thus the decomposition $\mathfrak{g}$ into eigenspaces is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, with $\mathfrak{p} = \{X \in \mathfrak{g}; \sigma_*(X) = -X\}$.

It follows from the condition $H = G_e^\sigma$ that $\mathfrak{p}$ is an $Ad_G(H)$-invariant subspace of $\mathfrak{g}$:

$$\begin{aligned}
\sigma_*(Ad_g(h)X) &= \frac{d}{dt}\left(\sigma(h\exp(tX)h^{-1})\right)_{t=0} \\
&= \frac{d}{dt}\left(h\sigma(\exp(tX))h^{-1}\right)_{t=0} \text{ since } H = G_e^\sigma \\
&= \frac{d}{dt}\left(h\exp(t\sigma_*(X))h^{-1}\right)_{t=0} \\
&= \frac{d}{dt}\left(h\exp(-tX)h^{-1}\right)_{t=0} \\
&= -Ad_G(h)(X)
\end{aligned}$$

We fix now an orthonormal basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{p}$. It allows us to identify $\mathfrak{p}$ with $\mathbb{R}^n$, and denote by Φ the isometry

$$\begin{aligned}\Phi : \mathfrak{p} &\to T_{[e]} {}^{G}\!/_{H} \\ X &\mapsto \frac{d}{dt}\Big([\exp(tX)]\Big)_{t=0}\end{aligned}$$

In the following, for simplicity, we denote by α the homomorphism :

$$\begin{aligned}\alpha : H &\to SO_n \\ h &\mapsto Ad_g(h)_{|\mathfrak{p}}\end{aligned}$$

Proposition 21. Let $P_{SO_n}M$ be the bundle of positive orthonormal frames of M. Let $\pi : G \to {}^{G}\!/_{H}$ be the canonical principal bundle over ${}^{G}\!/_{H}$ with structural space group H. Consider the associated SO_n-principal bundle given by $G \times_\alpha SO_n$. Then the principal bundles P_{SO_n} and $G \times_\alpha SO_n$ are isomorphic.

Proof. Let $[g]$ be some element in ${}^{G}\!/_{H}$ and let $b_{[g]}$ be a positive orthonormal frame at $[g]$, that is an isometry $\mathbb{R}^n \to T_{[g]}\left({}^{G}\!/_{H}\right)$ preserving the orientation. Let g be a representative of $[g]$, and denote by L_{g*} the tangent map at the point $p := [e]$ of L_g. Consider the map $u_g : \mathbb{R}^n \to \mathbb{R}^n \simeq \mathfrak{p}$ defined by :

$$u_g := \Phi^{-1} \circ L_{g*}^{-1} \circ b_{[g]} \tag{9}$$

that is

$$u_g : \mathbb{R}^n \xrightarrow{b_{[g]}} T_{[g]} {}^{G}\!/_{H} \xrightarrow{L_{g*}^{-1}} T_{[e]} {}^{G}\!/_{H} \xrightarrow{\Phi^{-1}} \mathfrak{p} \simeq \mathbb{R}^n$$

Since u_g is an isometry preserving the orientation of $\mathbb{R}^n$, u_g belongs to SO_n. Furthermore, for any $h \in H$, we have :

$$\begin{aligned}u_{gh} &= \Phi^{-1} \circ (L_{gh*})^{-1} \circ b_{[gh]} \\ &= \Phi^{-1} \circ L_{h*}^{-1} \circ L_{g*}^{-1} \circ b_{[g]} \\ &= \Phi^{-1} \circ L_{h*}^{-1} \circ \Phi \circ \Phi^{-1} \circ L_{g*}^{-1} \circ b_{[g]} \\ &= \underbrace{\Phi^{-1} \circ L_{h*}^{-1} \circ \Phi}_{=\alpha(h^{-1})} \circ\, u_{[g]}\end{aligned}$$

Hence, the element $[g, u_g]$ in the fibre of $G \times_\alpha SO_n$ at $[g]$ depends only in the equivalence class $[g]$ of g. The map :

$$\begin{aligned}P_{SO_n}M &\to G \times_\alpha SO_n \\ b_{[g]} &\mapsto [g, u_g]\end{aligned}$$

gives the claimed SO_n-isomorphism between $P_{SO_n}M$ and $G \times_\alpha SO_n$ (the inverse map is given by $[g, u] \to L_{g*} \circ \Phi \circ u$, where $(g, u) \in G \times SO_n$ is a representative of $[g, u]$). $\square$

Consider the vector bundle associated with the principal bundle $\pi : G \to {}^{G}\!/_{H}$ by the linear representation $\alpha : H \to SO_n \subset GL_n$, that is $G \times_\alpha \mathbb{R}^n$.

Proposition 22. The vector bundle TM and $G \times_\alpha \mathbb{R}^n$ are isomorphic.

Proof. The tangent bundle TM is the vector bundle associated to $P_{SO_n}M$ by the standard linear representation $P_{SO_n}M \times_\rho \mathbb{R}^n$.

Let $[g]$ be an element in ${}^{G}\!/_{H}$ and $[b_{[g]}, x]$ an element in the fibre of $P_{SO_n}M \times_\rho \mathbb{R}^n$ over $[g]$, where $b_{[g]}$ is a positive orthonormal frame at $[g]$, and $x \in \mathbb{R}^n$. We can see that the element $[g, u_g(x)]$ of the fibre of $G \times_\alpha \mathbb{R}^n$ at $[g]$, where u_g is the isometry of $\mathbb{R}^n$ defined previously (9), depends only in the equivalence class $[b_{[g]}, x]$ and the map

$$\begin{aligned} P_{SO_n}M \times_\rho \mathbb{R}^n &\to G \times_\alpha \mathbb{R}^n \\ [b_{[g]}, x] &\mapsto [g, u_g(x)] \end{aligned}$$

is the claimed isomorphism between the two vector bundles. The inverse map is given by

$$[g, x] \mapsto [L_{g*} \circ \Phi, x]$$

where $(g, x) \in G \times \mathbb{R}^n$ is a representative of $[g, x]$. □

Theorem 23. Let $M = {}^{G}\!/_{H}$ be a compact, simply connected irreducible symmetric space, with corresponding triple (G, H, σ) satisfying the conditions of Theorem 20. Then, M admits a spin structure if and only if the homomorphism

$$\begin{aligned} \alpha : H &\to SO_n \\ h &\mapsto Ad_G(h)_{|\mathfrak{p}} \end{aligned}$$

lifts to a homomorphism $\widetilde{\alpha} : H \to Spin_n$ such that the diagram
$$\begin{tikzcd} & Spin_n \arrow[d, "\xi"] \\ H \arrow[ur, "\widetilde{\alpha}"] \arrow[r, "\alpha"] & SO_n \end{tikzcd}$$
commutes.

In this case, the spin structure is G-invariant.

Proof. First, the condition is sufficient. Suppose α lifts to $\widetilde{\alpha} : H \to SO_n$. Let $P_{Spin_n}M$ be the principal bundle over M with structural group $Spin_n$, associated with the principal bundle $(G, \pi, {}^{G}\!/_{H})$ by the homomorphism $\widetilde{\alpha}$, that is :

$$P_{Spin_n}M = G \times_{\widetilde{\alpha}} Spin_n$$

and let ξ_M be the map :

$$\begin{aligned} \xi_M : P_{Spin_n}M = G \times_{\widetilde{\alpha}} Spin_n &\to G \times_\alpha SO_n \simeq P_{SO_n}M \\ [g, u] &\mapsto [g, \xi(u)] \end{aligned}$$

We can see that $(P_{Spin_n}M, \xi_M)$ is a spin structure on M. Furthermore this spin structure is G-invariant since the left action of the group G on $P_{Spin_n}M$ given by

$$g_0 \cdot [g, u] = [g_0 g, u]$$

and the right action of the group $Spin_n$ on $P_{Spin_n}M$ given by

$$[g, u] \cdot u_0 = [g, u u_0]$$

clearly commutes.

Finally, the condition is necessary. Suppose there exists a spin structure $(P_{Spin_n}M, \xi_M)$ on $M = {}^{G}\!/_{H}$. First, as G is supposed to be simply connected, the monodromy principle allows us to lift the action of G on $P_{SO_n}M$ to a G-action on $P_{Spin_n}M$ which commutes with the right action of $Spin_n$ on $P_{Spin_n}M$.

The induced action of H on $P_{Spin_n}M$ stabilizes the fibre $(P_{Spin_n}M)_{[e]}$ of $P_{Spin_n}M$ at the point $[e]$. Let $\widetilde{b}_{[e]}$ be a fixed element in $(P_{Spin_n}M)_{[e]}$. For any $h \in H$, denote by $h \cdot \widetilde{b}_{[e]}$, the action of h on $\widetilde{b}_{[e]}$.

Then define $\widetilde{\alpha}(h) \in Spin_n$ by the relations

$$h \cdot \widetilde{b}_{[e]} = \widetilde{b}_{[e]} \cdot \widetilde{\alpha}(h)$$

We now check that $\widetilde{\alpha}$ is an a homomorphism $H \to Spin_n$ which is a lift of α.

$$\begin{aligned}\widetilde{b}_{[e]} \cdot \widetilde{\alpha}(hh') &= hh' \cdot \widetilde{b}_{[e]} \\ &= h \cdot \widetilde{b}_{[e]} \cdot \widetilde{\alpha}(h') = \widetilde{b}_{[e]} \cdot \widetilde{\alpha}(h) \cdot \widetilde{\alpha}(h')\end{aligned}$$

So $\widetilde{\alpha}(hh') = \widetilde{\alpha}(h)\widetilde{\alpha}(h')$ It is a lift by definition (since we have lift the action of G on $P_{SO_n}M$) and we have

$$P_{Spin_n}M \simeq G \times_{\widetilde{\alpha}} Spin_n$$

□

Proposition 24. Let $\rho_n : Spin_n \to GL(\Sigma_n)$ be the spinor representation. Consider the vector bundle associated to $(G, \pi, {}^{G}\!/\!_{H})$ by the representation

$$\widetilde{\rho_n} := \rho_n \circ \widetilde{\alpha}$$

that is $G \times_{\widetilde{\rho_n}} \Sigma_n$. Then the spinor bundle ΣM is isomorphic to $G \times_{\widetilde{\rho_n}} \Sigma_n$.

Proof. The proof is analogous to that of Proposition 22. By definition 18, the spinor bundle ΣM is the associated bundle $P_{Spin_n}M \times_{\rho_n} \Sigma_n$. Let $[g]$ be an element in ${}^{G}\!/\!_{H}$ and $[\widetilde{b}_{[g]}, \psi]$ an element in the fibre $P_{Spin_n}M \times_{\rho_n} \Sigma_n$ over $[g]$, where $\widetilde{b}_{[g]}$ is an element of the fibre $P_{Spin_n}M$ over $[g]$ and $\psi \in \Sigma_n$.

But $\widetilde{b}_{[g]}$ has the form $[g, u]$ where g is a representative of $[g]$ and $u \in Spin_n$. Now it is straightforward that the element $[g, \rho_n(u)\psi]$ of the fibre of $G \times_{\widetilde{\rho_n}} \Sigma_n$ at $[g]$ depends only on the equivalence class $\big[[g, u], \psi\big]$. Indeed, in one hand, if we consider $\big[[gh, \widetilde{\alpha}(h^{-1})u], \psi\big] \in P_{Spin_n}M \times_{\widetilde{\rho_n}} \Sigma_n$, one has :

$$\begin{aligned}\Big[gh, \rho_n(\widetilde{\alpha}(h^{-1})u).\psi\Big] &= \Big[gh, \rho_n(\widetilde{\alpha}(h^{-1}))\rho_n(u).\psi\Big] \\ &= \Big[gh, (\widetilde{\rho_n}(h^{-1}))\rho_n(u).\psi\Big] = \Big[g, \rho_n(u)\psi\Big]\end{aligned}$$

In the other hand, if we consider $\Big[[g, u\widetilde{\alpha}(h^{-1})], \widetilde{\rho}_n(h)\psi\Big]$, one has :

$$\begin{aligned}\Big[g, \rho_n\left(u\widetilde{\alpha}(h^{-1})\right)\left(\widetilde{\rho}_n(h)\psi\right)\Big] &= \Big[g, \rho_n\left(u\widetilde{\alpha}(h^{-1})\right)\left(\rho_n(\widetilde{\alpha}(h))\psi\right)\Big] \\ &= \Big[g, \rho_n\left(u\widetilde{\alpha}(h^{-1})\widetilde{\alpha}(h)\right)\psi\Big] = \Big[g, \rho_n(u)\psi\Big]\end{aligned}$$

The map

$$\begin{aligned}P_{Spin_n}M \times_{\rho_n} \Sigma_n &\to X \times_{\widetilde{\rho}_n} \Sigma_n \\ \Big[[g, u], \psi\Big] &\mapsto \Big[g, \rho_n(u)\psi\Big]\end{aligned}$$

is the claimed isomorphism between the two vector bundles. The inverse map is given by

$$[g, \psi] \mapsto \big[[g, e], \psi\big]$$

for $g \in G, \psi \in \Sigma_n$. □

Proposition 25. Let $C_H^\infty(G, \Sigma_n)$ be the space of H-equivariant smooth functions $G \to \Sigma_n$, that is, the set of functions $f : G \to \Sigma_n$ satisfying

$$f(gh) = \widetilde{\rho}_n(h^{-1})f(g)$$

The space $\Gamma(\Sigma M)$ of smooth sections of the bundle ΣM is isomorphic to the space $C_H^\infty(G, \Sigma_n)$.

This means that a spinor field $\Psi \in \Gamma(\Sigma M)$ can be see as a H-equivariant function $\Psi : G \to \Sigma_n$.

Proof. The proof is very similar to that of the previous proposition. The H-equivariant functions f_Ψ corresponding to a spinor field Ψ is defined by

$$\Psi([g]) = [g, f_\Psi(g)]$$

$\square$

2.2 A general formula for the Dirac operator

In [2], C. Bär gives several important results about the Dirac operator among which an explicit general formula written in theorem 26, which will be used later to compute explicitly the eigenvalues on some examples of homogeneous spaces.

For $X, Y \in \mathfrak{g}$, let $[X, Y]_\mathfrak{p}$ be the $\mathfrak{p}$-component of $[X, Y]$. We define

$$\begin{aligned} \alpha_{ijk} &:= \frac{1}{4}\Big(\big\langle [X_i, X_j]_\mathfrak{p}, X_k\big\rangle + \big\langle [X_j, X_k]_\mathfrak{p}, X_i\big\rangle + \big\langle [X_k, X_i]_\mathfrak{p}, X_j\big\rangle\Big) \\ \beta_i &:= \frac{1}{2}\sum_{j=1}^n \big\langle [X_j, X_i]_\mathfrak{p}, X_j\big\rangle \end{aligned}$$

Recall that $\{X_1, \cdots, X_n\}$ is an orthonormal basis of $\mathfrak{p}$. From Proposition 25, we see that spinor fields are given by $\widetilde{\rho}_n$-equivariant maps $f : G \to \Sigma_n$. More precisely, the spinor field Ψ corresponding to f is given by $\Psi([g]) = [g, f(g)]$.

Let $\{E_1, \ldots, E_n\}$ be the standard basis of $\mathbb{R}^n$.

Theorem 26. For the Dirac operator, we have the formula :

$$\mathcal{D}[g, \Psi(g)]_{|g=g_0} = \left[g_0, \sum_{k=1}^n E_k \cdot X_k|_{g_0}(\Psi) + \left(\sum_{i=1}^n \beta_i E_i + \sum_{1\le i<j<k\le n} \alpha_{ijk} E_i \cdot E_j \cdot E_k\right) \cdot \Psi(g_0)\right]$$

Proof. Recall that $\mathcal{D} = \gamma \circ \nabla$.

Let $X \in \mathfrak{p}$, $\overline{X} = \Phi(x) \in T_pM (= T_{[e]} G/H)$, $g_0 \in G$. We choose $\Lambda(t) \in Spin_n$ such that $\Lambda(0) = 1_{Spin_n}$ and the curve $t \mapsto [e^{tX}, \Lambda(t)]$ is horizontal with respect to the distribution coming from the Levi-Civita connection (See Appendix B about horizontal tangent vectors). Since this distribution is invariant under the action of G on $P_{Spin_n}M (= G \times_{\widetilde{\alpha}} Spin_n)$, the curve $t \mapsto [g_0 e^{tX}, \Lambda(t)]$ is also horizontal.

Therefore the horizontal lift $(dg_0 \cdot \overline{X})^{Spin}$ of $dg_0 \cdot \overline{X} \in T_{g_0}M$ to $P_{Spin_n}M$ is given by

$$\left(dg_0 \cdot \overline{X}\right)^{Spin} ([g_0, 1_{Spin_n}]) = \frac{d}{dt}\left[g_0 e^{tX}, \Lambda(t)\right]_{|t=0}$$

and the horizontal lift $(dg_0 \cdot \overline{X})^{SO}$ of $dg_0 \cdot \overline{X} \in T_{g_0}M$ to $P_{SO_n}M (= G \times_\alpha SO_n)$ is given by

$$(dg_0 \cdot \overline{X})^{SO}\left([g_0, \overline{b}_0]\right) = \frac{d}{dt}\left[g_0 e^{tX}, \overline{b}_0(\xi \cdot \Lambda(t))\right]_{|t=0}$$

Let $[g] \mapsto [g, \Psi(g)] \cong [[g, \Lambda], \sigma(g, \Lambda)]$ be a spinor field, where $\cong$ denotes the isomorphism between $G \times_{\widetilde{\rho}_n} \Sigma_n$ and $P_{Spin_n}M \times_{\rho_n} \Sigma_n (= \Sigma M)$, and Ψ and σ are related by :

$$\begin{aligned}
\Psi(g) &= \sigma(g, 1_{Spin}) \\
\sigma(g, \Lambda) &= \rho_n(\Lambda^{-1})\Psi(g) \ , \ g \in G, \Lambda \in Spin_n
\end{aligned}$$

We calculate now $\nabla^{\Sigma}_{dg_0 \cdot \overline{X}}$ where ∇^{Σ} is the spinor connection induced by the Levi-Civita connection (see 17 in Appendix B.2).

$$\begin{aligned}
\nabla^{\Sigma}_{dg_0 \cdot \overline{X}} [[g, \Lambda], \sigma(g, \Lambda)] &= \Big[[g_0, 1_{Spin}], \left(dg_0 \cdot \overline{X}\right)^{Spin}|_{[g_0,1]}(\sigma)\Big] \\
&\cong \Big[g_0, \left(dg_0 \cdot \overline{X}\right)^{Spin}|_{[g_0,1]}(\sigma)\Big] \\
\left(dg_0 \cdot \overline{X}\right)^{Spin} |_{[g_0,1]}(\sigma) &= \frac{d}{dt}\sigma\left(g_0 e^{tX}, \Lambda(t)\right)|_{t=0} \\
&= \frac{d}{dt}\rho\left(\Lambda(t)^{-1}\right) \cdot \Psi(g_0 e^{tX})|_{t=0} \\
&= -\rho_*(\dot{\Lambda}(0))\Psi(g_0) + dL_{g_0}X(\Psi) \\
&= -\rho_*(\dot{\Lambda}(0))\Psi(g_0) + X|_{g_0}(\Psi)
\end{aligned}$$

Now, we have to calculate $\dot{\Lambda}(0)$. We know that the curve $t \to [e^{tX}, \Theta \cdot \Lambda(t)]$ is horizontal in $P_{SO_n}M$, so we get :

$$\frac{\nabla}{dt}\left[e^{tX}, \left(\Theta \cdot \Lambda(t)\right)\right]|_{t=0} = 0$$

But we have also (by the proof of Proposition 21)

$$\begin{aligned}
\frac{\nabla}{dt}\left[e^{tX}, \left(\Theta \cdot \Lambda(t)\right)\right]|_{t=0} &= \frac{\nabla}{dt}\left(de^{tX}.\overline{b}_0.\Theta(\Lambda(t))\right)|_{t=0} \\
&= d\overline{b}_0 \cdot \left(\Theta_* \cdot \dot{\Lambda}(0)\right) + \frac{\nabla}{dt}\left(de^{tX} \cdot \overline{b}_0\right)|_{t=0}
\end{aligned}$$

where $\overline{b}_0$ denotes the mapping from SO_n to $P_{SO_n}M$ given by multiplication by $\overline{b}_0$, and $d\overline{b}_0$ denotes its differential which is an isomorphism form $\mathfrak{so}(n)$ to the vertical subspace of $T_{\overline{b}_0}P_{So_n}M$, de^{tX} denotes the differential of the left action of G on T_pM by left multiplication (here we take $t = 0$, so $e^{tX} = p$), so $(de^{tX} \cdot \overline{b}_0) \in P_{So_n}M$.

Therefore, we have an expression for $\dot{\Lambda}(0)$

$$\dot{\Lambda}(0) = -\Theta_*^{-1} \cdot (d\overline{b}_0)^{-1}\frac{\nabla}{dt}(de^{tX} \cdot \overline{b}_0)|_{t=0}$$

Let's compute this equation step by step. First, let $c(s)$ be an integral curve for $\overline{X}_i$ in M. Then

$$\begin{aligned}
\frac{\nabla}{dt}de^{tX} \cdot c'(0)|_{t=0} &= \frac{\nabla}{\partial t}\frac{\partial}{\partial s}e^{tX} \cdot c(s)|_{t=s=0} \\
&= \frac{\nabla}{\partial s}\frac{\partial}{\partial t}e^{tX} \cdot c(s)|_{t=s=0} \\
&= \frac{\nabla}{\partial s}\overline{X}(c(s))|_{s=0} \\
&= \nabla_{\overline{X}_i}\overline{X} = \sum_{j=1}^{n}\langle\nabla_{\overline{X}_i}\overline{X}, \overline{X}_j\rangle\overline{X}_j \\
&= \sum_{j=1}^{n} c_{ji}(X)\overline{X}_j
\end{aligned}$$

where

$$c_{ji}(X) = \frac{1}{2}\left(-\left\langle\overline{[X_i, X]_{\mathfrak{p}}}, \overline{X}_j\right\rangle + \left\langle\overline{[X_j, X_i]_{\mathfrak{p}}}, \overline{X}\right\rangle + \left\langle\overline{[X_j, X]_{\mathfrak{p}}}, \overline{X}_i\right\rangle\right)$$

Now, for a matrix $A = (a_{ij}) \in \mathfrak{so}_n$ we have

$$\begin{aligned} d\bar{b}_0 \cdot A &= \frac{d}{dt}\bar{b}_0 \cdot e^{tA}|_{t=0} \\ &= \frac{d}{dt}\bar{b}_0 \cdot (1 + t \cdot A + O(t^2))|_{t=0} = \bar{b}_0 \cdot A \\ &= (\overline{X}_1, \cdots, \overline{X}_n) \cdot A = \left(\sum_j a_{j1} \cdot \overline{X}_j, \ldots, \sum_j a_{jn}\overline{X}_j\right) \end{aligned}$$

Thus $(d\bar{b}_0)^{-1}\frac{\nabla}{dt}\left(de^{tX} \cdot \bar{b}_0\right)|_{t=0} = (c_{ij}(X)) \in \mathfrak{so}_n$ and therefore

$$\dot{\Lambda}(0) = -\Theta_*^{-1}(c_{ij}(X)) = \frac{1}{4}\sum_{i,j} c_{ij}(X)E_i \cdot E_j \in \mathfrak{spin}_n$$

the factor $\frac{1}{4}$ coming from the fact that $Spin_n$ is a double covering of SO_n.

Now that we have the result for $\dot{\Lambda}(0)$, putting it in $\nabla^{\Sigma}_{dg_0 \cdot \overline{X}}$, we get :

$$\begin{aligned} \nabla^{\Sigma}_{dg_0 \cdot \overline{X}}[g, \Psi(g)] &= \left[g_0, X|_{g_0}(\Psi) - \rho_*\left(\frac{1}{4}\sum_{ij} c_{ij}(X)E_i \cdot E_j\right) \cdot \Psi(g_0)\right] \\ &= \left[g_0, X|_{g_0}(\Psi) - \frac{1}{4}\sum_{ij} c_{ij}(X)E_i \cdot E_j \cdot \Psi(g_0)\right] \end{aligned}$$

In the last line, we omit ρ_*, seeing the element $E_i \cdot E_j$ of $\mathfrak{spin}(n)$ as the element $E_i \cdot E_j \in Cl_n$.

Now that we have an expression for ∇^{Σ}, we calculate the formula for the Dirac operator. By definition, we have :

$$\begin{aligned} \mathcal{D}[g, \Psi(g)]|_{g=g_0} &= \sum_{k=1}^{n}(dg_0 \cdot \overline{X}_k) \cdot \nabla^{\Sigma}_{dg_0 \cdot \overline{X}_k}[g, \Psi(g)] \\ &= \left[g_0, \sum_{k=1}^{n} E_k \cdot \left(X_k|_{g_0}(\Psi) - \frac{1}{4}\sum_{i,j} c_{ij}(X_k)E_i \cdot E_j \cdot \Psi(g_0)\right)\right] \end{aligned}$$

We can see that for $i \neq j$, $c_{ij}(X) = -c_{ji}(X)$ and for all i, $c_{ii}(X) = 0$. So

$$c_{ij}(X)E_i \cdot E_j + c_{ji}(X)E_j \cdot E_i = 2c_{ij}E_i \cdot E_j$$

since $E_i \cdot E_j = -E_j \cdot E_i$. Putting all this together, we have :

$$-\frac{1}{4}\sum_{i,j} c_{ij}(X_k)E_i \cdot E_j \cdot \Psi(g_0) = -\frac{1}{2}\sum_{1 \le i < j \le n} c_{ij}(X_k)E_i \cdot E_j \cdot \Psi(g_0)$$

and

$$\mathcal{D}[g, \Psi(g)]|_{g=g_0} = \left[g_0, \sum_{k=1}^{n} E_k \cdot X_k|_{g_0}(\Psi) + \sum_{k=1}^{n} E_k \cdot \left(\left(-\frac{1}{2}\right)\sum_{1 \le i < j \le n} c_{ij}(X_k)E_i \cdot E_j \cdot \Psi(g_0)\right)\right]$$

We denote $c_{ijk} = c_{ij}(X_k)$. Then, we have

$$\sum_{k=1}^{n} E_k \cdot \left(-\frac{1}{2} \sum_{i<j} c_{ijk} E_i \cdot E_j \right) = -\frac{1}{2} \Bigg(\sum_{k<i<j} c_{ijk} E_k \cdot E_i \cdot E_j + \sum_{i<j} c_{iji}(-E_j) + \sum_{i<k<j} c_{ijk}(-E_i \cdot E_k \cdot E_j)$$
$$+ \sum_{i<j} c_{ijj} E_i + \sum_{i<j<k} c_{ijk} E_i \cdot E_j \cdot E_k \Bigg)$$
$$= -\frac{1}{2} \left(\sum_{i<j<k} (c_{jki} - c_{ikj} + c_{ijk}) E_i \cdot E_j \cdot E_k + \sum_{i \neq j} c_{ijj} E_j \right)$$

But we see

$$c_{ijj} = \frac{1}{2} \left(- \left\langle \overline{[X_j, X_j]_{\mathfrak{p}}}, \overline{X}_i \right\rangle + \left\langle \overline{[X_i, X_j]_{\mathfrak{p}}}, \overline{X}_j \right\rangle + \left\langle \overline{[X_i, X_j]_{\mathfrak{p}}}, \overline{X}_j \right\rangle \right)$$
$$= \left\langle \overline{[X_i, X_j]_{\mathfrak{p}}}, \overline{X}_j \right\rangle$$

So

$$\beta_i = \frac{1}{2} \sum_{j=1}^{n} \langle [X_j, X_i]_{\mathfrak{p}}, X_j \rangle = -\frac{1}{2} \sum_{j=1}^{n} c_{ijj}$$

By the same calculus, we have

$$c_{jki} - c_{ikj} + c_{ijk} = \frac{1}{2} \left(\left\langle \overline{[X_i, X_j]_{\mathfrak{p}}}, \overline{X}_k \right\rangle + \left\langle \overline{[X_j, X_k]_{\mathfrak{p}}}, \overline{X}_i \right\rangle + \left\langle \overline{[X_k, X_i]_{\mathfrak{p}}}, \overline{X}_j \right\rangle \right) = 2\alpha_{ijk}$$

The $\overline{X}$ became X by the isomorphism between $\mathfrak{p}$ and T_pM, and since we take the scalar product, we have $\langle \overline{X}, \overline{Y} \rangle = \langle X, Y \rangle$

So, taking all the results together, it gives

$$\mathcal{D}[g, \Psi(g)]|_{g=g_0} = \left[g_0, \sum_{k=1}^{n} E_k \cdot X_k|_{g_0}(\Psi) + \left(\sum_{i=1}^{n} \beta_i E_i + \sum_{1 \leq i<j<k \leq n} \alpha_{ijk} E_i \cdot E_j \cdot E_k \right) \cdot \Psi(g_0) \right]$$

□

As remarkable as this explicit formula is, we will not use it on a random spinor field. We will rather focus on the decomposition of $\Gamma(\Sigma M)$ into $\mathcal{D}$-invariant subspaces, given in Proposition 29, which will make the computations in the next subsections easier.

Definition 27. We denote by $\widehat{G}$ the set of equivalence classes of unitary irreducible finite dimensional complex representation of G. Any representative of $\widehat{G}$ is denoted by (ρ_γ, V_γ).

Theorem 28 (Frobenius reciprocity)**.** The unitary representation splits into the Hilbert sum

$$L^2_H(G, \Sigma_n) = \underset{\gamma \in \widehat{G}}{\oplus} (V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)) \tag{10}$$

Proposition 29. The Dirac operator $\mathcal{D}$ leaves invariant the space $V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)$, and

$$\mathcal{D}_{V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)} = Id \otimes \mathcal{D}_\gamma$$

where

$$\mathcal{D}_\gamma : Hom_H(V_\gamma, \Sigma_n) \to Hom_H(V_\gamma, \Sigma_n)$$
$$A \mapsto -\sum_{k=1}^{n} E_k \cdot A \cdot (\rho_\gamma)_*(X_k) + \left(\sum_{i=1}^{n} \beta_i E_i + \sum_{1 \leq i<j<k \leq n} \alpha_{ijk} E_i \cdot E_j \cdot E_k \right) \cdot A$$

Proof. Let $v \otimes A \in (V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n))$. For more clarity, let

$$L := \sum_{i=1}^{n} \beta_i E_i + \sum_{1\leq i<j<k\leq n} \alpha_{ijk} E_i \cdot E_j \cdot E_k$$

and let

$$\begin{aligned} \Psi : M &\to \Sigma_n \\ [g] &\mapsto A\rho_\gamma(g^{-1})v \end{aligned}$$

From Theorem 26, we know that

$$\mathcal{D}[g, \Psi(g)]|_{g=g_0} = \left[g_0, \sum_{k=1}^{n} E_k \cdot X_k|_{g_0}(\Psi) + L \cdot \Psi(g_0) \right]$$

We have

$$\begin{aligned} X_k|_{g_0}(\Psi) &= \frac{d}{dt} \Psi \left(g_0 e^{tX_k} \right) |_{t=0} \\ &= \frac{d}{dt} A\rho_\gamma \left(e^{-tX_k} g_0^{-1} \right) v|_{t=0} \\ &= -A(\rho_\gamma)_*(X_k)\rho_\gamma(g_0^{-1})v \end{aligned}$$

Moreover, $L \cdot \Psi(g_0) = L \cdot A\rho_\gamma(g_0^{-1})v$.
So we have :

$$\mathcal{D}[g, \Psi(g)]|_{g=g_0} = \left[g_0, \left(-\sum_{k=1}^{n} E_k \cdot A(\rho_\gamma)_*(X_k) + L \cdot A \right) \rho_\gamma(g_0^{-1})v \right]$$

Thus, we see that

$$\mathcal{D}A = \left(-\sum_{k=1}^{n} E_k \cdot A(\rho_\gamma)_*(X_k) + L \cdot A \right)$$

which is exactly the proposition. □

2.3 Spectrum on quotient of the 3-sphere

2.3.1 On the Berger sphere $S^3(T)$.

Let's apply the results obtained previously on concrete examples, namely on quotient of the 3-sphere. In the spirit of Theorem 28 and Proposition 29, the work now is to find the irreducible representations of the subgroup H in order to determine the subspaces $Hom_H(V_\gamma, \Sigma_n)$, and to compute the Dirac operator on those spaces.

Consider the group SU_2, which have a double role in the following : it is the group G in the space $M = {}^{G}\!/_{H}$, and it is also the spin group $Spin_3 = SU_2$ (see section 1.2). We take H finite, so $\mathfrak{h} = 0$ and $\mathfrak{g} = \mathfrak{p} = \mathfrak{su}_2$

We define the inner product on $\mathfrak{p}$ by declaring the basis

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \frac{1}{T} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{11}$$

to be orthonormal, where $T > 0$ is a fixed parameter. The Lie algebra structure is given by

$$[X_1, X_2] = 2TX_3,\ [X_2, X_3] = \frac{2}{T}X_1,\ [X_3, X_1] = \frac{2}{T}X_2$$

By definition of β_i and α_{ijk}, we have

$$\beta_1 = \beta_2 = \beta_3 = 0 \text{ and } \alpha_{123} = \frac{T}{2} + \frac{1}{T}$$

The spinor space Σ_3 is a 2-dimensional complex vector space ($2^{\lfloor \frac{3}{2} \rfloor} = 2$). We choose a basis $\{Z_1, Z_2\}$ of Σ_3 such that the Clifford multiplication by the standard basis of $\mathbb{R}^3$ is given by the following matrices :

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\ E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with $E_1E_2E_3 = 1$. It means that, if $z = (z_1, z_2) \in \Sigma_3$ (in the basis $\{Z_1, Z_2\}$), then $\rho(e_i)(z) = E_i \cdot z$

The irreducible unitary representations of SU_2 are given by (π_n, V_n) where V_n is the vector space of all complex homogeneous polynomials P of degree n in two variables z_1, z_2. The group SU_2 acts via

$$\pi_n(g)P(z) = P(zg)$$

for $P \in V_n$, $g \in SU_2$ and $z = (z_1, z_2)$. We will use the polynomials $P_k = z_1^{n-k} z_2^k$, with $k = 0, \ldots, n$ as a basis of V_n.

We start by considering the case that H is trivial. Then $M = SU_2$ is diffeomorphic to S^3, and we denote the resulting homogeneous Riemannian manifold by $S^3(T)$. We will calculate $\mathcal{D}_n$ which is acting on $Hom(V_n, \Sigma_3)$. From Proposition 29, since $\beta_1 = \beta_2 = \beta_3 = 0$ and $\alpha_{123} = \frac{T}{2} + \frac{1}{T}$, we have

$$\mathcal{D}_n A = -\sum_{k=1}^{n} E_k \cdot A \cdot (\pi_n)_*(X_k) - \left(\frac{T}{2} + \frac{1}{T}\right) A$$

Since the term $\left(\frac{T}{2} + \frac{1}{T}\right)$ only shift the spectrum, we will focus on the operator

$$\mathcal{D}'_n A = -\sum_{k=1}^{n} E_k \cdot A \cdot (\pi_n)_*(X_k)$$

First of all, let's calculate the endomorphisms $(\pi_n)_*(X_k)$.

$$\begin{aligned}
\left(\pi_{n^*}(X_1)P_k\right)(z) &= \left(\frac{d}{ds}(id + sX_1)P_k|_{s=0}\right)(z) \\
&= \frac{d}{ds} P_k \left((z_1, z_2) \begin{pmatrix} 1 & is \\ is & 1 \end{pmatrix} \right) |_{s=0} \\
&= \frac{d}{ds}(z_1 + isz_2)^{n-k}(isz_1 + z_2)^k|_{s=0} \\
&= i(n-k) z_1^{n-k-1} z_2^{k+1} + ikz_1^{n-k+1} z_2^{k-1} \\
&= i(n-k)P_{k+1}(z) + ikP_{k-1}(z)
\end{aligned}$$

So we have :

$$\left(\pi_{n^*}(X_1)P_k\right)(z) = i(n-k)P_{k+1}(z) + ikP_{k-1}(z) \tag{12}$$

Therefore, we can represent $\pi_{n^*}(X_1)$ by the matrix

$$\pi_{n^*}(X_1) = i \begin{bmatrix} 0 & 1 & 0 & & & \\ n & 0 & 2 & & 0 & \\ 0 & n-1 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & n \\ & & & & 1 & 0 \end{bmatrix}$$

By the same calculus, we find

$$\pi_{n^*}(X_2) = \begin{bmatrix} 0 & -1 & 0 & & & \\ n & 0 & -2 & & 0 & \\ 0 & n-1 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & -n \\ & & & & 1 & 0 \end{bmatrix}$$

and

$$\pi_{n^*}(X_3) = \frac{i}{T} \begin{bmatrix} n & & & & \\ & n-2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & -n \end{bmatrix}$$

Now that we have a formula for $(\pi_n)_*(X_k)$, let's finish the computation of $D'_n A$.

Since V_n and Σ_3 are complex vector spaces of dimension $n+1$ and 2 respectively, $Hom(V_n, \Sigma_3)$ is a complex vector space of dimension $2(n+1)$, for which we consider the following basis :

$$A_k(P_l) = \begin{cases} Z_1 & \text{if } k = l, \ k \text{ even} \\ Z_2 & \text{if } k = l, \ k \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$B_k(P_l) = \begin{cases} Z_1 & \text{if } k = l, \ k \text{ odd} \\ Z_2 & \text{if } k = l, \ k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

for $k = 0, \cdots, n$

We will compute $\mathcal{D}'_n A_k$ and $\mathcal{D}'_n B_k$, for k even and k odd separately.

- For k even, we get by equation (12) :

$$E_1 \cdot A_k \cdot (\pi_n)_*(X_1)(P_l) = E_1 \cdot A_k \left(i(n-l)P_{l+1} + ilP_{l-1} \right)$$

By definition of A_k, the only l for which this computation is non zero are $l = k \pm 1$. So we have for $l = k-1$ odd

$$\begin{aligned} E_1 \cdot A_k \cdot (\pi_n)_*(X_1)(P_{k-1}) &= E_1 \cdot A_k \left(i(n-k+1)P_k + i(k-1)P_{k-2}\right) \\ &= E_1 \cdot A_k \left(i(n-k+1)P_k\right) = i(n-k+1)E_1 \cdot Z_1 \\ &= i(n-k+1)(iZ_2) = (k-n-1)Z_2 = (k-n-1)A_{k-1}(P_{k-1}) \end{aligned}$$

By the same reasoning, we get for $l = k+1$

$$E_1 \cdot A_k \cdot (\pi_n)_*(X_1)(P_{k+1}) = -(k+1)Z_2 = -(k+1)A_{k+1)(P_{k+1}}$$

so, considering $A_k = 0$ for $k < 0$ and $k > n$, we have

$$E_1 \cdot A_k \cdot (\pi_n)_*(X_1) = (k-n-1)A_{k-1} - (k+1)A_{k+1}$$

and we also have, by the same computations

$$\begin{aligned} E_2 \cdot A_k \cdot (\pi_n)_*(X_2) &= (n-k+1)A_{k-1} - (k+1)A_{k+1} \\ E_3 \cdot A_k \cdot (\pi_n)_*(X_3) &= \frac{1}{T}(2k-n)A_k \end{aligned}$$

Therefore, putting together these results, we get

$$\mathcal{D}'_n A_k = -\sum_{k=1}^{n} E_k \cdot A_k \cdot (\pi_n)_*(X_k) = \frac{1}{T}(n-2k)A_k + 2(k+1)A_{k+1} \tag{13}$$

- <u>If k is odd</u>, we obtain

$$\begin{aligned} E_1 \cdot A_k \cdot (\pi_n)_*(X_1) &= (k-n-1)A_{k-1} - (k+1)A_{k+1} \\ E_2 \cdot A_k \cdot (\pi_n)_*(X_2) &= (k-n-1)A_{k-1} + (k+1)A_{k+1} \\ E_3 \cdot A_k \cdot (\pi_n)_*(X_3) &= \frac{1}{T}(n-2k)A_k \end{aligned}$$

and therefore

$$\mathcal{D}'_n A_k = -\sum_{k=1}^{n} E_k \cdot A_k \cdot (\pi_n)_*(X_k) = 2(n+1-k)A_{k-1} + \frac{1}{T}(2k-n)A_k \tag{14}$$

Moreover, since we will have the same computations for B_k, with the cases k even and k odd interchanged, we have

$$\begin{aligned} \mathcal{D}'_n B_k &= 2(n+1-k)B_{k-1} + \frac{1}{T}(2k-n)B_k \quad \text{if } k \text{ even} \\ \mathcal{D}'_n B_k &= \frac{1}{T}(n-2k)B_k + 2(k+1)B_{k+1} \quad \text{if } k \text{ odd} \end{aligned}$$

Now that we have these explicit formulas for $\mathcal{D}'_n$, we focus on the eigenvalues.

We see that the subspace of $Hom(V_n, \Sigma_3)$ spanned by the A_k is $\mathcal{D}'_n$-invariant, as well as the subspace spanned by the B_k.

- Considering the endomorphism $\mathcal{D}'_n$ on the subspace spanned by the A_k,, and with respect to this basis, it can be represented by a block-diagonal matrix with 2×2 blocks on the diagonal. The 2×2 block have the form

$$\begin{pmatrix} \frac{1}{T}(n-2l) & 2(n-l) \\ 2(l+1) & \frac{1}{T}(2l+2-n) \end{pmatrix}$$

$$l = 0, 2, 4, \cdots, \begin{cases} n-1 & \text{for } n \text{ odd} \\ n-2 & \text{for } n \text{ even} \end{cases}$$

In the case n even, there is one more 1×1 block, with entry $-\frac{n}{T}$, corresponding to (13) for $k = n$:

$$\mathcal{D}'_n A_n = \frac{1}{T}(n-2n)A_n + 2(n+1)A_{n+1} = -\frac{n}{T}A_n$$

(Recall that there is $n+1$ elements in the basis $\{A_0, \cdots, A_n\}$, so there is $n+1$ lines in the matrix corresponding to $\mathcal{D}_n$).

- In the subspace spanned by the B_k, we have also a block-diagonal matrix. First, we have a 1×1 block, with entry $-\frac{n}{T}$, corresponding to

$$\mathcal{D}'_n B_0 = 2(n+1-0)B_{-1} + \frac{1}{T}(0-n)B_0 = -\frac{n}{T}B_0$$

Then, since the computations are the same as for A_k, we have the same 2×2 blocks

$$\begin{pmatrix} \frac{1}{T}(n-2l) & 2(n-l) \\ 2(l+1) & \frac{1}{T}(2l+2-n) \end{pmatrix}$$

this time for

$$l = 1, 3, 5, \cdots, \begin{cases} n-1 & \text{for } n \text{ even} \\ n-2 & \text{for } n \text{ odd} \end{cases}$$

If n is odd, there is one more 1×1 block with entry $-\frac{n}{T}$, corresponding to (14) with $k = n$:

$$\mathcal{D}'_n B_n = \frac{1}{T}(n-2n)B_n + 2(n+1)B_{n+1} = -\frac{n}{T}B_n$$

Summing everything up, we have two cases :

- For n even :

$$\mathcal{D}'_n = \begin{pmatrix} \alpha_0 &&&&&&&&& \\ & \alpha_2 &&&&&&&& \\ && \ddots &&&&&&& \\ &&& \alpha_{n-2} &&&&&& \\ &&&& \alpha_n &&&&& \\ &&&&& \beta_0 &&&& \\ &&&&&& \beta_1 &&& \\ &&&&&&& \beta_3 && \\ &&&&&&&& \ddots & \\ &&&&&&&&& \beta_{n-1} \end{pmatrix} \tag{15}$$

where

$$\alpha_j = \begin{pmatrix} \frac{1}{T}(n-2j) & 2(n-j) \\ 2(j+1) & \frac{1}{T}(2j+2-n) \end{pmatrix} \text{ for } j = 0, 2, \cdots, n-2$$

$$\alpha_n = \beta_0 = -\frac{n}{T}$$

$$\beta_k = \begin{pmatrix} \frac{1}{T}(n-2k) & 2(n-k) \\ 2(k+1) & \frac{1}{T}(2k+2-n) \end{pmatrix} \text{ for } k = 1, 3, \cdots, n-1$$

- <u>For n odd</u> :

$$\mathcal{D}'_n = \begin{pmatrix} \alpha_0 & & & & & & & & & \\ & \alpha_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \alpha_{n-1} & & & & & & \\ & & & & \beta_0 & & & & & \\ & & & & & \beta_1 & & & & \\ & & & & & & \beta_3 & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \beta_{n-2} & \\ & & & & & & & & & \beta_n \end{pmatrix} \tag{16}$$

where

$$\alpha_j = \begin{pmatrix} \frac{1}{T}(n-2j) & 2(n-j) \\ 2(j+1) & \frac{1}{T}(2j+2-n) \end{pmatrix} \text{ for } j = 0, 2, \cdots, n-1$$

$$\beta_0 = \beta_n = -\frac{n}{T}$$

$$\beta_k = \begin{pmatrix} \frac{1}{T}(n-2k) & 2(n-k) \\ 2(k+1) & \frac{1}{T}(2k+2-n) \end{pmatrix} \text{ for } k = 1, 3, \cdots, n-2$$

Seeing $\mathcal{D}'_n$ under the form of a matrix, it is easier to compute its eigenvalues.

First, we see that in the matrices, there are two 1×1 block with entry $-\frac{n}{T}$, which give two times the eigenvalue $-\frac{n}{T}$.

The 2×2 block

$$\begin{pmatrix} \frac{1}{T}(n-2l) & 2(n-l) \\ 2(l+1) & \frac{1}{T}(2l+2-n) \end{pmatrix}$$

appear for $l = 0, 1, 2, \cdots, n-1$. The eigenvalues of this block are

$$\lambda_{1,2} = \frac{1}{T} \pm \left[\left(\frac{1}{T^2} - 1 \right) (n-2l+1)^2 + (n+1)^2 \right]^{\frac{1}{2}}$$

It is then easy to determine the eigenvalues of $\mathcal{D}_n$, it suffices to add the term $-\left(\frac{1}{T} + \frac{1}{2}T\right)$ to obtain the eigenvalues.

Eigenvalue	Multiplicity
$-\frac{n+1}{T}-\frac{1}{2}T$	$2(n+1)$, $n=0,1,2,\ldots$
$-\frac{1}{2}T \pm \left[\left(\frac{1}{T^2}-1\right)(n-2l+1)^2+(n+1)^2\right]^{\frac{1}{2}}$	$n+1$, $n=1,2,\ldots$ and $l=0,1,\ldots,n-1$

The term $(n+1)$ in each multiplicity comes from $\dim(V_n)=(n+1)$. Indeed, we saw that on $V_n \otimes Hom_H(V_n,\Sigma_3)$, we have $\mathcal{D}=Id\otimes\mathcal{D}_n$. So if λ is an eigenvalue of $\mathcal{D}_n$ of multiplicity μ, then λ is an eigenvalue of $\mathcal{D}$ of multiplicity $\mu\dim(V_n)=\mu(n+1)$.

To be more clear, let $m=n+1$ and $k=l+1$. Then, we have the following theorem.

Theorem 30. On the sphere $S^3(T)$, endowed with the metric coming from (11), the Dirac operator $\mathcal{D}$ has the following eigenvalues :

$-\dfrac{m}{T}-\dfrac{T}{2}$	with multiplicity $2m$, for $m=1,2,\ldots$
$-\dfrac{1}{2}T \pm \left[\left(\dfrac{1}{T^2}-1\right)(m-2k)^2+m^2\right]^{\frac{1}{2}}$	with multiplicity m, for $k=1,2,\ldots,m-1$ and $m=2,3,\ldots$

2.3.2 On lens spaces $\mathscr{L}(N,T)$

We can also find the eigenvalues of the Dirac operator on lens spaces, that is quotient of the sphere

$$\mathscr{L} = {S^{2m-1}}\diagup{\Gamma}$$

where Γ is a subgroup of SO_{2m}. A general study of the Dirac operator on those spaces can be found in [3].

Here, we will focus on quotients of the 3-sphere of the form

$$\mathscr{L}(N,T) = {S^3(T)}\diagup{\mathbb{Z}_N} = {SU_2}\diagup{\mathbb{Z}_N}$$

(that is, we take $H=\mathbb{Z}_N$) where the embedding of $\mathbb{Z}_N$ in SU_2 is

$$e^{it} \mapsto \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

Recall that we chose a basis of SU_2 :

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_3 = \frac{1}{T}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

We saw that the spin structures are in $1-1$ correspondence with the lifts of the homomorphism

$\alpha : H \to SO_n$ (see Theorem 23). Take a look at the isotropy representation of $\mathbb{Z}_N$

$$
\begin{aligned}
Ad_{e^{it}}X_1 &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} 0 & ie^{2it} \\ ie^{-2it} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i\cos(2t) - \sin(2t) \\ i\cos(-2t) - \sin(-2t) & 0 \end{pmatrix} = \cos(2t)X_1 + \sin(2t)X_2 \\
Ad_{e^{it}}X_2 &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} = \begin{pmatrix} 0 & -e^{2it} \\ e^{-2it} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\cos(2t) - i\sin(2t) \\ \cos(-2t) + i\sin(-2t) & 0 \end{pmatrix} = -\sin(2t)X_1 + \cos(2t)X_2 \\
Ad_{e^{it}}X_3 &= \frac{1}{T} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \\
&= \frac{1}{T} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = X_3
\end{aligned}
$$

Summing up, the isotropy representation $\mathbb{Z}_N \to SO(3)$ is given by

$$
e^{it} \mapsto \begin{pmatrix} \cos(2t) & -\sin(2t) & 0 \\ \sin(2t) & \cos(2t) & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Then, by identifying $Spin_3$ with SU_2, we can lift the isotropy representation, that is

$$
\begin{aligned}
\mathbb{Z}_N &\to SU_2 \simeq Spin_3 \\
e^{it} &\mapsto \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}
\end{aligned}
$$

As in the previous computations, we want to determine the space

$$
Hom_{\mathbb{Z}_N}(V_n, \Sigma_3) \subset Hom(V_n, \Sigma_3)
$$

So we want to know how $\mathbb{Z}_N$ acts on V_n and on Σ_3. For that, we decompose the spinor representation and the representations π_n, restricted to $\mathbb{Z}_N$, into $\mathbb{Z}_N$-irreducible components.

First, the irreducible representations of $\mathbb{Z}_N$ are

$$
\begin{aligned}
\rho_m : \mathbb{Z}_N &\to U_1 \\
z &\mapsto z^m
\end{aligned}
$$

Two such representations ρ_m and ρ_p are equivalent if and only if $m \equiv p \mod N$.

Moreover, since the spinor representation of $Spin_3 \simeq SU_2$ is exactly the standard representation of SU_2 (since for the standard representation, the image of $-Id$ is $-Id \neq Id$), we get for its restriction to $\mathbb{Z}_N$ the representation $\rho_1 \otimes \rho_{-1}$. Indeed the standard representation of SU_2 is

$$
\begin{aligned}
SU_2 &\to GL(\mathbb{C}^2) \\
A &\mapsto A
\end{aligned}
$$

Then its restriction to $\mathbb{Z}_N$ is given by :

$$\begin{aligned}\mathbb{Z}_N &\to GL(\mathbb{C}^2)\\ e^{it} &\mapsto \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{pmatrix} = \begin{pmatrix} \rho_1(e^{it}) & 0 \\ 0 & \rho_{-1}(e^{it})\end{pmatrix}\end{aligned}$$

So we see that the spinor representation, restricted to $\mathbb{Z}_N$, is $\rho_1 \otimes \rho_{-1}$.

Finally, the action of $\mathbb{Z}_N$ on V_n is

$$\begin{aligned}\left(e^{it}\cdot P_k\right)(z) &= P_k\left((z_1,z_2)\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{pmatrix}\right)\\ &= P_k(e^{it}z_1, e^{-it}z_2) = e^{(n-k)it}z_1^{n-k}e^{-kit}z_2^k\\ &= e^{(n-2k)it}P_k(z) = \rho_{n-2k}(e^{it})P_k(z)\end{aligned}$$

Thus, $\mathbb{Z}_N$ acts on $\mathbb{C}\times P_k$ via ρ_{n-2k}. So, in order to calculate $Hom_{\mathbb{Z}_N}(V_n,\Sigma_3)$, we have to find the homomorphisms $A : V_n \to \Sigma_3$ such that

$$\begin{aligned} e^{it}\cdot A(P_k) &= A\left(e^{it}\cdot P_k\right)\\ \begin{pmatrix} \rho_1(e^{it}) & 0 \\ 0 & \rho_{-1}(e^{it})\end{pmatrix} A(P_k) &= A\left(\rho_{n-2k}(e^{it})P_k\right)\end{aligned}$$

where the dot $\cdot$ in the first line denotes the action of e^{it} on respectively Σ_3 and V_n.

We will search such A in the basis $\{A_0,\dots,A_n,B_0,\dots,B_n\}$. By definition $A_k(P_l)\neq 0$ and $B_k(P_l)\neq 0$ if and only if $k=l$. So we have to find for which k, ρ_1 and ρ_{-1} are sent to ρ_{n-2k}, or at least to equivalent representations.

We will start by studying the case $N=2$ and then the case $N\geq 3$.

The case $N=2$

When $N=2$, the lens space $\mathscr{L}(2,T) = {S^3}/{\mathbb{Z}_2}$ is just the real projective space $\mathbb{R}P^3$. Also, the representation ρ_1 and ρ_{-1} are equivalent. So we just have to determine k such that $n-2k\equiv 1 \mod 2$. But since $2k$ is even for all k, $n-2k\equiv 1 \mod 2$ if and only if n is odd.

So, if n is even, there is not any k for which $n-2k\equiv 1 \mod 2$ holds. It means that $Hom_{\mathbb{Z}_N}(V_n,\Sigma_3)=0$.

If n is odd, then for all k, we have $n-2k\equiv 1 \mod 2$. So $Hom_{\mathbb{Z}_N}(V_n,\Sigma_3) = Hom(V_n,\Sigma_3)$.

If we look at Theorem 30, the eigenvalues of the Dirac operator on the lens space $\mathscr{L}(2,T)$ are the eigenvalues with $m=n+1$ even, that is $m=2m'$.

Proposition 31. The eigenvalues of the Dirac operator on the real projective space $\mathbb{R}P^3$ $(=\mathscr{L}(2,T))$ are

Eigenvalue	Multiplicity
$-\dfrac{2m'}{T}-\dfrac{1}{2}T$	$4m'$, $m'=1,2,\dots$
$-\dfrac{1}{2}T \pm 2\left[\left(\dfrac{1}{T^2}-1\right)(m'-k)^2+m'^2\right]^{\frac{1}{2}}$	$2m'$, $m'=1,2,\dots$ and $k=1,\dots,2m'-1$

The case $N \geq 3$

For $N \geq 3$, the representations ρ_1 and ρ_{-1} are not equivalent. We first focus on the subspace spanned by the A_l. For l even, we have

$$A_l\left(e^{it} \cdot P_l\right) = A_l\left(e^{(n-2l)it} P_l\right) = e^{(n-2l)it} A_l(P_l) = e^{(n-2l)it} Z_1$$
$$e^{it} \cdot A_l(P_l) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} Z_1 = e^{it} Z_1$$

We see that for l even, A_l is $\mathbb{Z}_N$-invariant if and only if $n - 2l \equiv 1 \mod N$. By the same calculus, we can see that for l odd, A_l is $\mathbb{Z}_N$-invariant if and only if $n - 2l \equiv -1 \mod N$.

In the case l even, and $n - 2l \equiv 1 \mod N$, we have then $n - 2(l+1) \equiv -1 \mod N$. It means that for l even, if A_l is $\mathbb{Z}_N$-invariant then A_{l+1} also is $\mathbb{Z}_N$-invariant.

So, if we look back at the blocks we found in our study of the Berger sphere $S^3(T)$ (see (15) and (16)), a 2×2 block α_l

$$\begin{pmatrix} \frac{1}{T}(n-2l) & 2(n-l) \\ 2(l+1) & \frac{1}{T}(2l+2-n) \end{pmatrix}$$

will "survive" if and only if $n - 2l \equiv 1 \mod N$, with l even, and so the Dirac operator will admit the corresponding eigenvalues.

The 1×1 block α_n with entry $-\frac{n}{T}$, in the case n even, survives if and only if $n - 2n \equiv 1 \mod N$, that is $n \equiv -1 \mod N$.

If we take $m = n+1$ and $k = l+1$ (as we did in Theorem 30), the condition for the 2×2 blocks becomes $m - 2k \equiv 0 \mod N$, and for the 1×1 block $m \equiv 0 \mod N$, m odd.

If we make the same reasoning for the subspace spanned by the B_l, then we obtain that the 2×2 blocks β_l survives if and only if $n - 2l \equiv 1 \mod N$, for l odd. The first 1×1 block β_0 remains if $n \equiv 1 \mod N$, and the second 1×1 block β_n if $n \equiv 1 \mod N$, for n odd. Again, by setting $m = n+1$ and $k = l+1$, we have the 2×2 blocks if $m - 2k \equiv 0 \mod N$, the first 1×1 block if $m \equiv 0 \mod N$, and the second one if $m \equiv 0 \mod N$, m even.

Finally, we get to the result. Putting everything together, we obtain that the 2×2 blocks remain if $m - 2k \equiv 0 \mod N$, and two 1×1 blocks if $m \equiv 0 \mod N$.

Proposition 32. The eigenvalues of the Dirac operator on the lens space $\mathscr{L}(N,T)$ for $N \geq 3$ are

Eigenvalue	Multiplicity
$-\frac{iN}{T} - \frac{1}{2}T$	$2iN$, with $i = 1, 2, \ldots$
$-\frac{1}{2}T \pm 2\left[\left(\frac{1}{T^2} - 1\right)i^2N^2 + m^2\right]^{\frac{1}{2}}$	m, with $m = 2, 3, \ldots$ and $-\frac{m+1}{N} < i \leq \frac{m-2}{N}$

The case N even, $N = 2N'$.

For $N = 2N'$ even, $\mathscr{L}(N,T)$ admits one more spin structure, corresponding to the lift

$$\begin{aligned}
&\mathbb{Z}_N \to SU_2\\
&e^{\frac{2i\pi q}{N}} \mapsto (-1)^q \begin{pmatrix} e^{\frac{2i\pi q}{N}} & 0 \\ 0 & e^{-\frac{2i\pi q}{N}} \end{pmatrix} = \begin{pmatrix} e^{\frac{2i\pi q}{N}} e^{\frac{i\pi q N}{N}} & 0 \\ 0 e^{-\frac{2i\pi q}{N}} e^{\frac{i\pi q N}{N}} & \end{pmatrix}\\
&\quad = \begin{pmatrix} e^{\frac{2i\pi q(N'+1)}{N}} & 0 \\ 0 & e^{\frac{2i\pi q(N'-1)}{N}} \end{pmatrix}
\end{aligned}$$

This map is not well-defined if N is odd. Indeed, we have $e^{\frac{2i\pi 0}{N}} = e^{\frac{2i\pi N}{N}}$, but

$$\begin{aligned}
e^{\frac{2i\pi 0}{N}} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\
e^{\frac{2i\pi N}{N}} &\mapsto (-1)^N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

We see that $\mathbb{Z}_N$ acts on Σ_3 via $\rho_{N'+1} \otimes \rho_{N'-1}$. If we make the same computations as in the previous case, we show that

- For $N = 2$, the eigenvalues of $\mathscr{L}(2,T) = \mathbb{R}P^3$ are the eigenvalues of $S^3(T)$ for m odd.
- For $N = 2N'$, $N' \geq 2$. In this case, $\rho_{N'-1}$ and $\rho_{N'+1}$ are not equivalent. The eigenvalues come from the 2×2 blocks with $m - 2k \equiv N' \mod N$, and from the 1×1 block with $m \equiv N' \mod N$.

All these results are summarized in the following theorems.

Theorem 33. The real projective space $\mathbb{R}P^3$ $(= \mathscr{L}(2,T))$ has exactly two spin structures. With respect to the first one, the eigenvalues of the Dirac operator are

$-\frac{2m'}{T} - \frac{1}{2}T$	with multiplicity $4m'$, where $m' = 1, 2, \dots$
$-\frac{1}{2}T \pm 2\left[\left(\frac{1}{T^2} - 1\right)(m'-k)^2 + m'^2\right]^{\frac{1}{2}}$	with multiplicity $2m'$, where $m' = 1, 2, \dots$ and $k = 1, \dots, 2m'-1$

With respect to the second spin structure, the eigenvalues of the Dirac operator are

$-\frac{2m'+1}{T} - \frac{1}{2}T$	with multiplicity $2(2m'+1)$, where $m' = 0, 1, 2, \dots$
$-\frac{1}{2}T \pm \left[\left(\frac{1}{T^2} - 1\right)(2m'+1-2k)^2 + (2m'+1)\right]^{\frac{1}{2}}$	with multiplicity $2m'+1$, where $m' = 1, 2, \dots$ and $k = 1, 2, \dots, 2m'$.

Theorem 34. The lens space $\mathscr{L}(N,T)$, for $N \geq 3$, has one spin structure, for which the Dirac operator admits the following eigenvalues :

$-\dfrac{iN}{T}-\dfrac{1}{2}T$	with multiplicity $2iN$, where $i=1,2,\dots$
$-\dfrac{1}{2}T\pm\left[\left(\dfrac{1}{T^2}-1\right)i^2N^2+m^2\right]^{\frac{1}{2}}$	with multiplicity m, where $m=2,3,\dots$ and $-\dfrac{m+1}{N}<i\leq\dfrac{m-2}{N}$

If $N=2N'$ even, $\mathscr{L}(N,T)$ has one more spin structure, for which the Dirac operator admits the following eigenvalues :

$-\dfrac{N'+iN}{N}-\dfrac{1}{2}T$	with multiplicity $2(N'+iN)$, where $i=0,1,\dots$
$-\dfrac{1}{2}T\pm\left(\left(\dfrac{1}{T^2}-1\right)(N'+iN)^2+m^2\right)^{\frac{1}{2}}$	with multiplicity m, where $m=2,3,\dots$ and $\dfrac{(1-m-N')}{N}<i\leq\dfrac{(m-2-N')}{N}$

3 Computations on more generic homogeneous spaces

In the previous section, we calculate the eigenvalues for some homogeneous spaces $M = {}^{SU_2}\!/_{H}$, with $H = \mathbb{Z}_N$, or with $H = 0$, so $M = S^3(T)$ and M is diffeomorphic to S^3. But this computations were made with a specific metric, which change the eigenvalues of the Dirac operator. In the following, we will give the eigenvalues on the n-sphere, with the induced metric of $\mathbb{R}^{n+1}$.

We start with a very important result about the symmetry of the set of eigenvalues of the Dirac operator, and the determination of its spectrum by its square. The following proposition plays a major role in the calculus that will follow.

Proposition 35. The spectrum of $\mathcal{D}$ is symmetric with respect to the origin. Furthermore, it is completely determined by the spectrum of $\mathcal{D}^2$

Proof. Let λ be an eigenvalue of $\mathcal{D}$, and Ψ a non trivial function in $C^\infty_H(G, \Sigma_n) \simeq \Gamma(\Sigma M)$ such that $\mathcal{D}\Psi = \lambda\Psi$. Consider the function $\sigma^*\Psi$, where σ is the involutive automorphism of G defining the symmetric structure (see Theorem 20). Since $X_i \in \mathfrak{p}$, we know that

$$\sigma\left(e^{tX_i}\right) = e^{t\sigma_*(X_i)} = e^{-tX_i}$$

It follows

$$\begin{aligned}
\widetilde{X}_i(\sigma^*\Psi)(g) &= \frac{d}{dt}\left(\psi\Big(\sigma(ge^{tX_i})\Big)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\psi\Big(\sigma(g)\sigma(e^{tX_i})\Big)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\psi\Big(\sigma(g)e^{-tX_i}\Big)\right)\Big|_{t=0} \\
&= -\widetilde{X}_i(\Psi)(\sigma(g))
\end{aligned}$$

Hence

$$\mathcal{D}(\sigma^*\Psi)(g) = -\mathcal{D}\Psi(\sigma(g)) = -\lambda\Psi(\sigma(g)) = -\lambda\sigma^*\Psi(g)$$

So $\sigma^*\Psi$ is an eigenfunction for the eigenvalue $-\lambda$. It means that the spectrum of $\mathcal{D}$ is symmetric with respect to the origin.

Furthermore, by Proposition 29, $\mathcal{D}$ leaves invariant the space $V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)$, and so does $\mathcal{D}^2$. Thus we have

$$\mathcal{D}^2|_{V_\gamma \otimes Hom_H(V_\gamma,\Sigma_n)} = Id \otimes \mathcal{D}^2_\gamma$$

Thus

$$\mathrm{Spec}(\mathcal{D}^2) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}\left(\mathcal{D}^2|_{V_\gamma\otimes Hom_H(V_\gamma,\Sigma_n)}\right) = \bigcup_{\gamma\in\widehat{G}} \mathrm{Spec}(\mathcal{D}^2_\gamma)$$

We admit that $\mathcal{D}$ is a formally self-adjoint operator, so its restriction to any non-trivial space $V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)$ is a Hermitian operator on the finite dimensional Hermitian space

$$\left(V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n), \langle.|.\rangle_{V_\gamma\otimes Hom_H(V_\gamma,\Sigma_n)}\right)$$

hence, it is diagonalizable. So there exists an orthonormal basis of $V_\gamma \otimes Hom_H(V_\gamma, \Sigma_n)$ which diagonalizes $\mathcal{D}|_{V_\gamma\otimes Hom_H(V_\gamma,\Sigma_n)}$; but it also diagonalizes $\mathcal{D}^2|_{V_\gamma\otimes Hom_H(V_\gamma,\Sigma_n)}$.

This leads to the conclusion that

$$\mathrm{Spec}(\mathcal{D}) = \bigcup_{\gamma \in \widehat{G}} \left\{ \pm\sqrt{\mu}; \mu \in \mathrm{Spec}(\mathcal{D}^2) \right\}$$

□

3.1 The eigenvalues on the classical spherical 3-sphere.

Theorem 36. The classical Dirac operator on the sphere S^n of constant sectional curvature 1 has the eigenvalues

$$\pm\left(\frac{n}{2} + k\right) \;\; k \leq 0$$

with multiplicities

$$2^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1}{k}$$

Proof. This computation of the eigenvalues is different from the one above for $S^3(T)$, since here we don't have the homogeneous space structure. We will use instead Killing spinors.

A Killing spinor with Killing constant μ is a spinor field Ψ satisfying the equation

$$\widetilde{\nabla}_X \Psi = \nabla_X \Psi - \mu X \cdot \Psi = 0$$

for all tangent vectors X. Killing spinors are very useful in this context, because of the following result, which we will admit :

Proposition 37. The spinor bundle ΣS^n can be trivialized by Killing spinors for $\mu = \frac{1}{2}$ and $\mu = -\frac{1}{2}$.

So, from now on, let $\mu = \pm\frac{1}{2}$ Now, we will prove the Lichnerowicz formula, which relates the connection $\widetilde{\nabla}$ over the sphere to the Dirac operator :

Proposition 38.

$$(\mathcal{D} + \mu)^2 = \widetilde{\nabla} * \widetilde{\nabla} + \frac{1}{4}(n-1)^2$$

Proof. Let $p \in S^n$, let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame near p such that $\nabla_{e_i}(p) = 0$. At p, we get

$$\begin{aligned}
(\mathcal{D} + \mu)^2 - \widetilde{\nabla} * \widetilde{\nabla} &= \left(\sum_{i=1}^{n} e_i \nabla_{e_i} + \mu\right)\left(\sum_{j=1}^{n} e_j \nabla_{e_j} + \mu\right) + \sum_{j=1}^{n} \widetilde{\nabla}_{e_j} \widetilde{\nabla}_{e_i} \\
&= \sum_{1 \leq i,j \leq n} e_i \cdot e_j \nabla_{e_i} \nabla_{e_j} + 2\mu\mathcal{D} + \frac{1}{4} + \sum_{j=1}^{n} \left(\nabla_{e_j} - \mu e_j\right)\left(\nabla_{e_j} - \mu e_j\right) \\
&= \sum_{j=1}^{n} e_j \cdot e_j \nabla_{e_j} \nabla_{e_j} + \sum_{1 \leq i < j \leq n} e_i \cdot e_j \nabla_{e_i} \nabla_{e_j} + \sum_{i>j} e_i \cdot e_j \nabla_{e_i} \nabla_{e_j} + 2\mu\mathcal{D} + \frac{1}{4} \\
&\quad + \sum_{j=1}^{n} \nabla_{e_j} \nabla_{e_j} - 2\mu\mathcal{D} - \frac{1}{4} n
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{D}+\mu)^2 - \widetilde{\nabla} * \widetilde{\nabla} &= -\sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j} + \sum_{1\le i<j\le n} e_i e_j \left(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right) + \sum_{j=1}^{n} \nabla_{e_j}\nabla_{e_j} + \frac{1}{4} - \frac{n}{4} \\
&= \sum_{1\le i<j\le n} e_i \cdot e_j R^{\Sigma}(e_i, e_j) - \frac{n-1}{4} \\
&= \frac{1}{4}\sum_{1\le i<j\le n} e_i \cdot e_j \left(e_i \cdot e_j - e_j \cdot e_i\right) - \frac{1}{4}(n-1) \\
&= \frac{1}{4}\sum_{1\le i<j\le n} 2 - \frac{1}{4}(n-1) \\
&= \frac{1}{4}n(n-1) - \frac{1}{4}(n-1) = \frac{1}{4}(n-1)^2
\end{aligned}
$$

where R^{Σ} is the curvature of the spinor bundle, and

$$
R^{\Sigma}(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = \frac{1}{4}(YX - XY)
$$

□

The last equality comes from the constant curvature (see [4], Equation (5), p71).

Now, we choose an orthogonal basis $f_0 \equiv 1, f_1, \ldots,$ of the L^2-functions on S^n, $L^2(S^n, \mathbb{R})$, consisting of eigenfunctions of the Laplace operator $\Delta = d * d$, with $\Delta f_i = \lambda_i f_i$. We see that $f_i \Psi_j$ form a basis of the L^2-spinor fields $L^2(S^n, \Sigma S^n)$, where $\Psi_1, \ldots, \Psi_{2^{\lfloor \frac{n}{2} \rfloor}}$ are a trivialization of the spinor bundle by Killing spinors with Killing constant μ.

The following lemma tells us that we have found an eigenbasis for the operator $(\mathcal{D}+\mu)^2$

Lemma 39.

$$
(\mathcal{D}+\mu)^2 (f_i\Psi_j) = \left(\lambda_i + \frac{(n-1)^2}{4}\right) f_i\Psi_j
$$

Proof. From Proposition 38, we have

$$
(\mathcal{D}+\mu)^2 (f_i\Psi_j) = \left(\widetilde{\nabla} * \widetilde{\nabla} + \frac{1}{4}(n-1)^2\right)(f_i\Psi_j)
$$

We compute

$$
\begin{aligned}
\widetilde{\nabla}_{e_k} f_i \Psi_j &= \nabla_{e_k} f_i \Psi_j - \mu e_k f_i \Psi_j \\
&= df_i \Psi_j + f_i \nabla_{e_k}\Psi_j - \mu e_k f_i \Psi_j \\
&= df_i \Psi_j + f_i \mu e_k \Psi_j - \mu e_k f_i \Psi_j \quad \text{since } \Psi_j \text{ is a Killing spinor} \\
&= df_i \Psi_j
\end{aligned}
$$

So

$$
\widetilde{\nabla} * \widetilde{\nabla}(f_i\Psi_j) = d^2 f_i \Psi_j = \lambda_i f_i \Psi_j
$$

Thus, we have

$$
(\mathcal{D}+\mu)^2 (f_i\Psi_j) = \left(\lambda_i + \frac{(n-1)^2}{4}\right) f_i\Psi_j
$$

□

The eigenvalues of the Laplace operator are given in the following proposition, which we will admit.

Proposition 40. The eigenvalues of the Laplace operator on S^n are

$$k(n+k-1), \text{ for } k \geq 0, \text{ with multiplicity } m_k = \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}$$

This proposition leads to the following corollary

Corollary 41. $(\mathcal{D}+\mu)^2$ has the eigenvalues

$$k(n+k-1) + \frac{(n-1)^2}{4}, \text{ for } k \geq 0, \text{ with multiplicity } 2^{\lfloor \frac{n}{2} \rfloor} m_k$$

The next step is the calculation of the eigenvalues of $\mathcal{D}+\mu$. First, if an operator A has a vector u satisfying

$$A^2 u = \lambda^2 u$$

then for $v^\pm = \pm\lambda u + Au$, we have

$$Av^\pm = \pm A\lambda u + A^2 u = \pm\lambda Au + \lambda^2 u = \pm\lambda\left(\pm\lambda u + Au\right) = \pm\lambda v^\pm$$

Hence, if $v^\pm \neq 0$, then $\pm\lambda$ is an eigenvalue of A. In our case, $A = \mathcal{D}+\mu$. Let us first look at the case $k=0$, that is $u = \Psi_j$ and $\lambda = -\mu(n-1)$.

$$\begin{aligned}
v^+ &= -\mu(n-1)\Psi_j + (\mathcal{D}+\mu)\Psi_j \\
&= -\mu(n-1)\Psi_j + \sum_{i=1}^n e_i \nabla_{e_i} \Psi_j + \mu\Psi_j = -\mu(n-1)\Psi_j + \sum_{i=1}^n e_i \mu e_i \Psi_j + \mu\Psi_j \\
&= -\mu(n-1)\Psi_j - \mu\Psi_j \sum_{i=1}^n 1 + \mu\Psi_j = -\mu(n-1)\Psi_j - n\mu\Psi_j + \mu\Psi_j \\
&= -2\mu(n-1)\Psi_j
\end{aligned}$$

and by the same computation, $v^- = 0$. Hence $v^+ \neq 0$, so $\lambda = -\mu(n-1)$ is an eigenvalue of $\mathcal{D}+\mu$ of multiplicity at least $2^{\lfloor \frac{n}{2} \rfloor}$, since it is an eigenvalue for all Ψ_j, $j = 1, \ldots, 2^{\lfloor \frac{n}{2} \rfloor}$. Since the multiplicity of $\frac{(n-1)^2}{4}$ of $(\mathcal{D}+\mu)^2$ is $2^{\lfloor \frac{n}{2} \rfloor}$, the eigenvalue $-\mu(n-1)$ of $\mathcal{D}+\mu$ has exactly multiplicity $2^{\lfloor \frac{n}{2} \rfloor}$.

Now, we look at the case $k \geq 1$, that is

$$u = f_i \Psi_j \text{ for } i \geq 1, \text{ and } \lambda = \sqrt{k(n+k-1) + \frac{(n-1)^2}{4}} = k + \frac{n-1}{2}$$

We know all the eigenvalues of $\mathcal{D}$, namely $-\mu n$ with multiplicity $2^{\lfloor \frac{n}{2} \rfloor}$ (for $k=0$) and the other eigenvalues are $-\mu \pm (k + \frac{n-1}{2})$, for $k \geq 1$, and we must determine their multiplicity.

To do this, let us recall that we may choose $\mu = \pm\frac{1}{2}$. We start with $\mu = -\frac{1}{2}$. Then $-\mu + (k + \frac{n-1}{2}) = k + \frac{n}{2}$ and $-\mu - (k + \frac{n-1}{2}) = 1 - k - \frac{n}{2}$. We introduce the following notations for the eigenvalues of $\mathcal{D}$.

$$\lambda_0^+ = \frac{n}{2}, \;\; \lambda_k^+ = \frac{n}{2} + k, \;\; k \geq 1$$
$$\lambda_{-k}^+ = 1 - \frac{n}{2} - k, \;\; k \geq 1$$

We know that the multiplicity of λ_0^+ $(= -\mu n)$, namely $m\left(\frac{n}{2}\right) = 2^{\lfloor \frac{n}{2} \rfloor}$, and from the previous corollary, we know that $m\left(\lambda_k^+\right) + m\left(\lambda_{-k}^+\right) = 2^{\lfloor \frac{n}{2} \rfloor} m_k$.

The same reasoning applied to $\mu = \frac{1}{2}$ gives

$$\lambda_0^- = \frac{n}{2}, \quad \lambda_k^- = -1 + \frac{n}{2} + k, \quad k \geq 1$$
$$\lambda_{-k}^- = -\frac{n}{2} - k, \quad k \geq 1$$

and $m\left(-\frac{n}{2}\right) = 2^{\lfloor \frac{n}{2} \rfloor}$ and $m\left(\lambda_k^-\right) + m\left(\lambda_{-k}^-\right) = 2^{\lfloor \frac{n}{2} \rfloor} m_k$

Proposition 42. We have for $k \geq 0$

$$m\left(\lambda_k^+\right) = m\left(\lambda_{-k}^-\right) = 2^{\lfloor \frac{n}{2} \rfloor} \binom{k+n-1}{k}$$

Proof. We will prove this proposition by induction on k. We saw already that the claim is true for $k = 0$. For the induction step $k \to k+1$:

$$\begin{aligned}
m\left(\lambda_{k+1}^+\right) &= 2^{\lfloor \frac{n}{2} \rfloor} m_{k+1} - m\left(\lambda_{-k-1}^+\right) \\
&= 2^{\lfloor \frac{n}{2} \rfloor} m_{k+1} - m\left(\lambda_{-k}^-\right) \quad (\text{since } \lambda_{-k-1}^+ = \lambda_{-k}^-) \\
&= 2^{\lfloor \frac{n}{2} \rfloor} m_{k+1} - 2^{\lfloor \frac{n}{2} \rfloor} \binom{n+k-1}{k} \\
&= 2^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n+k}{k+1} \frac{n+2k+1}{n+k} - \binom{n+k-1}{k} \right) \\
&= 2^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n+k}{k+1} \frac{n+2k+1}{n+k} - \binom{n+k}{k+1} \frac{k+1}{n+k} \right) \\
&= 2^{\lfloor \frac{n}{2} \rfloor} \binom{n+k}{k+1}
\end{aligned}$$

and we do the same for $\lambda_{-(k+1)}^-$. $\square$

Summing up everything we get :

- $\dfrac{n}{2}$ is an eigenvalue of multiplicity $2^{\lfloor \frac{n}{2} \rfloor}$
- $\dfrac{n}{2} + k$ and $-\dfrac{n}{2} - k$, for $k \geq 1$, are eigenvalues of multiplicity $2^{\lfloor \frac{n}{2} \rfloor} \dbinom{n+k-1}{k}$.

which is what we wanted to prove. $\square$

3.2 The eigenvalues on lens spaces.

A more general result about the eigenvalues of the Dirac operator can be given on lens spaces, that is

$$M = {S^n}/{\Gamma}$$

where $n = 2m-1$ is odd, and $\Gamma \subset SO_{n+1}$ is a fixed point free subgroup.

The Dirac eigenvalues, according to the previous theorem, are of the form $\pm\left(\frac{n}{2} + k\right)$. The same holds for the lens space M, but the multiplicities will be smaller than those for S^n. Let's compute these.

Definition 43. Using the same notations as before for the multiplicity, let

$$F_+(z) = \sum_{k=0}^{+\infty} m\left(\frac{n}{2} + k, \mathcal{D}\right) z^k$$
$$F_-(z) = \sum_{k=0}^{+\infty} m\left(-\frac{n}{2} - k, \mathcal{D}\right) z^k$$

The Theorem 36 gives the following lemma.

Lemma 44. $F_+(z)$ and $F_-(z)$ converge absolutely for $|z| < 1$

Now, in even dimension $2m$, the complex spinor representation of $Spin_{2m}$ on Σ_{2m} decomposes into two irreducible half spin representations (see 13) :

$$\rho^+ : Spin_{2m} \to Aut(\Sigma_{2m}^+)$$
$$\rho^- : Spin_{2m} \to Aut(\Sigma_{2m}^-)$$

Let $\chi^\pm : Spin_{2m} \to \mathbb{C}$ be the character of $\rho^\pm$. The main result is

Theorem 45. Let ${S^{2m-1}}/{\Gamma}$ be a spherical lens space with spin structure given by

$$\varepsilon : \Gamma \to Spin_{2m}$$

Then the eigenvalue of the Dirac operator are $\pm\left(\frac{n}{2} + k\right)$, $k \geq 0$ with multiplicities determined by

$$F_+(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^-\left(\varepsilon(\gamma)\right) - z.\chi^+\left(\varepsilon(\gamma)\right)}{\det\left(1_{2m} - z.\gamma\right)}$$
$$F_-(z) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi^+\left(\varepsilon(\gamma)\right) - z.\chi^-\left(\varepsilon(\gamma)\right)}{\det\left(1_{2m} - z.\gamma\right)}$$

As a consequence, two lens spaces can be linked according to the power series they have, or we can obtain information on their structure, for example if it is homogeneous. If you are interested, read the article of S. Boldt [3].

Conclusion

This report shows that the study of the Dirac operator and the computation of its eigenvalues is easier on homogeneous spaces thanks to their specific structure : $1-1$ correspondence between spin structures and lifts of the isotropy representations, isomorphisms between $P_{SO_n}M$ ans $G \times_\alpha SO_n$, between ΣM and $G \times_{\widetilde{\rho}_n} \Sigma_n$ ... As we have seen, one can have an explicit formula of the operator, find the eigenvalues and, in some cases, the exact multiplicity of each eigenvalues, or in some other cases, the multiplicities under the form of a power series. Of course, one have to be careful of the chosen metric, which may shift the spectrum, and change the multiplicities.

Usually, the study of the eigenvalues of the Dirac operator is much more complicated, so one can hardly find the exact eigenvalues, but can determine bounds. For instance, on a compact Riemannian manifold, we still have the Schrödinger-Lichnerowicz formula (Proposition 38). So, if $\mathcal{D}\varphi = \lambda\varphi$, we get

$$\begin{aligned} \lambda^2\langle\varphi,\varphi\rangle = \langle\mathcal{D}^2\varphi,\varphi\rangle &= \left(\left(\nabla * \nabla + \frac{R}{4}\right)\varphi,\varphi\right) \\ &= \langle\nabla\varphi,\nabla\varphi\rangle + \frac{R}{4}\langle\varphi,\varphi\rangle \geq \frac{R}{4}\langle\varphi,\varphi\rangle \end{aligned}$$

where R denotes the Riemannian curvature. So if $R > S$, for some positive constant S, then

$$\lambda^2 \geq \frac{S}{4}\langle\varphi,\varphi\rangle$$

Such bounds can sometimes be improved, depending on the manifold. For example, Hijazi proved the following theorem for compact manifold.

Theorem 46. Let M be a compact Riemannian spin manifold of dimension $n \geq 3$. Then all Dirac eigenvalues λ of M satisfy

$$\lambda^2 \geq \frac{n}{n-1}\frac{\mu_1(Y)}{4}$$

where $\mu_1(Y)$ denotes the smallest eigenvalue of the Yamabe operator

$$Y = 4\frac{n-1}{n-2}\Delta + R$$

Also, Bär gave a lower bound for the eigenvalues on S^2.

Theorem 47. Let $M = S^2$ be equipped with any Riemannian metric. Then all Dirac eigenvalues satisfy

$$\lambda^2 \geq \frac{4\pi}{\text{area}(M)}$$

Equality holds for the smallest eigenvalue if and only if M has constant curvature.

The Dirac operator is therefore the object of a great study, especially with its consequences in physics.

Appendices

A Principal and associated bundles

A.1 Principal bundles

This introduction to principal bundles is taken from [5], Appendix A, p370.

Let X be a paracompact Hausdorff space and G a topological space. A principal G-bundle P over X is essentially a bundle of "affine G-spaces" over X. To be more precise, it is a fibre bundle $\pi : P \to X$ together with a continuous, right action of G on P which preserves the fibres (if $y \in P_x$, then $\forall g \in G$, $yg \in P_x$), and acts simply and transitively on them. Thus the fibres are exactly the orbits of G.

Moreover, every point in X has a neighbourhood U and a homomorphism

$$\begin{aligned} h_U : \pi^{-1}(U) &\to U \times G \\ p &\mapsto (\pi(p), \gamma(p)) \end{aligned}$$

with the property that $h(pg) = (\pi(p), \gamma(p)g)$, $\forall g \in G$.

Examples.

- Let $\pi : P \to X$ be a 2-sheeted covering space of X, and let $G = \mathbb{Z}/2\mathbb{Z}$. The group G acts on P by interchanging the sheets, which preserves the fibres. That is clearly a principal $\mathbb{Z}/2\mathbb{Z}$-bundle.
- Let E be a real n-dimensional vector bundle over X, and let $P_{GL_n}E$ be the bundle of bases of E, that is the bundle whose fibre at $x \in X$ is the set of all bases for the vector space E_x (the fibre over x in E). This is a GL_n principal bundle.

  Fix a matrix $g = (a_{ij}) \in GL_n$. Then given a basis $v = (v_1, \dots, v_n)$ of E_x at a point $x \in X$, we set $vg = (v'_1, \dots, v'_n)$ where

  $$v'_j = \sum_{k=1}^{n} v_k a_{kj}$$

  This action is clearly continuous, and it is simple and transitive on the fibres.

A.2 Associated bundles

More about associated bundles can be found in [6], Chapter I, p54-55.

Let $P(M, G)$ be a principal fibre bundle over M with group G and let F be a manifold on which G acts on the left. We shall construct a fibre bundle $E(M, F, P, G)$ associated with P with standard fibre F.

On the product manifold $P \times F$, we let G acts on the right as follows : an element $a \in G$ maps the element $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$.

At this moment, E is just a set, but we can introduce a differentiable structure. The mapping

$$\begin{aligned} P \times F &\to M \\ (u, \xi) &\mapsto \pi(u) \end{aligned}$$

induces a mapping π_E, called the projection of E onto M. For each $x \in M$, the set $\pi_E^{-1}(x)$ is called the fibre of E over x. Every point $x \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(U)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$b : (x, a, \xi) \mapsto (x, ab, b^{-1}\xi) \text{ for } (x, a, \xi) \in U \times G \times F \text{ and } b \in G$$

It follows that the isomorphism $\pi^{-1}(U) \simeq U \times G$ induces an isomorphism $\pi_E^{-1}(U) \simeq U \times F$. We can therefore introduce a differentiable structure on E by the requirement that $\pi_E^{-1}(U)$ is an open submanifold of E which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E^{-1}(U) \simeq U \times F$. The projection π_E is then a differentiable mapping of E onto M. We call E the fibre bundle over the base M, with (standard) fibre F and (structure) group G, which is associated with the principal fibre bundle P.

Example. Let M be a manifold of dimension n. Let $L(M)$ be the set of all linear frames u at all points of M. It is a $GL_n(\mathbb{R})$ principal bundle over M. Then the tangent bundle TM is an associated bundle with $L(M)$, with standard fibre $\mathbb{R}^n$.

B Connections

B.1 On principal bundles

For proofs and further results about connections, see [6], chapter II, p63-64.

Let $P(M, G)$ be a principal bundle over a manifold M with group G. For each $u \in P$, let T_uP be the tangent space of P at u and G_u the subspace of T_uP consisting of vectors tangent to the fibre through u.

Definition 48. A connection Γ on P is an assignment of a subspace Q_u of T_uP to each $u \in P$ such that :

1. $T_uP = G_u \oplus Q_u$ (direct sum);
2. $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in G$, where R_a is the transformation of P induced by $R_a u = ua$;
3. Q_u depends differentiably on u.

Condition 2 means that the distribution $u \mapsto Q_u$ is invariant by G. We call G_u the vertical subspace and Q_u the horizontal subspace of T_uP.

Definition 49. A vector $X \in T_uP$ is called vertical (resp. horizontal) if it lies in G_u (resp. Q_u). By 1, every vector can be uniquely written as $X = Y + Z$, where $Y \in G_u$ and $Z \in Q_u$. We call Y (resp. Z) the vertical (resp. horizontal) component of X and denote it by vX (resp. hX).

Given a connection Γ in P, we define a 1-form on P with value in the Lie algebra $\mathscr{G}$ of G as follows. Every $A \in \mathscr{G}$ induces a vector field A^* on P by setting, for $u \in P$,

$$(A^*)_u = \frac{d}{dt}\left(u.e^{tA}\right)|_{t=0}$$

called the fundamental vector field corresponding to A, and that

$$A \mapsto (A^*)_u$$

is a linear isomorphism of $\mathscr{G}$ onto G_u for each $u \in P$.

For each $X \in T_uP$, we define $\omega(X)$ to be the unique $A \in \mathscr{G}$ such that $(A^*)_u$ is equal to the vertical component of X. So $\omega(X) = 0$ if and only if X is horizontal. The form ω is called the connexion form of the given connection Γ.

Proposition 50. The connection form ω of a connection Γ satisfies the following conditions :

1. $\omega(A^*) = A$, for every $A \in \mathscr{G}$;
2. $(R_a)^*\omega = Ad(a^{-1})\omega$, that is $\omega\left((R_a)_*X\right) = Ad(a^{-1})\omega(X)$, for every $a \in G$ and every vector field X on P.

   Conversely, given a $\mathscr{G}$-valued 1-form ω on P satisfying 1 and 2, there is a unique connection Γ on P whose connection form is ω.

The projection $\pi : P \to M$ induces a linear mapping $\pi : T_uP \to T_xM$ for each $u \in P$, where $x = \pi(u)$. When a connection is given, π maps the horizontal subspace Q_u isomorphically onto T_xM.

The horizontal lift (or simply lift) of a vector field X on M is the unique vector field X^* on P which is horizontal and which projects onto X, that is $\pi(X^*_u) = X_{\pi(u)}$, $\forall u \in P$.

Proposition 51. Given a connection in P and a vector field X on M, there is a unique horizontal lift X^* of X. The lift X^* is invariant by R_a, $\forall a \in G$. Conversely, every horizontal vector field X^* on P invariant by G is the lift of a vector field X on M.

Proposition 52. Let X^* and Y^* be the horizontal lifts of X and Y respectively. Then

1. $X^* + Y^*$ is the horizontal lift of $X + Y$;
2. For every function f on M, $f^*.X^*$ is the horizontal lift of fX where f^* is the function on P defined by $f^* = f \circ \pi$;
3. The horizontal component of $[X^*, Y^*]$ is the horizontal lift of $[X, Y]$.

B.2 On associated bundles

The following comes from [7], 10.4, p346-347.

Let $P(M, G)$ be a G-bundle with the projection π_P. Let us take a chart U_i of M and a section σ_i over U_i. We take the canonical trivialization $\phi_i(p, e) = \sigma_i(p)$. Let $\widetilde{\gamma}$ be a horizontal lift of a curve $\gamma : [0, 1] \to U_i$. We denote $\gamma(0) = p_0$ and $\widetilde{\gamma}(0) = u_0$. Associated with P is a vector bundle $E = P \times_\rho V$ with the projection π_E. Let $X \in T_{p_0}M$ be a tangent vector to $\gamma(t)$ at p_0. Let $s \in \Gamma(M, E)$ be a section, or a vector field, on M. Write an element of E as $[(u, v)] = \{(ug, \rho(g)^{-1}v) | u \in P, v \in V, g \in G\}$. We choose the following form :

$$s(p) = [\sigma_i(p), \xi(p)]$$

as a representative.

Now, along a curve $\gamma : [0, 1] \to M$, we have $s(t) = [\widetilde{\gamma}(t), \eta(t)]$, where $\widetilde{\gamma}(t)$ is an arbitrary horizontal lift of $\gamma(t)$. The covariant derivative of $s(t)$ along $\gamma(t)$ at $p_0 = \gamma(0)$ is defined by

$$\nabla_X s = \left[\widetilde{\gamma}(0), \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right] \tag{17}$$

where X is the tangent vector to $\gamma(t)$ at p_0. For the covariant derivative to be really intrinsic, it should not depend on hte horizontal lift. Let $\widetilde{\gamma}'(t) = \widetilde{\gamma}(t)a$ $(a \in G)$ be another horizontal lift of γ. If $\widetilde{\gamma}'(t)$ is chosen to be the horizontal lift, we have a representative $[(\widetilde{\gamma}'(t), a^{-1}\eta(t))]$. The covariant derivative is now given by

$$\left[\widetilde{\gamma}'(0), \frac{d}{dt}(a^{-1}\eta(t))\big|_{t=0}\right] = \left[\widetilde{\gamma}'(0)a^{-1}, \frac{d}{dt}\eta(t)\big|_{t=0}\right]$$

which agrees with 17. Hence $\nabla_X s$ depends only on the tangent vector X and the section $s \in \Gamma(M, E)$ and not on the horizontal lift $\widetilde{\gamma}(t)$. The definition depends only on a curve γ and a connection and not on local trivializations.

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