

Mémoire de Master 2

Groupes Ext dans la catégorie des bimodules sur une algèbre de Leibniz simple.

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1 Introduction

Leibniz algebras are a generalization of Lie algebras, where we do not require the bracket $[-, -] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ to be antisymmetric. They were introduced in the 1960's by A. Bloh. J.-L. Loday and his collaborators used them in the early 1990's to study, among other things, the cyclic homology of associative algebras. Our goal is to use a result from J. Feldvoss and F. Wagemann ([3]) to obtain a version of Theorem 3.1 of [7] from J.-L. Loday and T. Pirashvili in the case where we have a Leibniz algebra \mathfrak{h} which is not a Lie algebra, and to compute the Ext groups in the category of finite-dimensional \mathfrak{h} -bimodules.

1.1 Leibniz Algebras

Definition 1.1. A (left) Leibniz algebra over a field k is a vector space \mathfrak{h} equipped with a bilinear map :

$$[-,-]:\mathfrak{h}\times\mathfrak{h}\longrightarrow\mathfrak{h}$$

called Leibniz bracket, that satisfies the (left) Leibniz identity :

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \forall x, y, z \in \mathfrak{h}$$
(1)

Remark 1.2. We can also define a right Leibniz algebra by asking our bracket to satisfy the right Leibniz identity instead : [[x, y], z] = [[x, z], y] + [x, [y, z]], but we will only be concerned with left Leibniz algebras.

Example 1.3. Every Lie algebra is also a Leibniz algebra. It is easy to see that the Jacobi and Leibniz identities are equivalent if we impose the antisymmetry of the bracket.

For every Leibniz algebra \mathfrak{h} , we have short exact sequence :

$$0 \longrightarrow \mathfrak{Leib}(\mathfrak{h}) \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h}_{Lie} \longrightarrow 0$$
⁽²⁾

where $\mathfrak{Leib}(\mathfrak{h})$ is the *Leibniz kernel* of \mathfrak{h} , that is the two-sided ideal generated by the elements [x, x] for $x \in \mathfrak{h}$; and $\mathfrak{h}_{Lie} := \mathfrak{h}/\mathfrak{Leib}(\mathfrak{h})$.

By what has been said in Example 1.3, \mathfrak{h}_{Lie} is a Lie algebra, called the *canonical Lie algebra associated to* \mathfrak{h}

Definition 1.4. A left Leibniz algebra is called semisimple if 0, $\mathfrak{Leib}(\mathfrak{h})$, and \mathfrak{h} are the only two sided ideals of \mathfrak{h} , and $\mathfrak{Leib}(\mathfrak{h}) \subsetneq [\mathfrak{h}, \mathfrak{h}]$.

Proposition 1.5. If \mathfrak{h} is a simple Leibniz algebra, then \mathfrak{h}_{Lie} is a simple Lie algebra and $\mathfrak{Leib}(\mathfrak{h})$ is a simple \mathfrak{h}_{Lie} -module.

This proposition is a direct consequence of the correspondence theorem for ideals. For a deeper study of Leibniz algebras, see for example [2].

1.2 Lebniz bimodules

Definition 1.6. Let \mathfrak{h} be a Leibniz algebra. A \mathfrak{h} -bimodule is a vector space M over k equipped with two bilinear maps :

$$[-,-]_L:\mathfrak{h}\times M\longrightarrow M$$

and

$$[-,-]_R: M \times \mathfrak{h} \longrightarrow M$$

which satisfy the following relations $\forall x, y \in \mathfrak{h}, \forall m \in M$:

$$[x, [y, m]_L]_L = [[x, y], m]_L + [y, [x, m]_L]_L$$
(LLM)

$$[x, [m, y]_R]_L = [[x, m]_L, y]_R + [m, [x, y]]_R$$
(LML)

$$[m, [x, y]]_R = [[m, x]_R, y]_R + [x, [m, y]_R]_L$$
(MLL)

Remark 1.7. If *M* is a \mathfrak{h} -bimodule, *M* has a natural \mathfrak{h}_{Lie} -module structure (in the Lie sense). Indeed one can define a left action of \mathfrak{h}_{Lie} as follows :

$$\begin{split} \mathfrak{h}_{Lie} \times M &\longrightarrow M \\ (\bar{x}, m) &\longmapsto [x, m]_L \end{split}$$

Note that this is well defined, since two lifts of \bar{x} differ by an element of $\mathfrak{Leib}(\mathfrak{h})$, which acts in a trivial way by the identity (LLM) :

$$[x, [x, m]_L]_L = [[x, x], m]_L + [x, [x, m]_L]_L$$

implying :

$$\left[\left[x,x\right],m\right]_{L}=0$$

We now define some particular classes of bimodules that will be of use in what follows.

Definition 1.8. Let \mathfrak{h} be a Leibniz algebra, and M a Leibniz bimodule. If

 $[x,m]_L = - [m,x]_R \ \forall x \in \mathfrak{h}, \forall m \in M$

then M is said to be symmetric and denoted M^s . If

 $[m, x]_{R} = 0 \; \forall x \in \mathfrak{h}, \forall m \in M$

then M is said to be **antisymmetric** and denoted M^a . If M is both symmetric and antisymmetric, then M is **trivial**.

For every \mathfrak{h} -bimodule M, there is a short exact sequence of \mathfrak{h} -bimodules :

 $0 \longrightarrow M_0 \longrightarrow M \longrightarrow M/M_0 \longrightarrow 0$

where $M_0 = Span_k([x, m]_L + [m, x]_R)$.

Note that by construction M/M_0 is a symmetric \mathfrak{h} -bimodule, and that M_0 is an antisymmetric \mathfrak{h} -bimodule. Indeed, by summing the relations (LML) and (MLL) of the Definition 1.6, we obtain $\forall x, y \in \mathfrak{h}, \forall m \in M$:

$$[[x,m]_L,y]_R + [[m,x]_R,y]_R = 0$$

proving that the right action of \mathfrak{h} on the generators of M_0 is trivial.

Remark 1.9. We can also define a left \mathfrak{h} -module as being a vector space M over k equipped with a bilinear map :

$$[-,-]_L:\mathfrak{h}\times M\longrightarrow M$$

satisfying the relation (LLM) of Definition 1.4.

We will sometimes be interested in viewing such a module as a \mathfrak{h} -bimodule, and a natural way to do this is to turn it into either a symmetric or antisymmetric bimodule, the right action then being respectively described by the left one or trivial.

With this we have all we need to find the simple objects in the category of finite-dimensional \mathfrak{h} bimodules.

Theorem 1.10. The simple objects in the category of \mathfrak{h} -bimodules of finite dimension are exactly the modules of the form M^a and M^s , where M is a simple \mathfrak{h}_{Lie} -module.

Démonstration. We first show that all the simple objects are of this type.

Let M be a simple \mathfrak{h} -bimodule. Since we have the exact sequence :

 $0 \longrightarrow M_0 \longrightarrow M \longrightarrow M/M_0 \longrightarrow 0$

and M is simple, then the \mathfrak{h} -subbimodule M_0 of M is either M or 0. In the first case M is an antisymmetric bimodule, and in the second M is a symmetric bimodule. We now need to show that M is also a simple left \mathfrak{h}_{Lie} -module.

If M is antisymmetric, then we are not concerned with the right action, and by construction of the \mathfrak{h}_{Lie} -module structure given in the Remark 1.5, we see that M can not have nontrivial \mathfrak{h}_{Lie} -submodules.

If M is symmetric then the same argument holds since the right action being defined by the left one means that for all $x \in \mathfrak{h}$, the $[x, -]_L$ - and $[-, x]_R$ -invariants subspaces coincide.

Now we need to prove the converse. Let M be a simple \mathfrak{h}_{Lie} -module. We can see M as a \mathfrak{h} -module, via the projection on \mathfrak{h}_{Lie} , and then endow it with a natural structure of symmetric or antisymmetric \mathfrak{h} -bimodule as per Remark 1.7. It is then clear that since M has no nontrivial \mathfrak{h}_{Lie} -submodules, M can not have a nontrivial \mathfrak{h} -subbimodule either.

1.3Universal enveloping algebra

Definition 1.11. Let \mathfrak{h} be a Leibniz algebra. Given two copies \mathfrak{h}^l and \mathfrak{h}^r of \mathfrak{h} generated respectively by the elements l_x and r_x for $x \in \mathfrak{h}$, we define the universal enveloping algebra of \mathfrak{h} as the unital associative algebra :

$$UL(\mathfrak{h}) := T(\mathfrak{h}^l \oplus \mathfrak{h}^r)/\mathfrak{I}$$

where $T(\mathfrak{h}^l \oplus \mathfrak{h}^r) := \bigoplus_{n=0}^{\infty} (\mathfrak{h}^l \oplus \mathfrak{h}^r)^{\otimes n}$ is the tensor algebra of $\mathfrak{h}^l \oplus \mathfrak{h}^r$ and \mathfrak{I} is the two-sided ideal of \mathfrak{h} generated by the elements :

$$l_{[x,y]} - l_x \otimes l_y + l_y \otimes l_x$$
$$r_{[x,y]} - l_x \otimes r_y + r_y \otimes l_x$$
$$r_y \otimes (l_x + r_x)$$

For a Lie algebra \mathfrak{g} , there is an equivalence between being a \mathfrak{g} -module and being a U(\mathfrak{g})-module, where $U(\mathfrak{g})$ is the universal algebra of \mathfrak{g} . The following theorem allows us to establish the same kind of connection between the structure of \mathfrak{h} -bimodule and left UL(\mathfrak{h})-module.

Theorem 1.12. Let \mathfrak{h} be a Leibniz algebra. There is an equivalence of categories between the category of \mathfrak{h} -bimodules and the category of UL(\mathfrak{h})-modules.

Démonstration. Let M be a \mathfrak{h} -bimodule. We will define, step by step, a morphism of unital and associative algebras $UL(\mathfrak{h}) \longrightarrow End(M)$. First define a linear map :

First define a linear map :

$$\mathfrak{h}^{l} \oplus \mathfrak{h}^{r} \longrightarrow End(M) l_{x} + r_{y} \longmapsto (m \mapsto [x, m]_{L} + [m, y]_{R})$$

We can extend this map in a unique way to a morphism of algebras $T(\mathfrak{h}^l \oplus \mathfrak{h}^r) \longrightarrow End(M)$. Now we see that the axioms (LLM) and (LML) imply that the first two families of generators of the ideal \mathfrak{I} are sent to zero. Moreover by summing the relations (LML) and (MLL) we saw that we obtain the relation

$$[[x,m]_L,y]_R + [[m,x]_R,y]_R = 0$$

showing that the last family of generators is also sent to zero. This proves that we obtain an algebra homomorphism $UL(\mathfrak{h}) \longrightarrow End(M)$.

Conversely, if M is a lef $UL(\mathfrak{h})$ -module, we can define two linear maps :

$$\begin{split} [-,-]_L: \mathfrak{h} \times M \longrightarrow M \\ (x,m) \longmapsto [x,m]_L = l_x.m \end{split}$$

and

$$\begin{split} [-,-]_R &: M \times \mathfrak{h} \longrightarrow M \\ & (m,x) \longmapsto [m,x]_R = r_x.m \end{split}$$

And we can check that these two maps verify the axioms (LLM), (LML), (MLL).

We included the proof of this result, because given one of the two structures, it explicitly tells us how to obtain the other : the action of l_x corresponds to the left action $[x, -]_L$ while the action r_y corresponds to the right action $[-, y]_R$.

Proposition 1.13. Let \mathfrak{h} be a Leibniz algebra. There is a $U(\mathfrak{h}_{Lie})$ -module isomorphism :

$$\eta: U(\mathfrak{h}_{Lie}) \oplus U(\mathfrak{h}_{Lie}) \otimes \mathfrak{h} \longrightarrow UL(\mathfrak{h})$$
$$\bar{x} \longmapsto l_x$$
$$1 \otimes y \longmapsto r_y$$

Under this isomorphism the product structure on $U(\mathfrak{h}_{Lie}) \oplus U(\mathfrak{h}_{Lie}) \otimes \mathfrak{h}$ is induced by the product structure of $U(\mathfrak{h}_{Lie})$ and the formulas $\forall x, y \in \mathfrak{h}$:

$$(1 \otimes x)\overline{y} = \overline{y} \otimes x - 1 \otimes [y, x]$$
$$(1 \otimes x)(1 \otimes y) = -\overline{y} \otimes x$$

For the proof of this proposition, see [6], Proposition (2.4). Be careful, for the authors work with right Leibniz algebras.

Finally we want to establish a connection between $U(\mathfrak{h}_{Lie})$ -modules and $UL(\mathfrak{h})$ -modules. To this end we define the following algebras homomorphisms :

$$d_0: UL(\mathfrak{h}) \longrightarrow U(\mathfrak{h}_{Lie})$$
$$d_0(l_x) = \bar{x}$$
$$d_0(r_x) = 0$$

and :

$$d_1: UL(\mathfrak{h}) \longrightarrow U(\mathfrak{h}_{Lie})$$
$$d_1(l_x) = \bar{x}$$
$$d_1(r_x) = -\bar{x}$$

These definitions give well defined algebras homomorphisms. We shall only check it for d_0 . We have :

$$d_0(l_{[x,y]} - l_x \otimes l_y + l_y \otimes l_x) = \overline{[x,y]} - \overline{x}\overline{y} + \overline{y}\overline{x} = 0$$

$$d_0(r_{[x,y]} - l_x \otimes r_y + r_y \otimes l_x) = 0$$

$$d_0(r_y \otimes (l_x + r_x)) = 0$$

With these, given a $U(\mathfrak{h}_{Lie})$ -module, we can see it as a $UL(\mathfrak{h})$ -module either via d_0 or via d_1 . The former gives an antisymmetric \mathfrak{h} -bimodule, while the latter gives a symmetric \mathfrak{h} -bimodule. Moreover, since they are surjective (their image contains the generators of $U(\mathfrak{h}_{Lie})$), this allows us to consider $U(h_{Lie})$ as the quotient $UL(\mathfrak{h})/Ker(d_i)$ for $i \in \{0, 1\}$.

1.4 Leibniz cohomology

Let \mathfrak{h} be a Leibniz algebra, and M be a \mathfrak{h} -module. We define a cochain complex

 $CL^n(\mathfrak{h}, M), dL^n\}_{n>0}$

by :

$$CL^{n}(\mathfrak{h}, M) = Hom(\mathfrak{h}^{\otimes n}, M)$$
$$dL^{n}: CL^{n}(\mathfrak{h}, M) \longrightarrow CL^{n+1}(\mathfrak{h}, M)$$

with :

$$dL^{n}\omega(x_{0},...,x_{n}) = \sum_{i=0}^{n-1} (-1)^{i} [x_{i},\omega(x_{0},...,\hat{x}_{i},...,x_{n})]_{L} + (-1)^{n-1} [\omega(x_{0},...,x_{n-1}),x_{n}]_{R} + \sum_{0 \le i < j \le n} (-1)^{i+1} \omega(x_{0},...,\hat{x}_{i},...,x_{j-1},[x_{i},x_{j}],x_{j+1},...,x_{n})$$

Definition 1.14. Let \mathfrak{h} be a Leibniz algebra, and M be a \mathfrak{h} -module. The cohomology of \mathfrak{h} with coefficients in M is the cohomology of the cochain complex $\{CL^n(\mathfrak{h}, M), dL^n\}_{n\geq 0}$.

$$HL^{n}(\mathfrak{h}, M) = H^{n}(\{CL^{n}(\mathfrak{h}, M), dL^{n}\}_{n \ge 0}) \quad \forall n \ge 0$$

Remark 1.15. By definition $CL^0(\mathfrak{h}, M) = M$ and $dL^0m(x) = -[m, x,]_R$. Therefore, we have :

$$HL^{0}(\mathfrak{h}, M) = \{ m \in M, \ [m, x]_{R} = 0 \ \forall x \in \mathfrak{h} \}$$

This is the submodule of right invariants. Note that if M is antisymmetric, then $HL^0(\mathfrak{h}, M) = M$.

2 Ext in the category of Leibniz bimodules

We are now interested in computing the Ext groups in the category of \mathfrak{h} -bimodules. From now on, we will consider a finite-dimensional left Leibniz algebra \mathfrak{h} over a field of characteristic zero k.

2.1 Change of rings spectral sequence

In this section, we will give details on the construction of two spectral sequences, yielding the following proposition :

Proposition 2.1. Let \mathfrak{h} be a Leibniz algebra, let X be a \mathfrak{h} -bimodule, and Y and Z be left \mathfrak{h} -modules. There are two spectral sequences :

$$E_2^{pq} = H^p(\mathfrak{h}_{Lie}, Hom(Y, HL^q(\mathfrak{h}, X))) \Longrightarrow Ext_{UL(\mathfrak{h})}^{p+q}(Y^a, X)$$
$$E_2^{pq} = H^p(\mathfrak{h}_{Lie}, Hom(Z, Ext_{UL(\mathfrak{h})}^q(U(\mathfrak{h}_{Lie})^s, X))) \Longrightarrow Ext_{UL(\mathfrak{h})}^{p+q}(Z^s, X)$$

Démonstration. We will focus on the first of the two, following the construction given in the subsections 1 to 4 of Chapter XVI from [1]. We are mostly interested in constructing the spectral sequence given in Case 4 page 350 of the book.

Let Y be a left $U(\mathfrak{h}_{Lie})$ -module, and X be a left $UL(\mathfrak{h})$ -module. Given our algebra homomorphism

$$d_0: UL(\mathfrak{h}) \longrightarrow U(\mathfrak{h}_{Lie})$$

we can see Y as a $UL(\mathfrak{h})$ -module, via :

$$x.m := d_0(x).m \quad \forall x \in UL(\mathfrak{h}), \ \forall m \in Y$$

When we do so, we will write \widetilde{Y} .

Given X, we can also construct a new $U(\mathfrak{h}_{Lie})$ -module ${}^{(d_0)}X := Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}), X)$, and we have an adjunction :

$$Hom_{UL(\mathfrak{h})}(\widetilde{Y}, X) = Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}) \otimes_{U(\mathfrak{h}_{Lie})} Y, X)$$
$$= Hom_{U(\mathfrak{h}_{Lie})}(Y, {}^{(d_0)}X)$$

Now, using a projective resolution $P^*_{U(\mathfrak{h}_{Lie})} \longrightarrow Y$ of Y, and an injective resolution $X \longrightarrow I^*_{UL(\mathfrak{h})}$ of X, we obtain the bicomplex :

$$A^{*,*} = Hom_{UL(\mathfrak{h})}(P^{*}_{U(\mathfrak{h}_{Lie})}, I^{*}_{UL(\mathfrak{h})})$$

of which we can compute cohomology in different ways.

First, we consider $H_I(A^{*,*})$ the cohomology of $A^{*,*}$ with respect to the first variable. It is another double complex, with horizontal differential zero, and vertical differential induced by the differential of $I^*_{UL(\mathfrak{h})}$.

We can also look at $H_{II}(A^{*,*})$, the cohomology of $A^{*,*}$ with respect to the second variable, which is a double complex with vertical differential zero, and horizontal differential induced by the differential of $P^{*}_{U(\mathfrak{h}_{Lie})}$.

We can go one step further and consider $H_{II}H_I(A^{*,*})$, and $H_IH_{II}(A^{*,*})$, and by doing this, we claim that we are able to compute the E_2 term of our spectral sequence.

To see this, notice that the previous adjunction gives us two different ways to write a left exact functor, which is contravariant in the first variable and covariant in the second variable :

$$T(Y,X) = Hom_{UL(\mathfrak{h})}(\widetilde{Y}^{a},X) = Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}) \otimes_{U(\mathfrak{h}_{Lie})} Y,X)$$
$$= Hom_{U(\mathfrak{h}_{Lie})}(Y,^{(d_{0})}X)$$

where we consider \widetilde{Y}^a , because the left $UL(\mathfrak{h})$ action on \widetilde{Y} is given by d_0 . Applying it to our resolutions, we can then compute :

$$\begin{split} H_{II}^{p,q}(T(P_{U(\mathfrak{h}_{Lie})}^{p},I_{UL(\mathfrak{h})}^{q})) &= H_{II}^{p,q}(Hom_{U(\mathfrak{h}_{Lie})}(P_{U(\mathfrak{h}_{Lie})}^{*},Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}),I_{UL(\mathfrak{h})}^{*}))) \\ &= H^{q}(Hom_{U(\mathfrak{h}_{Lie})}(P_{U(\mathfrak{h}_{Lie})}^{p},Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}),I_{UL(\mathfrak{h})}^{*}))) \\ &\simeq Hom_{U(\mathfrak{h}_{Lie})}(P_{U(\mathfrak{h}_{Lie})}^{p},H^{q}(Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie}),I_{UL(\mathfrak{h})}^{*}))) \\ &= Hom_{U(\mathfrak{h}_{Lie})}(P_{U(\mathfrak{h}_{Lie})}^{p},Ext_{UL(\mathfrak{h})}^{q}(U(\mathfrak{h}_{Lie}),X))) \end{split}$$

Where the isomorphism is given by Proposition 6.1a pp65-66 in Chapter IV of [1]. This in turn yields :

$$H_{I}^{p,q}H_{II}(T(P_{U(\mathfrak{h}_{Lie})}^{p}, I_{UL(\mathfrak{h})}^{q})) \simeq Ext_{U(\mathfrak{h}_{Lie})}^{p}(M, Ext_{UL(\mathfrak{h})}^{q}(U(\mathfrak{h}_{Lie})^{a}, X))$$

By Theorem (3.4) of [7], we have an isomorphism

$$HL^*(\mathfrak{h}, X) \simeq Ext^*_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a, X)$$

Moreover, we have that

$$\begin{aligned} Ext_{U(\mathfrak{h}_{Lie})}^{p}(Y, HL^{q}(\mathfrak{h}, X)) &= Ext_{U(\mathfrak{h}_{Lie})}^{p}(k \otimes Y, HL^{q}(\mathfrak{h}, X)) \\ &= Ext_{U(\mathfrak{h}_{Lie})}^{p}(k, Hom(Y, HL^{q}(\mathfrak{h}, X))) \\ &\simeq H^{p}(\mathfrak{h}_{Lie}, Hom(Y, HL^{q}(\mathfrak{h}, X))) \end{aligned}$$

And $H_I^{p,q} H_{II}(T(P_{U(\mathfrak{h}_{Lie})}^p, I_{UL(\mathfrak{h})}^q))$ gives us the left term of our first spectral sequence

$$H^p(\mathfrak{h}_{Lie}, Hom(Y, HL^q(\mathfrak{h}, X)))$$

To see that it converges to $Ext_{UL(\mathfrak{h})}^{p+q}(Y^a, X)$, we need to study $H_{II}^{p,q}H_I(T(P_{U(\mathfrak{h}_{Lie})}^p, I_{UL(\mathfrak{h})}^q)))$. Similar computations give that

$$H_{II}^{p,q}H_I(T(P_{U(\mathfrak{h}_{Lie})}^p, I_{UL(\mathfrak{h})}^q) \simeq Ext_{UL(\mathfrak{h})}^q(Tor_p^{U(\mathfrak{h}_{Lie})}(U(\mathfrak{h}_{Lie}), Y), X)$$

But, since $U(\mathfrak{h}_{Lie})$ is a free $U(\mathfrak{h}_{Lie})$ -module, it is flat, and we have

$$Tor_{p}^{U(\mathfrak{h}_{Lie})}(U(\mathfrak{h}_{Lie}),Y) = U(\mathfrak{h}_{Lie}) \otimes_{U(\mathfrak{h}_{Lie})} Y \qquad \text{for } \mathbf{p} = 0$$
$$= 0 \qquad \text{for } \mathbf{p} \ge 1$$

in turn yielding

$$H_{II}^{p,q}H_{I}(T(P_{U(\mathfrak{h}_{Lie})}^{p}, I_{UL(\mathfrak{h})}^{q})) \simeq Ext_{UL(\mathfrak{h})}^{q}(\widetilde{Y}^{a}, X) \qquad \text{for } \mathbf{p} = 0$$
$$\simeq 0 \qquad \qquad \text{for } \mathbf{p} \ge 1$$

Now, since we are working with first quadrant spectral sequences, we have the convergence :

$$H_I^{p,q}H_{II}(T(P_{U(\mathfrak{h}_{Lie})}^p, I_{UL(\mathfrak{h})}^q)) \Longrightarrow H^{p+q}(A^*)$$

and

$$H^{p,q}_{II}H_I(T(P^p_{U(\mathfrak{h}_{Lie})}, I^q_{UL(\mathfrak{h})})) \Longrightarrow H^{p+q}(A^*)$$

where A^* is the single complex associated to $A^{*,*}$.

This allow us to conclude, because the computations for $H_{II}^{p,q}H_I(T(P_{U(\mathfrak{h}_{Lie})}^p, I_{UL(\mathfrak{h})}^q))$ show that the spectral sequence collapses, and we finally find that

$$E_2^{pq} = H^p(\mathfrak{h}_{Lie}, Hom(Y, HL^q(\mathfrak{h}, X))) \Longrightarrow Ext_{UL(\mathfrak{h})}^{p+q}(Y^a, X)$$

For the second spectral sequence, the work is the same, apart from the fact we use d_1 instead of d_0 , thus making symmetric bimodules appear instead of antisymmetric bimodules.

2.2 Study of $Ext^q_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, X))$

In the previous proposition, we were able to identify $Ext^*_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a, X)$ to the Leibniz cohomology $HL^*(\mathfrak{h}, X)$. What about $Ext^*_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, X)$? In this section, we will give a proof of a generalization of Proposition 2.3 of [7], in order to give a relation between $Ext^*_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, X)$ and Leibniz cohomology. In order to do so, we will have to introduce a shift in the homological degree which will be responsible for nontrivial Ext groups in what will follow.

Proposition 2.2. Let \mathfrak{h} be a Leibniz algebra, and M be a \mathfrak{h} -bimodule. There are isomorphisms :

$$Ext_{UL(\mathfrak{h})}^{i+1}(U(\mathfrak{h}_{Lie})^s, M) \simeq Hom(\mathfrak{h}, HL^i(\mathfrak{h}, M)) \qquad \text{for } i > 0$$
$$\simeq Cokerf \qquad \text{for } i = 0$$
$$\simeq Kerf \qquad \text{for } i = -1$$

where $f: M \longrightarrow Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M))$ is given by :

$$f(m)(h) = [h,m]_L + [m,h]_R \quad \forall h \in \mathfrak{h}, \ \forall m \in M$$

Démonstration. Let M be a \mathfrak{h} -bimodule, and

$$f: M \longrightarrow Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M))$$
$$f(m)(h) = [h, m]_L + [m, h]_R$$

We first want to show that $Ext^0_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M) = Kerf$. But by definition

$$Ext^{0}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M) = Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M)$$

We then define the map :

$$ev: Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M) \longrightarrow M$$
$$\varphi \longmapsto \varphi(1)$$

And we have :

Lemma 2.3. $Im(ev) \subset Kerf$, and the corestriction $ev|^{Kerf}$ of ev to Kerf is an isomorphism of inverse :

$$\mu: Kerf \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M)$$
$$m \longmapsto \varphi_m: (1 \mapsto m)$$

Démonstration. — $Im(ev) \subset Kerf$:

First notice that $Kerf = \{m \in M, [h,m]_L + [m,h]_R = 0 \forall h \in \mathfrak{h}\}$, which in terms of the action of $UL(\mathfrak{h})$ translate to : $\{m \in M, l_h.m = -r_h.m \forall h \in \mathfrak{h}\}$. Now let $m \in Im(ev)$, there exists $\varphi \in Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M)$ such that $m = \varphi(1)$. From there we get $\forall h \in \mathfrak{h}$:

$$l_h.m = l_h.\varphi(1) = \varphi(l_h.1)$$

$$(*) = \varphi(-r_h.1)$$

$$= -r_h.\varphi(1)$$

$$= -r_h.m$$

Where the (*) equality comes from the fact that we are considering $U(\mathfrak{h}_{Lie})$ as the left $UL(\mathfrak{h})$ -module $U(\mathfrak{h}_{Lie})^s$. This means that $m \in Kerf$.

 $- \mathrm{ev}|^{\mathrm{Kerf}}$ is an isomorphism :

For all $m \in M$, φ_m defines a $UL(\mathfrak{h})$ -module homomorphism and one can check that $ev|^{Kerf} \circ \mu = id$ and $\mu \circ ev|^{Kerf} = id$, therefore $ev|^{Kerf}$ is an isomorphism with inverse μ .

We have proved the degree zero equality of the proposition. We now want to show that $Ext^{1}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M) = Cokerf.$ Consider $UL(\mathfrak{h}) \otimes \mathfrak{h}$ as a left $UL(\mathfrak{h})$ -module with the following action $\forall x \in \mathfrak{h}, \forall r, s \in UL(\mathfrak{h})$:

$$s.(r \otimes x) = sr \otimes x$$

Define a homomorphism of left $UL(\mathfrak{h})$ -modules by :

$$f_1: UL(\mathfrak{h}) \otimes \mathfrak{h} \longrightarrow UL(\mathfrak{h})$$
$$1 \otimes h \longmapsto l_h + r_h$$

Then f_1 factors through $f_2 : U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h} \longrightarrow UL(\mathfrak{h})$. Indeed we have the following commutative diagram :

and define $f_2(d_0(x) \otimes h) := f_1(x \otimes h)$ which is well-defined : if $x, y \in UL(\mathfrak{h})$ are such that $d_0(x) = d_0(y)$, then $f_1(x \otimes h) = f_1(y \otimes h)$. Indeed if $x - y \in Kerd_0$, then $x = y + \overline{z}$ with $\overline{z} \in \langle r_z, z \in \mathfrak{h} \rangle$. Therefore, the relation $r_y(l_x + r_x) = 0$ in $UL(\mathfrak{h})$ implies that $f_1(x \otimes h) = f_1(y \otimes h)$.

We now want to show the injectivity of f_2 . Considering the diagram (D), it is sufficient to prove :

Lemma 2.4. $Kerf_1 = Ker(d_0 \otimes id)$

Démonstration. — $\operatorname{Ker}(\mathbf{d}_0 \otimes \operatorname{id}) \subset \operatorname{Ker} \mathbf{f}_1$: Let $\alpha \in \operatorname{Ker}(d_0 \otimes \operatorname{id})$. Then $\alpha = \sum_i (a_i \otimes x_i)$ with $a_i \in \langle r_z, z \in \mathfrak{h} \rangle$ and $x_i \in \mathfrak{h}$. By linearity, we can assume without loss of generality that $a_i = b_i r_{z_i} c_i$ with $b_i, c_i \in UL(\mathfrak{h})$ and $z_i \in \mathfrak{h}$. Then

$$f_1(\alpha) = \sum_i a_i (l_{x_i} + r_{x_i})$$
$$= \sum_i b_i r_{z_i} c_i (l_{x_i} + r_{x_i})$$

But $c_i \in UL(\mathfrak{h})$, and we can also assume that it is a monomial in some l_{y_j} and r_{y_j} . We are then in one of three cases :

- 1. $c_i = (...)r_{y_j}$ In this case the relation $r_y(l_x + r_x) = 0$ gives us immediatly that $b_i r_{z_i} c_i(l_{x_i} + r_{x_i}) = 0$.
- 2. c_i contains at least one element of the type r_{y_i} but not in last position.

We will proceed by induction on the position of the last element of the type r_{y_j} in c_i . The base step having been treated in case 1.

Consider the rightmost such element, say r_{y_m} . It is therefore followed by a l_{y_n} . We can then apply the relation $r_{[x,y]} - l_x r_y + r_y l_x = 0$ in $UL(\mathfrak{h})$, which in our case reads : $r_{y_m} l_{y_n} = l_{y_n} r_{y_m} - r_{[y_n, y_m]}$.

This yields two monomials, whose rightmost element of the type r_{y_j} is one place closer to the right than in c_i . By an iterating this process, we are left with some numbers of monomials but whose last element is always a r_{y_j} . Case 1 then gives us that $b_i r_{z_i} c_i (l_{x_i} + r_{x_i}) = 0$.

3. c_i contains no element of the type r_{y_i}

Then consider $r_{z_i}c_i$ instead of just c_i , and apply case 2 to it.

This shows that $Ker(d_0 \otimes id) \subset Kerf_1$.

- $\mathbf{Kerf_1} \subset \mathbf{Ker}(\mathbf{d_0} \otimes \mathbf{id}) :$

Let
$$\alpha \in Kerf_1 \subset UL(\mathfrak{h}) \otimes \mathfrak{h}$$
. This means that $\alpha = \sum_i a_i \otimes x_i$ with $a_i \in UL(\mathfrak{h})$ $x_i \in \mathfrak{h}$.

We need to show that $a_i \in \langle r_z, z \in \mathfrak{h} \rangle$, assuming (which we can) that a_i is a monomial in some r_{y_i} and l_{y_i} .

The a_i that contain an element of the type r_{y_j} are in $\langle r_z, z \in \mathfrak{h} \rangle$, and the first part of the proof tells us that for these $f_1(a_i \otimes x_i) = 0$. We are therefore left only with a_i of the type : $a_i = \lambda_{i_1...i_p} l_{y_{i_1}}...l_{y_{i_p}}, \lambda_{i_1...i_p} \in k$. We want to show that all of the coefficients $\lambda_{i_1...i_p}$ are zero. Since $\alpha \in Kerf_1$, we are in the situation :

$$\sum_{i} \lambda_{i_1 \dots i_p} l_{y_{i_1}} \dots l_{y_{i_p}} (l_{x_i} + r_{x_i}) = 0 \tag{(*)}$$

Using the isomorphism in Proposition 1.13, (*) can be reduced to :

$$\begin{cases} \sum_{i} \lambda_{i_1\dots i_p} l_{y_{i_1}}\dots l_{y_{i_p}} l_{x_i} = 0\\ \sum_{i} \lambda_{i_1\dots i_p} l_{y_{i_1}}\dots l_{y_{i_p}} r_{x_i} = 0 \end{cases}$$

Here we are only interested in the second equation. Note that without loss of generality, we can suppose the $(l_{y_{i_1}}...l_{y_{i_p}})_i$ to be linearly independent (we use the Poincaré-Birkhoff-Witt Theorem to get a basis of $U(\mathfrak{h}_{Lie})$ which we then transfer in $UL(\mathfrak{h})$ using Proposition 1.13). This implies that the $(l_{y_{i_1}}...l_{y_{i_p}}r_{x_i})_i$ are linearly independent, which in turn implies that all the $\lambda_{i_1...i_p}$ are zero. This means that all of the a_i in α are in $\langle r_z, z \in \mathfrak{h} \rangle$, which means that $Kerf_1 \subset \langle r_z, z \in \mathfrak{h} \rangle \otimes \mathfrak{h} = Ker(d_0 \otimes id)$.

We have therefore showed that $Kerf_1 = Ker(d_0 \otimes id)$.

This lemma implies that f_2 is injective. This therefore gives us the following short exact sequence :

$$0 \longrightarrow U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h} \xrightarrow{f_2} UL(\mathfrak{h}) \longrightarrow Cokerf_2 \longrightarrow 0$$

But by construction, $\text{Im}(f_2)$ is the left ideal $\langle l_x + r_x, x \in \mathfrak{h} \rangle$, which is equal to $\text{Ker}(d_1)$ (see Section 1.3). This implies that $\text{Coker}(f_2)$ is the quotient $UL(\mathfrak{h})/Ker(d_1)$, that is $\text{Im}(d_1)$, and the short exact sequence above becomes :

$$0 \longrightarrow U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h} \xrightarrow{f_2} UL(\mathfrak{h}) \longrightarrow U(\mathfrak{h}_{Lie})^s \longrightarrow 0$$

This short exact sequence yields the following long exact sequence in cohomology :

$$0 \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M) \longrightarrow Hom_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a} \otimes \mathfrak{h}, M) \\ \longrightarrow Ext^{1}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M) \longrightarrow Ext^{1}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) \longrightarrow Ext^{1}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a} \otimes \mathfrak{h}, M) \\ \longrightarrow Ext^{2}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, M) \longrightarrow (...)$$

Now, by noticing the obvious identification $Hom_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) = M$, and the fact that, $UL(\mathfrak{h})$ being a free $UL(\mathfrak{h})$ -module, it is projective, and therefore $Ext^{1}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) = 0$, we can extract the following exact sequence :

$$0 \to Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M) \to M \to Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}, M) \to Ext^1_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M) \to 0$$

To obtain the desired isomorphism, we want to relate it to the exact sequence we get from f:

$$0 \longrightarrow Ker(f) \longrightarrow M \longrightarrow Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M)) \longrightarrow Coker(f) \longrightarrow 0$$

and conclude by using the 5-lemma. We can send M onto M via the identity map. We then construct an isomorphism

$$Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}, M) \longrightarrow Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M))$$

Notice that since $U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}$ is a quotient of $UL(\mathfrak{h}) \otimes \mathfrak{h}$, it is generated, as a $UL(\mathfrak{h})$ -module, by the elements $1 \otimes h$, for $h \in \mathfrak{h}$. We can now define a map :

$$Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}, M) \longrightarrow Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M))$$
$$\varphi \longmapsto \tilde{\varphi}$$

where $\tilde{\varphi}(h) := \varphi(1 \otimes h)$, for $h \in \mathfrak{h}$. The image of $\tilde{\varphi}$ lies in $HL^0(\mathfrak{h}, M)$, for :

$$\begin{bmatrix} \tilde{\varphi}(h), h' \end{bmatrix}_R = \begin{bmatrix} \varphi(1 \otimes h), h' \end{bmatrix}_R$$
$$= \varphi(r_{h'}.(1 \otimes h))$$
$$= 0$$

using the fact that φ is a $UL(\mathfrak{h})$ -morphism, and the fact that we are considering the $UL(\mathfrak{h})$ -module $U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}$.

We can then construct its inverse, by :

$$Hom(\mathfrak{h}, HL^{0}(\mathfrak{h}, M)) \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a} \otimes \mathfrak{h}, M)$$
$$u \longmapsto \varphi_{u}$$

with $\varphi_u : \bar{x} \otimes h \mapsto x.u(h)$ where \bar{x} denotes the class of $x \in UL(\mathfrak{h})$ in the quotient $U(\mathfrak{h}_{Lie})^a$ (see Section 1.3).

This yields the following diagram :

where the arrow $(*): Ext^1_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^s, M) \longrightarrow Coker(f)$ is given by functoriality of the Coker.

To conclude, we just need to prove that this diagramm is commutative. It is sufficient to show that it is the case for the square :

$$M \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}, M)$$

$$\| \qquad \qquad \qquad \downarrow \sim$$

$$M \xrightarrow{f} Hom(\mathfrak{h}, HL^0(\mathfrak{h}, M))$$

Notice that for the arrow $M \longrightarrow Hom_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}, M)$ we identified

$$M \simeq Hom_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M)$$

via the map $m \mapsto (\psi_m : u \mapsto u.m)$. This arrow is therefore given by $\psi_m \mapsto \psi_m \circ f_2$, that is :

$$\bar{u} \otimes x \longmapsto \psi_m(f_2(\bar{u} \otimes x)) = \psi_m(f_2(d_0(u) \otimes x))$$
$$= \psi_m(f_1(u \otimes x))$$
$$= \psi_m(u(l_x + r_x))$$
$$= u(l_x + r_x).m$$

Since $U(\mathfrak{h}_{Lie})^a \otimes \mathfrak{h}$ is generated as a $UL(\mathfrak{h})$ -module by the elements $1 \otimes x$ for $x \in \mathfrak{h}$, we can check the commutativity of the diagramm only on these elements. By explicitly writing the maps in question we get :

which by Theorem 1.12 proves the commutativity of the square, and therefore of the diagramm. The 5 lemma then tells us the arrow (*) is an isomorphism, and we obtain the second isomorphism of the proposition.

To get the higher degree isomorphisms, notice that the long exact sequence in cohomology we found earlier goes as follow :

$$\dots \to Ext^{i}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) \longrightarrow Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a} \otimes \mathfrak{h}, M) \longrightarrow Ext^{i+1}_{UL(\mathfrak{h})^{s}}(U(\mathfrak{h}_{Lie}), M)$$
$$\longrightarrow Ext^{i+1}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) \to \dots$$

But $Ul(\mathfrak{h})$ being a free $Ul(\mathfrak{h})$ -module, it is projective, hence

$$Ext^{i}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M) = Ext^{i+1}_{UL(\mathfrak{h})}(UL(\mathfrak{h}), M)$$
$$= 0$$

and this for all i. We thus obtain :

$$0 \longrightarrow Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a} \otimes \mathfrak{h}, M) \longrightarrow Ext^{i+1}_{UL(\mathfrak{h})^{s}}(U(\mathfrak{h}_{Lie}), M) \longrightarrow 0$$

Now, in order to conclude, we use the fact that :

$$Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a}\otimes\mathfrak{h},M)=Hom(\mathfrak{h},Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a},M))$$

which is obtain from the classical Hom / Tens adjunction, and the fact that :

$$Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{a}, M) = HL^{i}(\mathfrak{h}, M)$$

For a proof of the last equality, see [6] Theorem (3.4). Note that the authors work with right Leibniz algebras.

This gives us all the promised isomorphisms, therefore concluding the proof.

2.3 Computing the Ext groups

In this section, we want to give a generalized version of Theorem 3.1 of [7], and give an explicit computation of the groups $Ext^*_{UL(\mathfrak{h})}(M, N)$ for simple finite-dimensional \mathfrak{h} -bimodules. To do so, we use the following Theorem from [3] :

Theorem 2.5. Let \mathfrak{h} be a finite-dimensional semisimple left Leibniz algebra over a field of characteristic zero, and let M be a finite-dimensional \mathfrak{h} -module. Then $HL^n(\mathfrak{h}, M) = 0$ for every integer $n \ge 2$, and there is a five-term exact sequence :

 $0 \longrightarrow M_0 \longrightarrow HL^0(\mathfrak{h}, M) \longrightarrow M^{\mathfrak{h}_{Lie}}_{sym} \longrightarrow Hom_{\mathfrak{h}}(\mathfrak{h}_{ad,l}, M_0) \longrightarrow HL^0(\mathfrak{h}, M) \longrightarrow 0$

Moreover, if M is symmetric, then $HL^{n}(\mathfrak{h}, M) = 0$ for every integer $n \geq 1$.

With this result, we can now prove :

Theorem 2.6. Let \mathfrak{h} be a finite dimensional simple Leibniz algebra over a field of characteristic zero k. All groups $Ext^2_{UL(\mathfrak{h})}(M, N)$ between simple finite dimensional \mathfrak{h} -bimodules are zero, except $Ext^2_{UL(\mathfrak{h})}(\mathfrak{M}^s, \mathfrak{N}^a)$, with $\mathfrak{M} \in {\mathfrak{Leib}(\mathfrak{h})^*, \mathfrak{h}^*_{Lie}}$ and $\mathfrak{N} \in {\mathfrak{Leib}(\mathfrak{h}), \mathfrak{h}_{Lie}}$ which is one dimensional. Moreover, we have that :

 $- Ext^{1}_{UL(\mathfrak{h})}(\mathfrak{M}^{s}, k), \text{ and } Ext^{1}_{UL(\mathfrak{h})}(k, \mathfrak{N}^{a}) \text{ are one dimensional, for } \mathfrak{M} \text{ and } \mathfrak{N} \in \{\mathfrak{Leib}(\mathfrak{h}), \mathfrak{h}_{Lie}\}; \\ - Ext^{1}_{UL(\mathfrak{h})}(M^{s}, N^{a}) \simeq Hom_{U(\mathfrak{h}_{Lie}}(M, \widehat{N}), \text{ where}$

$$\hat{N} := Coker(h: N \longrightarrow Hom(\mathfrak{h}, N)) \quad h(n)(x) := [n, x]_R$$

— All other groups $Ext^{1}_{UL(\mathfrak{h})}(M, N)$ between simple finite-dimensional \mathfrak{h} -bimodules M and N are zero.

Démonstration. We will compute $Ext^*_{UL(\mathfrak{h})}(M, N)$ for every combination of finite-dimensional \mathfrak{h} -bimodules M and N.

— Case 1 : M = N = k is the trivial \mathfrak{h} -bimodule.

We apply Proposition 2.1 to Y = X = k. By Theorem 2.5, $HL^q(\mathfrak{h}, k) = 0$ for $q \ge 1$, since k being trivial, it is also symmetric. Therefore, we obtain :

$$Ext^*_{UL(\mathfrak{h})}(k,k) \simeq H^*(\mathfrak{h}_{Lie},k)$$

— Case 2 : M = k is the trivial \mathfrak{h} -bimodule, and N is a nontrivial simple symmetric \mathfrak{h} -bimodule. We apply Proposition 2.1 to Y = k, and X = N. Once again by Theorem 2.5, we get :

$$Ext_{UL(\mathbf{h})}^n(k, N^s) = 0 \qquad \qquad \text{for } n \ge 1$$

— Case 3 : M = k is the trivial \mathfrak{h} -bimodule, and N is a nontrivial simple antisymmetric \mathfrak{h} -bimodule. We have :

$$\begin{split} HL^q(\mathfrak{h},N^a) &\simeq 0 & ext{for } q > 1, ext{ by Theorem } 2.5 \\ &\simeq Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h},N) & ext{for } q = 1 \\ &\simeq N & ext{for } q = 0, ext{ since N is antisymmetric} \end{split}$$

Since N is a nontrivial simple antisymetric \mathfrak{h} -bimodule, it is also a nontrivial simple \mathfrak{h}_{Lie} -module, and therefore $H^*(\mathfrak{h}_{Lie}, N) = 0$ by Whitehead's theorem. Now using Proposition 2.1, we find :

$$Ext^*_{UL(\mathfrak{h})}(k, N^a) \simeq H^{*-1}(\mathfrak{h}_{Lie}, Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N))$$
$$\simeq H^{*-1}(\mathfrak{h}_{Lie}, k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N)$$

The second isomorphism is given in [4], Theorem 2.1.8 pp74-75, or in [5], Theorem 13. Since \mathfrak{h} might not be a simple \mathfrak{h}_{Lie} -module, we can not just apply Schur's lemma to the group $Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h})$. But this is where the short exact sequence

$$0 \longrightarrow \mathfrak{Leib}(\mathfrak{h}) \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h}_{Lie} \longrightarrow 0 \tag{3}$$

comes in handy. As a sequence of left \mathfrak{h}_{Lie} -modules it actually splits, yielding the decomposition

$$\mathfrak{h} = \mathfrak{Leib}(\mathfrak{h}) \oplus \mathfrak{h}_{Lie}$$

and since \mathfrak{h} is a simple Leibniz algebra, this is the decomposition of \mathfrak{h} in simple \mathfrak{h}_{Lie} -modules. Now, since M is also a simple \mathfrak{h}_{Lie} -module, we get that if $M \simeq \mathfrak{Leib}(\mathfrak{h})$ or $M \simeq \mathfrak{h}_{Lie}$ (as a left \mathfrak{h}_{Lie} -module), then

$$H^{*-1}(\mathfrak{h}_{Lie},k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}) \simeq H^{*-1}(\mathfrak{h}_{Lie},k)$$

If this is not the case, then

$$H^{*-1}(\mathfrak{h}_{Lie},k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}) \simeq 0$$

— Case 4 : M is a nontrivial simple antisymmetric \mathfrak{h} -bimodule, and N is a simple symmetric \mathfrak{h} -bimodule.

Using Theorem 2.5, we have $HL^q(\mathfrak{h}, N^s) = 0$ for $q \ge 1$. Moreover, because $HL^0(\mathfrak{h}, N^a) = N^{\mathfrak{h}}$ is a trivial \mathfrak{h} -bimodule, and since we can identify $Hom(M, HL^0(\mathfrak{h}, N^s) \simeq M^* \otimes N^g$ with the direct sum of dim (N^g) copies of M^* we find that $H^p(\mathfrak{h}_{Lie}, Hom(M, HL^0(\mathfrak{h}, N^a))) \simeq H^p(\mathfrak{h}_{Lie}, M^*) \oplus ... \oplus$ $H^p(\mathfrak{h}_{Lie}, M^*) = 0$, since M being a simple nontrivial \mathfrak{h}_{Lie} -module, so is M^* . Thus yielding :

$$Ext^*_{UL(\mathfrak{h})}(M^a, N^s) = 0$$

— Case 5: M is a nontrivial simple antisymmetric representation, and N is simple and antisymmetric. Here, Theorem 2.5 apply again, and we have that $HL^q(\mathfrak{h}, N^a) \neq 0$ only when $q \in \{0, 1\}$. We check that $HL^1(\mathfrak{h}, N^a)$ is a trivial left \mathfrak{h} -module. By definition of the chain complex defining Leibniz cohomology, we have that $CL^1(\mathfrak{h}, N^a) = Hom(\mathfrak{h}, N)$. Now for a morphism $\varphi \in Hom(\mathfrak{h}, N)$ to be annihilated by the differential dL^1 means satisfying :

$$dL^{1}\varphi(x,y) := [x,\varphi(y)]_{L} - \varphi([x,y]) = 0 \qquad \qquad \forall x,y \in \mathfrak{h}$$

Which is exactly to say that the left action of \mathfrak{h} on the module $Hom(\mathfrak{h}, N)$ is trivial. Therefore, the same arguments used in Case 4 still apply, and we get that $E_2^{pq} = 0$ for q > 0, and :

$$Ext^*_{UL(\mathfrak{h})}(M^a, N^a) = 0$$

— Case 6 : M is a nontrivial simple symmetric representation, and N = k is the trivial \mathfrak{h} -bimodule. We apply Proposition 2.2 to k to find :

$$Ext^{i}_{UL(\mathfrak{h})}((U(\mathfrak{h}_{Lie})^{s},k) \simeq 0 \qquad \qquad ext{if } i > 1$$
 $\simeq \mathfrak{h} \qquad \qquad ext{if } i = 1$
 $\simeq k \qquad \qquad ext{if } i = 0$

because since in this case, the f in Proposition 2.2 is zero. We can now plug this in the second spectral sequence of Proposition 2.1, with X = k, and Z = M, to obtain :

$$Ext^*_{UL(\mathfrak{h})}(M^s,k) \simeq H^{*-1}(\mathfrak{h}_{Lie},Hom(M,\mathfrak{h}))$$
$$\simeq H^{*-1}(\mathfrak{h}_{Lie},k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h})$$

Using the same arguments as in Case 3, we get that if $M \simeq \mathfrak{Leib}(\mathfrak{h})$ or $M \simeq \mathfrak{h}_{Lie}$, then

$$H^{*-1}(\mathfrak{h}_{Lie},k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}) \simeq H^{*-1}(\mathfrak{h}_{Lie},k)$$

If this is not the case, then

$$H^{*-1}(\mathfrak{h}_{Lie},k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}) \simeq 0$$

— Case 7 : M and N are both simple nontrivial symmetric \mathfrak{h} -bimodules. Applying Proposition 2.2 to N^s , and because N is a symmetric \mathfrak{h} -bimodule, we find that

Now using the second spectral sequence of Proposition 2.1, we get :

$$Ext^*_{UL(\mathfrak{h})}(M^s, N^s) \simeq H^*(\mathfrak{h}_{Lie}, Hom(M, N))$$
$$\simeq H^*(\mathfrak{h}_{Lie}, k) \otimes Hom_{\mathfrak{h}_{Lie}}(M, N)$$

And once again, since M and N are simple \mathfrak{h}_{Lie} -modules, this vector space is nonzero only if $M \simeq N$, in which case, it is isomorphic to $H^*(\mathfrak{h}_{Lie}, k)$.

- Case 8: M is a simple nontrivial symmetric \mathfrak{h} -bimodule, and N is a simple nontrivial antisymmetric \mathfrak{h} -bimodule.

By Proposition 2.2, we have :

$$\begin{aligned} Ext^{i}_{UL(\mathfrak{h})}(U(\mathfrak{h}_{Lie})^{s}, N^{a}) &\simeq 0 & \text{for i} > 2 \\ &\simeq Hom(\mathfrak{h}, Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N)) & \text{for i} = 2 \\ &\simeq Coker(h) & \text{for i} = 1 \\ &\simeq Ker(h) & \text{for i} = 0 \end{aligned}$$

The *h* appearing here is due to the fact that *N* is an antisymmetric \mathfrak{h} -bimodule. Moreover, since *N* is supposed to be nontrivial and *h* is a \mathfrak{h} -module homomorphism, $\operatorname{Ker}(h) = 0$. Therefore we have that $E_2^{pq} = 0$ for q > 2 and q = 0. For the remaining values of *q*, we have isomorphisms

$$E_2^{p_1} \simeq H^p(\mathfrak{h}_{Lie}, Hom(M, \widehat{N}))$$
$$\simeq H^p(\mathfrak{h}_{Lie}, k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M, \widehat{N})$$

and

$$E_2^{p_2} \simeq H^p(\mathfrak{h}_{Lie}, Hom(M, Hom(\mathfrak{h}, Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N))))$$

$$\simeq H^p(\mathfrak{h}_{Lie}, k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M, Hom(\mathfrak{h}, Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N)))$$

The first isomorphism tells us that $Ext^{1}_{UL(\mathfrak{h})}(M^{s}, N^{a}) \simeq Hom_{U(\mathfrak{h}_{Lie})}(M, \widehat{N})$. To use the second isomorphism, we need to proceed as in Case 6, since \mathfrak{h} is not a priori a simple \mathfrak{h}_{Lie} -module, although with a bit more cases.

- If $N \not\simeq \mathfrak{Leib}(\mathfrak{h})$ or $N \not\simeq \mathfrak{h}_{Lie}$: Then $Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N) \simeq 0$, yielding $E_2^{p^2} = 0$ - If $N \simeq \mathfrak{Leib}(\mathfrak{h})$ or $N \simeq \mathfrak{h}_{Lie}$: Then $Hom_{U(\mathfrak{h}_{Lie})}(\mathfrak{h}, N) \simeq k$, and we have

$$E_2^{p_2} \simeq H^p(\mathfrak{h}_{Lie}, Hom(M, \mathfrak{h}^*))$$
$$\simeq H^p(\mathfrak{h}_{Lie}, k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M, \mathfrak{h}^*)$$

Now we need to do the same work for $Hom_{U(\mathfrak{h}_{Lie})}(M, \mathfrak{h}^*)$. Since \mathfrak{h} is a simple Leibniz algebra, so is its dual \mathfrak{h}^* . Moreover, the exactness of the functor Hom(M, -) gives us the short exact sequence

$$0 \longrightarrow \mathfrak{Leib}(\mathfrak{h})^{\star} \longrightarrow \mathfrak{h}^{\star} \longrightarrow \mathfrak{h}_{Lie}^{\star} \longrightarrow 0$$

and the decomposition of $\mathfrak{h}^* = \mathfrak{Leib}(\mathfrak{h})^* \oplus \mathfrak{h}_{Lie}^*$ as a left \mathfrak{h}_{Lie} -module. We therefore are in one of the following cases :

- If $M \not\simeq \mathfrak{Leib}(\mathfrak{h})^*$ or $M \not\simeq \mathfrak{h}_{Lie}^*$: Then $Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}^*) \simeq 0$, and $E_2^{p2} = 0$. - If $M \simeq \mathfrak{Leib}(\mathfrak{h})^*$ or $M \simeq \mathfrak{h}_{Lie}^*$: Then $Hom_{U(\mathfrak{h}_{Lie})}(M,\mathfrak{h}^*) \simeq k$, and we get

$$E_2^{p^2} \simeq H^p(\mathfrak{h}_{Lie}, k) \otimes Hom_{U(\mathfrak{h}_{Lie})}(M, \mathfrak{h}^{\star})$$
$$\simeq H^p(\mathfrak{h}_{Lie}, k)$$

In order to get the promised vanishing of the Ext groups, we just use the fact that

$$H^1(\mathfrak{h}_{Lie},k)\simeq H^2(\mathfrak{h}_{Lie},k)\simeq 0$$

and this concludes our proof.

Remark 2.7. Notice that the only difference between our proof and the one of Theorem 3.1 in [7] is our treatment of the cases where the first variable of the Ext functor is a symmetric bimodule, and the second is an antisymmetric bimodule, where we can not just use Schur's Lemma as the authors did.

Furthemore, we can see that the nontrivial Ext groups arise only when the shift in homological degree from Proposition 2.2 appears.

Moreover as in [7], this Theorem actually gives us a way to express all the Ext groups between simple \mathfrak{h} -bimodules in terms of the usual Chevalley-Eileberg cohomology groups.

Références

- Samuel Eilenberg and Henri Paul Cartan. *Homological algebra*, volume 238. Princeton University Press Princeton-New Jersey, 1956.
- [2] Jörg Feldvoss. Leibniz algebras as non-associative algebras. Nonassociative Mathematics and its Applications, 721 :115, 2019.
- [3] Jörg Feldvoss and Friedrich Wagemann. On Leibniz cohomology. arXiv preprint arXiv :1902.06128, 2019.
- [4] Dmitriĭ Borisovich Fuks. Cohomology of infinite-dimensional Lie algebras. Springer Science & Business Media, 2012.
- [5] Gerhard Hochschild and Jean-Pierre Serre. Cohomology of Lie algebras. Annals of Mathematics, pages 591–603, 1953.
- [6] Jean-Louis Loday and Teimuraz Pirashvili. Universal enveloping algebras of Leibniz algebras and (co) homology. *Mathematische Annalen*, 296(1):139–158, 1993.
- [7] Jean-Louis Loday and Teimuraz Pirashvili. Leibniz representations of Lie algebras. Journal of Algebra, 181(2):414-425, 1996.