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INTERNSHIP REPORT

Linearized wave-damping structure of Vlasov-Poisson in $\mathbb{R}^3$

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Preface

This preface will mainly be devided into two parts. First we will speak about plasma, the fourth state of matter before giving a brief explanation about Landau-damping phenomenom.

What is a plasma ?

Plasma is a state of matter that is often thought of as a subset of gases, but the two states behave very differently. Like gases, plasmas have no fixed shape or volume, and are less dense than solids or liquids. But unlike ordinary gases, plasmas are made up of atoms in which some or all of the electrons have been stripped away. When a solid is heated sufficiently, usually a liquid is formed. When a liquid is heated enough that atoms vaporize off the surface faster than they recondense, a gaz is formed. When a gaz is heated enough a plasma is formed : the so called 'fourth state of matter'.
The name plasma which means 'moldable substance' or 'jelly' was given by Irving Langmuir,the Nobel laureate who pioneered the scientific study of ionized gaz.

How is a plama made ?

A plasma is not usually made simply by heating a container of gaz. The problem is that container cannot be as hot as a plasma needs to be in order to be ionized or the container itself would vaporise and become plasma as well. Typically, in the laboratory, a small amount of gaz is heated and ionized by using electricity, or by shining radio waves into it. Either the thermal capacity of the container is used to keep it from getting hot enough to melt during a short heating pulse, or the container is actively cooled (with water for example) for longer pulse operation. Generally, these means of plasma formation give energy to free electrons in the plasma directly, and these electron-atom collision liberate more electrons, and the process cascades until the desired degree of ionization is achieved.

On the other hand, Landau damping, named after its discoverer, Russian physicist Lev Davidovich Landau, is the damping phenomenon (exponential decay as a function of time) of longitudinal oscillations in the electric field. This corresponds to a transfer of energy between an electromagnetic wave and electrons. It was then proposed by Lynden-Bell that a similar phenomenon took place in galaxy dynamics, where the gas of electrons interacting through electrical forces is replaced by a “star gas” interacting with gravitational forces.

Landau damping is due to the exchange of energy between a wave of phase velocity v, and a particle in a plasma whose velocity is approximately equal to v. The particles whose speed is slightly lower than the phase speed of the wave will be accelerated by the electric field of the wave to reach the phase speed. On the contrary, the particles whose speed is slightly higher than the phase speed of the wave will be decelerated, giving up their energy to the wave. In a non-collisional plasma where the velocities of the particles are distributed as a Maxwellian function, the number of particles whose velocity is slightly lower than the phase velocity of the wave is greater than the number of particles whose velocity is slightly bigger. Thus, there are more particles which gain energy from the wave than particles which give up. Therefore, the wave giving up energy, it is damped. (For more details about this phenomenom, check [1].)

Abstract

The aim of this report is to study the linearized Vlasov equation, and prove that the electric field can be decomposed into a part following a Klein Gordon type equation for long waves and a part submitted to a Landau damping phenomenon.

As an introduction, we start by taking a look at the Vlasov equation and then linearize it around an homogeneous Maxwellian. Using the method of characteristcs we can solve the free transport equation to switch to a Volterra equation. This leads us to a complex function $L(z, k)$ which will interfere with its poles $p_{\pm}(k)$ later in the solution of our Volterra equation. We then solve the Volterra equation, decompose its resolvent into multiple parts and then give the decomposition of the electric field into two main parts E_{KG} and E_{LD}. In the last part, we prove that E_{KG} nearly solves a Klein-Gordon equation and that E_{LD} verifies Landau-damping type decay estimates.

Finally, in the appendix, we recall some results about Cauchy integrals , and explain the method of characteristics.

This internship is an adaptation of the article [7] by Bedrossian, Masmoudi and Mouhot.

Acknowledgment

At the end of this internship, it is my duty to present my deep gratitudes to all those who helped, encouraged and guided me for the development of this work.

I would like to thank Professor Frédéric Hérau because he have guided me for the last 3 months. His advice was so important and effective to be able to accomplish this report. I really appreciate the valuable knowledge and skills he has given me.

I thank the entire Master 2 teaching team for the treasures it offers to listeners in order to help us progress, reach a higher level and to deepen our knowledge.

Finally, I want to thank my family who have never stopped believing in me throughout my studies. Their encouragement is the reason of what I have become today.

I apologize in advance to those I forgot to mention and I rearm them by elsewhere my gratitude.

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1 Introduction

The general model describing inhomogeneous kinetic equations is as follows. We have an evolution equation of unknown f, a probability density function of presence of particules in position and space. The function f depends on the time t, the position $x \in \mathbb{R}^d$ and the velocity $v \in \mathbb{R}^d$, and $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v)dxdv = 1$. We assume as well that the particles undergo shocks modeled by a collision kernel $Q(f,f)$ and that the particles are submitted to the action of an external force $F \in \mathbb{R}^d$. The equation satisfied by f defined on $\mathbb{R}_t^+ \times \mathbb{R}_x^d \times \mathbb{R}_v^d$ is

$$\partial_t f + v.\nabla_x f + F(t,x).\nabla_v f = Q(f,f)$$

with initial data $f_{/t=0} = f_0$ where we have omitted the dependance on (t,x,v) for readability. We are interested in Cauchy problem for this equation in the case of a self consistent Poisson force $Q = 0$. In 1930s and 1940s, Vlasov suggested to neglect collisions and derive the so-called Vlasov-Poisson equation for long range interactions (1.1). All the material in this internship comes from [7] with slight changes here and there.

1.1 Notation

Fourier transform

For a function $f : \mathbb{R}^d \to \mathbb{R}$, we define

$$\widehat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-ik.x} dx.$$

Then we have the usual formulae

$$\widehat{\nabla f}(k) = ik\widehat{f}(k), \text{ and } \widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k).$$

Denote $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and $\langle \nabla \rangle$ the Fourier multiplier defined as follow

$$\widehat{\langle \nabla \rangle f}(k) = \langle k \rangle \widehat{f}(k).$$

For a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we define

$$\widehat{f}(k,\eta) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) e^{-ik.x} e^{-i\eta.v} dxdv$$

Similarly, we denote $\langle (x,v) \rangle = (1 + |x|^2 + |v|^2)^{\frac{1}{2}}$ and $\langle \nabla_{x.v} \rangle$ the Fourier multiplier defined as follow

$$\widehat{\langle \nabla_{x,v} \rangle f}(k,\eta) = \langle (k,\eta) \rangle \widehat{f}(k,\eta).$$

Sobolev spaces

We shall use the following norms, defined for $\sigma, m, p \in \mathbb{R}$ and (x, v) in $\mathbb{R}^3 \times \mathbb{R}^3$, by

$$||f||_{L^p_{x,v}} := \left(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(x,v)|^p dx dv \right)^{\frac{1}{p}} \quad , \quad ||g||_{L^p_x} := \left(\iint_{\mathbb{R}^3} g(x) dx \right)^{\frac{1}{p}}$$

$$||f||_{W^{\sigma,p}_m} := ||\langle v \rangle^m \langle \nabla_{x,v} \rangle^\sigma f||_{L^p}.$$

Laplace transform

Let $f : [0, \infty[\to \mathbb{C}$ satisfying $e^{-\mu t} f(t) \in L^1$ for some $\mu \in \mathbb{R}$. Then for all complex numbers $z \in \mathbb{C}$ such that $\Re(z) \geq \mu$, we define Fourier-Laplace transform by

$$\hat{f}(z) := \int_0^\infty e^{-zt} f(t) dt.$$

For $\gamma > \mu$, the inverse Laplace transfom by

$$\check{f}(t) := \int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt} f(z) dz.$$

1.2 The Vlasov equation

In Vlasov-Poisson model, shocks between particles are neglected and we impose throughout this work $Q(f, f) = 0$. The external force is an electrostatic force created by the spatial distribution function of particles which is defined by

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv.$$

The force $F(t, x)$ is then given by

$$F(t, x) := E(t, x) = -\nabla_x W *_x \rho(t, x)$$

with $W(x) = \dfrac{q^2}{4\pi\epsilon_0 m_e |x|}$, q the electron charge, m_e the electron mass, and ϵ_0 the vacuum permittivity. When studying the Vlasov-Poisson equation, it is interesting to start with the simpliest case, called free transport.

For $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, the Vlasov equation is

$$\partial_t f + v.\nabla_x f = 0,$$

with inital data $f(0,x,v)=f_{in}(x,v)$. It is the fundamental kinetic equation which describes a system for which the particles do not undergo any forces or shocks. Let us take $f_{in}\in\mathcal{C}^1$, and define for $(x,v)\in\mathbb{R}^{2d}$ and $s,t\in\mathbb{R}^+$,

$$\mathrm{x}(s,t)=x+v(s-t) \qquad \mathrm{v}(s,t)=v.$$

We obtain that the function $f(t,x,v)=f_{in}(\mathrm{x}(0,t),\mathrm{v}(0,t))$ verifies $\partial_t f + v\nabla_x f = 0$. Furthermore we notice that

$$\frac{d\mathrm{x}}{ds}(s,t)=\mathrm{v} \qquad and \qquad \frac{d\mathrm{v}}{ds}(s,t)=0,$$

with initial data $\mathrm{x}(t,t)=x$ and $\mathrm{v}(t,t)=v$. In fact, we recognize here exactly Newton first Law governing trajectory of a particle of mass 1 and subjected to no external force.

More generally, we will prove (check 4) that under some hypothesis on F and f_{in}, the solution of

$$\partial_t f + v.\nabla_x f + F.\nabla_v f = 0$$

are given by $f(t,x,v)=f_{in}(\mathrm{x}(0,t),\mathrm{v}(0,t))$ where (x,v) is the solution of

$$\frac{d\mathrm{x}}{ds}(s,t)=\mathrm{v} \qquad and \qquad \frac{d\mathrm{v}}{ds}(s,t)=F(t,x),$$

with initial data at $s=t$ which is the Newton first law for a particle of mass 1 subjected to the force F.

In this report we work on the following set of equations

$$\begin{cases} \partial_t f + v.\nabla_x f + E(t,x).\nabla_v f = 0, \\ E(t,x) = -\nabla_x W *_x \rho(t,x), \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)dv - n_0, \\ f(t=0,x,v) = f_{in}(x,v), \end{cases} \tag{1.1}$$

for the time dependent probability of presence function $f(t,x,v)\geq 0$ of the electron in the phase space $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$. Here n_0 is the density of the constant ion background, and W is the kernel of Coulomb interaction defined before. We will consider (1.1) linearized around the homogeneous Maxwellian background with fixed temperature T

$$f^0(v) := n_0\Big(\frac{m_e}{2\pi T}\Big)^{3/2} e^{\frac{-m_e|v|^2}{2T}}. \tag{1.2}$$

1.3 The linearized Vlasov equation

We linearize the Vlasov-Poisson equation (1.1) on $\mathbb{R}^3_x \times \mathbb{R}^3_v$ around the homogeneous background $f^0(v) \geq 0$. First we check that $f^0(v)$ in a solution to (1.1). In fact $f^0(v)$ is independent from t and x with $\displaystyle\int_{\mathbb{R}^3} f^0(v)dv = n_0$ which leads us to $E(f^0) = 0$ so that $f^0(v)$ is a solution. Let $h(t,x,v) = f(t,x,v) - f^0(v)$, we have

$$\partial_t f = \partial_t h, \qquad \nabla_x h = \nabla_x f.$$

$$E(f).\nabla_v(f) = E(f^0+h)\nabla_v(f^0+h) = E(h)\nabla_v(h)+E(h)\nabla_v(f^0)+E(f^0)\nabla_v(h)+E(f^0)\nabla_v(f^0).$$

We neglect the bilinear term $E(h)\nabla_v(h)$ and since f^0 is a solution, $E(f^0) = 0$. Furthermore observe that

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv - n_0 = \int_{\mathbb{R}^3} h(t,x,v)dv.$$

This gives the linearized Vlasov equation

$$\begin{cases} \partial_t h + v.\nabla_x h + E(t,x).\nabla_v f^0(v) = 0, \\ E(t,x) = -\nabla_x W *_x \rho(t,x), \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^3} h(t,x,v)dv, \\ h(t=0,x,v) = h_{in}(x,v). \end{cases} \tag{1.3}$$

Here by assuming $\displaystyle\iint_{\mathbb{R}^3\times\mathbb{R}^3} h_{in}(x,v)dxdv = 0$. For the following, we define the standard plasma constant (number density, plasma frequency and temperature)

$$n_0 := \widehat{f^0}(0), \quad w_p^2 = \frac{q^2 n_0}{\epsilon_0 m_e}, \quad T := \int_{\mathbb{R}^3} |v|^2 f^0(v)dv.$$

1.4 Main results

Theorem 1 *Suppose* $\displaystyle\int_{\mathbb{R}^3} h_{in}dxdv = 0$ *and let* h *and* E *be a solution to* (1.3). *There exists a decomposition of the electric field* $E = E_{KG} + E_{LD}$ *between a Klein-Gordon and Landau damped parts with* E_{LD} *decomposed as* $E_{LD} = E^{(1)}_{LD} + E^{(2)}_{LD}$ *and* $E^{(2)}_{LD}$ *satisfies the following Landau-damping-type decay estimates for any* $\sigma,\ a \in \mathbb{N}$

$$||\langle \nabla_x, t\nabla_x\rangle^\sigma E^{(2)}_{LD}||_{L^\infty_x} \lesssim \frac{1}{\langle t\rangle^4}||h_{in}||_{W_0^{\sigma+3+a,1}}$$

Furthermore E_{KG} *decomposes as* $E_{KG} = E^{(1)}_{KG} + E^{(2)}_{KG}$, *where* $E^{(1)}_{KG}$ *solves a weakly damped Klein-Gordon type equation in the following sense : there are bounded, smooth functions* λ, Ω

such that

$$\widehat{E^{(1)}_{KG}}(t,k) = \widehat{E}_{in}(k)e^{-\lambda(k)t}cos(\Omega(k)t) - w_0 e^{-\lambda(k)t}\frac{ik}{|k|^2}\big(k.\nabla_\eta \widehat{h_{in}}(k,0)\big)\frac{sin(\Omega(k)t}{\Omega(k)}$$
$$+\mathcal{O}(|k|)e^{-\lambda(k)t+i\Omega(k)t} + \mathcal{O}(|k|)e^{-\lambda(k)t-i\Omega(k)t}$$

where $\Omega^2(k) = w_p^2 + \dfrac{9T}{m_e w_p^2}|k|^2 + \mathcal{O}(|k|^4), \quad \lambda(k) > 0, \quad \lambda(k) = \mathcal{O}(|k|^\infty)$ *specifically, for any* $N \geq 0$, *the norm of* $\lambda(k)$ *is less than* $|k|^N$*.(For more details check* (2.14)

Theorem 2 *Consider and initial data* $h_{in}(x,v) = \epsilon^3 H_0(\epsilon x, v)$ *such that* H_0 *has zero averaging and* $H_0 \in W^{5,1}_5$*. Denote*

$$E_0 = \frac{q^2 n_0}{\epsilon_0 m_e}\nabla_x(\Delta_x)^{-1}\int_{\mathbb{R}^3} H_0(.,v)dv.$$

Let $\mathcal{E} \in L^\infty_t H^1_x$ *and* $\partial_t\mathcal{E} \in L^\infty_t L^2_x$ *be initial data of the following Klein-Gordon equation*

$$\begin{cases} \partial_t^2\mathcal{E}(t,x) + (w_p^2 - \dfrac{9T}{m_e}\Delta)\mathcal{E}(t,x) = 0, \\ \mathcal{E}(0,x) = E_0(x), \\ \partial_t\mathcal{E}(0,x) = -n_0\nabla_x\big(\displaystyle\int v h_{in} dv\big). \end{cases} \tag{1.4}$$

Then for $0 < \epsilon \ll, \ and\ 0 < t < \epsilon^{-N}$ *there holds*

$$\epsilon||E^1_{KG,\epsilon}(t) - \mathcal{E}_\epsilon(t)||_{H^{-s}} \lesssim ||H_0||_{W^{0,1}_4}\epsilon^{s-\frac{3}{2}}\langle t\rangle$$

2 Decomposition of the electric field

2.1 Volterra equation

One can reduce the problem to a Volterra equation. In fact Duhamel principle gives

$$h(t,x,v) = h_{in}(x-vt,v) + \int_0^t (\nabla_x W * \rho)(s, x - v(t-s)).\nabla_v f^0(v)ds,$$

and taking the Fourier transform in x gets :

$$\hat{h}(t,k,v) = e^{-vkt}\widehat{h_{in}}(k,v) + \int_0^t \widehat{\nabla_x W}(k)\widehat{\rho}(s,k)e^{-ivk(t-s)}\nabla_v f^0(v)ds.$$

Integrating in v yields (with $w_0 = w_p^2 n_0^{-1}$)

$$\widehat{\rho}(t,k) = \int \widehat{h_{in}}(k,v)e^{-ivkt}dv + \widehat{\nabla_x W}(k)\int_0^t \int \nabla_v f^0(v)e^{-ivk(t-s)}\widehat{\rho}(s,k)dvds$$

$$= \widehat{h_{in}}(k,kt) + \widehat{\nabla_x W}(k)\int_0^t \int ik(t-s)f^0(v)e^{-ivk(t-s)}\widehat{\rho}(s,k)dvds$$

Observe $W(x) = w_0 G(x)$ where G is the fundamental solution of the Laplacien in 3 dimensions. (Check [6] for more details about Laplacien fondamental solutions).

This gives $\quad \widehat{W(k)} = \dfrac{w_0}{|k|^2}, \quad$ and $\widehat{\nabla W}(k) = w_0 i\dfrac{k}{|k|^2}$,

and eventually

$$\widehat{\rho}(t,k) = \widehat{h_{in}}(k,kt) - w_0\int_0^t (t-\tau)\widehat{f}^0(k(t-\tau))\widehat{\rho}(\tau,k)d\tau, \tag{2.1}$$

Taking the Fourier-Laplace transform in time for $\Re(z)$ sufficiently large gives

$$\hat{\rho}(z,k) = H(z,k) + L(z,k)\hat{\rho}(z,k), \tag{2.2}$$

where $H(z,k)$ is the Fourier-Laplace transform of $t \to \widehat{h_{in}}(k,kt)$ and the disperssion function is

$$L(z,k) := -w_0\int_0^\infty t\widehat{f}^0(kt)e^{-zt}dt = -\frac{w_0}{|k|^2}\int_0^\infty e^{-\frac{z}{|k|}s}s\widehat{f}^0(\widehat{k}s)ds \tag{2.3}$$

with $\widehat{k} := \dfrac{k}{|k|}$.

2.2 Asymptotic expansions and lower bounds on the dispertion function L

Solving (2.2) for ρ works except where $L(z,k)$ gets close to one. The following lemma treat the case of not-so small spatial frequencies k.

Lemma 1 *There exists a $\lambda > 0$ such that for any $\nu_0 > 0$, $\exists\ \kappa > 0$ Verifying*

$$\forall |k| > \nu_0, \qquad \inf_{\Re(z) > -\lambda|k|}|1 - L(z,k)| > \kappa \tag{2.4}$$

Furthermore, the following estimate holds

$$\forall |k| > \nu_0,\ w\in\mathbb{R},\ \ |L(-\lambda|k| + iw,k)| \lesssim_\lambda \frac{1}{1+|k|^2+w^2}. \tag{2.5}$$

Proof. Starting with the case where the dimension is 1 and $k>0$, we have

$$L(-iw,k)=-\frac{w_0}{|k|^2}\int_0^\infty e^{i\frac{w}{k}t}t\widehat{f}^0(t)e^{-ivt}te^{i\frac{w}{k}t}dt=-\frac{w_0}{|k|^2}\lim_{\lambda\to0^+}\int\int f^0(v)e^{-ivt}te^{i\frac{w}{k}t}e^{-\lambda t}dvdt$$
$$=-\frac{w_0}{ik^2}\lim_{\lambda\to0^+}\int\int(f^0)'(v)e^{-ivt}e^{-\frac{w}{k}t}e^{-\lambda t}dvdt=-\frac{w_0}{ik^2}\lim_{\lambda\to0^+}\int\int(f^0)'(v)e^{-[iv-i\frac{w}{k}+\lambda]t}dtdv$$
$$=-\frac{w_0}{ik^2}\int\frac{f^0(v)}{iv-i\frac{w}{k}+\lambda}=-\frac{w_0}{k^2}\lim_{\lambda\to0^+}\int\frac{(f^0)'(v)}{-v+\frac{w}{k}+i\lambda}dv=\frac{w_0}{k^2}\lim_{\lambda\to0^+}\int\frac{(f^0)'(v)}{v-\frac{w}{k}-i\lambda}dv$$

which lead us by Plemelj Formula (see (4.3) in appendix) to the equality

$$L(-iw,k)=\frac{w_0}{k^2}P\int_{\mathbb{R}}\frac{(f^0)'(r)}{r-\dfrac{w}{k}}+i\frac{w_0\pi}{k^2}(f^0)'(\frac{w}{k}). \tag{2.6}$$

Since W is Real and even, $\widehat{W}$ is real-valued, so the above formula yields the decomposition of $L(-iw,k)$ into real and imaginary parts. The problem is to check that the real part cannot approach 1 at the same time as the imaginary part approaches 0. As soon as $(f^0)'(v)=\mathcal{O}\left(\dfrac{1}{|v|}\right)$, we have

$$\int_{\mathbb{R}}\frac{(f^0)'(v)}{v-w}dv=\mathcal{O}\left(\frac{1}{|w|}\right),\quad as\ |w|\to\infty, \tag{2.7}$$

so the real part in the right-hand side of (2.6) becomes small when $|w|$ is large, and we can restrict to a bounded interval. Then the imaginary part, $\frac{w_0\pi}{k^2}(f^0_k)'(\frac{w}{k})$can become small only in the limit $k\longrightarrow\infty$ (but then also the real part becomes small) or if w approaches one of the zeroes of $(f^0)'$. Since w varies in a compact set, by continuity it will be sufficient to check the condition (2.7) only at the zeroes of $(f^0)'$. In the end, we have obtained the following stability criterion (known as Penrose Criterion) :

$$(f^0)'(w)=0\Rightarrow\left(\frac{w_0}{k^2}\int_{\mathbb{R}}\frac{(f^0)'(r)}{r-\dfrac{w}{k}}\right)\neq1,\ for\ all\ w\ \in\mathbb{R} \tag{2.8}$$

Now if $k<0$, we can restart the computation and the change of variable $v\to-v$ bring us back to the previous computation with k replaced by $|k|$ and $f^0(v)$ replaced by $f^0(-v)$. However, it is immediately checked that (2.8) is invariant under reversal of velocities, that is, if $f^0(v)$ is replaced by $f^0(-v)$.
Finally, we can generalize this to several dimensions. (check [1] for more details).

Turn now to (2.5). From (2.3) we know that

$$L(z,k)=-\frac{w_0}{|k|^2}\int_0^\infty e^{-\frac{z}{|k|}}s\widehat{f}^0(\widehat{k}s)ds.$$

When $z = -\lambda|k| + iw$,

$$L(-\lambda|k|+iw,k) = -\frac{w_0}{|k|^2}\int_0^\infty e^{\lambda s - i\frac{w}{|k|}s} s\widehat{f}^0(\widehat{k}s)ds = -\frac{w_0}{|k|^2}\int_0^\infty \frac{1}{(\lambda - i\frac{w}{|k|})^2}\partial_s^2(e^{\lambda s - i\frac{w}{|k|}s})s\widehat{f}^0(\widehat{k}s)ds$$

$$= -\frac{w_0}{|k|^2(\lambda - i\frac{w}{|k|})^2}\left[s\widehat{f}^0(\widehat{k}s)\partial_s(e^{\lambda s - i\frac{w}{|k|}s})\Big)_{s=0}^{s=\infty} - \int_0^\infty \partial_s(e^{\lambda s - i\frac{w}{|k|}s})\partial_s(s\widehat{f}^0(\widehat{k}s))ds\right]$$

$$= \frac{w_0}{|k|^2(\lambda - i\frac{w}{|k|})^2}\left[\partial_s(s\widehat{f}^0(\widehat{k}s)e^{\lambda s - i\frac{w}{|k|}s}\Big)_{s=0}^{s=\infty} - \int_0^\infty e^{\lambda s - i\frac{w}{|k|}s}\partial_s^2(s\widehat{f}^0(\widehat{k}s)ds\right]$$

$$= \frac{w_0}{|k|^2(\lambda - i\frac{w}{|k|})^2}\left[-n_0 - \int_0^\infty e^{\lambda s - i\frac{w}{|k|}s}\partial_s^2(s\widehat{f}^0(\widehat{k}s)ds\right]$$

since $\partial_s(s\widehat{f}^0(\widehat{k}s)) = \widehat{f}^0(\widehat{k}s) + s\widehat{k}\nabla\widehat{f}^0(\widehat{k}s)$, so $\partial_s(s\widehat{f}^0(\widehat{k}s)_{/s=0} = n_0$. Then

$$\big|L(-\lambda|k|+iw,k)\big| \leq \frac{w_0}{(\lambda^2|k|^2+w^2)}\left|\left[n_0 + \int_0^\infty |e^{\lambda s - i\frac{w}{|k|}s}\partial_s^2(s\widehat{f}^0(\widehat{k}s)|ds\right]\right|.$$

$$\lesssim_\lambda \frac{1}{1+|k|^2+w^2}.$$

since $\dfrac{w_0}{\lambda^2|k|^2+w^2} \lesssim_\lambda \dfrac{1}{1+|k|^2+w^2}$.

In fact, $\lambda^2|k|^2+w^2 \geq \lambda^2\nu_0^2 \quad$ and $\quad \lambda^2|k|^2+w^2 \geq \min(\lambda^2,1)(|k|^2+w^2)$

so for $C_\lambda := min(\lambda^2\nu_0^2, \lambda^2, 1)$, we have

$$\lambda^2|k|^2+w^2 \geq C_\lambda$$

and $\lambda^2|k|^2+w^2 \geq C_\lambda(|k|^2+w^2)$. We get $\lambda^2|k|^2+w^2 \geq \dfrac{C_\lambda}{2}(1+|k|^2+w^2)$ which yields the result.

Next we turn to the low frequency estimates. For $\delta, \delta' > 0$, define the following region in the complex plane

$$\Lambda_{\delta,\delta'} = \{z = \lambda + iw \in \mathbb{C} \;:\; \lambda > -\min\big[(1-\delta)|w|, \delta'|k|\big]\}.$$

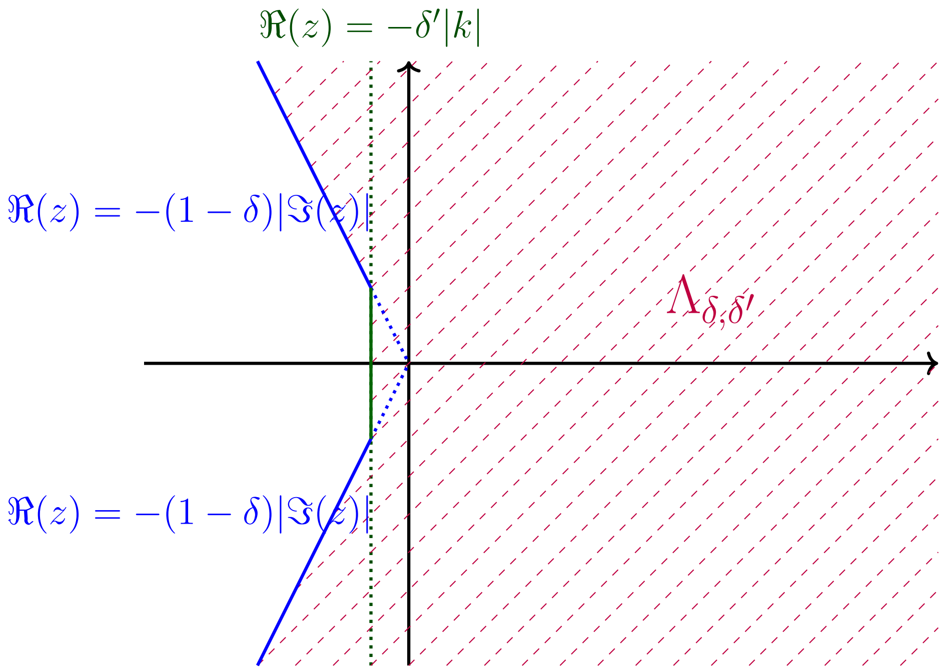


FIGURE 1 – The region $\Lambda_{\delta,\delta'}$

We will next show that $L(z,k)$ stays uniformly away from one in the region $\Lambda_{\delta,\delta'}\backslash\{|z \pm i w_p| < \epsilon\}$ where w_p is the cold plasma frequency. The proof relies on two representation : (1) an expansion obtained by successive integrations by parts in time meaningful for large values of $\frac{z}{|k|}$, (2) an approximation argument using the explicit formula obtained from Plemelj formula at the imaginary line $\Re(z) = 0$, that provides estimates near this line. The first representation is given by the following lemma.

Lemma 2 *(Asymptotic expansion of $L(z,k)$ for $|k| \ll |z|$). Given $\delta' > 0$ sufficiently small depending only on f^0,*

$$\forall z \in \Lambda_{\delta,\delta'}, \quad L(z,k) = -\frac{w_p^2}{z^2}\left[1 + \frac{9T|k|^2}{m_e z^2} + \mathcal{O}\left(\frac{|k|^4}{|z|^4}\right)\right], \quad as \quad \frac{|z|}{|k|} \to \infty \tag{2.9}$$

Proof. By integrating by parts consecutively

$$L(z,k) = -\frac{w_0}{|k|z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s(s\widehat{f}^0(\widehat{k}s))ds = -\frac{w_0}{z^2}\widehat{f}^0(0) - \frac{w_0|k|}{z^2}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^3(s\widehat{f}^0(\widehat{k}s))ds,$$

since the map $s \to \partial_s^2(s\widehat{f}^0(\widehat{k}s)$ is odd . Integrating by parts two more times one gets.

$$L(z,k) = \frac{-w_0}{z^2}\widehat{f}^0(0) - \frac{w_0|k|}{z^3}\left(\left[-\frac{|k|}{z}e^{-\frac{z}{k}s}\partial_s^3(s\widehat{f}^0(\widehat{k}s))\right]_{s=0}^{s=\infty} + \frac{|k|}{z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^4(s\widehat{f}^0(\widehat{k}s))ds\right)$$

$$= -\frac{w_0}{z^2}\widehat{f}^0(0) - \frac{w_0|k|^2}{z^4}\left(\partial_s^3(s\widehat{f}^0(\widehat{k}s))_{/s=0}\right) - \frac{w_0|k|^2}{z^4}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^4(s\widehat{f}^0(\widehat{k}s))ds.$$

Now $\partial_s^3(s\widehat{f}^0(\widehat{k}s)) = 3\widehat{k}.\nabla^2\widehat{f}^0(\widehat{k}s).\widehat{k} = 3\widehat{k}\otimes\widehat{k} : \nabla^2\widehat{f}^0(\widehat{k}s)$ because $\nabla^2\widehat{f}^0$ is symmetric. One gets therefore

$$L(z,k) = -\frac{w_0}{z^2}\widehat{f}^0(0) - 3\frac{w_0}{z^4}|k|^2\left(\widehat{k}\otimes\widehat{k} : \nabla^2\widehat{f}^0(\widehat{k}s)\right) - \frac{w_0}{z^4}|k|^2\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^4(s\widehat{f}^0(\widehat{k}s))ds$$

$$= -\frac{w_0}{z^2}\widehat{f}^0(0) - \frac{3w_0}{z^4}|k|^2\left(\widehat{k}\otimes\widehat{k} : \nabla^2\widehat{f}^0(0)\right) - \frac{3w_0|k|^2}{z^4}\zeta(z,k),$$

since $w_0\widehat{f}^0 = w_0 n_0 = w_p^2$ and $3n_0T := m_e\widehat{k}\otimes\widehat{k} : \nabla^2\widehat{f}^0(0)$.

It remains to show that for $z\in\Lambda_{\delta,\delta'}$ there holds $|\zeta(z,k)| \lesssim_{\delta'} \dfrac{|k|^2}{|z|^2}$. First consider the region $\Re(z) \geq -\delta'|k|$.

$$\forall\, \Re z \geq -\delta'|k|, \quad \zeta(z,k) = \int_0^\infty e^{-\frac{z}{|k|}}\partial_s^4(s\hat{f}^0(\hat{k}s))ds$$

$$= \frac{|k|}{z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^5(s\hat{f}^0(\hat{k}s))ds$$

$$= \frac{|k|^2}{z^2}(\partial_s^5(s\hat{f}^0(\hat{k}s)_{/s=0} + \int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^6(s\hat{f}^0(\hat{k}s))ds).$$

It implies $|\zeta(z,k)| \leq \frac{|k|^2}{|z|^2}\left(|\partial_s^5\hat{f}^0(\hat{k}s)_{/s=0}| + \displaystyle\int_0^\infty e^{\delta' s}|\partial_s^6(s\hat{f}^0(\hat{k}s))|ds\right) \lesssim_{\delta'} \dfrac{|k|^2}{|z|^2}$ $\big($since $\Re(z) \geq -\delta'|k| \Rightarrow \frac{\Re(z)}{|k|}s \leq \delta' s$ for s positive$\big)$.

Turn next to the region $\Re(z) < -\delta'|k|$ with $\Re(z) > -(1-\delta)|\Im z|$. Observe then $\arg z^2 \in [\frac{\pi}{2}+\beta, \frac{3\pi}{2}-\beta]$ for a small $\beta > 0$ depending on δ. Write

$$\zeta(z,k) = \int_{-\infty}^\infty e^{-\frac{z}{|k|}s}\partial_s^4\big(s\widehat{f}^0(\widehat{k}s)\big)ds - \int_{-\infty}^0 e^{-\frac{z}{|k|}s}\partial_s^4\big(s\widehat{f}^0\widehat{k}s)\big)ds := \zeta_1 + \zeta_2$$

On the one hand ζ_2 is bounded as in the region $\Re(z) \geq -\delta'|k|$ because $s < 0$ and we are working in the region $\Re(z) < -\delta'|k|$. On the other hand due to (1.2),

$$\zeta_1 = \int_{-\infty}^\infty e^{-\frac{z}{|k|}s}\partial_s^4\big(s\widehat{f}^0(\widehat{k}s)\big) = \frac{z^4}{|k|^4}\int_{-\infty}^\infty e^{-\frac{z}{|k|}s}s\widehat{f}^0(\widehat{k}s)ds$$

$$= \frac{n_0}{(2\pi)^{3/2}}\frac{z^4}{|k|^4}\int_{-\infty}^\infty se^{-\frac{z}{|k|}s - s^2\frac{T}{2m_e}ds} = \frac{n_0}{(2\pi)^{3/2}}\frac{z^4}{|k|^4}\int_{-\infty}^\infty se^{-\frac{T}{2m_e}(s+\dfrac{m_e z}{T|k|})^2 + \frac{m_e z^2}{2T|k|^2}}ds$$

$$= \frac{n_0}{(2\pi)^{3/2}} \frac{z^4}{|k|^4} e^{\frac{m_e z^2}{2T|k|^2}} \int_{-\infty}^{\infty} s e^{-\frac{T}{2m_e}(s+\frac{m_e z}{T|k|})^2} ds = -\frac{n_0 m_e z}{(2\pi)^{3/2} T|k|} \frac{z^4}{|k|^4} \sqrt{\frac{2m_e\pi}{T}} e^{\frac{m_e z^2}{2T|k|^2}}.$$

Now since we are working in the region $\Re(z) < -\delta'|k|$ with $\Re(z) > -(1-\delta)|\Im(z)|$ it holds $\Re(z^2) \lesssim_\delta -|\Im(z)|^2 \lesssim_\delta -|z|^2$ and this terms vanishes when $\dfrac{|z|}{|k|} \to \infty$.

For claiming this, just notice that, for $z = a + ib$ we have $z^2 = a^2 - b^2 + 2iab$ so that $a^2 - b^2 \leq -C_1 b^2 \leq -C_2(a^2+b^2)$ where $C_1 = C_1(\delta) > 0$ and $C_2 = C_2(\delta) > 0$.
For the first inequality, ($a^2 - b^2 \leq -C_1 b^2$) we are working in the area where

$$-(1-\delta)|b| < a < -\delta'|k| < 0 \;\; so \;\; a^2 < (1-\delta)^2 b^2 \;\; so \;\; a^2 - b^2 < -(1-(1-\delta))^2 b^2$$

and we obtain the first inéquality with $C(\delta) = 1 - (1-\delta)^2$. For the second inequality : as $z \in [\frac{\pi}{2} + \beta, \frac{3\pi}{2} - \beta]$ implies $z^2 \in \Lambda_{\delta,\delta'}$. So like before we have $\Re(z^2) = a^2 - b^2 < -\delta'|k| < 0$ so $a^2 < b^2$. But $C_1 b^2 \geq C_2(a^2+b^2) \Leftrightarrow C_1 b^2 - C_2 b^2 \geq C_2 a^2 \Leftrightarrow \left(\frac{C_1 - C_2}{C_2}\right) b^2 \geq a^2 \Leftrightarrow a^2 \leq \left(\frac{C_1}{C_2} - 1\right) b^2$ so in particular for C_2 which verifies $\frac{C_1}{C_2} - 1 > 1$ we get our inequality .

The next lemma gives and estimate of the resolvent for low frequencies in the half plane $\Re(z) \geq -\delta'|k|$ with δ' small enough and away from $\pm i w_p$. Given $\epsilon > 0$, define the following region

$$\mathbb{H}_{\epsilon,\delta'} := \{z = \lambda + iw \in \mathbb{C}; \lambda > -\delta'|k| \;\; and \;\; |z + \pm i w_p| \geqslant \epsilon\}.$$

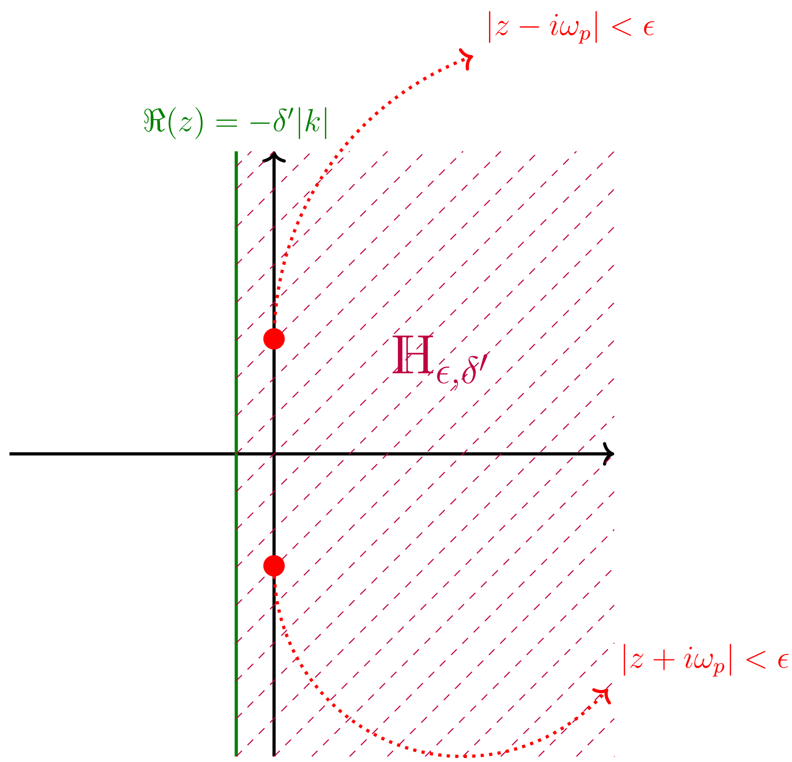


FIGURE 2 – The region $\mathbb{H}_{\epsilon,\delta'}$

Lemma 3 *(Resolvent estimates for low frequency). Given $\epsilon, \delta' > 0$, there are $\nu_0, \kappa > 0$ such that*

$$\forall |k| < \nu_0,\ \forall z \in \mathbb{H}_{\epsilon,\delta'}, |1 - L(z,k)| \geq \kappa.$$

Proof. Let $R > 0$ fixe independant of k.

Case 1 : $|z| > R|k|$. In this region the estimate follows from (2.9) taking R sufficiently large.

Case 2 : In this region the asymptotic expansion (2.9) is no longer useful and we use the Plemelj formula instead. Writing $z = \lambda + iw$, for $\lambda = 0$, we have by Plemelj formula in multidimension,

$$L(-iw,k) = \frac{w_0}{|k|^2}\int_{\mathbb{R}} \frac{(f_k^0)'(r)}{r - \frac{w}{|k|}}dr + i\frac{w_0\pi}{|k|^2}(f_k^0)'\left(\frac{w}{|k|}\right). \tag{2.10}$$

where, for any $k \neq 0$, the partial hyperplane average is defined as

$$\forall r \in \mathbb{R} \qquad f_k^0(r) := \int_{\frac{k}{|k|}r + k_\perp} f^0(v_*)dv_*$$

Subcase 2.1 : $|z| \leq R|k|$ and $c|k| \leq |w| \leq R|k|$. Given any $c > 0$, we deduce from decay and smoothness of f^0 that

$$\inf_{c<\frac{w}{|k|}<R} \big|(f_k^0)'\big(\frac{w}{|k|}\big)\big| \gtrsim_{c,R} \frac{1}{|k|^2} \geq 1.$$

Therefore $|\Im\big(L(z,k)\big)| \geq 1$ for $\Re(z) \in]-\delta'|k|, 0]$.The lengthier calculations for the last case are omitted for the sake of brevity.

Notice that thes different cases above together prove that $\dfrac{1}{1-L}$ is bounded from above on the strip $\Re(z) \in]\delta'|k|, 0]$ and outside $B(0, R|k|) \cap \{\Re(z) > 0\}$, and since there are no poles within the remaining region $B(0, R|k|) \cap \{\Re(z) > 0\}$, the function is holomorphic in this region and the upper bound is also valid there by the maximum principle.

2.3 Branches of poles of the dispertion function L(z,k)

From Lemma 2, $(1 - L(z, 0))^{-1}$ has 2 poles at $(iw_p, 0)$ and $(-iw_p, 0)$. Given $\epsilon > 0$, define

$$F := 1 - L(z, 0) \qquad G := 1 - L(z, k).$$

We know from Lemma 2 that

$$|F - G| = |L(z, k) - L(z, 0)| \lesssim |k|^2, \ \ \textit{on the set} \ \ |z \pm iw_p| \leq \epsilon$$

Indeed $\quad |L(z,k) - L(z,0)| = \left| \dfrac{9Tw_p^2|k|^2}{z^4 m_e} + \dfrac{w_p^2}{z^2}\mathcal{O}\Big(\dfrac{|k|^4}{z^4}\Big) \right| \lesssim |k|^2$ because $\dfrac{|z|}{|k|} \to \infty$.

On the other hand

$$|1 - L(z, 0)| = |1 + \frac{w_p^2}{z^2}| \geq \epsilon.$$

So we obtain,

$|F - G| = |L(z, k) - L(z, 0)| < |k|^2 < \epsilon < |1 - L(z, 0)| = |F|$ provided that $|k| \ll \epsilon$.

So $(1 - L(z, k))^{-1}$ and $(1 - L(z, 0))^{-1}$ have the same number of pole in $|z \pm iw_p| \leq \epsilon$

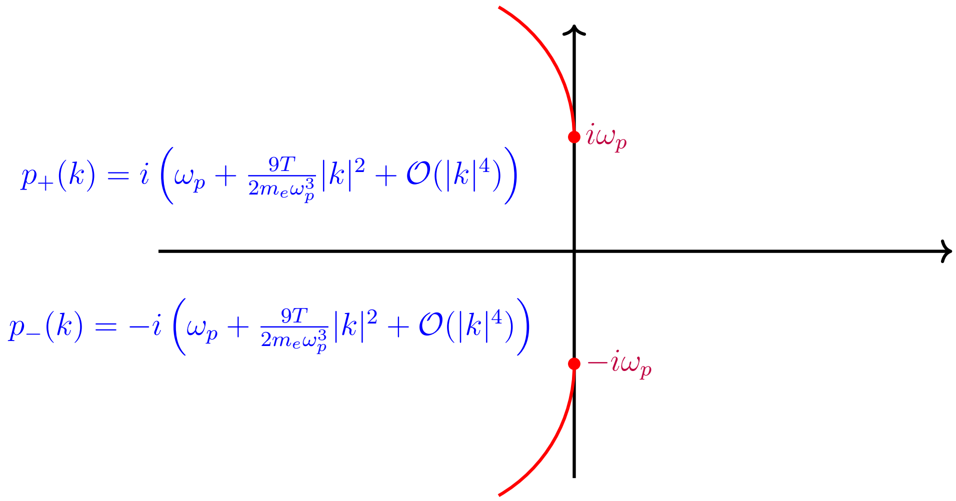


FIGURE 3 – The branch of poles $k \to p_\pm(k)$

We next use the implicit function theorem to construct the branches of poles $p_\pm(k)$.

Lemma 4 *There are $\epsilon, \nu_0 > 0$ such that for all $|k| < \nu_0$, there are unique $p_\pm(k) \in \mathbb{C}$ solutions to $L(p_\pm(k), k) = 1$ in $\{|z \mp iw_p| < \epsilon\}$. Moreover $k \to p_\pm(k) := -\lambda(k) \pm i\Omega(k)$ and $p_\pm(k) \sim_{k\to 0} \pm iw_p$ with the following expansions as $k \to 0$*

$$\Omega(k)^2 = w_p^2 + \frac{9T}{m_e w_p^2}|k|^2 + \mathcal{O}\big(|k|^4\big), \tag{2.11}$$

$$\nabla\Omega(k) = i\frac{3T}{m_e w_p}k + \mathcal{O}\big(|k|^3\big), \tag{2.12}$$

$$\nabla^2\Omega(k) = i\frac{3T}{m_e w_p}Id + \mathcal{O}\big(|k|^2\big), \tag{2.13}$$

$$|\nabla^j\lambda(k)| \lesssim_{j,N} |k|^N for \;\; any \; j, N \in \mathbb{N}. \tag{2.14}$$

Proof. Since $f^0(v)$ is an even real function, $\widehat{f^0}(k)$ is also real, and from the definition of $L(z,k)$ it is easy to check that if $L(p_+(k), k) = 1$ we have that $L(\bar{p}_+, k) = 1$ so $p_-(k) = \bar{p}_+(k)$ and it is enough to build the branch near iw_p. We aim to use the implicit function theorem applied to the function $L(z,k)$, the result will follow by verifying that $\partial_z L(iw_p, 0) \neq 0$, since L is smooth in this neighborhood. From (2.3) and integrating by parts, we get

$$\partial_z L(z,k) = \frac{w_0}{|k|^3}\int_0^\infty e^{-\frac{z}{|k|}s} s^2 \widehat{f^0}(\widehat{k}s)ds$$

$$= \frac{w_0}{|k|^3}\Big[-\frac{|k|}{z} e^{-\frac{z}{|k|}s} s^2 \widehat{f^0}(\widehat{k}s)ds\Big)_{s=0}^{s=\infty} + \frac{|k|}{z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s(s^2\widehat{f^0}(\widehat{k}s))ds\Big]$$

$$
\begin{aligned}
&= \frac{w_0}{|k|^2 z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s(s^2\widehat{f}^0(\widehat{k}s)ds\\
&= \frac{w_0}{|k|z}\Big[-\frac{|k|}{z}e^{-\frac{z}{|k|}s}\partial_s(s^2\widehat{f}^0(\widehat{k}s)\Big)_{s=0}^{s=\infty}+\frac{|k|}{z}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^2(s^2\widehat{f}^0(\widehat{k}s)ds\Big]\\
&= \ldots\\
&= \frac{2w_0 n_0}{z^3}+\frac{6w_0|k|}{z^4}\big[\widehat{k}.\nabla\widehat{f}^0(0)\big]+\frac{w_0|k|}{z^4}\int_0^\infty e^{-\frac{z}{|k|}s}\partial_s^4(s^2\widehat{f}^0(\widehat{k}s))ds.
\end{aligned}
$$

Eventually we get $\partial_z L(iw_p,0)=\dfrac{2i}{w_p}\neq 0$. Then by the implicit function theorem we get the existence of a unique smooth solution $k\to p_+(k)$ to the equation $L(p_+(k),k)=1$ in a neighborhood of iw_p. More precisely from Lemma 2

$$
L(z,k)=-\frac{w_p^2}{z^2}\Big[1+\frac{9T}{m_e z^2}|k|^2+\mathcal{O}\Big(\frac{|k|^4}{z^4}\Big)\Big].
$$

So $p_+^2=p_+^2L(p_+(k),k)=-w_p^2+\dfrac{9T}{m_e p_+^2}|k|^2+\mathcal{O}(|k|^4)=-w_p^2+\mathcal{O}(|k|^2)+\mathcal{O}(|k|^4)$

$$
\begin{aligned}
&= -w_p^2+\frac{9T}{m_e(-w_p^2+\mathcal{O}(|k|^2)}+\mathcal{O}(|k|^4)\\
&= -w_p^2-\frac{9T}{m_e w_p^2}|k|^2+\mathcal{O}(|k|^4)\\
&= (iw_p)^2\Big[1+\frac{9T}{m_e w_p^4}|k|^2+\mathcal{O}(|k|^4)\Big].
\end{aligned}
$$

Then,

$$
p_+(k)=iw_p\Big(1+\frac{9T}{m_e w_p^4}+\mathcal{O}(|k|^4)\Big)^{1/2}=i\Big(w_p+\frac{9T}{2m_e w_p^3}|k|^2+\mathcal{O}(|k|^4)\Big),
$$

where we have used that $(1+x)^{1/2}=1+\dfrac{1}{2}x+\mathcal{O}(x)$ near 0. Next observe that

$$
L(p_+(k),k)=1 \quad and \quad \nabla_k L(p_+(k),k)=0,
$$

This yields the following result

$$
\nabla p_+(k)=-\frac{\nabla_k L(p_+(k),k)}{\partial_z L(p_+(k),k)}. \tag{2.15}
$$

Now $L(z,k)=-w_0\displaystyle\int_0^\infty t\widehat{f}^0(kt)e^{-zt}dt$ so that

$$
\nabla_k L(z,k)=-w_0\int_0^\infty t^2\nabla_k\widehat{f}^0(kt)e^{-zt}dt=-w_0\int_0^\infty\frac{s^2}{|k|^2}\nabla_k\widehat{f}^0(\widehat{k}s)e^{-\frac{z}{|k|}s}\frac{1}{|k|}ds
$$

$$= -\frac{w_0}{|k|^3}\int_0^\infty s^2 \nabla_k \widehat{f}^0(\widehat{k}s)e^{-\frac{z}{|k|}s}ds.$$

By the same method used in Lemma 2 we obtain

$$\nabla_k L(z,k) = \frac{2w_0 k}{z^4}\left[\widehat{k}\otimes\widehat{k} : \nabla^2 \widehat{f}^0(0)\right] = \frac{2w_0 k}{z^4}\times\frac{3n_0 T}{m_e} + \mathcal{O}(|k|^3) = \frac{6w_p^2 T k}{z^4 m_e}.$$

This implies

$$\nabla_k L(p_+(k),k) = \frac{6Tk}{w_p^2 m_e} + \mathcal{O}(|k|^3),$$

and since $\partial_z L(p_+(k),k) = \dfrac{2i}{w_p}$, this yields

$$\nabla p_+(k) = -\frac{\nabla_k L(p_+(k),k)}{\partial_z L(p_+(k),k)} = i\frac{3T}{m_e w_p}k + \mathcal{O}(|k|^3).$$

Next observe that

$$\begin{aligned}\nabla_k\left[(\nabla_k L)(p_+,k)\right] &= (\nabla_k p_+).(\nabla_k \partial_z L)(p_+,k) + (\nabla_k^2 L)(p_+,k)\\ \nabla_k\left[(\partial_z L)(p_+,k)\right] &= \nabla_k p_+.\partial^2_{z^2}L)(p_+,k) + (\nabla_k\partial_z L)(p_+,k).\end{aligned}$$

We get from (2.12) that $\nabla^2 p_+$ is equal to the following expression

$$\frac{(\nabla_k^2 L)(P_+,k) + 2(\nabla_k\partial_z L)(P_+,k).\nabla p_+ + (\partial^2_{z^2}L)(p_+,k)|\nabla p_+|^2 + (\nabla_k\partial_z L)(P_+,k).(\nabla p_+)}{(\partial_z L)(p_+,k)}. \tag{2.16}$$

The proof of (2.16) is similar to (2.12) but using (2.16) instead of (2.15). If we continue the computations of $\nabla_k^m p_+(k)$ for $m \in \mathbb{N}$, we see that in $\nabla_k^m L(z,k)$ (respectively $\nabla_k^m \partial_z L(z,k)$), the leading order as $k \to 0$ is an even (respectively odd) power of z^{-1}. Thus at $z = \pm i w_p$ all derivatives $\nabla_k^m L(iw_p, 0)$ are purely real (respectively purely imaginary). We can see for exemple that if $m = 0$, $\partial_z L(iw_p,0) = \dfrac{2i}{w_p}$ which is purely imaginary and from the expression of $\nabla_k L(z,k)$ then $\nabla_k L(iw_p,0)$ is purely real which let us conclude that all the derivatives $\nabla_k^m p_+(0)$ are purely imaginary (for example see expressions (2.12) and (2.13). Since $p_+(k) = iw_p + \mathcal{O}(|k|^2) = -i\lambda(k) + i\Omega(k)$ we obtain that

$$|\nabla^j \lambda(k)| \lesssim |k|^N.$$

2.4 Extraction of Klein-Gordon waves

The equation (2.1) is equivalent to $\widehat{\rho}(t,k)+w_0\big(t\widehat{f^0}(kt)*\widehat{\rho}(t,k)\big) = \widehat{h_{in}}(k,kt)$. The solution to this equation is (the convolution is in the first variable)

$$\widehat{\rho}(t,k) = \widehat{h_{in}}(k,kt) + \mathcal{R}(t,k) * \widehat{h_{in}}(k,kt) \tag{2.17}$$

where $\mathcal{R}(t,k) = -r(t,k)$ and r is the solution of

$$r(t,k) + w_0\big(t\widehat{f^0}(kt)\big) * r(t,k) = w_0 t\widehat{f^0}(kt).$$

The Fourier-Laplace transform yields

$$\hat{r}(z,k) - L(z,k)\hat{r}(z,k) = -L(z,k).$$

It implies $\hat{r}(z,k) = -\dfrac{L(z,k)}{1-L(z,k)}$ and $\mathcal{R}(t,k) = \dfrac{1}{2i\pi}\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} \dfrac{L(z,k)}{1-L(z,k)}e^{zt}dt$, for a suitable Bromwish contour such that $z \to \dfrac{L(z,k)}{1-L(z,k)}$ is holomorphic for $\Re(z) \geq \gamma$. The calculations in the two previous subsections 2.2-2.3 show that for $|k| < \nu_0$ sufficiently small, $\dfrac{L(.,k)}{1-L(.,k)}$ is holomorphic in the region $\mathbb{H}_{\epsilon,\delta'}$ represented in figure 2 (the half plane $\Re(z) > -\delta'|k|$ minus a disc of raduis ϵ around each pole) with one isolated pole $p_{\pm}(k)$ in each disc, depending on k as studied in the last subsection.

Let us define f and Δ as follow

$$f(z,k) := \frac{L(z,k)}{1-L(z,k)}e^{zt},$$

and $\Delta = [\gamma' - i\infty, \gamma - i\infty] \cup [\gamma - i\infty, \gamma + i\infty] \cup [\gamma + i\infty, \gamma' + i\infty] \cup [\gamma' + i\infty, \gamma' - i\infty]$.

By Cauchy's Residue theorem, we get

$$\frac{1}{2i\pi}\int_{\Delta} f(z,k)dz = Res(f(z,k),p_+)Ind(f(z,k),p_+) + Res(f(z,k),p_-)Ind(f(z,k),p_-)$$
$$= Res(f(z,k),p_+) + Res(f(z,k),P_-),$$

so that

$$\mathcal{R}(t,k) = Res(f(z,k),p_+) + Res(f(z,k),p_-) - \frac{1}{2i\pi}\int_{\gamma'-i\infty}^{\gamma-i\infty} f(z,k)dz - \frac{1}{2i\pi}\int_{\gamma+i\infty}^{\gamma'+i\infty} f(z,k)dz$$
$$-\frac{1}{2i\pi}\int_{\gamma'+i\infty}^{\gamma'-i\infty} f(z,k)dz.$$

Now $Res(f(z,k),p_\pm(k)) = \underset{z\to p_\pm(k)}{lim} \dfrac{(z-p_\pm)L(z,k)e^{zt}}{1-L(z,k)}$

$$= e^{p_\pm(k)t}L(p_\pm(k),k) \underset{z\to p_\pm(k)}{lim} \frac{z-p_\pm(k)}{1-L(z,k)}$$

$$= e^{p_\pm(k)t}L(p_\pm(k),k) \underset{z\to p_\pm(k)}{lim} \frac{z-p_\pm(k)}{L(p_\pm(k),k)-L(z,k)}$$

$$= \frac{e^{p_\pm(k)t}}{-\partial_z L(p_\pm(k),k)},$$

Since the two other integrals vanishes, so

$$\begin{aligned}\mathcal{R}(t,k) &= \frac{e^{p_+(k)t}}{-\partial_z L(p_+(k),k)} + \frac{e^{p_-(k)t}}{-\partial_z L(p_-(k),k)} + \frac{1}{2i\pi}\int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{zt}\frac{L(z,k)}{1-L(z,k)}ds \\ &:= \mathcal{R}^+_{KG}(t,k) + \mathcal{R}^-_{KG}(t,k) + \mathcal{R}_{RFT}(t,k),\end{aligned}$$

for γ' so that the vertical line is to the left of the poles $p_\pm(k)$ but still in $\mathbb{H}_{\epsilon,\delta'}$. This decomposes the resolvent $\mathcal{R} = \mathcal{R}_{KG} + \mathcal{R}_{LD}$ into a Klein-Gordon part and a remainder free transport part. We will only do the calculus of the first integral since the computation of the second one is similar. Recall that

$$L(z,k) = -\frac{w_p^2}{z^2} + \mathcal{O}(|k|^2)$$

and parameterize the line between $\gamma + iR$ and $\gamma' + iR$ as follow

$\Gamma(x) = (1-x)(\gamma+iR) + x(\gamma'+iR) = \gamma(1-x) + x\gamma' + iR$, we get

$$\int_{\gamma+iR}^{\gamma'+iR} \frac{L(z,k)}{1-L(z,k)}e^{zt}dz = \int_0^1 e^{\Gamma(x)t}\frac{L(\Gamma(x),k)}{1-L(\Gamma(x),k)}\Gamma'(x)dx \underset{R\to\infty}{\longrightarrow} 0.$$

Note that we used that $|L(\Gamma(x),k)| \underset{R\to\infty}{\longrightarrow} 0$ which lead us to the corresponding expression of $\mathcal{R}(t,k)$. This yileds $\mathcal{R} = \mathcal{R}_{KG} + \mathcal{R}_{LD}$ where

$$R_{KG}(t,k) := \mathcal{R}^+_{KG}(t,k) + \mathcal{R}^-_{KG}(t,k) \qquad \text{and} \qquad \mathcal{R}_{LD}(t,k) := \mathcal{R}_{RFT}(t,k).$$

This also yields the corresponding decomposition of $\widehat{\rho}(t,k)$ through (2.17)

$$\begin{aligned}\widehat{\rho}(t,k) &= \widehat{h_{in}}(k,kt) + \int_0^t \mathcal{R}_{KG}(t-\tau,k)\widehat{h_{in}}(k,k\tau)d\tau + \int_0^t \mathcal{R}_{RFT}(t-\tau,k)\widehat{h_{in}}(k,k\tau)d\tau \\ &= \widehat{\rho}_{FT}(t,k) + \widehat{\rho}^+_{KG}(t,k) + \widehat{\rho}^-_{KG}(t,k) + \widehat{\rho}_{RFT}(t,k).\end{aligned}$$

We first prove a general expansion of $\widehat{\rho}^\pm_{KG}(t,k)$ by succesive integrations in time.

Lemma 5 *(Expansion of the Klein-Gordon density). For all $|k| < \nu_0$ and all $l \in \mathbb{N}$, we have*

$$\widehat{\rho}^{\pm}_{KG}(t,k) = \sum_{j=0}^{j=l} e^{p_\pm(k)t} A_j^{\pm}(k)\big[k^{\otimes j} : \nabla_\eta^j \widehat{h_{in}}(k,0)\big] - \sum_{j=0}^{j=l} A_j^{\pm}(k)\big[k^{\otimes j} : \nabla_\eta^j \widehat{h_{in}}(k,kt)\big]$$
$$+ \int_0^t e^{p_\pm(k)(t-\tau)} A_l^{\pm}(k)\Big[k^{\otimes(l+1)} : \nabla_\eta^{l+1}\widehat{h_{in}}(k,k\tau)\Big] d\tau, \tag{2.18}$$

where $\nabla_\eta \widehat{h_{in}}(k,\eta)$ is the differential in the second Fourier variable, and with the notation

$$A_j^{\pm}(k) := \frac{J_\pm(k)}{p_\pm(k)^{j+1}} \quad and \quad J_\pm(k) := -\frac{1}{\partial_z L(p_\pm(k),k))}.$$

Proof. $$\widehat{\rho^{\pm}_{KG}}(k,t) = \int_0^t \frac{e^{p_\pm(k)(t-\tau)}}{-\partial_z L(p_+(k),k)} \widehat{h_{in}}(k,k\tau)d\tau = \int_0^t J_\pm(k) e^{p_\pm(k)(t-\tau)} \widehat{h_{in}}(k,k\tau)$$
$$= -\int_0^t \frac{J_+(k)}{p_\pm(k)} \partial_\tau\big(e^{p_\pm(k)(t-\tau)}\big)\widehat{h_{in}}(k,k\tau)d\tau$$
$$= -\frac{J_\pm(k)}{p_\pm(k)}\Bigg(\big[\widehat{h_{in}}(k,k\tau)e^{p_\pm(k)(t-\tau)}\big]_{\tau=0}^{\tau=t} - \int_0^t e^{p_\pm(k)(t-\tau)}\partial_\tau \widehat{h_{in}}(k,k\tau)d\tau\big]\Bigg)$$
$$= \frac{J_\pm(k)}{p_\pm(k)}\widehat{h_{in}}(k,0)e^{p_\pm(k)t} - \frac{J_\pm(k)}{p_\pm(k)}\widehat{h_{in}}(k,kt) + \frac{J_\pm(k)}{p_\pm(k)}\int_0^t e^{p_\pm(k)(t-\tau)}\partial_\tau\big(\widehat{h_{in}}(k,k\tau)\big)d\tau.$$

And iterating finitely many times yields the result.

Furthermore note that $\overline{J_-(k)} = J_+(k)$ and $\overline{A_-(k)} = A_+(k)$, so that the computations made in the proof of Lemma 4 give $J_\pm(k) = \mp\frac{w_p}{2i} + \mathcal{O}(|k|^2)$. Precisely,

$$J_+(k) = -\frac{1}{\partial_z L(p_+(k),k)} = -\frac{p_+(k)^3}{2w_p^2} = -\frac{(iw_p + \mathcal{O}(|k|^2))^3}{2w_p^2} = \frac{iw_p^3(1+\mathcal{O}(|k|^2)}{2w_p^2}$$
$$= \frac{-w_p}{2i} + \mathcal{O}(|k|^2),$$

where we used the fact $(1+x)^a = 1 + ax + \mathcal{O}(x)$ near 0. A similar computation gives the expansion of $J_-(k)$. Denoting $p_\pm(k) = -\lambda(k) + i\Omega(k)$ with $\lambda(k) > 0$ and $\Omega(k) = w_p + \mathcal{O}(|k|^2$ it implies

Lemma 6 *One has as $k \to 0$,*

$$A_0^+(k) + A_0^-(k) = 1 + \mathcal{O}(|k|^2) \tag{2.19}$$
$$A_1^+(k) + A_1^-(k) = \mathcal{O}(|k|^2)$$
$$e^{p_+(k)t}A_0^+(k) + e^{p_-(k)t}A_0^-(k) = e^{-\lambda(k)t}\big[cos[\Omega(k)t] + \mathcal{O}(|k|^2)e^{i\Omega(k)t} + \mathcal{O}(|k|^2)e^{-i\Omega(k)t}\big]$$
$$e^{p_+(k)t}A_1^+(k) + e^{p_-(k)t}A_1^-(k) = e^{-\lambda(k)t}\left[\frac{sin[\Omega(k)t]}{\Omega(k)} + \mathcal{O}(|k|^2)e^{i\Omega(k)t} + \mathcal{O}(|k|^2)e^{-i\Omega(k)t}\right]$$

Proof. by definition of A,

$$A_0^+(k) + A_0^-(k) = \frac{J_+(k)}{p_+(k)} + \frac{J_-(k)}{p_-(k)} = \frac{\frac{-w_p}{2i} + \mathcal{O}(|k|^2)}{iw_p + \mathcal{O}(|k|^2)} + \frac{\frac{w_p}{2i} + \mathcal{O}(|k|^2)}{-iw_p + \mathcal{O}(|k|^2)}$$
$$= \frac{-w_p + \mathcal{O}(|k|^2)}{-2w_p + \mathcal{O}(|k|^2)} + \frac{w_p + \mathcal{O}(|k|^2)}{2w_p + \mathcal{O}(|k|^2)}$$
$$= \frac{-w_p + \mathcal{O}(|k|^2)}{-2w_p(1 + \mathcal{O}(|k|^2))} + \frac{w_p + \mathcal{O}(|k|^2)}{2w_p(1 + \mathcal{O}(|k|^2))},$$

and using that near 0, $\frac{1}{1 \pm x} = 1 \mp x + \mathcal{O}(x)$ we get the result.

For the second equality,

$$A_1^+(k) + A_1^-(k) = \frac{J_+(k)}{p_+(k)^2} + \frac{J_-(k)}{p_-(k)^2} = \frac{\frac{-w_p}{2i} + \mathcal{O}(|k|^2)}{\big(iw_p + \mathcal{O}(|k|^2)\big)^2} + \frac{\frac{w_p}{2i} + \mathcal{O}(|k|^2)}{\big(-iw_p + \mathcal{O}(|k|^2)\big)^2}$$
$$= \frac{-w_p + \mathcal{O}(|k|^2)}{-2iw_p^2(1 + \mathcal{O}(|k|^2))} + \frac{w_p + \mathcal{O}(|k|^2)}{-2iw_p^2(1 + \mathcal{O}(|k|^2))}$$
$$= \mathcal{O}(|k|^2),$$

and

$$e^{p_+(k)t}A_0^+(k) + e^{p_-(k)t}A_0^-(k) = e^{-\lambda(k)t + i\Omega(k)t}A_0^+(k) + e^{-\lambda(k)t - i\Omega(k)t}A_0^-(k)$$
$$= e^{-\lambda(k)t}\big[e^{i\Omega(k)t}A_0^+(k) + e^{-i\Omega(k)t}A_0^-(k)\big]$$
$$= e^{-\lambda(k)t}\big[e^{i\Omega(k)t}(1/2 + \mathcal{O}(|k|^2) + e^{-i\Omega(k)t}(1/2 + \mathcal{O}(|k|^2)\big]$$
$$= e^{-\lambda(k)t}\left[\frac{1}{2}e^{i\Omega(k)t} + \mathcal{O}(|k|^2)e^{i\Omega(k)t} + \frac{1}{2}e^{-i\Omega(k)t} + \mathcal{O}(|k|^2 e^{-i\Omega(k)t}\right]$$
$$= e^{-\lambda(k)t}[cos(\Omega(k)t) + \mathcal{O}(|k|^2)e^{i\Omega(k)t} + \mathcal{O}(|k|^2)e^{-i\Omega(k)t}].$$

$$e^{p_+(k)t}A_1^+(k) + e^{p_-(k)t}A_1^-(k) = e^{-\lambda(k)t + i\Omega(k)t}\left[\frac{1}{2iw_p} + \mathcal{O}(|k|^2)\right] + e^{-\lambda(k)t - i\Omega(k)t}\left[\frac{-1}{2iw_p} + \mathcal{O}(|k|^2)\right]$$
$$= \ldots$$
$$= e^{-\lambda(k)t}\left[\frac{sin[\Omega(k)t]}{\Omega(k)} + \mathcal{O}(|k|^2)e^{i\Omega(k)t} + \mathcal{O}(|k|^2)e^{-i\Omega(k)t}\right].$$

The following definition yields the decomposition in theorem 1.

Definition 1 (decomposition of the electric field.)

$$\widehat{E}^{(1;l)}_{LD}(t,k) = -w_0\frac{ik}{|k|^2}\left[\widehat{h_{in}}(k,kt) - \sum_{j=1}^{l} A^{\pm}_j(k)\big(k^{\otimes j} : \nabla^j_\eta \widehat{h_{in}}(k,kt)\big)\right]$$

$$\widehat{E}^{(2)}_{LD}(t,k) = -w_0\frac{ik}{|k|^2}\int_0^t \mathcal{R}_{RFT}(t-\tau,k)\widehat{h_{in}}(k,k\tau)d\tau$$

$$\widehat{E}^{(1;l)}_{KG}(t,k) = -w_0\frac{ik}{|k|^2}\left[\sum_{j=0}^{l} e^{p_+(k)t}A^{+}_j(k)\big(k^{\otimes j} : \nabla^j_\eta \widehat{h_{in}}(k,0)\big) + \sum_{j=0}^{l} e^{p(k)t}A_j(k)\big(k^{\otimes j} : \nabla^j_\eta \widehat{h_{in}}(k,0)\big)\right]$$

$$\widehat{E}^{(2;l)}_{KG}(t,k) = -w_0\frac{ik}{|k|^2}\int_0^t e^{p_\pm(k)(t-\tau)}A^{\pm}_l(k)\big[k^{\otimes(l+1)} : \nabla^{l+1}_\eta \widehat{h_{in}}(k,k\tau)\big]d\tau$$

and accordingly define the particular decomposion for $l = 4$

$$E^{(1)}_{LD} := E^{(1;4)}_{LD}, \qquad E^{(2)}_{LD} :=\text{as above} \qquad E_{LD} = E^{(1)}_{LD} + E^{(2)}_{LD},$$

$$E^{(1)}_{KG} := E^{(1;4)}_{KG}, \qquad E^{(2)}_{KG} := E^{(2;4)}_{KG}, \qquad E_{KG} = E^{(1)}_{KG} + E^{(2)}_{KG}.$$

Next, we estimate the remainder free transport part of the resolvent $\mathcal{R}_{RFT}$. The gain in powers of k present in lemma 7 is critical to the high quality decay rate of landau damping electric field (check lemma 10)

Lemma 7 *There exists $\lambda_0 > 0$ such that for all $|k| < \nu_0$ there holds,*

$$\forall |k| < \nu_0, \quad |\mathcal{R}_{RFT}(t,k)| \lesssim |k|^3 e^{-\lambda_0|k|t}.$$

Proof. Define $\alpha := 9Tw_p^2 m_e^{-1}$, and $\quad Q(z,k) = \dfrac{w_p^2}{z^2+w_p^2} + \dfrac{\alpha|k|^2}{(z^2+w_p^2)^2}.$

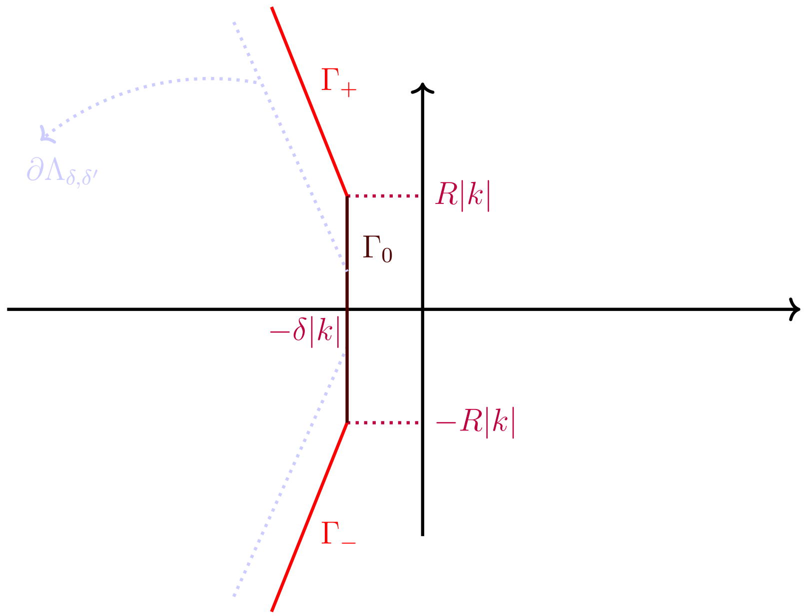


FIGURE 4

where

$$\Gamma_0 = \big\{ z = \lambda + iw : \lambda = -\delta|k|, w \in]-R|k|, R|k|[\big\}$$
$$\Gamma_+ = \big\{ z = \lambda - i(1+\delta)\lambda + i[R - (1+\delta)\delta]|k| : \lambda \in]-\infty, -\delta|k|] \big\}$$
$$\Gamma_- = \big\{ z = \lambda + i(1+\delta)\lambda - i[R - (1+\delta)\delta]|k| : \lambda \in]-\infty, -\delta|k|] \big\}.$$

The function $z \to e^{zt}Q(z,k)$ is holomorphic in the left plane $\Re z < -\gamma'|k|$, and hence we deduce by deforming the contour suitably,

$$\mathcal{R}_{RFT}(t,k) = \frac{1}{2i\pi}\Big(\int_{\Gamma_0} + \int_{\Gamma_+} + \int_{\Gamma_-} \Big) e^{zt} \Big(\frac{L(z,k)}{1 - L(z,k)} + Q(z,k) \Big) dz$$
$$=: \mathcal{R}^0_{RFT} + \mathcal{R}^+_{RFT} + \mathcal{R}^-_{RFT}.$$

Roughly speaking, we used Cauchy's Residue theorem and the fact that there is no poles to the left of γ'.

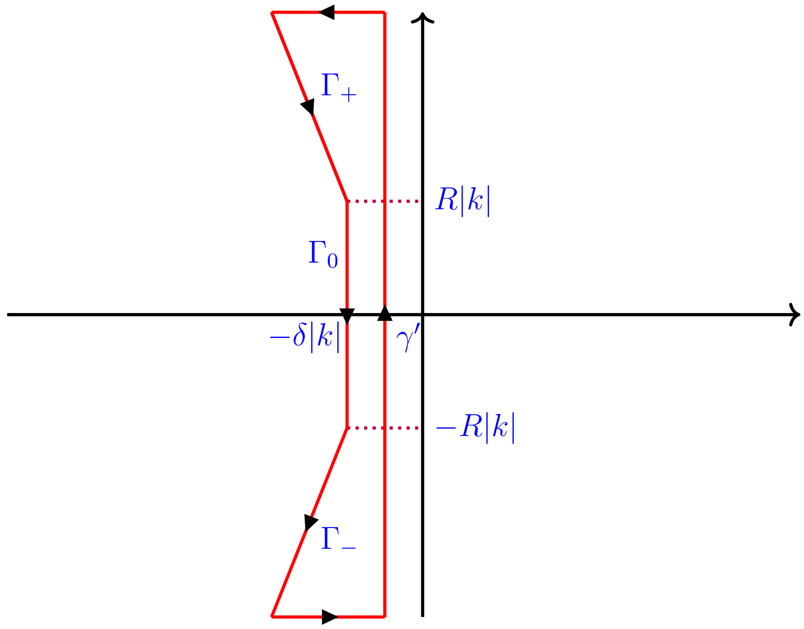


FIGURE 5 – The contour of integration

We separate cases as in Lemma 3. Consider first $z \in \Gamma_0$ with $|\Im(z)| < \delta'|k|$ so $z = -\delta|k| + iw$ and hence $|z| \leq \sqrt{\delta^2|k|^2 + \delta'^2|k|^2}$, Then $z = \mathcal{O}(|k|)$. From the proof of Lemma 3, for all $z \in \Gamma_0$, with $|\Im(z)| < \delta'|k|$, $L(-\Im(z), k) = \dfrac{w_0}{|k|^2} + \mathcal{O}\left(\dfrac{|\Im(z)|}{|k|^3}\right)$

so that for all $|z| \lesssim |k|$,

$$|L(z,k)| \leq |k|^{-2},$$

and by Taylor expansion near 0,

$$\begin{aligned}|\mathcal{R}_{RFT}(t,k)| &\leq \int_{\Gamma_0} \left| e^{zt} \frac{z^2 L(z,k) + w_p^2}{(1 - L(z,k))(z^2 + w_p^2} + \frac{\alpha|k|^2}{(z^2 + w_p^2)^2} \right| dz \\ &\lesssim e^{-\delta'|k|t} \int_{-\delta'|k|}^{\delta'|k|} \mathcal{O}(|k|^2) dw \\ &\lesssim |k|^3 e^{-\delta'|k|t}.\end{aligned}$$

Now we study the case $\delta'|k| \leq |\Im(z)| \leq R|k|$. From Lemma 3 we have $|1 - L(z,k)| \geq |k|^{-2}$ and $|L(z,k)| \lesssim |k|^{-2}$. So with a similar method done in the first case, the integrand in $R_{RFT}(t,k)$ is $\mathcal{O}(|k|^2)$ which lead us to result. This complete the estimates on Γ_0.

For Γ_+ and Γ_- we use a similar method as the one used in the decomposition of L in Lemma 2

$$L(z,k) = \frac{-w_p^2}{z^2} - \frac{9w_p^2 T|k|^2}{m_e z^4} + \mathcal{O}\left(\frac{|k|^4}{|z|^6}\right),$$

which yields the result.

In the next lemma we estimate the whole resolvent at bounded frequencies away from zero.

Lemma 8 *(Non-small frequencies resolvent estimate). Given any $\nu_0 > 0$ there is $\lambda > 0$ such that*

$$\forall |k| \geq \frac{\nu_0}{2}, \quad \mathcal{R}(t,k) \lesssim \frac{1}{|k|} e^{-\lambda |k| t}.$$

Proof. We take λ that verifies lemma 1 from which we know that there is no pole in the region $\Re(z) \geq -\lambda|k|$ and we deforme the contour to get

$$\mathcal{R}(t,k) = \frac{1}{2i\pi} \int_{-\lambda|k|-i\infty}^{-\lambda|k|+i\infty} e^{zt} \frac{L(z,k)}{1 - L(z,k)} dz.$$

Using the estimate of Lemma 1 gives

$$|\mathcal{R}(t,k)| \lesssim e^{-\lambda|k|t} \int_{-\infty}^{+\infty} \frac{1}{|k|^2 + |w|^2} dw \lesssim \frac{1}{|k|} e^{-\lambda|k|t}.$$

Since $L(-\lambda|k| + iw, k) \lesssim \dfrac{1}{1 + |k|^2 + w^2}$, we get

$$\left| \frac{L(-\lambda|k| + iw, k)}{1 - L(-\lambda|k| + iw, k)} \right| \lesssim \frac{1}{|k|^2 + |w|^2}.$$

2.5 The Klein-Gordon part of the electric field

In this part we will prove theorem 2, the proof is based on definition 1 and the previous estimates of this section.

Proof. For any field F we define $F_\epsilon(t,x) := \dfrac{1}{\epsilon^3} F(t, \dfrac{x}{\epsilon})$ with fourier transform

$$\widehat{F_\epsilon}(t,k) = \widehat{F}(t, \epsilon k).$$

Note that we have defined h_{in} in theorem 2 such that for all $\epsilon > 0, h_{in,\epsilon}(x,v) = \mathcal{H}_0(x,v)$. We also define

$$j_{in}(x) := \frac{1}{n_0} \int_{\mathcal{R}^3} v h_{in} dv$$

with fourier transform $\widehat{j_{in}}(k) = \dfrac{1}{n_0} i \nabla_\eta \widehat{h_{in}}(k, 0)$.

Denote $\widehat{J}(k) = \dfrac{1}{n_0} i\nabla_\eta \widehat{\mathcal{H}}_0(k)$ and recall the system(1.4)

$$\begin{cases} \partial_t^2 \mathcal{E}(t,x) + (w_p^2 - \dfrac{9T}{m_e}\Delta)\mathcal{E}(t,x) = 0, \\ \mathcal{E}(0,x) = E_0(x), \\ \partial_t \mathcal{E}(0,x) = -n_0 \nabla_x \big(\displaystyle\int v h_{in} dv \big) = -n_0 \nabla_x (n_0 j_{in}(x)) \end{cases}$$

with $E_0 = \dfrac{q^2 n_0}{\epsilon_0 m_e} \nabla_x (-\Delta_x)^{-1} \displaystyle\int_{\mathbb{R}^3} \mathcal{H}_0(.,v) dv.$

Applying Fourier transform to (1.4) yields

$$\begin{cases} \partial_t^2 \widehat{\mathcal{E}}(t,k) + (w_p^2 + \dfrac{9T}{m_e}|k|^2)\widehat{\mathcal{E}}(t,k) = 0, \\ \widehat{\mathcal{E}}(0,k) = \widehat{E_0}(k), \\ \partial_t \widehat{\mathcal{E}}(0,k) = -n_0^2 ik.\widehat{j_{in}}(k). \end{cases}$$

Since $E_0 = \dfrac{q^2 n_0}{\epsilon_0 m_e} \nabla_x (-\Delta_x)^{-1} \displaystyle\int_{\mathbb{R}^3} \mathcal{H}_0(.,v) dv$, we have $\widehat{E_0}(k) = \dfrac{q^2 n_0}{\epsilon_0 m_e} \dfrac{ik}{|k|^2} \displaystyle\int_{\mathbb{R}^3} \widehat{\mathcal{H}_0}(k,v) dv.$

The solution to this system is

$$\widehat{\mathcal{E}}(t,k) = Acos(\Omega_{KG}(k)t) + Bsin(\Omega_{KG}((k)t) \qquad \text{where } \Omega_{KG}(k) = \sqrt{w_p^2 + \frac{9T}{m_e}|k|^2}.$$

and $\qquad A = \widehat{\mathcal{E}}(0,k) = \dfrac{q^2 n_0}{\epsilon_0 m_e} i \dfrac{k}{|k|^2} \widehat{\mathcal{H}}_0(k,0), \qquad B = \dfrac{\partial_t \widehat{\mathcal{E}}(0,k)}{\Omega_{KG}(k)} = \dfrac{-n_0^2 ik\widehat{j_{in}}(k)}{\Omega_{KG}(k)}.$

So we get

$$\widehat{\mathcal{E}}(t,k) = \frac{q^2 n_0}{\epsilon_0 m_e} i \frac{k}{|k|^2} \widehat{\mathcal{H}_0}(k,0) cos(\Omega_{KG}(k)t) - n_0^2 ik\widehat{j_{in}}(k) \frac{sin(\Omega_{KG}(k)t)}{\Omega_{KG}(k)}.$$

We multiply by ϵ and appply this equality in ϵk instead of k to get

$$\epsilon \widehat{\mathcal{E}}_\epsilon(t,k) = iw_p^2 \frac{k}{|k|^2} \widehat{\mathcal{H}_0}(k,0) cos(\Omega_{KG}(\epsilon k)t) - i\epsilon^2 n_0^2 (k.\widehat{J}(k)) \frac{sin(\Omega_{KG}(\epsilon k)t)}{\Omega_{KG}(\epsilon k)}.$$

$$|\Omega(\epsilon k) - \Omega_{KG}(\epsilon k)| \lesssim \left| \sqrt{w_p^2 + \frac{9T}{m_e}|\epsilon k|^2 + \mathcal{O}(|\epsilon k|^4)} - \sqrt{w_p^2 + \frac{9T}{m_e}|\epsilon k|^2} \right|,$$

we deduce that

$$|\Omega(\epsilon k) - \Omega_{KG}(\epsilon k)| \lesssim \big| w_p\big(1 + \frac{9T}{2m_e w_p^2}|\epsilon k|^2 + \mathcal{O}(|\epsilon k|^4)\big) - w_p\big(1 + \frac{9T}{2m_e w_p^2}|\epsilon k|^2\big) \big| \lesssim |\epsilon k|^4.$$

and therefore

$$|sin[\Omega(\epsilon k)t] - sin[\Omega_{KG}(\epsilon k)t]| = |\Omega(\epsilon k)t - \Omega_{KG}(\epsilon k)t| = \mathcal{O}(|\epsilon k|^4).$$

For $t < 1$,

$$\Big|cos[\Omega(\epsilon k)t] - cos[\Omega_{KG}(\epsilon k)t]\Big| = \Big|\Omega^2(\epsilon k)t^2 - \Omega^2_{KG}(\epsilon k)t^2\Big| = \mathcal{O}(\epsilon|k|^4 t^2) = \mathcal{O}(\epsilon|k|^4 t).$$

Furthermore, using (2.14) we have for any $N > 0$ that $|1 - e^{-\lambda(\epsilon k)t}| \sim |\lambda(\epsilon k)t| \leq |\epsilon k|^N t$

so we can prove (but will admit) that

$$\epsilon|\widehat{E}^{(1)}_{KG,\epsilon} - \widehat{\mathcal{E}_\epsilon}| \lesssim (|\epsilon k|^4 t + \epsilon|\epsilon k|^2)||H_0||_{W^{0,1}_4}.$$

and thus,

$$\epsilon^2||E^{(1)}_{KG,\epsilon} - \mathcal{E}_\epsilon||^2_{H^{-s}} \lesssim ||H_0||^2_{W^{0,1}_4} \int_{|\epsilon k|\lesssim 1} \frac{\epsilon^2|\epsilon k|^4 + |\epsilon k|^8 t^2}{\langle k\rangle^{2s}} dk.$$

Now

$$\int_{|\epsilon k|\lesssim 1} \frac{\epsilon^2|\epsilon k|^4 + |\epsilon k|^8 t^2}{\langle k\rangle^{2s}} dk \leq \epsilon^2 \int_{|\epsilon k|\lesssim 1} \frac{|\epsilon k|^4}{|k|^{2s}} dk + \int_{|\epsilon k|\lesssim 1} \frac{|\epsilon k|^8 t^2}{|k|^{2s}} dk \lesssim \epsilon^2 \int_{|\epsilon k|\lesssim 1} \frac{dk}{|k|^{2s}} + t^2 \int_{|\epsilon k|\lesssim 1} \frac{dk}{|k|^{2s}}$$

Using spherical coordinates, we obtain

$$\epsilon^2 \int_{0<r<1/\epsilon} \frac{r^2}{r^{2s}} dr + t^2 \int_{0<r<1/\epsilon} \frac{r^2}{r^{2s}} dr \lesssim \epsilon^{2s-1} + t^2\epsilon^{2s-3} \lesssim \epsilon^{2s-3}(1+t^2) = \epsilon^{2s-3}\langle t\rangle^2$$

since $\epsilon \ll 1$ and $2s - 3 < 2s - 1$ so $\epsilon^{2s-1} < \epsilon^{2s-3}$.

we therefore get

$$\epsilon||E^{(1)}_{KG,\epsilon} - \mathcal{E}_\epsilon||_{H^{-s}} \lesssim ||H_0||_{W^{0,1}_4} \epsilon^{s-\frac{3}{2}} \langle t\rangle.$$

3 Landau damping estimates on the electric field

In this section we provide an estimate for E_{LD}. Denote the spatial density of the solution to the free transport equation

$$\mathfrak{H}(t,x) := \int_{\mathcal{R}^3} h_{in}(x - tv, v) dv$$

with fourier transform $\widehat{\mathfrak{H}}(t,k) = \widehat{h_{in}}(k, kt)$.

We will assume the following result.

Lemma 9 *For all $\sigma \geq 0$,*

$$||\langle \nabla_x, t\nabla_x\rangle^\sigma \mathfrak{H}(t,.)||_{L^1_x} \lesssim ||h_{in}||_{W_0^{\sigma,1}}.$$

Lemma 10 *There holds the following estimate*

$$||\langle \nabla_x, t\nabla_x\rangle^\sigma E^{(2)}_{LD}(t)||_{L^\infty_x} \lesssim \frac{1}{\langle t\rangle^4}||h_{in}||_{W_0^{\sigma+3+a,1}}.$$

Proof. Consider the low spatial frequencies $|k| \lesssim \nu_0$. Using lemma 7 for any $a > 0$, we have

$$\begin{aligned}
|\langle(\nabla_x, t\nabla_x)\rangle^\sigma E^{(2)}_{LD}(t,x)||_{L^\infty_x} &\lesssim ||\langle(k,tk)\rangle^\sigma \widehat{E^{(2)}_{LD}}(t,k)||_{L^1_k} \\
&\lesssim \int_{\mathbb{R}^3} \langle(k,tk)\rangle^\sigma |k|^{-1} \int_0^t \mathcal{R}_{RFT}(t-\tau,k)\widehat{h}_{im}(k,k\tau)d\tau dk \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} \langle(k,tk)\rangle^\sigma |k|^{-1} |\mathcal{R}_{RFT}(t-\tau,k)||\widehat{h}_{in}(t,k\tau)|dkd\tau.
\end{aligned}$$

Using the fact $\langle(k,tk)\rangle \lesssim \langle(k,(t-\tau)k)\rangle\langle(k,\tau k)\rangle$, we get

$$||\langle(\nabla_x, t\nabla_x)\rangle^\sigma E^{(2)}_{LD}(t,x)||_{L^\infty_x} \lesssim \int_0^t \int_{\mathbb{R}^3} \langle(t,(t-\tau)k)\rangle^\sigma |k|^{-1}|\mathcal{R}_{RFT}(t-\tau,k)|\langle(k,\tau k))\rangle^\sigma |\widehat{h}_{in}(k,k\tau)|dkd\tau.$$

Lemma 9 and Lemma 7 yield,

$$\begin{aligned}
|\langle(\nabla_x, t\nabla_x)\rangle^\sigma E^{(2)}_{LD}(t,x)||_{L^\infty_x} &\lesssim \int_0^t \int_{\mathbb{R}^3} \langle(k,(t-\tau)k)\rangle^\sigma |k|^2 \langle(k,\tau k)\rangle^{-3-a} dkd\tau ||h_{in}||_{W_0^{\sigma+3+a,1}} \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} |k|^2 \langle\tau k\rangle^{-3-a}\langle(t-\tau)k\rangle^{-3-a} dkd\tau ||h_{in}||_{W_0^{\sigma+3+a,1}}.
\end{aligned}$$

Since $\langle(k,(t-\tau)k\rangle^\sigma \langle(k,\tau k)\rangle^{-3-a} \lesssim \langle k\rangle^\sigma \langle(t-\tau)k\rangle^\sigma \langle k\rangle^{-3-a}\langle\tau k\rangle^{-3-a}$

and using $|k| \leq \nu_0$,

$$\lesssim \langle(t-\tau)k\rangle^\sigma \langle\tau k\rangle^{-3-a}$$

$$\lesssim \langle(t-\tau)k\rangle^{-3-a} \langle\tau k\rangle^{-3-a}$$

because $\langle(t-\tau)k\rangle^\sigma = \langle(t-\tau)k\rangle^{\sigma+3+a}\langle(t-\tau)k\rangle^{-3-a} \lesssim \langle(t-\tau)k\rangle^{-3-a}$.

We split the integral

$$\int_0^t \Big(\int_{\mathbb{R}^3} |k|^2 \langle \tau k\rangle^{-3-a} \langle (t-\tau)k\rangle^{-3-a} dk \Big) d\tau = \Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \Big) \big(\int_{\mathbb{R}^3} |k|^2 \langle \tau k\rangle^{-a-3} \langle (t-\tau)k\rangle^{-3-a} dk \big) d\tau,$$

and use the change of variable $k' = \tau k$ in the first integral and $k' = (t-\tau)k$ in the second one, we get

$$\int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} |k|^2 \langle \tau k\rangle^{-a-3} \langle (t-\tau)k\rangle^{-3-a} dk \big) d\tau = \int_0^{\frac{t}{2}} \tau^{-3} \int_{\mathbb{R}^3} \tau^{-2} |\tau k|^2 \langle \tau k\rangle^{-3-a} \langle (t-\tau)k\rangle^{-3-a} dk' d\tau$$
$$= \int_0^{\frac{t}{2}} \tau^{-5} \int_{\mathbb{R}^3} |k'|^2 \langle k'\rangle^{-3-a} \langle (\frac{t}{\tau} - 1)k'\rangle^{-3-a} dk' d\tau.$$

Since $\tau \in [0, \frac{t}{2}]$ we have $\frac{t}{\tau} \in [2, \infty[$ and then $\langle \big(\frac{t}{\tau} - 1\big)k'\rangle^{-3-a} \leq \langle k'\rangle^{-3-a}$. This yields

$$\int_0^{\frac{t}{2}} \int_{\mathbb{R}^3} |k|^2 \langle \tau k\rangle^{-a-3} \langle (t-\tau)k\rangle^{-3-a} dk \big) d\tau = \int_0^{\frac{t}{2}} \tau^{-5} \Big(\int_{\mathbb{R}^3} |k'|^2 \langle k'\rangle^{-3-a} \langle k'\rangle^{-3-a} dk' \Big) d\tau$$
$$\lesssim t^{-4}.$$

For the second integral, we have similarly

$$\int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} |k|^2 \langle \tau k\rangle^{-a-3} \langle (t-\tau)k\rangle^{-3-a} dk \big) d\tau = \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} (t-\tau)^{-5} |k'|^{-2} \langle \frac{\tau k'}{t-\tau}\rangle^{-3-a} \langle k'\rangle^{-3-a} dk' d\tau$$
$$\lesssim \int_{\frac{t}{2}}^t \Big(\int_{\mathbb{R}^3} (t-\tau)^{-5} |k'|^2 \langle k'\rangle^{-3-a} dk' \Big) d\tau \lesssim t^{-4}.$$

4 Appendix

In this section we will prove some results that we used in the main part of this report. We will start by stating Green theorem which will helps us to prove Cauchy integral Theorem.

Green Theorem

Let γ be a positeviley oriented, smooth, simple closed curve in a plane, and let D be the region bounded by γ. If u and v are functions of (x, y) defined on an open region containing D and having continiuous partial derivatives then

$$\int_{\gamma} udx + vdy = \iint_{D} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

4.1 Cauchy Integral Theorem

Let $U \subseteq \mathbb{C}$ be a simply connected open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $\gamma : [a, b] \to U$ be a smooth closed curve. then

$$\int_{\gamma} f(z)dz = 0$$

Proof. For simplicity, we will assume that the partial derivatives of a holomorphic function are continuous, then the Cauchy theorem can be proved as a direct consequence of Green theorem and the fact that the real and imaginary parts of $f = u + iv$ must satisfy the Cauchy-Riemann equations in the region bounded by Γ, moreover in the open neighborhood U of this region. We can break the integral of $f(z)$, as well as the differentiel dz into their real and imaginary part component

$$f(z) = u(x, y) + iv(x, y)$$

$$dz = dx + idy$$

In this case, we have

$$\begin{aligned}
\int_{\gamma} f(z)dz &= \int_{\gamma} \big(u(x, y) + iv(x, y)\big)(dx + idy) \\
&= \int_{\gamma} \big(u(x, y)dx - v(x, y)dy\big) + i \int_{\gamma} v(x, y)dx + u(x, y)dy
\end{aligned}$$

By Green theorem, we may then replace the integrals around the closed contour γ with an area integral throughout th domain D that is enclosed by γ as follows

$$\int_\gamma \Big(u(x,y)dx - v(x,y)dy \Big) = \iint_D \Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \Big) dxdy$$
$$\int_\gamma \Big(v(x,y)dx + u(x,y)dy \Big) = \iint_D \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Big) dxdy.$$

But as the real and imaginary part of a holomorphic function in the domain D, u and v must satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad and \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

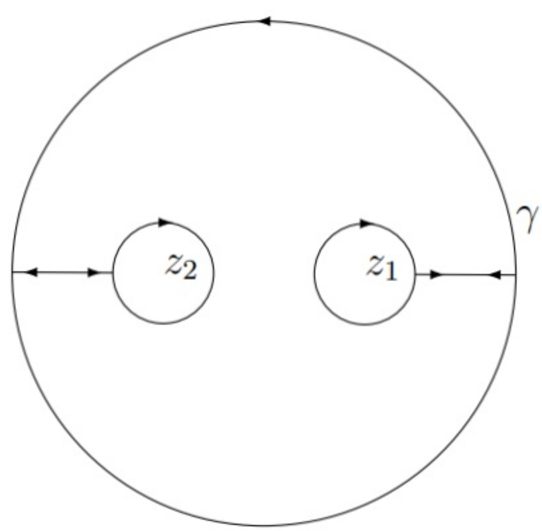
We therefore find that both integrands (and hence their integrals) are zero, which lead us to

$$\int_\gamma f(z)dz = 0.$$

4.2 Cauchy Residue theorem

Suppose $f(z)$ is analytic in U except for a set of n isolated singularities. Suppose also that γ is a simple closed curve in U that doesnt go trough any of the singularites of f and is oriented positevily (counterclockwise) and suppose that the singularites lies inside of γ then

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res(f, z_j)$$



Proof. Let $\tilde{\gamma} = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_2 + \gamma_4 + \gamma_5 - \gamma_6 - \gamma_5$. γ the biggest circle, Where γ_3 is the circle surrounding the point z_1 and γ_5 the circle surrounding the point z_2. We have $\gamma = \gamma_1 + \gamma_4$ and since there is no pole inside $\tilde{\gamma}$ we get by Cauchy integral theorem

$$\int_{\tilde{\gamma}} f(z)dz = \int_{\gamma_1 ... - \gamma_5} f(z)dz) = 0$$

Dropping γ_2 and γ_5 which are both added and substracted, we get

$$\int_{\gamma_1+\gamma_4} f(z)dz = \int_{\gamma_3+\gamma_6} f(z)dz \tag{4.1}$$

If $f(z) = ... + \dfrac{a_2}{(z-z_1)^2} + \dfrac{a_1}{(z-z_1)} + a_0 + a_1(z-z_1) + ...$ is the Laurent expansion of f around z_1 then

$$\int_{\gamma_3} f(z)dz = \int_{\gamma_3} ...+\frac{a_2}{(z-z_1)^2}+\frac{a_1}{(z-z_1)}+a_0+a_1(z-z_1)+...\,dz = 2\pi i a_{-1} = 2\pi i Res(f, z_1).$$

Because for $n \neq 1$, $\displaystyle\int_{\gamma_j} \frac{1}{(z-z_j)^n} = 0$ where z_j is in the region bounded by γ_j

In fact $\gamma_j = z_j + Re^{it} \qquad t \in [0, 2\pi]$ so

$$\int_{\gamma_j} \frac{1}{(z-z_j)^n} dz = \int_0^{2\pi} \frac{Rie^{it}}{(Re^{it})^n} dt = iR^{n-1}{}_0^{2\pi} e^{i(n-1)t} dt = 0.$$

Likewize $\displaystyle\int_{\gamma_6} f(z)dz = 2i\pi Res(f, z_2)$ Using these residues and the fact that $\gamma = \gamma_1 + \gamma_4$ (4.1) becomes

$$\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{j=2} Res(f, z_j)$$

That proves the residue theorem for the case of two poles, which gives the result by the same method for n poles.

4.3 Method of characteristics

We consider the general problem. For $0 < t < T, x \in \mathbb{R}^n$

$$\partial_t u + a(t.x)\nabla_x u = 0 \tag{4.2}$$

Known as transport equation and we assume that $a :]a, T[\times \mathbb{R}^n \to \mathbb{R}^n$ is smooth.

Definition Let $t \geq 0$ and $x \in \mathbb{R}^n$. We call characteristics of (4.2) a solution $s \to \mathrm{x}(s.t) \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that

$$\frac{d\mathrm{x}}{ds}(s.t) = a(s, \mathrm{x}(s,t)) \text{ and } \mathrm{x}(t,t) = x.$$

Theorem A1
Let $a \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ differentiable in x with $\partial_x a \in \mathcal{C}([0,T] \times \mathbb{R}^n)$ and

$$\forall t \in [0,T], \;\; x \in \mathbb{R}^n, \;\; |a(t,x)| \leq \kappa(1 + |x|).$$

so for all $t \in$ [0,T] and $x \in \mathbb{R}^n$, there exists a unique characteristic defined for $s \in [0,T]$ and such that $\mathrm{x}(t,t) = x$, which we will denote as $\mathrm{x}(s,t,x)$ and we have that

$$\mathrm{x} \in \mathcal{C}^1([0,T]_s \times [0,T]_t \times \mathbb{R}^n_{\mathrm{x}}).$$

Furthermore $\partial_s \partial_x \mathrm{x}$ and $\partial_x \partial_s \mathrm{x}$ existe and are continiuos on $[0,T]_s \times [0,T]_t \times \mathbb{R}^n_x$

Proof We can check [5]

Proposition 1 under the assumptions of theorem A1 we have

(i) $\forall r, s, t \in [0,T], s \in \mathbb{R}^n \mathrm{x}(t, s.\mathrm{x}(s,r,x)) = \mathrm{x}(t,r,x)$

(ii) $\forall s, t \in [0,T]$, the map $x \to \mathrm{x}(s,t,x)$ is $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^n$ with inverse $\mathrm{x}(t,s,.)$

Proof (i) from the definition of x, we have that both terms of the equality verifies $\mathrm{x}(r,r) = x$ and from the unicity of Cauchy problem solution we get the result.

(ii) The regularity is given from Theorem A1. On the other hand, if we denote

$$f_1 : \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad f_2 : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \to \mathrm{x}(s,t,x) \qquad\qquad x \to \mathrm{x}(t,s,x)$$

we have $\big(f_1 \circ f_2(x) = f_1(\mathrm{x}(t,s,x)) = \mathrm{x}(s,t,\mathrm{x}(t,s,x))\big) = \mathrm{x}(s,s,x) = \mathrm{x}(s,s) = x$

and $\big(f_2 \circ f_1\big)(x) = f_2(\mathrm{x}(s,t,x)) = \mathrm{x}(t,s,\mathrm{x}(s,t,x)) = \mathrm{x}(t,t,x) = x(t,t) = x$

Proposition 2 The characteristic satisfy the following equation

$$\frac{d\mathrm{x}}{dt}(s,t,x) + a(t,x)\nabla_x \mathrm{x}(s,t,x) = 0.$$

Proof We differentiate $\mathrm{x}(t,s,\mathrm{x}(s,r,x)) = \mathrm{x}(t,r,x)$ with respect to s to get

$$\begin{gathered}\partial_t \mathrm{x}(t,s,\mathrm{x}(s,r,x)) + \frac{\partial \mathrm{x}}{\partial_s}(s.t)\frac{\partial \mathrm{x}}{\partial_x}(t,s,\mathrm{x}(s,r,x)) = 0,\\ \partial_t \mathrm{x}(t,s,\mathrm{x}(s,r,x)) + a(s,\mathrm{x}(s,t,x))\nabla_x \mathrm{x}(t,r,x) = 0,\\ \partial_t \mathrm{x}(t,r,x) + a(s,\mathrm{x}(s,t,x))\nabla_x \mathrm{x}(t,r,x)) = 0.\end{gathered}$$

Applying it for $r = s$ and switching the roles between t and s we get

$$\partial_t \mathrm{x}(s,t,x) + a(t,x)\nabla_x \mathrm{x}(s,t,x) = 0 \quad \text{since } \mathrm{x}(t,t,x) = x)$$

Theorem A2 (Existence and unicity of the solution.)
Under the assumptions of theorem A1, and if we suppose that $u_0 \in \mathcal{C}^1(\mathbb{R}^n)$, then there exists a unique solution to (**??**) with initial data $u(0,x) = u_0$ and u is given by

$$u(t,x) = u_0(\mathrm{x}(0,t,x)).$$

Proof We will start by proving that u given in the statement in a solution.

From proposition 2,

$$\frac{d\mathrm{x}}{dt}(s,t,x) + a(t,x)\nabla_x \mathrm{x}(s,t,x) = 0,$$

then

$$\nabla_x u_0(\mathrm{x}(0,t,x))\frac{d\mathrm{x}}{dt}(s,t,x) + a(t,x)\nabla_x u_0(\mathrm{x}(0,t,x))\nabla_x \mathrm{x}(s,t,x) = 0.$$

So $s = 0$,

$$\nabla_x u_0(\mathrm{x}(0,t,x))\frac{d\mathrm{x}}{dt}(0,t,x) + a(t,x)\nabla_x u_0(\mathrm{x}(0,t,x))\nabla_x \mathrm{x}(0,t,x) = 0$$

which lead us to the following result

$$\frac{d}{dt}\big(u_0(\mathrm{x}(0,t,x))\big) + a(t,x)\frac{d}{dx}\big(u_0(\mathrm{x}(0,t,x))\big) = 0$$

and $u_0(\mathrm{x}(0,t,x)$ is a solution.

In the opposite way , let $u \in (C)^1$ a solution to **??**. For all (t_0, x_0) we have

$$\begin{aligned}\frac{d}{ds}[u(s,\mathrm{x}(s,t_0,x_0))] &= \partial_t u(s,\mathrm{x}(s,t_0,x_0) + \nabla_x u(s,\mathrm{x}(s,t_0,x_0))\partial_s \mathrm{x}(s,t_0,x_0)\\ &= \partial_t u(s,\mathrm{x}(s,t_0,x_0)) + \nabla_x u(s,\mathrm{x}(s,t_0,x_0))a(s,\mathrm{x}(s,t_0,x_0)\\ &= 0.\end{aligned}$$

And we deduce that $u(s, \mathrm{x}(s, t_0, x_0)) = u_0(\mathrm{x}(0, t_0, x_0))$ Now let $x = \mathrm{x}(0, s, x))$. we get

$$u(s, x) = u_0(\mathrm{x}(0, t_0, x_0)$$

because $\quad u_0(\mathrm{x}(0, s, x)) = u_0\mathrm{x}(0, s, \mathrm{x}(s, t_0, x_0)) = u_0(0, t_0, x_0).$

4.4 The Sokhotski-Plemelj Formula

The Sokhotski-Plemelj formula is a relation between distributions,

$$\lim_{\epsilon \to 0} \frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x), \tag{4.3}$$

where $\epsilon > 0$ is an infinitesimal quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. We shall also assume that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ in order that integrals evaluated over the entire real line are convergent.

To establish (4.3), we shall prove that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \frac{f(x)dx}{x \pm i\epsilon} = P\int_{-\infty}^{+\infty} \frac{f(x)dx}{x} \mp i\pi f(0), \tag{4.4}$$

where the cauchy principal value integral is defined as

$$P\int_{-\infty}^{+\infty} \frac{f(x)dx}{x} := \lim_{\delta \to 0} \left\{ \int_{-\infty}^{-\delta} \frac{f(x)dx}{x} + \int_{\delta}^{\infty} \frac{f(x)dx}{x} \right\}, \tag{4.5}$$

assuming $f(x)$ is regular in a neighborhood of the real axis and vanishes as $x \to \pm\infty$.

Proof of (4.4) We begin with the identity,

$$\frac{1}{x \pm i\epsilon} = \frac{x \mp i\epsilon}{x^2 + \epsilon^2},$$

where ϵ is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the region.

$$\int_{-\infty}^{\infty} \frac{f(x)dx}{x \pm i\epsilon} = \int_{-\infty}^{\infty} \frac{xf(x)dx}{x^2 + \epsilon^2} \mp i\epsilon \int_{-\infty}^{\infty} \frac{f(x)dx}{x^2 + \epsilon^2}. \tag{4.6}$$

The first integral on the right hand side of equation (4.6),

$$\int_{-\infty}^{\infty} \frac{xf(x)dx}{x^2 + \epsilon^2} = \int_{-\infty}^{-\delta} \frac{xf(x)dx}{x^2 + \epsilon^2} + \int_{\delta}^{\infty} \frac{xf(x)dx}{x^2 + \epsilon^2} + \int_{-\delta}^{\delta} \frac{xf(x)dx}{x^2 + \epsilon^2}. \tag{4.7}$$

In the first two integrals on the right hand side of equation (4.7), we can take the limit $\epsilon \to 0$. In the last integal on the right hand side of the same equation, if δ is small enough, then we can approximate $f(x) \simeq f(0)$ for values of $|x| < \delta$. Therefore, equation (4.7) yields,

$$\int_{-\infty}^{\infty} \frac{xf(x)dx}{x^2+\epsilon^2} = \lim_{\delta\to 0}\left\{ \int_{-\infty}^{-\delta} \frac{f(x)dx}{x} + \int_{\delta}^{\infty} \frac{f(x)dx}{x} \right\} + f(0)\int_{\delta}^{\delta} \frac{xdx}{x^2+\epsilon^2}. \tag{4.8}$$

However,

$$\int_{\delta}^{\delta} \frac{xdx}{x^2+\epsilon^2} = 0,$$

since the integrand in an odd function of x that is being integrated symmetrically about the origin, and

$$P\int_{-\infty}^{+\infty} \frac{f(x)dx}{x} := \lim_{\delta\to 0}\left\{ \int_{-\infty}^{-\delta} \frac{f(x)dx}{x} + \int_{\delta}^{\infty} \frac{f(x)dx}{x} \right\},$$

defines the principal value integral. Hence, equation (4.8) yields

$$\int_{-\infty}^{\infty} \frac{xf(x)dx}{x^2+\epsilon^2} = P\int_{-\infty}^{\infty} \frac{f(x)dx}{x}. \tag{4.9}$$

Next, we consider the second integral on the right hand side of equation (4.6). We assume that the only significant contribution from

$$\epsilon\int_{-\infty}^{\infty} \frac{f(x)dx}{x^2+\epsilon^2}$$

can comme from the integration where $x \simeq 0$. Thus, we can again approximate $f(x) \simeq f(0)$, in which case we obtain

$$\epsilon\int_{-\infty}^{\infty} \frac{f(x)dx}{x^2+\epsilon^2} \simeq \epsilon f(0)\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \pi f(0), \tag{4.10}$$

where we have made use of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \frac{1}{\epsilon} tan^{-1}(\frac{x}{\epsilon})\Big)_{-\infty}^{\infty} = \frac{\pi}{\epsilon}.$$

Now using the results of equations $(4.9$ and (4.10), we see that equation (4.6) yields

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \frac{f(x)dx}{x \pm i\epsilon} = P\int_{-\infty}^{\infty} \frac{f(x)dx}{x} \mp i\pi f(0),$$

which establishes equation (4.4). Note that (4.3) can be generalized as follows,

$$\lim_{\epsilon\to 0}\frac{1}{x-x_0 \pm i\epsilon} = P\frac{1}{x-x_0} \mp i\pi\delta(x-x_0),$$

where

$$P\int_{-\infty}^{\infty} \frac{f(x)dx}{x-x_0} := \lim_{\delta\to 0}\left\{ \int_{-\infty}^{x_0-\delta} \frac{f(x)dx}{x-x_0} + \int_{x_0+\delta}^{\infty} \frac{f(x)dx}{x-x_0} \right\}.$$

The proof is similar for what we have done for (4.4).

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