Bohr-Sommerfeld conditions for several commuting Hamiltonians

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Abstract

The goal of this paper is to find the quantization conditions of Bohr-Sommerfeld of several quantum Hamiltonians $Q_1(h), ..., Q_k(h)$ acting on $\mathbb{R}^n$, depending on a small parameter h, and which commute with each other. That is we determine, around a regular energy level $E_0 \in \mathbb{R}^k$ the principal term of the asymptotics in h of eigenvalues $\lambda_j(h)$, $1 \leq j \leq k$ of the operators $Q_j(h)$ that are associated to a common eigenfunction. Thus we localize the so-called joint spectrum of the operators.

Under the assumption that the classical Hamiltonian flow of the joint principal symbol q_0 is *periodic with constant periods on the one energy level* $q_0^{-1}(E_0)$, we prove that the part of the joint spectrum lying in a small neighborhood of E_0 is localized near a lattice of size h determined in terms of actions and Maslov indices. The multiplicity of the spectrum is also determined.

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Key words : semiclassical techniques, Fourier integral operators, Hamiltonian flow.

Contents

1 Introduction

The joint spectrum of several commuting operators arises naturally if suitable symmetries are present. Let us consider for instance the Schrödinger operator, acting on $\mathbb{R}^3$

$$A_1(h) \;=\; -h^2\Delta \;+\; V(x) \;,$$

and suppose that the potential is real valued, smooth, and spherically symmetric. Then $A_1(h)$ commutes with the kinetic momentum

$$A_2(h) \;=\; -ih\;(x_2\partial_{x_3} \;-\; x_3\partial_{x_2}).$$

The bounded states of $A_1(h)$ are common eigenfunctions of $A_1(h)$ and $A_2(h)$, and if we consider a third operator

$$A_3(h) \;=\; -h^2\;\{(x_2\partial_{x_3} \;-\; x_3\partial_{x_2})^2 \;+\; (x_3\partial_{x_1} \;-\; x_1\partial_{x_3})^2 \;+\; (x_1\partial_{x_2} \;-\; x_2\partial_{x_1})^2\;\}$$

which commutes with the other two operators, these bounded states are common eigenfunctions of $A_1(h)$, $A_2(h)$ and $A_3(h)$, and their joint spectrum consists of the associated 3-up-lets of eigenvalues,

$$\lambda(h) \;=\; (\lambda_1(h), \lambda_2(h), \lambda_3(h)) \in \mathbb{R}^3.$$

They have a multiplicity equal to 1, the bounded states are said to be completely separated.
In fact this example is integrable (the number of operators equals the dimension) and by the Liouville theorem the Hamiltonian flow is periodic, (see [6] for the explicit calculation of the joint flow for the harmonic oscillator). This example can naturally be extended to all dimensions. A few new examples of integrable systems on manifolds, inspired by the classical case can be found in the works of J.A. Toth [25] and [26].

On the other hand with a potential of the form $V(x,y) = f(\|x\|^2) + g(\|y\|^2)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$ we can construct examples with less symmetries, and which are no more integrable.

1.1 Pseudo-differential calculus

More generally we consider k quantum Hamiltonians $Q_1(h), ..., Q_k(h)$ acting on $\mathbb{R}^n$, depending on a small parameter h, and which commute with each other. The $Q_1(h), ..., Q_k(h)$ are supposed to be h-admissible pseudo-differential operators, as introduced by B. Helffer and D. Robert in [17] and [18] (see [24] for the general theory).
Let p be a weight function on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, *i.e.* a continuous function from $\mathbb{R}^{2n}$ into $\mathbb{R}^+$ which satisfies the following property :
there exist constants $C > 0,\ m > 0$ such that

$$\forall (x,\xi), (x',\xi') \in \mathbb{R}^{2n}\; p(x,\xi) \;\leq\; C\;p(x',\xi')\;(1 + |x - x'|^2 + |\xi - \xi'|^2)^m \quad .$$

We denote by S_p the space of all the functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for all $(\alpha,\beta) \in \mathbb{N}^{2n}$ there exists $C_{\alpha,\beta} > 0$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \;\leq\; C_{\alpha,\beta}\; p(x,\xi) \quad \forall (x,\xi) \in \mathbb{R}^{2n}. \tag{1.1}$$

To such a symbol a we associate a h-pseudo-differential operator $\mathcal{A}(h) = Op_h^w(a)$ acting on functions $\psi \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of smooth functions rapidly decreasing at infinity together with their derivatives, defined by the Weyl quantization :

$$\mathcal{A}(h)\psi(x) \;=\; (2\pi h)^{-n} \int\int e^{ih^{-1}\langle x-y,\xi\rangle}\, a(\frac{x+y}{2},\xi)\, \psi(y)\, dy\, d\xi \; .$$

More generally we may assume that the symbol a depends on h : if there exists a sequence $\{a_i\}_{i\in\mathbb{N}} \in S_p$ such that

$$a(h,x,\xi) \;\sim\; \sum_{i\geq 0} h^i a_i(x,\xi) \tag{1.2}$$

where $\sim$ means that for all $N \in \mathbb{N}$ the difference

$$r_{N+1} = h^{-(N+1)} \;(\; a(h,x,\xi) \;-\; \sum_{i=0}^{N} h^i a_i(x,\xi)\;)$$

defines a family $\mathcal{R}_{N+1}(h) = Op_h^w(r_{N+1})$, $h \in]0,h_0]$ of *bounded* operators which is bounded (uniformly in h) in $\mathcal{L}(L^2(\mathbb{R}^n))$; then we say that the operator $\mathcal{A}(h)$ is *h-admissible.*

We suppose that we have $Q_j(h) = Op_h^w(q_j)$ are h-admissible, so there exists a sequence $\{q_{ij}\}_{i\in\mathbb{N}} \in S_p$ such that

$$q_j(h,x,\xi) \;\sim\; \sum_{i\geq 0} h^i q_{ij}(x,\xi). \tag{1.3}$$

Then $Q_j(h)$ belongs to $\mathcal{L}(\mathcal{S}(\mathbb{R}^n),\mathcal{S}(\mathbb{R}^n))$ for each $j = 1,\ldots,k$.

We also assume that each operator $Q_j(h)$ admits a *self-adjoint extension* to $L^2(\mathbb{R}^n)$. It is the case if the principal symbol q_{0j} is itself a weight and if $q_{ij} \in S_{q_{0j}}$ for all i, and more generally if q_{0j} is bounded from below [24].

In our case, we always can reduce to this situation by means of the functional calculus, if the weight is $p = \|q_0\|^2 + 1$, and if the operator $\sum_{j=1}^k {Q_j}^2(h)$ is h-admissible : let f be a C^∞ function on $\mathbb{R}^+$ such that $f(x) = x$ for $x \in [0,c]$, (with $0 < c < 1$) f is increasing and asymptotic to 1 at infinity and define $g(x) = f(x)/x$ a C^∞ function on $\mathbb{R}^+$, equal to 1 on the interval $[0,c]$. Then, for $E_0 \in \mathbb{R}^k$, the new operators defined vectorially by

$$P(h) = g(\|Q(h) - E_0\|)(Q(h) - E_0)$$

are h-admissible pseudo-differential operators (remark that the operators $g(\|Q(h) - E_0\|)$ are bounded) with bounded symbols, so every principal symbol is a weight bounded from below. Moreover the $P_j(h)$ commute with each other if the $Q_j(h)$ have this property ; furthermore, on the domain $\{(x,\xi), \|q(h)(x,\xi) - E_0\| \leq c\}$ one has $p(h) = q(h) - E_0$, so $p(h)$ is proper if $q(h)$ is.

1.2 Commutativity and Hamiltonian flow

We suppose that the operators $Q_j(h)$ commute with each other

$$[Q_i(h), Q_j(h)] = 0 \text{ on } \;\mathcal{S}(\mathbb{R}^n)\ \forall\ i,j = 1,\ldots,k\ ,\ \forall\ h > 0 \quad . \tag{1.4}$$

We denote by $q_0 = (q_{01}, q_{02}, \ldots, q_{0k}) \;:\; T^*\mathbb{R}^n \;\to\; \mathbb{R}^k$ the joint principal symbol of the operator $Q(h) = (Q_1(h), Q_2(h), \ldots, Q_k(h))$, and by $q_1 = (q_{11}, q_{12}, \ldots, q_{1k})$ its sub-principal symbol.

As a consequence of the commutativity, one has $\{q_{0i}, q_{0j}\} = 0$, and the associated symplectic or classical flows Φ_j^t also commute.

The Φ_j^t are defined as follows : with the canonical symplectic form ω of $T^*\mathbb{R}^n$: $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ one can define the Hamiltonian field H_j of q_{0j} by :

$$dq_{0j} \; = -H_j \;\lrcorner\; \omega \tag{1.5}$$

then the Φ_j^t are the symplectic transformations satisfying :

$$\Phi_j^0 = Id \; ; \; \frac{\partial}{\partial t}\Phi_j^t = H_j \circ \Phi_j^t. \tag{1.6}$$

We want to look at spectral properties for the operators Q_j in a neighborhood of a value $E_0 \in \mathbb{R}^k$ on which we make the following assumption :

(H_1) *E_0 is a regular value of q_0 and q_0 is proper in a neighborhood of E_0 : there exists a compact neighborhood K_1 of E_0 in $\mathbb{R}^k$ such that ${q_0}^{-1}(K_1)$ is compact.*

Note that this assumption is fulfilled for the example above if

$$-\infty < \liminf_{|x|\to\infty} V(x).$$

Moreover, the conditions above imply $k \leq n$.

The joint spectrum $\Lambda^{Q(h)}$ of commuting operators, as $Q_j(h)$, has been defined in [5], and it has been proved that, under the hypothesis (H_1), the part of the joint spectrum contained in any compact K included in the interior of K_1 consists in finitely many joint eigenvalues of finite multiplicity, *i.e.*

$$\Lambda^{Q(h)} \bigcap K = \{(\lambda_q(h))_{q\in I(h)}; \exists \Psi_q^h \in L^2(\mathbb{R}^n) : Q_j(h)\Psi_q^h = \lambda_q^j(h)\Psi_q^h \; \forall j = 1,.,k\} \tag{1.7}$$

where I(h) is finite.

The asymptotics of these eigenvalues can be precised with an assumption of periodicity of the classical flow. We will make a very weak assumption as follows. Suppose that K is sufficiently small to be composed of regular values of q_0. As a consequence of (H_1), for all E belonging to K, the set $\Sigma_E = {q_0}^{-1}(E)$ has the structure of a sub-manifold of $\mathbb{R}^{2n}$ of co-dimension k, Σ_E is compact and invariant under the action of the Hamiltonian flow which is symplectic. Moreover the Hamiltonian fields H_j are independent, and $H_j \;\lrcorner\; \omega$ vanishes in restriction to the tangent plane of Σ_E, so this manifold is co-isotropic.
The Hamiltonian fields H_j commute so the joint flow defines finally an action ρ_E of $\mathbb{R}^k$ on Σ_E :

$$\begin{array}{rrcl} \rho_E \; : & \mathbb{R}^k \times \Sigma_E & \to & \Sigma_E \\ & ((t_1, \ldots, t_k), \nu) & \mapsto & \Phi_1^{t_1} \circ \ldots \circ \Phi_k^{t_k}(\nu). \end{array} \tag{1.8}$$

We will suppose that the joint flow is periodic with constant periods on *one* level $\Sigma_0 = \Sigma_{E_0}$. More precisely we make the following hypothesis :

(H_2) *All the points of $\Sigma_0 = {q_0}^{-1}(E_0)$ are periodic under the action of the joint flow $\rho_0 = \rho_{E_0}$ with the same lattice of periods.*

Let $(e_1,\ldots,e_k)$ be a basis of this lattice ; it is a basis of $\mathbb{R}^k$ verifying for all $\nu\in\Sigma_0$ and for all $z=(z_1,\ldots,z_k)\in\mathbb{Z}^k$: $\rho_0\ (2\pi\sum_{j=1,\ldots,k}\ z_je_j\ ,\ \nu)\ =\ \nu$.
So ρ_0 can be regarded as the action of a torus $\mathbb{T}^k$ on Σ_0, that we will still denote by ρ_0. By the last hypothesis *this action is free*, *i.e.* without fixed points.
Comparison with the hypothesis of [5] and notations. — In this previous work it was made an assumption of periodicity on a open set of energy. More precisely it was supposed that there exists a function $f\in C_0^\infty(\mathbb{R}^k,\mathbb{R}^k)$ such that the principal symbols of the operators $f(Q(h))$ are periodic in a neighborhood of the energy level.
In our case let $(\varepsilon_j)_{1\leq j\leq k}$ be the canonical basis of $\mathbb{R}^k$ and define $\mathbf{a}\in End(\mathbb{R}^k)$ by $e_j=\mathbf{a}(\varepsilon_j)=\Sigma_i\alpha_{ij}\varepsilon_i$.
Then the new symbol $p_0=(p_{01},\ldots,p_{0k})=\mathbf{a}(q_0)$ satisfies

$$dp_{0j}=\Sigma_i\ \alpha_{ij}\ dq_{0i}=-\Sigma_i\ \alpha_{ij}\ H_i\ \lrcorner\ \omega$$

and the corresponding Hamiltonian flow Ψ_j is 2π-periodic on the energy level $\mathbf{a}(E_0)$ in view of (H_2).
Consequently the operators $P(h)\ =\ \mathbf{a}(Q(h))$ satisfy the point-wise equivalent of the hypothesis of [5].
We will denote $F=\mathbf{a}(E_0)$ and $K_j\ =\ \Sigma_i\ \alpha_{ij}\ H_i$ the Hamiltonian field of p_{0j}.

We can now give our main result :

Theorem 1 . — *Let $Q_j(h)=Op_h^w(q_j)$, $1\leq j\leq k$ be k commuting h-admissible pseudo-differential operators essentially self adjoint and satisfying the following hypotheses :*
(H_1) the joint principal symbol q_0 is proper in a neighborhood of a regular value E_0,
(H_2) the classical flow is periodic with constant period on the energy level $\Sigma_0=q_0^{-1}(E_0)$,
(H_3) the sub-principal q_1 vanishes,
(H_4) the surface Σ_0 is connected.

Then the part of the joint spectrum $\Lambda^{Q(h)}$ lying in any k-cube $\prod_{j=1}^{k}\,]E_{0j}-hc_j\ ,\ E_{0j}+hc_j[$ centered at E_0 is discrete and localized modulo $O(h^2)$ near a lattice

$$E_0\ +\ \mathbf{a}^{-1}\Big((-\frac{1}{4}\mu_1h-\frac{\alpha_1}{2\pi}+\mathbb{Z}h)\oplus\ldots\oplus(-\frac{1}{4}\mu_kh-\frac{\alpha_k}{2\pi}+\mathbb{Z}h)\Big),$$

where the μ_j are the Maslov indices of the basic cycles of the torus acting on Σ_0, α_j are the action of these cycles and $\mathbf{a}\in Gl(k)$.

Comments. — The hypothesis (H_3) can be weaked in

(H_3') *The integral of the sub-principal symbol q_1 on a closed trajectory of the joint Hamiltonian flow is independent of the point on the energy level Σ_0, it depends only on the period.*

see the Theorem 2, subsection 2.6. below.

When the hypothesis (H_4) fails each connected component gives a part of the discrete spectrum.
About the method. — We will look at the operators $P(h)\ =\ \mathbf{a}(Q(h))$. For these ones the Hamiltonian flow of each component p_{0j} of the principal symbols is 2π-periodic, we remark that μ_j the Maslov index of the 2π-periodic trajectories of the Hamiltonian flow of p_{0j} is constant on Σ_0 as an invariant of homotopy ; as well the action of such a trajectory,

α_j, is constant on Σ_0. We will denote by μ respectively α the k-vector defined by the Maslov index, respectively the action, of the basic cycles.

In order to localize the eigenvalues of $P(h)$ lying in the k-cube $\Pi_{j=1}^k]F_j - ch, F_j + ch[$ modulo $O(h^2)$ we suppose for the moment that the sub-principal symbols are null and look at the following operator $\zeta(\frac{P(h)-F}{h})\ \theta(P(h))$, where $\hat{\zeta}$ and θ belong to $C_0^\infty(\mathbb{R}^k)$ and $\theta(\lambda) = 1$ for λ in a smaller neighborhood of F and compare it with $\zeta(\frac{P(h)-F}{h}) \exp -\frac{i}{h} < T, P(h) >\ \theta(P(h))$ where $T \in 2\pi\mathbb{Z}^k$, the lattice of periods.
We will do this by approaching these operators by Fourier Integral Operator. We will see that their Lagrangian manifold are identical and their principal symbol differ by a scalar which is determined by the Maslov index and the action of the closed trajectories, consequently the L^2 norm of the difference between $\zeta(\frac{P(h)-F}{h})\ \theta(P)$, and the operator $\zeta(\frac{P(h)-F}{h}) \exp -\frac{i}{h}(< T, P(h) > +\alpha + \frac{\pi}{2}\mu h)\ \theta(P)$, is only $O(h)$.

This method, consisting on expressing the evolution operator $e^{-ih^{-1}<t,P(h)>}$ by means of the theory of Fourier Integral Operators was initiated by Duistermaat-Guillemin [14] and Colin de Verdière [8] for compact manifolds, and Helffer and Robert [19] in the semi-classical case. We especially tried to make clear the appearance of the Maslov Index. On the other hand one can find in [1], a construction, by symplectic geometry, of new symbols with the strong property of periodicity and which approach the first ones. But the new operators do not commute anymore.

This paper is organized as follows : in the second section we will prove the theorem 1, and in the third one we will look at the multiplicities of the joint eigenvalues.

2 Proof of theorem 1

Let ζ be a function belonging to $\mathcal{S}(\mathbb{R}^k)$ with a compactly-supported Fourier-transform $\hat{\zeta}$, and θ belonging to $C_0^\infty(\mathbb{R}^k)$ with a compact support lying in a neighborhood of F and verifying $\theta(\lambda) = 1$ for λ in a smaller neighborhood of F. We can write

$$\zeta(\frac{F - P(h)}{h})\ \theta(P(h)) \ = \ \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} e^{-ih^{-1}<t,(P(h)-F)>}\hat{\zeta}(t)\theta(P(h))dt \quad .$$

Using the functional calculus developed in [5], we consider $\theta(P(h))$ as a pseudo-differential operator with a classical symbol supported in a compact neighborhood of $p_0^{-1}(F)$.

2.1 The evolution operator

It is proved in [7], section 2, that there exists a Fourier Integral Operator (FIO) $\tilde{U}_h$ which approaches modulo $O(h^\infty)$, for $t \in [-T, T]^k$, the operator

$$U_{\theta,h}(t) = e^{-ih^{-1}<t,P(h)>}\theta(P).$$

The general "semi-classical FIO" theory used by Charbonnel and Popov in [7] is based on the presentation of Duistermaat [13]. (Remark that, in the context of first order classical elliptic operators on a compact manifold, Guillemin and Uribe have developed, in [16], a FIO calculus for systems of commuting operators quite similar after the pioneer works of Colin de Verdière [9] and [10]).

More precisely $\tilde{U}_h \in I^{-k/4}(\mathbb{R}^{2n+k}, \Lambda; h)$ with $C = \Lambda'$ the canonical relation

$$C \; = \; \{ \; \Big((x,\xi),(y,\eta),(t,\tau)\Big) \in \; T^*(\mathbb{R}^{2n+k}) = T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^k \; ; \\ \tau = -p_0(y,\eta) \; , \quad (x,\xi) = \Psi^t(y,\eta); \; (y,\eta) \in \mathcal{O} \; \} \tag{2.9}$$

where Ψ^t is the joint flow of p_0 :

$$\Psi^t = \Psi_1^{t_1} \circ \ldots \circ \Psi_k^{t_k}$$

and $\mathcal{O} = q_0^{-1}(K)$ is a compact neighborhood of Σ_0 which is invariant by the flow.
We parametrize C by $(t,y,\eta) \in [-T,T]^k \times \mathcal{O}$; then the principal symbol of $\tilde{U}_h$ is written :

$$\sigma(\tilde{U}_h)(t,y,\eta) = \exp\left(ih^{-1}(-\langle p_0(y,\eta),t\rangle + A(\gamma^t(y,\eta)))\right)\sigma_1 \otimes \sigma_2 \tag{2.10}$$

where σ_1 is the half density $|dt \wedge dy \wedge d\eta|^{1/2}$ and σ_2 is a "fixed section of the Maslov bundle", see below, $\gamma^t(y,\eta)$ is the path

$$\gamma^t(y,\eta) = \{\Psi^{st}(y,\eta); 0 \le s \le 1\}$$

and A is the action of the path in $T^*(\mathbb{R}^n)$: $A(\gamma^t) = \int_{\gamma^t} \xi dx$.
Note that the function θ does not appear either in the Lagrangian manifold, nor in the principal symbol of $U_{\theta,h}(t)$, for $\theta(P)$ is a pseudo-differential operator, so its Lagrangian Manifold is the graph of the identity, and its principal symbol is equal to one in a small neighborhood of Σ_0.

2.2 The action

The manifold $\mathbb{R}^k.(y,\eta) = \{\Psi^t(y,\eta); t \in \mathbb{R}^k\}$ generated by the flow from one point is isotropic, because its tangent space admits the K_j, $1 \le j \le k$ as a basis ; on the other hand $d(\xi dx) = \omega$ the symplectic form of $T^*(\mathbb{R}^n)$; we conclude then from the Stokes formula that the action is constant on the homotopy class of a path in $\mathbb{R}^k.(y,\eta)$ and

$$A(\gamma^t(y,\eta)) = \sum_{j=1}^{j=k} A(\gamma^{t_j}(\nu_j))$$

if $t = (t_1,\ldots,t_k)$; $\nu_1 = (y,\eta)$; $\nu_{j+1} = \Psi_j^{t_j}(\nu_j)$ and $\gamma^{t_j}(\nu_j) = \{\Psi_j^s(\nu_j); \, 0 \le s \le t_j\}$. But Σ_0 is a connected manifold and two points $\nu_0, \nu_1 \in \Sigma_0$ can be connected by a path ν_s. On $\{\Psi_j^{t_j}(\nu_s); \, t_j \in \mathbb{R}, 0 \le s \le 1\}$ the symplectic form is null (because $K_j \lrcorner\, \omega = 0$ on Σ_0). Now if $T_j \in 2\pi\mathbb{Z}$ is a period the Stokes formula gives that the action of the path $\gamma^{t_j}(\nu_s)$, $t_j \in [0,T_j]$ does not depend on $s \in [0,1]$ and we have proved the

Lemma 2.1 . — *For all pair of points ν_0 and (y,η) in Σ_0 and for all periods $T = (T_1,\ldots,T_k) \in 2\pi\mathbb{Z}^k$ we have*

$$A(\gamma^T(y,\eta)) = \sum_{j=1}^{j=k} A(\gamma^{T_j}(\nu_0)). \tag{2.11}$$

2.3 The Maslov bundle

We first recall the results of Arnol'd [2]. Let $\mathbb{L}(n)$ be the Grassmannian manifold of the Lagrangian subspaces of $T^*\mathbb{R}^n$ and make the identification $\mathbb{L}(n) = U(n)/O(n)$. The application Det^2 is well define on $\mathbb{L}(n)$. It is proved in [2] that any path $\gamma : \mathbb{S}^1 \to \mathbb{L}(n)$ such that $Det^2 \circ \gamma : \mathbb{S}^1 \to \mathbb{S}^1$ generates $\Pi_1(\mathbb{S}^1)$ gives a generator of $\Pi_1(\mathbb{L}(n))$. Consequently $\Pi_1(\mathbb{L}(n)) \simeq \mathbb{Z}$ and the co-cycle μ_0 defined by

$$\forall \gamma \in \Pi_1(\mathbb{L}(n)) \quad \mu_0(\gamma) = \text{Degree}(Det^2 \circ \gamma)$$

gives a generator of $H^1(\mathbb{L}(n)) \simeq \mathbb{Z}$. We can define a canonical *Maslov bundle* $\mathbb{M}(n)$ on $\mathbb{L}(n)$ by the representation $\exp(i\frac{\pi}{2}\mu_0) = i^{\mu_0}$ of $\Pi_1(\mathbb{L}(n))$. This bundle is a bundle of torsion because $\mathbb{M}(n)^{\otimes 4}$ is trivial.

Now the Maslov bundle of a Lagrangian submanifold $\mathcal{L}$ of $T^*\mathbb{R}^n$ is the pull back of $\mathbb{M}(n)$ by the natural map

$$\begin{aligned} \varphi_n : \mathcal{L} &\to \mathbb{L}(n) \\ \nu &\mapsto T_\nu \mathcal{L}. \end{aligned}$$

Arnol'd shows actually that $\mu = {\varphi_n}^* \mu_0$ is the Maslov index of $\mathcal{L}$. It can be written

$$\begin{aligned} \mu : \Pi_1(\mathcal{L}) &\to \mathbb{Z} \\ [\gamma] &\mapsto <\mu_0, \varphi_n \circ \gamma> = \text{Degree}(Det^2 \circ \varphi_n \circ \gamma). \end{aligned} \tag{2.12}$$

We have to take care of the structural group of this bundle. As a $U(1)$ it is always trivial. But we will consider it at a $\mathbb{Z}_4 = \{1, i, -1, -i\}$-bundle. Actually we can see with the expression of the Maslov co-cycle σ_{jk} given in [21] (3.2.15) that the Maslov bundle has a trivial Chern class but σ_{jk} can not be in general written as a co-boundary of a *constant* co chain.

We recall now the result of the Proposition 3.2. p.132 of [15].

Proposition 2.1 (Guillemin, Sternberg) . — *Let Δ be an isotropic subspace of dimension m in $T^*\mathbb{R}^{(n+m)}$ and define $S_\Delta = \{\lambda \in \mathbb{L}(n+m)/ \ \lambda \supset \Delta\}$. Then S_Δ is a sub-manifold of $\mathbb{L}(n+m)$ of codimension $(n+m)$, and if the map ρ is defined by*

$$\begin{aligned} \mathbb{L}(n+m) &\overset{\rho}{\to} \mathbb{L}(n) \\ \lambda &\mapsto \lambda \cap \Delta^\omega / \lambda \cap \Delta \end{aligned}$$

(Δ^ω denote the orthogonal of Δ for the canonical symplectic form ω) then ρ, which is not continue on $\mathbb{L}(n+m)$ itself, is smooth restricted to $\mathbb{L}(n+m) - S_\Delta$ and make this space as a fiber bundle on $\mathbb{L}(n)$ with fiber $\mathbb{R}^{(n+m)}$.

Moreover the image of the generator of $\Pi_1(\mathbb{L}(n+m))$ is sent by ρ to the generator of $\Pi_1(\mathbb{L}(n))$.

This last result is easily seen if one choose symplectic coordinates (x, ξ) such that $\Delta = \{x = 0, \xi' = 0\}$. A generator of $\Pi_1(\mathbb{L}(n+m))$ is given by $U_{(n+m)}(t)(\lambda_0)$, $0 \le t \le 1$ where $\lambda_0 = \{\xi = 0\}$ and $U_{(n+m)}(t)$ is given in the complex coordinates $z_j = x_j + i\xi_j$ by $U_{(n+m)}(t)(z_1, \ldots, z_{(n+m)}) = (e^{i\pi t} z_1, z_2, \ldots, z_{(n+m)})$. The (x', ξ') give symplectic coordinates of $T^*\mathbb{R}^n$ and $\rho(U_{(n+m)}(t)\lambda_0) = U_n(t)\lambda'_0$.

If we return now to Λ, we remark, as [23] p. 264 following [14] p.65, that

Lemma 2.2 *the Maslov bundle of Λ is trivial (as a $\mathbb{Z}_4$-bundle).*

proof. — Actually $h_s(t,y,\eta) = (st,y,\eta)$ for $0 \leq s \leq 1$ makes a retrack Λ_s of Λ on Λ_0 such that $\Lambda_0' = \{\Big(\nu,\nu,(0,-q_0(\nu))\Big) ; \nu \in T^*\mathbb{R}^n\}$ and Λ_0 can be seen as a Lagrangian manifold in $T^*\mathbb{R}^{2n}$. We now apply the proposition 2.1 with $\Delta = \Big\{\Big(0,0,(0,V)\Big) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^k\Big\}$. The image $\varphi_{2n+k}(\Lambda_s)$ never meets S_Δ so the Maslov bundle of Λ is as well the pull-back by $\rho \circ \varphi_{2n+k}$ of $\mathbb{M}(2n)$, but there is an homotopy between $\rho \circ \varphi_{2n+k}(\Lambda)$ and $\varphi_{2n}(\Lambda_0)$ and the application φ_{2n} is constant on $\Lambda_0 \subset T^*\mathbb{R}^{2n}$. ■

Then $\sigma_2 = 1$ in the formula of the principal symbol (2.10).

2.4 Composition

Now we have to compose the FIO $\tilde{U}_h(t)$ with the operator $B(h)$ defined by :

$$B(h)\ u\ (x) \ = \ \frac{1}{(2\pi)^k} \int e^{ih^{-1}<t,F>}\hat{\zeta}(t)\ u(x,t)\ dt \tag{2.13}$$

$B(h)$ is a Fourier Integral Operator from $\mathbb{R}^k$ to $\mathbb{R}^0$ if we take x as a parameter. Its canonical relation is equal to

$$C_B = \Lambda_B' \ = \ \{\ (t,\tau) \in \ T^*(\mathbb{R}^k)\ ; \tau = -F\}. \tag{2.14}$$

We remark that the Maslov bundle of C_B is trivial because the application φ_k is constant on it. Its principal symbol is equal to $\sigma(t,-F) = \frac{1}{(2\pi)^k}\ e^{ih^{-1}<t,F>}\hat{\zeta}(t)|dt|^{1/2}$ and $B(h) \in I^{-k/4}(\mathbb{R}^k, \Lambda_B; h)$. We have now to compose this two FIO.

Proposition 2.2 *We can approximate modulo $O(h^\infty)$ the operator $\zeta(\frac{F-P}{h})\ \theta(P)$ by a FIO in $I^{-k/2}(\mathbb{R}^{2n}, \Lambda_1; h)$ with*

$$\Lambda_1' = \ C \circ C_B = \ \{\ (x,\xi,y,\eta) \in \ T^*(\mathbb{R}^n) \times T^*(\mathbb{R}^n)\ ; \exists t \in \mathbb{R}^k : (x,\xi) = \Psi^t(y,\eta), \\ p_0(y,\eta) = F\} \tag{2.15}$$

and if we write for (y,η) fixed, applying the hypothesis (H3),

$$\{t\, ;\ (x,\xi) = \Psi^t(y,\eta)\} = t_0 + \oplus_{1\leq j\leq k} 2\pi\mathbb{Z}$$

the principal symbol is

$$\begin{aligned}\sigma_0(x,\xi,y,\eta) \ &= \sum_{\{t\,;(x,\xi)=\Psi^t(y,\eta)\}} \frac{1}{(2\pi)^k}\ (2\pi h)^{-n/2}\ e^{ih^{-1}<t,F>}\hat{\zeta}(t) \times \\ &\qquad \exp\Big(ih^{-1}(-<p_0(y,\eta),t> + A(\gamma^t))\Big) \exp\Big(i\frac{\pi}{2}\mu(\gamma^t - \gamma^{t_0})\Big)\ \sigma_1 \\ &= \sum_{\{t\,;(x,\xi)=\Psi^t(y,\eta)\}} \frac{1}{(2\pi)^k}\ (2\pi h)^{-n/2}\ \hat{\zeta}(t) \times \\ &\qquad \exp\Big(ih^{-1}A(\gamma^t)\Big) \exp\Big(i\frac{\pi}{2}\mu(\gamma^t - \gamma^{t_0})\Big)\ \sigma_1.\end{aligned} \tag{2.16}$$

The sum in (2.16) is discrete and locally finite, because the support of $\hat{\zeta}$ is compact ; σ_1 is the canonical half density.

proof. — To understand the introduction of the Maslov term we have to make some recall on the composition of canonical relations.

If C_1 is a canonical relation in $T^*\mathbb{R}^m \times T^*\mathbb{R}^n$ and C_2 a canonical relation in $T^*\mathbb{R}^n \times T^*\mathbb{R}^p$ the composition $C_1 \circ C_2$ can be defined as a canonical relation in $T^*\mathbb{R}^m \times T^*\mathbb{R}^p$ if $C_1 \times C_2$ intersects transversally $T^*\mathbb{R}^m \times \Delta_{T^*\mathbb{R}^n} \times T^*\mathbb{R}^p$ (with the notation $\Delta_{T^*\mathbb{R}^n} = \{(Y,Y);\, Y \in T^*\mathbb{R}^n\} \subset (T^*\mathbb{R}^n)^2$). In fact it can be defined when the intersection is clean but in our case *this intersection is transversal* :
let $(\Psi^t(y,\eta),(y,\eta),(t,-F),(t,-F))$ be a point of this intersection. We will first calculate the orthogonal, for the canonical scalar product, of the sum of the tangent spaces. If $(X,Y,T,T') \in T(T^*\mathbb{R}^n)\times T(T^*\mathbb{R}^n)\times T(T^*\mathbb{R}^k)\times T(T^*\mathbb{R}^k)$ is orthogonal to $T(T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times \Delta_{T^*\mathbb{R}^k})$ then it is of the form $(0,0,T,-T)$ now if it is orthogonal to $(\sum u_j K_j, 0, (U,0), 0)$ obtained by moving only the variable t on Λ, and writing $U = (u_1,\dots,u_k)$ then $T = (0,T_2)$, finally by moving only (y,η) our vector must be orthogonal to $(d\Psi^t(V), V, (0, -dp_0(V)), 0)$ for any $V \in T_{(y,\eta)}T^*\mathbb{R}^n$ but dp_0 is on to by hypothesis (H1) so $T_2 = 0$. We will then follow [21].

If we denote by $C.C_B$ this intersection, then $C \circ C_B = \pi(C.C_B)$ where

$$\begin{array}{rcl} \pi : T^*\mathbb{R}^{2n+k} \times T^*\mathbb{R}^k & \to & T^*\mathbb{R}^{2n} \\ (X,Y,T,T') & \mapsto & (X,Y) \end{array}$$

and

$$\pi \ : C.C_B \to C \circ C_B$$

is a covering map.

Indeed in our case

$$\begin{array}{rcl} C.C_B & = & \{(\Psi^t(y,\eta),(y,\eta),(t,-F),(t,-F))\,;\ p_0(y,\eta) = F\} \\ C \circ C_B & = & \{((x,\xi),(y,\eta))\,;\ p_0(y,\eta) = F \text{ and } \exists\, t;\ (x,\xi) = \Psi^t(y,\eta)\} \end{array}$$

So the fiber of π is isomorphic to $2\pi\mathbb{Z}^k$ (ie. $C \circ C_B = C.C_B/2\pi\mathbb{Z}^k$) by hypothesis (H3), and π realizes an injection of $\Pi_1(C.C_B)$ in $\Pi_1(C \circ C_B)$ denoted by π_*. We have then an exact sequence :

$$0 \to \Pi_1(C.C_B) \xrightarrow{\pi_*} \Pi_1(C \circ C_B) \to 2\pi\mathbb{Z}^k \to 0. \tag{2.17}$$

(See Theorem 3, Ch 4 19 in [12].)

Now if M_C, M_{C_B} and $M_{C\circ C_B}$ are the Maslov bundles of C, C_B and $C \circ C_B$ respectively, one can make the following construction on $C.C_B$: denoting by π_1, π_2 the projection of $C \times C_B$ on each factor, let

$$M = \pi_1^*(M_C) \otimes \pi_2^*(M_{C_B})_{|C.C_B}$$

be the restriction to $C.C_B$ of $\pi_1^*(M_C) \otimes \pi_2^*(M_{C_B})$, which is the Maslov bundle of $C \times C_B$. By construction the bundle M corresponds to the representation i^λ of $\Pi_1(C.C_B)$ in $\mathbb{C}^*$ with :

$$\forall \gamma \in \Pi_1(C.C_B)\ ;\ \lambda(\gamma) = \mu(\pi_1 \circ \gamma) + \mu(\pi_2 \circ \gamma).$$

But we have noted that M_C and M_{C_B} are trivial, it means that $\lambda(\gamma) = 0$ in our case.

Lemma 2.3 . — *There is a natural relation*

$$M \simeq \pi^* M_{C\circ C_B}.$$

proof. — We can consider this result as Theorem 21.6.7 in [22] ; actually define $\Delta \subset T^*\mathbb{R}^{2n+2k}$

$$\Delta = \{(0,0,V,V) \in T^*\mathbb{R}^n \times T^*\mathbb{R}^n \times T^*\mathbb{R}^k \times T^*\mathbb{R}^k\}$$

Δ is isotropic, one can identify $\Delta^\omega/\Delta \simeq T^*\mathbb{R}^{2n}$ as a symplectic space and we can apply the proposition 2.1. Consider the following diagram

$$\begin{array}{ccc} C_1.C_2 & \xrightarrow{\pi} & C_1 \circ C_2 \\ \varphi_{2n+2k}\big\downarrow & & \varphi_{2n}\big\downarrow \\ \mathbb{L}(2n+2k) & \overset{\rho}{\leadsto} & \mathbb{L}(2n) \end{array}$$

where $\varphi_{2n+2k}(\nu) = T_\nu(C \times C_B)$. The map ρ is defined as follow : $\rho(\lambda) = \lambda \cap \Delta^\omega / \lambda \cap \Delta$ for $\lambda \in \mathbb{L}(2n+2k)$. This map makes the diagram commutative.

So the lemma is proved if one can see that the range of φ_{2n+2k} is included in $\mathbb{L}(2n+2k) - S_\Delta$. Indeed for $\nu = (\Psi^t(y,\eta),(y,\eta),(t,-F),(t,-F))$ a vector in $T_\nu(C \times C_B) = T_{\pi_1(\nu)}C \times T_{\pi_2(\nu)}C_B$ is a sum of terms of three types : $\Big(\sum \alpha_j K_j(\Psi^t(y,\eta)), 0, (0,\alpha), (0,0)\Big)$ for $\alpha \in \mathbb{R}^k$, $\Big(d\Psi^t(Y), Y, (0, -dp_0(Y)), (0,0)\Big)$ for $Y \in T_{(y,\eta)}T^*\mathbb{R}^n$ and $\Big(0,0,0,(\beta,0)\Big)$ for $\beta \in \mathbb{R}^k$. But it is impossible to write an element of Δ in such a way and we have $\varphi_{2n+2k}(\nu) \cap \Delta = \{0\}$. ■

Consequence of the lemma :

$$\forall \gamma \in \Pi_1(C.C_B) \; ; \; \mu(\pi_*(\gamma)) = \lambda(\gamma) = 0. \tag{2.18}$$

Then all the non triviality of $M_{C\circ C_B}$ comes from the action of the Hamiltonian flow, by (2.17). It means, using the exact sequence (2.17) that any section of $M_{C\circ C_B}$ can be represented by a $\mathbb{C}$-value function f on $C.C_B$ which satisfies the equivariant relation :

$$\forall T \in 2\pi\mathbb{Z}^k \; ; \quad f(t+T,y,\eta) = i^{-\mu(\gamma^T)} f(t,y,\eta) \tag{2.19}$$

if we parametrize $C.C_B$ by $(t,y,\eta) \in \mathbb{R}^k \times \Sigma_0$ and if we notice that, because of the connexity of Σ_0, the Maslov index of the loop $\gamma^T = \{(\Psi^{sT}(y,\eta),(y,\eta), 0 \le s \le 1\}$ is independent of the point (y,η) ; because of the homotopy of the loops γ^T and $\gamma^{2\pi T_1 e_1} + \ldots + \gamma^{2\pi T_k e_k}$ if $T = 2\pi(T_1e_1 + \ldots + T_k e_k)$ we can write :

$$\forall\, T \in \oplus_{1\le j\le k} 2\pi\mathbb{Z} \; ; \; \mu(\gamma^T) = \sum_{j=1}^{j=k} T_j \mu_j \;\text{ with }\; \mu_j = \mu(\gamma^{2\pi e_j}). \tag{2.20}$$

Let now $((x,\xi),(y,\eta)) \in C \circ C_B$. At each time that we choose $t \in \mathbb{R}^k$ such that $(x,\xi) = \Psi^t(y,\eta)$ we have a natural local isomorphism between M and $M_{C\circ C_B}$ as described in [21] p.181 ; but when we change t, say we take t_0 and $t_0 + T$ with T a period of our lattice, it corresponds to a change of trivialization of the bundle $M_{C\circ C_B}$ around our point. By definition of the Maslov bundle, the transition function is $i^{\mu(\gamma)}$ where γ is the loop

$$\gamma(s) = \big(\Psi^{t_0+sT}(y,\eta),(y,\eta)\big) \qquad \gamma = \gamma^{t_0+T} - \gamma^{t_0}.$$

Notice that $\mu(\gamma)$ does not depend on the path that we draw in $C.C_B$ joining the point $((x,\xi),(y,\eta),(t_0,-F),(t_0,-F))$ to $((x,\xi),(y,\eta),(t_0+T,-F),(t_0+T,-F))$ because of (2.18).

The conclusion is that to define the product of the two symbols, we fix t_0 and then multiply the product of the two symbols at $((x,\xi),(y,\eta),(t_0+T,-F),(t_0+T,-F))$ by $i^{\mu(\gamma^{t_0+T}-\gamma^{t_0})}$ and make the sum for all periods T. The result of this calculus is just (2.16) in the sens that the function $\sigma_0(t_0,y,\eta)$ defined by the formula (2.16) is a $\mathbb{C}-$value function on $C.C_B$ witch satisfies the equivariance (2.19). ∎

Remark. — This lemma 2.3 is certainly the way to understand that "there is an isomorphism between the two bundles" (with our notations M and $M_{C\circ C_B}$) as in [21] p.181 : in our case one is trivial but the second is not trivial in general so they cannot be really "isomorphic". It would be compared also with Lemma 4.2 of [4] given without proof for the Maslov factor.

2.5 end of the proof

Let T_0 be a period of our lattice ; we want now to compare $\zeta(\frac{F-P(h)}{h})\theta(P)$ with $\zeta(\frac{F-P(h)}{h})\exp-\frac{i}{h}(<T_0,P(h)>)\theta(P)$. One has

$$\begin{aligned}\zeta(\frac{F-P(h)}{h})e^{-\frac{i}{h}<T_0,P(h)>}\ \theta(P) &= \frac{e^{-\frac{i}{h}<T_0,F>}}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-\frac{i}{h}<t+T_0,(P(h)-F)>}\hat{\zeta}(t)\theta(P)dt\\ &= \frac{e^{-\frac{i}{h}<T_0,F>}}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-\frac{i}{h}<t,(P(h)-F)>}\hat{\zeta}(t-T_0)\theta(P)dt.\end{aligned}$$

We can conclude by the previous calculus that $\zeta(\frac{F-P(h)}{h})\exp-\frac{i}{h}<T_0,P(h)>\theta(P)$ can be approached by a FIO with the same Lagrangian than $\zeta(\frac{F-P(h)}{h})\theta(P)$ and with principal symbol

$$\begin{aligned}\sigma_{T_0}(x,\xi,y,\eta) &= e^{-\frac{i}{h}<T_0,F>}\sum_{\{t\,;\,(x,\xi)=\Psi^t(y,\eta)\}}\frac{1}{(2\pi)^k}\ (2\pi h)^{-n/2}\ \hat{\zeta}(t-T_0)\times\\ &\qquad \exp\left(ih^{-1}A(\gamma^t)\right)\exp\left(i\frac{\pi}{2}\mu(\gamma^t-\gamma^{t_0})\right)\ \sigma_1\\ &= e^{-\frac{i}{h}<T_0,F>}\sum_{\{t\,;\,(x,\xi)=\Psi^t(y,\eta)\}}\frac{1}{(2\pi)^k}\ (2\pi h)^{-n/2}\ \hat{\zeta}(t)\times\\ &\qquad \exp\left(ih^{-1}A(\gamma^{t+T_0})\right)\exp\left(i\frac{\pi}{2}\mu(\gamma^{t+T_0}-\gamma^{t_0})\right)\ \sigma_1 \qquad (2.21)\\ &= e^{i\left(h^{-1}(A(\gamma^{T_0})-<T_0,F>)+\frac{\pi}{2}\mu(\gamma^{T_0})\right)}\sigma_0(x,\xi,y,\eta). \qquad (2.22)\end{aligned}$$

We conclude that the two FIO $e^{-i\left(h^{-1}(A(\gamma^{T_0})-<T_0,F>)+\frac{\pi}{2}\mu(\gamma^{T_0})\right)}\zeta(\frac{F-P(h)}{h})e^{-\frac{i}{h}(<T_0,P(h)>)}\theta(P)$ and $\zeta(\frac{F-P(h)}{h})\theta(P)$ have the same Lagrangian and the same principal symbol, if we remark that $A(\gamma^{T_0})$ and $\mu(\gamma^{T_0})$ are numbers depending only on T_0 but not on $(y,\eta)\in\Sigma_0$. So their difference is a FIO for which the amplitude has a compact support and a factor h ; we conclude by the theorem of L_2 continuity (we can use the theorem of Asada-Fujiwara [3]) that there exists a constant $C>0$ such that

$$\|\zeta(\frac{F-P(h)}{h})\left(\mathrm{Id}-e^{-\frac{i}{h}(<T_0,P(h)-F>+A(\gamma^{T_0})+h\frac{\pi}{2}\mu(\gamma^{T_0}))}\right)\theta(P)\|_{L_2}\leq hC. \qquad (2.23)$$

Remark. — We will give a precise estimate of the constant C below with Lemma 3.2.

Now if f is a common eigenfunction of our $P_j(h)$ with joint eigenvalue $\lambda = (\lambda_1, \dots, \lambda_k)$ and if we suppose $\|F - \lambda\| \leq ch$ and $|\zeta| \geq d > 0$ on the set $\{x \in \mathbb{R}^k;\ \|x\| \leq c\}$, the previous evaluation gives

$$|1 - e^{-\frac{i}{h}(<T_0, \lambda - F> + A(\gamma^{T_0}) + h\frac{\pi}{2}\mu(\gamma^{T_0}))}| \leq h\frac{C}{d}.$$

This gives the theorem by taking for T_0 the basis of the lattice. ∎

2.6 extension

What happens if one relaxes the hypothesis (H_3) ? Following the formula (2.11) of [7] we see that appears a new term in the principal symbol of the evolution operator :

$$\exp\left(-i\int_0^1 < p_1(\Psi^{st}(y,\eta)), t > ds\right)$$

and in the comparison between the principal symbol of $\zeta(\frac{F-P(h)}{h})e^{-\frac{i}{h}(<T_0, P(h)>)}\theta(P)$ and $\zeta(\frac{F-P(h)}{h})\ \theta(P)$ will appear a new term which a priori depends on (y,η) :

$$\exp\left(-i\int_0^1 < p_1(\Psi^{sT_0}(y,\eta)), t > ds\right).$$

The hypothesis (H_3') assures that this term is constant, let us denote it by $e^{-i\delta(T_0)}$. Following the preceding proof we obtain

Theorem 2 . — *Under the assumptions H_1, H_2, H_3' and H_4, the part of the joint spectrum $\Lambda^{Q(h)}$ lying in any k-cube $\prod_{j=1}^{k}\,]E_{0j} - hc_j\ ,\ E_{0j} + hc_j[$ centered at E_0 is localized modulo $O(h^2)$ near a lattice*

$$E_0\ +\ \mathbf{a}^{-1}\Big(((\frac{\delta_1}{2\pi} - \frac{\mu_1}{4})h - \frac{\alpha_1}{2\pi} + \mathbb{Z}h) \oplus \ldots \oplus ((\frac{\delta_k}{2\pi} - \frac{\mu_k}{4})h - \frac{\alpha_k}{2\pi} + \mathbb{Z}h)\Big),$$

where δ_j are the integral of the sub-principal symbol on the basic cycles of the torus acting on Σ_0, the μ_j are the Maslov indices of these cycles and α_j are the action of these cycles.

3 Multiplicity

For simplicity we maintain in this section the hypothesis (H_3).

Theorem 3 . — *For $\mathbf{n} \in \mathbb{Z}^k$ denote by $I_\mathbf{n}(h)$ the k-cube with size $2Ch^2$ centered at $E_0 + \mathbf{a}^{-1}(-\frac{\alpha}{2\pi} - \frac{\mu}{4}h + \mathbf{n}h)$ and suppose that $I_\mathbf{n}(h) \subset \prod_{j=1}^{k}]E_{0j} - c_jh, E_{0j} + c_jh[$. Then the number*

$$N_\mathbf{n}(h) = \sharp\Big(I_\mathbf{n}(h) \cap \mathrm{Spec}(P(h))\Big)$$

admits the following behavior on h : there exists a sequence $\Big(l_j(\mathbf{n})\Big)_{j\in\mathbb{N}}$ such that for any $m \in \mathbb{N}$

$$N_\mathbf{n}(h) = h^{k-n}\sum_0^{m-1} l_j(\mathbf{n})h^j + O(h^{m-n}) \tag{3.24}$$

and $l_0(\mathbf{n}) = l_0 = \frac{1}{(2\pi)^n}\int_{\Sigma_0} d\nu$ where $d\nu$ denote the Liouville density of Σ_0.

As before, we work with the new operators $P(h)$. The Liouville density of Σ_0 is defined as follows : the map $p_0 : T^*\mathbb{R}^n \to \mathbb{R}^k$ is a submersion on a neighborhood of Σ_0 because of hypothesis (H_1) and then defines a density on Σ_0 by "dividing" the euclidean density $dx_{T^*\mathbb{R}^n}$ by the pullback $p_0^*(dx_{\mathbb{R}^k})$ of the euclidean density of $\mathbb{R}^k$ by the submersion. Actually $p_0^*(dx_{\mathbb{R}^k})$ is a well defined density on a transversal of Σ_0 and $d\nu$ must satisfy at the points of Σ_0

$$dx_{T^*\mathbb{R}^n} = d\nu \wedge p_0^*(dx_{\mathbb{R}^k}).$$

If we use the Riemannian structure of the sub-manifold Σ_0 of the euclidean space $T^*\mathbb{R}^n$ and denote by dS the associated Riemannian density, then

$$dS = \|dp_{01} \wedge \ldots \wedge dp_{0k}\| d\nu.$$

proof. — We will just sketch it because we just follow [11] herself inspired by [4]. We want to approach $N_{\mathbf{n}}(h)$ by a trace

$$\mathrm{Tr}_\zeta(h) = \mathrm{Tr}\Big(\zeta(\frac{F-P(h)}{h})\ \theta(P(h))\Big) = \mathrm{Tr}\Big(\frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-\frac{i}{h}<t,(P(h)-F)>}\hat{\zeta}(t)\theta(P(h))dt\Big)$$

with ζ to be chosen satisfying two conditions : $\hat{\zeta}(0) = 1$ and the support of $\hat{\zeta}$ is a small compact such that 0 is the only one period of the joint flow belonging to $\mathrm{Supp}(\hat{\zeta})$.

On one hand we can calculate $\mathrm{Tr}_\zeta(h)$ using the stationary phase theorem and obtain the development (3.24) with the expression of l_0 as mentioned in [7] Theorem 5.2.

On the other hand we cut $\mathrm{Tr}_\zeta(h)$ in three terms

$$\mathrm{Tr}_\zeta(h) = \Big(\sum_{\lambda;\ \forall j |\lambda_j - F_j| < hc_j} + \sum_{\substack{\lambda;\ \forall j |\lambda_j - F_j| < h^{1-\epsilon}c_j \\ \exists j |\lambda_j - F_j| \geq hc_j}} + \sum_{\lambda; \exists j |\lambda_j - F_j| \geq h^{1-\epsilon}c_j}\Big)\zeta(\frac{F-\lambda}{h})\theta(\lambda)$$

and we will choose ϵ and ζ. The third term can be controlled by $O_\zeta(h^{\epsilon N})$ for any N because $\zeta \in \mathcal{S}(\mathbb{R}^k)$. We just have to take N such that $\epsilon N > m + k - n$ when ϵ is fixed. If we want that the first term approaches $N_{\mathbf{n}}(h)$ we must choose ζ.

Lemma 3.1 . — *For any $N \in \mathbb{N}$ there exists a function $\varphi_N : \mathbb{R}^k \to \mathbb{R}$ satisfying*

$$\hat{\varphi}_N \in C_0^\infty(\mathbb{R}^k), \quad \varphi_N(t) \overset{0}{\simeq} 1 + O(\|t\|^N) \text{ and} \tag{3.25}$$

$$\exists c, \forall \mathbf{m} \in \mathbb{Z}^k - \{0\} \quad |\varphi_N(t)| \leq c\|t - \mathbf{m}\|^N. \tag{3.26}$$

proof. — Remark first that for a function $f, \hat{f} \in C_0^\infty(\mathbb{R})$ and $f(0) = 1$, the function $\varphi(x) = f(x)\frac{\sin(2\pi x)}{2\pi x}$ satisfies

$$\hat{\varphi} \in C_0^\infty(\mathbb{R})\,; \quad \varphi(x) \overset{0}{\simeq} 1 + O(|x|)\,; \quad \exists c > 0\,; \forall n \in \mathbb{Z} - \{0\},\ |\varphi(x)| \leq c|x - n|$$

(see Lemma 2.5.1 of [11]). The function

$$\varphi_N(t_1, \ldots, t_k) = \prod_{j=1}^k \Big(1 - (1 - \varphi(t_j))^N\Big)^N$$

satisfies the properties of the lemma. ■

Define now for any $\mathbf{m} \in \mathbb{Z}^k$

$$\beta_{\mathbf{m}} = \frac{1}{2\pi h}\alpha + \frac{1}{4}\mu - \mathbf{m}, \quad \text{and} \quad \zeta(t) = \varphi_N(t - \beta_{\mathbf{n}}) \tag{3.27}$$

then, by the Theorem 1

$$\begin{aligned}\sum_{\lambda;\ \forall j |\lambda_j - F_j| < hc_j} \zeta(\frac{F-\lambda}{h})\theta(\lambda) &= \sum_{\mathbf{m}\in\mathbb{Z}^k}\left(\sum_{\lambda; \|\lambda-(F-h\beta_{\mathbf{m}})\|<Ch^2} \zeta(\frac{F-\lambda}{h})\theta(\lambda)\right) \\ &= \sum_{\lambda; \|\lambda-(F-h\beta_{\mathbf{n}})\|<Ch^2} \zeta(\frac{F-\lambda}{h})\theta(\lambda) + O(h^{N+k-n})\end{aligned}$$

if we use (3.26) to bound each other term and Lemma 5.4 of [7] to bound the number of such terms by $O(h^{k-n})$. Then using (3.25) we obtain

Corollary 3.1 . — *For any $m \in \mathbb{N}$ if we define ζ by the formula above with $N > m$ then*

$$| \ N_{\mathbf{n}}(h) - \sum_{\lambda;\ \forall j |\lambda_j - F_j| < hc_j} \zeta(\frac{F-\lambda}{h})\theta(\lambda) \ | \leq Ch^{m+k-n}$$

For the second term we need a refined version of Theorem 1 by the control of the constant C in the inequality (2.23).

Lemma 3.2 . — *For a function ζ such that $\hat{\zeta} \in C_0^\infty(\mathbb{R}^k)$ and $b \in \mathbb{R}^{>0}$ define $\zeta_b(t) = \frac{1}{b^k}\zeta(\frac{t}{b})$. There exists $b_0, h_0, C > 0$ such that for any $h < h_0$ and $b > b_0$,*

$$\|\zeta_b(\frac{F-P(h)}{h})\left(\mathrm{Id} - e^{-\frac{i}{h}(<T_0,P(h)-F>+A(\gamma^{T_0})+h\frac{\pi}{2}\mu(\gamma^{T_0}))}\right)\theta(P)\|_{L_2} \leq b^{2-k}hC.$$

proof. — We will write here the kernel of the evolution operator as an oscillatory integral. If we remark that $\hat{\zeta_b}(t) = \hat{\zeta}(bt)$, the kernel of the operator $\zeta_b(\frac{F-P(h)}{h})\left(\mathrm{Id} - e^{-\frac{i}{h}(<T_0,P(h)-F>+A(\gamma^{T_0})+h\frac{\pi}{2}\mu(\gamma^{T_0}))}\right)\theta(P)$ can be written (locally in (x,y))

$$\int \hat{\zeta}(bt)e^{\frac{i}{h}(\phi(t,x,y,\theta)+<t,F>)}a(t,x,y,\theta,h)dt\ d\theta = \int e^{\frac{i}{h}\tilde{\phi}(t,x,y,\theta)}\tilde{a}(t,x,y,\theta,h)dt\ d\theta.$$

The relation between the phase function and the amplitude with the geometric objects already described are : $C = \{(x,y,t,\phi'_x,\phi'_y,\phi'_t); \phi'_\theta(t,x,y,\theta) = 0\}$. The amplitude $\tilde{a}$ has compact support and by the previous section we know that $\tilde{a}(t,x,y,\theta,h) = h^{-n}\sum h^j\tilde{a}_j(t,x,y,\theta)$ and $\tilde{a}_0(t,x,y,\theta)$ vanishes on the canonical relation of our operator : Λ'_1. Moreover for any integer l there exists a constant $C_l > 0$ such that

$$\|D^l\tilde{a}_0\|_\infty \leq b^lC_l \tag{3.28}$$

where D^l is any composition of l partial derivatives. Finally we can remark that, the phase function $\tilde{\phi}$ is non-degenerated in the sense that the function $(x,y,t,\theta) \to (\phi'_t + F, \phi'_\theta)$ is a submersion. This fact is a consequence of the description of ϕ given in [7] Theorem 4.2. Because the phase function is non-degenerated and $\tilde{a}_0$ vanishes on Λ'_1 which can be described as $\Lambda'_1 = \{(x,y,\tilde{\phi}'_x,\tilde{\phi}'_y); \tilde{\phi}'_t(t,x,y,\theta) = 0, \tilde{\phi}'_\theta(t,x,y,\theta) = 0\}$, there exist C^∞-functions with compact support γ_j and χ_l such that $\tilde{a}_0 =< \gamma, \tilde{\phi}'_t > + < \chi, \tilde{\phi}'_\theta >$, and as they are defined in terms of derivatives of $\tilde{a}_0$ and as $\tilde{a}_0$ satisfies (3.28) their derivatives of order l are controlled by b^{l+1}. Finally an integration by parts gives

$$\begin{aligned}\int e^{\frac{i}{h}\tilde{\phi}(t,x,y,\theta)}\left(<\gamma,\tilde{\phi}'_t> + <\chi,\tilde{\phi}'_\theta>\right) &= \\ \frac{-h}{i}\int e^{\frac{i}{h}\tilde{\phi}(t,x,y,\theta)}\Big(\sum_j \frac{\partial\gamma_j}{\partial t_j} + \sum_l \frac{\partial\chi_l}{\partial\theta_l}\Big) &= O(hb^{2-k})\end{aligned}$$

if we remember that $\mathrm{Supp}(\hat{\zeta}_b) \subset]\frac{-1}{b},\frac{1}{b}[^k$. ■

Corollary 3.2 . — *For any $\epsilon \ll 1/2$ if $\lambda = \lambda(h)$ is a joint eigenvalue of $P(h)$ such that $\|\lambda - F\|_\infty \leq ch^{1-\epsilon}$ then there exist $C > 0$ and $\mathbf{m} \in \mathbb{Z}^k$ such that*

$$\|\lambda - (F - h\beta_{\mathbf{m}})\| \leq Ch^{2-2\epsilon}$$

with $\beta_{\mathbf{m}}$ defined by (3.27).

proof. — We use Lemma 3.2 with $b = h^{-\epsilon}$ remarking that $|\zeta_b(t)| \geq \frac{d}{b^k}$ if $\|t\|_\infty \leq bc$. It follows that if $\|\lambda - F\|_\infty \leq ch^{1-\epsilon}$ then

$$|1 - e^{-\frac{i}{h}(<T_0,\lambda-F>+A(\gamma^{T_0})+h\frac{\pi}{2}\mu(\gamma^{T_0}))}| \leq h^{1-2\epsilon}\frac{C}{d}.$$

The conclusion goes as for Theorem 1. ■

Consequence. — If λ is a joint eigenvalue occurring in the second term of $\mathrm{Tr}_\zeta(h)$ for the special ζ defined by (3.27), then it can be written $F - h\beta_{\mathbf{m}} + O(h^{2-2\epsilon})$ with $\mathbf{m} \neq \mathbf{n}$ an $\zeta((\frac{F-\lambda}{h})\theta(\lambda) = O(h^{N(1-2\epsilon)})$ by (3.25) ; as previously we can bound the number of terms occurring in this sum by $O(h^{k-n})$, we obtain finally

$$\Big|\sum_{\substack{\lambda;\ \forall j|\lambda_j-F_j|<h^{1-\epsilon}c_j \\ \exists j|\lambda_j-F_j|\geq hc_j}} \zeta(\frac{F-\lambda}{h})\theta(\lambda)\Big| \leq Ch^{N(1-2\epsilon)+k-n}$$

it can be written as $O(h^{m+k-n})$ if $N(1-2\epsilon) > m$. Therefore we have proved that for any integer m there exist $C > 0$ and a function ζ such that $\hat{\zeta}(0) = 1$ and 0 is the only period of the joint flow belonging to the (compact) support of $\hat{\zeta}$ and such that

$$|\ \mathrm{Tr}_\zeta(h) - N_{\mathbf{n}}(h)\ | \leq Ch^{m+k-n}$$

for h small enough. This concludes the proof of Theorem 3. ■

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