A SHIFT BETWEEN DIRICHLET AND NEUMANN SPECTRUM FOR GENERALIZED LINEAR ELASTICITY

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Presentation : defining the equation of linear elasticity on a general riemannian manifold with boundary, we prove a formula relating the counting functions of the Neumann and the Dirichlet problem to the counting function of the Dirichlet2Neumann operator. Namely the difference of the two counting functions at α equals the number of negative eigenvalues of the Dirichlet2Neumann operator related to the resolvant at α.
With this formula we can show that this difference is always bigger than one in the homogeneous case (i.e. when the Lamé functions λ, μ are constant) for bounded domains of symmetric spaces of non compact type and rank bigger than 2, for instance the euclidean space, and for rank 1 if the dimension of the nilpotent part is less than $\frac{8\mu}{\lambda+2\mu}$.

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1 Introduction

The operator of linear elasticity is a good example of an non scalar operator : its principal symbol is not an homothety.

In the theory of Elasticity one studies the deformation due to displacement of solid bodies regarded as continuous media. Let (M, g) be a riemannian manifold with boundary of dimension $m \geq 2$, and ∇ the Levi-Civita covariant derivative. A deformation of the "shape" of M under an infinitesimal displacement given by the vector field X is then given by the Lie derivative : $\mathcal{L}_X(g)$, the so called *strain-tensor*. Recall that the metric defines a scalar product on every tensor bundle, we shall note it $(.,.)_g$. Its defines also a natural isomorphism between the tangent bundle and the cotangent bundle. Recall the musical symbols [BGM] : for X a tangent vector $X^\flat(Y) = g(X,Y)$ defines a linear form and the inverse map is $\xi \mapsto \xi^\sharp$. If X is a vector field on M, then ∇X defines a section of the bundle $\mathrm{End}(TM)$, on the other hand the metric defines an adjoint on this bundle : $A \in \mathrm{End}(TM)$, $g(A^tX, Y) = g(X, AY)$. Let S be the differential operator acting on the sections of TM by the formula :

$$SX = \nabla X + (\nabla X)^t$$

but if we use the metric to identify $T^*M \otimes T^*M$ with $\mathrm{End}(TM)$ then $SX = \mathcal{L}_X(g)$.

The *stress tensor* Σ expresses the pressure n the boundary following upon the deformation. The hypothesis of elasticity is precisely that the stress tensor , which is symmetric, only depends on the strain tensor SX. On linear elasticity one supposes that this dependency is linear ; finally the elasticity is said to be isotropic when this dependency does not depend on the orthonormal frame. So we look for an endomorphism of $Sym(\mathbb{R}^n)$ which commute with the action of $O(n)$ (by conjugation). The irreducible subspaces of this action are : $\mathbb{R}.\mathrm{I}$, the line generate by the matrix identity, and $\ker(\mathrm{Tr})$ the subspace of matrix of trace null. This endomorphism must be homothetic on each part and every matrix is decomposed on these orthogonal subspaces by :

$$A = \frac{\mathrm{Tr}(A)}{n}\mathrm{I} + (A - \frac{\mathrm{Tr}(A)}{n}\mathrm{I}).$$

But $\mathrm{Tr}(SX) = 2\,\mathrm{Tr}(\nabla X) = 2\,\mathrm{div}(X)$ we obtain then at each point :

$$\Sigma = \lambda \mathrm{div}(X)\mathrm{I} + \mu SX.$$

The density of work is then given at each point by

$$(\Sigma, \nabla X)_g = \lambda \mathrm{div}(X)^2 + \mu(SX, \nabla X)_g = \lambda \mathrm{div}(X)^2 + \frac{\mu}{2}(SX, SX)_g$$

while symmetric and skewsymmetric matrices are orthogonal.

The *operator of elasticity* L with Lamé functions λ, μ which are supposed to be smooth and positive, is the operator received from the polarisation of the quadratic form

$$\mathcal{A}(X,Y) = \int_M \left(\lambda\ \mathrm{div}(X)\mathrm{div}(Y) + \frac{\mu}{2}\ (SX, SY)_g\right)$$

defined on vector fields.

The operator L defines, via the polarisation formula, two boundary conditions : $X = 0$ on ∂M and $BX = 0$ on ∂M which appear to define two selfadjoint elliptic operators [proposition 1]. The aim of this work is to compare the spectrum of these two operators. For this, we establish a formula [Proposition 5] which calculate the difference of their counting functions. It is shown that the difference of the two counting functions at α equals the number of negative eigenvalues of the Dirichlet2Neumann operator related to the resolvant at α. With this formula we can prove a shift between these spectra for domains of symmetric spaces of non compact type [Theorem 6] : if the symmetric space is of rank bigger than 2 or if the rank is 1 and the dimension of the nilpotent part is less or equal to $\frac{8\mu}{\lambda+2\mu}$ then $N(\alpha, L, B) - N(\alpha, L, 0) \geq 1$. When these hypothesis are not satisfied the conclusion is still true for α big enough ; the theorem gives also an upper bound of the first eigenvalue of the Neumann problem.

2 Elliptic boundary conditions

2.1 polarisation formula

We use equally for any function f and any vector field X the notation $\mathrm{d}_X f = \imath(X)\mathrm{d}f = X \cdot f$ if $\imath(.)$ denotes the inner product.

1. Let $\vec{n}_e$ be the exterior normal of the boundary ∂M of M. From the Stokes formula we derive for any vector field X on M

$$\int_M \operatorname{div}(X) = \int_{\partial M} (X, \vec{n}_e)_g$$

if the relation between the volume forms ω_M on M and $\omega_{\partial M}$ on ∂M is :

$$\omega_{\partial M} = \imath(\vec{n}_e)\ \omega_M$$

We have then

$$\begin{aligned}
\int_M \Big(\lambda \operatorname{div}(X)\operatorname{div}(Y)\Big) &= \int_M \Big(\operatorname{div}(\lambda \operatorname{div}(X)\ Y) - \mathrm{d}_Y(\lambda \operatorname{div}(X))\Big) \\
&= \int_{\partial M} \Big(\lambda \operatorname{div}(X)(Y, \vec{n}_e)\Big) - \int_M \Big(\mathrm{d}_Y(\lambda \operatorname{div}(X))\Big)
\end{aligned}$$

2. For the term involving the function μ we introduce a local orthonormal frame $e_1, \cdots, e_m$. Let A be a section of $\operatorname{End}(TM)$ and Y a vector field. We have locally, with the notation $a_{ji} = (Ae_i, e_j)_g$

$$\begin{aligned}
(A, \nabla Y)_g \omega_M &= \sum_{i,j} a_{ji}(\nabla_{e_i} Y, e_j)_g \omega_M \\
&= \sum_{i,j} e_i \cdot \Big(a_{ji}(Y, e_j)_g\Big)\ \omega_M - \cdots \\
&\qquad \ldots \sum_{i,j} \Big(e_i \cdot a_{ji}\ (Y, e_j)_g + a_{ji}(Y, \nabla_{e_i} e_j)_g\Big)\ \omega_M \\
&= \mathrm{d}\Big(\sum_{i,j} a_{ji}(Y, e_j)_g\ \imath(e_i)\omega_M\Big) + (\nabla^* A, Y)_g \omega_M \\
&= \mathrm{d}\Big(\imath(A^t Y)\omega_M\Big) + (\nabla^* A, Y)_g \omega_M
\end{aligned}$$

From this calculation we receive the local expression of ∇^* :

$$\nabla^* A = -\sum_i (\nabla_{e_i}(Ae_i)) + A(\sum_i \nabla_{e_i} e_i) \tag{1}$$

(see the details in appendix). On the other hand, integrating this formula over M we obtain

$$\int_M (A, \nabla Y)_g = \int_{\partial M} (A^t Y, \vec{n}_e)_g + \int_M (\nabla^* A, Y)_g$$

We can apply this expression on the second term of the quadratic form $\mathcal{A}$

$$\int_M \Big(\frac{\mu}{2}\,(SX,SY)_g\Big) = \int_M \Big(\mu\,(SX,\nabla Y)_g\Big)$$

because symmetric and skewsymmetric matrices are orthogonal. Setting $A = \mu\, SX$ we obtain

$$\begin{aligned}\int_M \Big(\frac{\mu}{2}\,(SX,SY)_g\Big) &= \int_M (\nabla^*(\mu\ SX),Y)_g + \int_{\partial M} \mu\ (SX(Y),\vec{n}_e)_g \\ &= \int_M (\nabla^*(\mu\ SX),Y)_g + \cdots \\ &\quad \cdots \int_{\partial M} \mu\ \Big(Y, SX(\vec{n}_e)_g\Big)\end{aligned}$$

because SX is symmetric.

3. If we add these two results we obtain

$$\mathcal{A}(X,Y) = \int_M \Big(LX,Y\Big)_g + \int_{\partial M} \Big(BX,Y\Big)_g$$

where $LX = -\Big(\mathrm{d}(\lambda\ \mathrm{div}(X))\Big)^\sharp + \nabla^*(\mu\ SX)$ is the operator of elasticity and $BX = \lambda\ \mathrm{div}(X)\vec{n}_e + \mu\ SX(\vec{n}_e)$ is the boundary condition. We have then for any $Y \in TM$ the expression of B

$$(BX,Y)_g = \lambda\ \mathrm{div}(X)(\vec{n}_e,Y)_g + \mu\Big((\nabla_Y X,\vec{n}_e)_g + (Y,\nabla_{\vec{n}_e}X)_g\Big). \qquad (2)$$

Proposition 1 . — *The operator L defines two selfadjoint elliptic operators with the two boundary conditions : $X = 0$ on ∂M and $BX = 0$ on ∂M. We call them respectively the Dirichlet and Neumann boundary conditions and refer to them by $(L,0)$ and (L,B).*

This proposition is certainly obvious for the specialists of elasticity, [CP] [NU] [V] ..., even if they work on domains of $\mathbb{R}^m$.

Proof. — From the polarisation formula we have only to prove ellipticity. We refer here to the Chapter XX, §1 in the Hörmander's book [H3]. We need first the principal symbol of the operators.

2.2 principal symbols

A straightforward calculation shows that $p(x,\xi)$ the principal symbol of L is for $X \in T_xM$ a tangent vector, and $\xi \in T_x^*M$

$$p(x,\xi)(X) = \mu(x)|\xi|^2 X + (\lambda(x)+\mu(x))\xi(X)\xi^\sharp$$

while the principal symbol $b(x,\xi)$ of B is

$$b(x,\xi)(X) = i\lambda(x)\xi(X)\vec{n}_e + i\mu(x)\Big(\xi(\vec{n}_e)X + (\vec{n}_e, X)_g\xi^\sharp\Big)$$

and the principal symbol of the boundary operator corresponding to the Dirichlet problem is

$$b_0(x,\xi)(X) = X.$$

If $\xi \neq 0$ then $p(x,\xi)$ is invertible : on $\ker\xi$ the homomorphism $p(x,\xi)$ acts as $X \mapsto \mu(x)|\xi|^2 X$ while $p(x,\xi)(\xi^\sharp) = (\lambda(x)+2\mu(x))|\xi|^2\xi^\sharp$. Finally the *Lopatinski condition* is satisfied for both :

2.3 Lopatinski condition

Recall that, with $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $\vec{n}_i = -\vec{n}_e$ the *interior* normal derivative, one defines for $x \in \partial M$, $\xi \in T_x^* M$ so that $\xi(\vec{n}_i) = 0$ the space

$$\begin{aligned} M^+_{(x,\xi)} = \{u \in C^\infty(\mathbb{R}, T_x M \otimes \mathbb{C}) \ ; \ u \text{ bounded on } \mathbb{R}^+ \cdots \\ \cdots \text{ and } p(x, \xi + D_t \vec{n}_i^\flat)u = 0\} \end{aligned}$$

The Lopatinski condition is satisfied if the map

$$\begin{array}{ccc} M^+_{(x,\xi)} & \xrightarrow{\Phi_b} & T_x M \otimes \mathbb{C} \\ u & \longmapsto & b(x, \xi + D_t\vec{n}_i^\flat)u[t=0] \end{array}$$

is an isomorphism.

2.3.1 description of $M^+_{(x,\xi)}$.

In the following we omit the variable x. The elements of $M^+_{(x,\xi)}$ must satisfy the equation

$$\begin{aligned} &\mu\Big(|\xi|^2 - \frac{\partial^2}{\partial t^2}\Big)u + \dots \\ &\qquad (\lambda+\mu)\Big(\xi(u)\xi^\sharp + \frac{1}{i}\xi(\frac{\partial}{\partial t}u)\vec{n}_i + \frac{1}{i}(\vec{n}_i, \frac{\partial}{\partial t}u)_g\xi^\sharp - (\vec{n}_i, \frac{\partial^2}{\partial t^2}u)_g\vec{n}_i\Big) = 0 \end{aligned}$$

if we make the orthogonal decomposition $u = u_0 + u_1\xi^\sharp + u_n\vec{n}_i$ with u_1 and u_n scalar functions and $(u_0,\xi^\sharp)_g = (u_0,\vec{n}_i)_g = 0$ then we have to solve the system

$$\begin{aligned} \Big(|\xi|^2 - \frac{\partial^2}{\partial t^2}\Big)u_0 &= 0 \\ \mu\frac{\partial^2}{\partial t^2}u_1 - (\lambda+2\mu)|\xi|^2 u_1 - \frac{\lambda+\mu}{i}\frac{\partial}{\partial t}u_n &= 0 \\ (\lambda+2\mu)\frac{\partial^2}{\partial t^2}u_n - \mu|\xi|^2 u_n - \frac{\lambda+\mu}{i}|\xi|^2\frac{\partial}{\partial t}u_1 &= 0 \end{aligned}$$

The first equation with the extra condition to be bounded on $\mathbb{R}^+$ gives $m-2$ independent solutions : $u_\perp(t) = e^{-|\xi|t}\vec{u}_\perp$ with $\vec{u}_\perp \in \ker\xi \cap \vec{n}_i^\perp$. We have then to solve a linear constant system of rank 2 and order 2, if we keep only the solutions which are bounded on $\mathbb{R}^+$ we receive the two functions :

$$\begin{array}{rcll} u_1(t) & = & e^{-|\xi|t}\vec{u}_1 & \text{where } \vec{u}_1 = \xi^\sharp + i|\xi|\vec{n}_i \\ u_2(t) & = & tu_1(t) + e^{-|\xi|t}\vec{u}_2 & \text{where } \vec{u}_2 = i\dfrac{\lambda+3\mu}{\lambda+\mu}\vec{n}_i \end{array}$$

2.3.2 The boundary condition $(L, 0)$ is elliptic

we calculate $\Phi_0 = \Phi_{b_0}$ on this base :

$$\begin{array}{ll} \Phi_0(u_\perp) & = \vec{u}_\perp \\ \Phi_0(u_1) & = \vec{u}_1 \\ \Phi_0(u_2) & = \vec{u}_2 \end{array}$$

so Φ_0 transforms a base on a base, it is an isomorphism.

2.3.3 The boundary condition (L, B) is elliptic

we identify an element $u \in M^+_{(x,\xi)}$ with its Cauchy data $U_0 = u(0)$ and $U_1 = D_t u(0)$, the expression of Φ_b is then

$$\Phi_b(u) = -i\Big(\lambda\xi(U_0)\vec{n}_i + (\lambda+\mu)(\vec{n}_i, U_1)_g\vec{n}_i + \mu(\vec{n}_i, U_0)_g\xi^\sharp + \mu U_1\Big)$$

(see the appendix for details) and we find

$$\begin{array}{rcl} \Phi_b(u_\perp) & = & \mu|\xi|\vec{u}_\perp \\ \Phi_b(u_1) & = & 2\mu|\xi|\vec{u}_1 \\ \Phi_b(u_2) & = & \dfrac{2\mu}{\lambda+\mu}\Big(\mu\xi^\sharp + i(\lambda+2\mu)|\xi|\vec{n}_i\Big) \\ & = & \dfrac{2\mu}{\lambda+\mu}\Big(\mu\vec{u}_1 + i(\lambda+\mu)|\xi|\vec{n}_i\Big) \end{array}$$

so Φ_b is also an isomorphism. ◁

Remark 1 . — *The ellipticity is preserved if* $\lambda = 0$.

3 The Dirichlet to Neumann Operator

By the last proposition we know that the two operators with boundary conditions $(L, 0)$and (L, B) can be diagonalised on an orthonormal base of $L^2(M, TM)$ with eigenvalues accumulating only on $+\infty$. Each spectrum is non negative because these problems come from a non negative quadratic form.

Lemma 2 . — *The spectrum of $(L, 0)$ is positive.*

Proof. — Suppose that the vector field X satisfies $LX = 0$ and $X_{|\partial M} = 0$, then $\mathcal{A}(X, X) = 0 \Rightarrow SX = 0$. Locally one can choose an orthonormal frame $e_1, \ldots, e_n$ such that $\nabla_{e_n} e_j = 0$ and $x \in \partial M \Rightarrow e_n(x) = \vec{n}_i(x)$ by this way : we fix $e_1, \ldots, e_{n-1}$ locally on the boundary and transport $e_1, \ldots, e_{n-1}$ and $\vec{n}_i$ by the parallel transport along the normal geodesics. We write t the normal parameter of these geodesics, with this choice $\frac{\partial}{\partial t} = e_n$. Then $SX = 0$ means

$$\forall i, j \quad (\nabla_{e_i} X, e_j)_g + (\nabla_{e_j} X, e_i)_g = 0$$

in particular $(\nabla_{e_n} X, e_n) = 0$ or $\frac{\partial}{\partial t}(X, e_n)_g = 0$ but $(X, e_n)_g = 0$ on the boundary, so $(X, e_n)_g = 0$ on a neighbourhood of the boundary. Consequently

$$\begin{aligned} \forall 1 \leq j \leq n-1 \qquad \frac{\partial}{\partial t}(X, e_j)_g &= (\nabla_{e_n} X, e_j)_g \\ &= -(\nabla_{e_j} X, e_n)_g \\ &= (X, \nabla_{e_j} e_n)_g \\ &= \sum_{k=1}^{k=n-1} (X, e_k)_g \, (e_k, \nabla_{e_j} e_n)_g \end{aligned}$$

The vector $[(X, e_j)_g]_{1 \leq j \leq n-1}$ viewed as a function of t, with parameter the points of ∂M is solution of an ordinary first order differential linear equation ; it is null at $t = 0$ so it is null on a neighbourhood of ∂M, then X is solution of the elliptic equation $LX = 0$ null on an open set, so every where null, by the principle of unique continuation. $\triangleleft$

For each $\alpha \notin \mathrm{Spec}(L, 0)$ we can define the Dirichlet to Neumann operator $DN(\alpha)$ as follows : let f be a section of the restriction $TM_{|\partial M}$ of the fiber bundle TM on the boundary of M. There exists a unique solution u of the problem

$$\begin{aligned} Lu &= \alpha u \text{ on } M \\ u &= f \quad \text{on } \partial M \end{aligned}$$

let $\gamma(u)$ be the Cauchy data $(u_{|\partial M}, (\frac{1}{i}\nabla_{\vec{n}_i} u)_{|\partial M})$, then

$$DN(\alpha)(f) = B(u) = B(\gamma(u))$$

while in fact B can be regarded as an operator on the boundary Cauchy data.

Following carefully the construction of the parametrix of $(L - \alpha, 0)$ by the method of [H3] we can calculate the principal symbol of DN.

Proposition 3 . — *Let $x \in \partial M$ and $\xi \in T_x^* M - \{0\}$ so that $\xi(\vec{n}_i) = 0$, then the principal symbol dn of $DN(\alpha)$ is diagonal on the orthogonal decomposition of*

$$T_x M \otimes \mathbb{C} = (\ker \xi \cap \vec{n}_i^{\perp}) \oplus \mathbb{C}.(\xi + i|\xi|\vec{n}_i) \oplus \mathbb{C}.(\xi - i|\xi|\vec{n}_i)$$

with eigenvalues $\mu|\xi|, 2\mu|\xi|$ and $2\mu \left(\frac{\lambda+\mu}{\lambda+3\mu}\right)|\xi|$.

Proof. — We recall here the main steps of [H3], Chap.XX §1. We have first to take some notations : identifying a solution u of $p(x,\xi+D_t\vec{n}_i)u=0$ with its Cauchy data $\gamma(u)=(u_{|\partial M},(\frac{1}{i}\nabla_{\vec{n}_i}u)_{|\partial M})$, we have the decomposition $(T_xM\otimes\mathbb{C})^2=M^+_{(x,\xi)}\oplus M^-_{(x,\xi)}$ if M^- denotes the set of solutions bounded on $\mathbb{R}^-$.

The *Calderon projector* Q is a pseudo-differential operator defined by $Q=\gamma TL^c$ if T is a properly supported parametrix of L on an extension $\tilde{M}$ of the manifold M and L^c is defined by $L^c\circ\gamma(u)=L(u^o)-(Lu)^o$ for any $u\in C^\infty(M,TM)$ if u^o denotes the distribution on $\tilde{M}$ equal to u on M and 0 on $\tilde{M}-M$.

The operator Q is approximated by the projector on $M^+_{(x,\xi)}$ in the direction of $M^-_{(x,\xi)}$ (Theorem 20.1.3) ; the formula (20.1.12) gives a parametrix of the boundary problem $(L-\alpha,0)$, in particular if u satisfies $Lu-\alpha u=0$ and $u_{|\partial M}=f$ then an approximation of u is given by $u=TL^cSf$ where S is characterised by $QS=S$ and $B_0S=\mathrm{Id}$ on $T_xM\otimes\mathbb{C}$.

We have then $\gamma(u)=QS(f)=S(f)$, on the other hand, for the Dirichlet problem we have $B_0(U_0,U_1)=U_0$ and $S(U)$ can be approximated by $(U,s(U))$ so that $(U,s(U))$ belongs to $M^+_{(x,\xi)}$. We receive from the description of $M^+_{(x,\xi)}$

$$\begin{aligned}
U\in(\ker\xi\cap\vec{n}_i^\perp)\Rightarrow s(U) &= i|\xi|U\\
s(\vec{n}_i) &= -\frac{\lambda+\mu}{\lambda+3\mu}\xi^\sharp+i\frac{2\mu}{\lambda+3\mu}|\xi|\vec{n}_i\\
s(\xi^\sharp) &= i|\xi|\frac{2(\lambda+2\mu)}{\lambda+3\mu}\xi^\sharp-\frac{\lambda+\mu}{\lambda+3\mu}|\xi|^2\vec{n}_i
\end{aligned}$$

Finally $DN(\alpha)=B\circ S$, and its principal symbol is yet calculated : it is $\Phi_b(U,s(U))$.

From the calculation of the section 2.3.3 we receive

$$\begin{aligned}
U\in(\ker\xi\cap\vec{n}_i^\perp)\Rightarrow dn(U) &= \mu|\xi|U\\
dn(\vec{u}_1) &= 2\mu|\xi|\vec{u}_1\\
dn(\vec{u}_2) &= \frac{2\mu^2}{\lambda+\mu}\vec{u}_1+i2\mu|\xi|\vec{n}_i\\
&= \frac{2\mu^2}{\lambda+\mu}\vec{u}_1+2\mu|\xi|\frac{\lambda+\mu}{\lambda+3\mu}\vec{u}_2
\end{aligned}$$

it is then an easy calculation to show that

$$dn(\xi-i|\xi|\vec{n}_i)=2\mu|\xi|\frac{\lambda+\mu}{\lambda+3\mu}(\xi-i|\xi|\vec{n}_i)$$

from this triangular form of dn. ◁

Corollary 4 . — *The operator DN has only finitely many negative eigenvalues.*

We note $n(\alpha)$ the number of these eigenvalues (i.e. which are < 0). Let also

$$N(L,B)(\alpha) = \#\Big( ]-\infty, \alpha[\cap \mathrm{Spec}(L,B) \Big)$$

and idem $N(L,0)(\alpha)$ for the Dirichet problem.

Proposition 5 . — *This three counting functions are related by the formula*

$$n(\alpha) = N(L,B)(\alpha) - N(L,0)(\alpha)$$

Proof. — In [F] the proof of this formula is given for the Laplace operator, one can find a geometric interpretation in [M]. We follow here the ideas of [F] excepted for the step 5, which is not very precise in [F], see remark 2 below. Denote by $\alpha_0 < \alpha_1 < \alpha_2 \ldots$ the spectrum, not repeated with the multiplicity, of $(L,0)$. For α not in this spectrum let $\mathcal{P}(\alpha)(X)$ be the solution of $L(u) = \alpha u$, $u_{|\partial M} = X$ and by $(L-\alpha)^{-1}$ the inverse of $(L-\alpha)$ with the Dirichlet boundary conditions. The polarisation formula can be written

$$\mathcal{A}(\mathcal{P}(\alpha)(X), \mathcal{P}(\alpha)(Y)) = \alpha \int_M (\mathcal{P}(\alpha)(X), \mathcal{P}(\alpha)(Y))_g + \int_{\partial M} (DN(\alpha)(X), Y)_g$$

We conclude that the elliptic pseudo-differential operator of degree 1 and with compact resolvent $DN(\alpha)$ is selfadjoint .

Step 1 . — *The eigenvalues of $DN(\alpha)$ define continuous decreasing functions of α in an interval $]\alpha_k, \alpha_{k+1}[$ and consequently $n(\alpha)$ is increasing semi continuous from below on $]\alpha_k, \alpha_{k+1}[$.*

In fact we have the formula

$$\mathcal{P}(\beta) - \mathcal{P}(\alpha) = (\beta - \alpha)(L-\beta)^{-1}\mathcal{P}(\alpha)$$

so $\mathcal{P}(\alpha)$ is analytic on an interval $]\alpha_k, \alpha_{k+1}[$ and also $DN(\alpha)$ which satisfies

$$DN(\beta) - DN(\alpha) = (\beta - \alpha)B(L-\beta)^{-1}\mathcal{P}(\alpha)$$

and then the derivative with respect to α is

$$\dot{DN}(\alpha) = B(L-\alpha)^{-1}\mathcal{P}(\alpha)$$

Then for any sections X, Y of $TM_{|\partial M}$ the polarisation formula gives

$$< \dot{DN}(\alpha)X, Y > = < B(L-\alpha)^{-1}\mathcal{P}(\alpha)X, Y >$$
$$= \mathcal{A}((L-\alpha)^{-1}\mathcal{P}(\alpha)X, \mathcal{P}(\alpha)Y) - < L(L-\alpha)^{-1}\mathcal{P}(\alpha)X, \mathcal{P}(\alpha)Y >$$
$$= \mathcal{A}((L-\alpha)^{-1}\mathcal{P}(\alpha)X, \mathcal{P}(\alpha)Y) - < \mathcal{P}(\alpha)X + \alpha(L-\alpha)^{-1}\mathcal{P}(\alpha)X, \mathcal{P}(\alpha)Y >$$
$$= < (L-\alpha)^{-1}\mathcal{P}(\alpha)X, (L-\alpha)\mathcal{P}(\alpha)Y > - < \mathcal{P}(\alpha)X, \mathcal{P}(\alpha)Y >$$

by a second application of the polarisation formula ; at this stage there is no boundary term because $(L-\alpha)^{-1}\mathcal{P}(\alpha)X$ is null on the boundary. But $(L-\alpha)\mathcal{P}(\alpha)Y=0$ we have then

$$< \dot{DN}(\alpha)X, X >= - < \mathcal{P}(\alpha)X, \mathcal{P}(\alpha)X >$$

Thus for any X the Reileigh-Ritz quotient $< DN(\alpha)X, X > / < X, X >$ is a decreasing function of α which proves the assertion of the step 1. $\bowtie$

Step 2 . — $\lim_{\beta\to\alpha^+} n(\beta) > n(\alpha)$ *means that* $0 \in \mathrm{Spec} DN(\alpha)$. *Consequently* $\lim_{\beta\to\alpha^+} n(\beta) - n(\alpha)$ *equals the multiplicity of* $\alpha \in \mathrm{Spec}(L,B)$.

This is obvious : $DN(\alpha)X = 0$ means that $B\mathcal{P}(\alpha)(X) = 0$ on the boundary, on the other and $(L-\alpha)\mathcal{P}(\alpha)(X) = 0$. So $\mathcal{P}(\alpha)(X)$ is an eigenvector of (L,B) with eigenvalue α. $\bowtie$

As a consequence we have

$$\alpha < \beta \in]\alpha_k, \alpha_{k+1}[\Rightarrow n(\beta) - n(\alpha) = \sharp\Big(\mathrm{Spec}(L,B) \cap [\alpha, \beta[\Big)$$

Step 3 . — *If* $\alpha \leq 0$ *then* $n(\alpha) = 0 = N(L,B)(\alpha)$.

We have by the polarisation formula

$$\int_{\partial M} (DN(\alpha)(X), X)_g = \mathcal{A}(\mathcal{P}(\alpha)(X), \mathcal{P}(\alpha)(X)) - \alpha \int_M |\mathcal{P}(\alpha)(X)|_g^2$$

so if $\alpha \leq 0$ then $\int_{\partial M}(DN(\alpha)(X), X)_g \geq 0$ and DN has no negative eigenvalues.

At this point the formula is proved for $\alpha \leq 0$. We study now the behaviour of $n(\alpha)$ near α_k. Following [F] we introduce a new parameter $a \geq 0$: let L_a be the elliptic operator L with boundary value condition $(B+a)(X) = 0$. Hence $L_0 = (L,B)$ and $L_\infty = (L,0)$ This operator is elliptic, because the principal symbol of the boundary operator is the same as B ; we see also that it is self adjoint and non negative if $a \geq 0$, by the polarisation formula.

Step 4 . — *Let* $\mathcal{P}_B X$ *denote, for* $X \in C^\infty(M, TM)$, *the solution of the problem*

$$\begin{aligned} (L+1)u &= 0 \\ Bu &= X \text{ on the boundary.} \end{aligned}$$

then$(1+L_a)^{-1} = (1+a\mathcal{P}_B)^{-1}(1+L_0)^{-1}$ *is a bounded operator which depends analytically on* a.

This formula is obvious if we know that $(1 + a\mathcal{P}_B)$ is invertible. By the theorem 20.1.7 of [H3] we know that $(1 + a\mathcal{P}_B)$ is bounded from the Sobolev space $H^2(M, TM)$ into itself. It is also an isomorphism :

$$\begin{aligned}(1 + a\mathcal{P}_B)(X) = 0 \quad &\Leftrightarrow \quad X = -a\mathcal{P}_B(X)\\ &\Rightarrow \quad (L+1)(X) = 0 \text{ and } (B+a)(X) = 0 \text{ on } \partial M\\ &\Rightarrow \quad X = 0\end{aligned}$$

because L_a is non negative ; so $(1 + a\mathcal{P}_B)$ is one-to-one. It is also onto : for $Y \in H^2(M, TM)$ let $\tilde{X}$ be the solution of

$$\begin{aligned}(L+1)\tilde{X} &= 0\\ (B+a)(\tilde{X}) &= Y \text{ on the boundary.}\end{aligned}$$

then $\tilde{X} = \mathcal{P}_B(Y - a\tilde{X})$ and

$$(1 + a\mathcal{P}_B)(Y - a\tilde{X}) = (Y - a\tilde{X}) + a\tilde{X} = Y.$$

$\bowtie$

Step 5 . — *The eigenvalues of the operator L_a are strictly increasing functions of the parameter a.*

We take for this step the notation

$$\Phi(a) = (1 + L_a)^{-1} = (1 + a\mathcal{P}_B)^{-1}(1 + L_0)^{-1}$$

and represent with a dot the derivative with respect to a. A simple calculation shows that

$$\dot{\Phi}(a) = -(1 + a\mathcal{P}_B)^{-1}\mathcal{P}_B\Phi(a).$$

Let $u \in L^2(TM)$ and denote $Y = \Phi(a)(u), X = \mathcal{P}_B(Y)$ and $v = (1 + a\mathcal{P}_B)^{-1}X$. Then Y, X, v satisfy

$$\begin{aligned}(L+1)(Y) &= u, \quad (B+a)(Y) = 0 \text{ on } \partial M\\ (L+1)(X) &= 0, \quad B(X) = Y \text{ on } \partial M\\ v = X - a\tilde{X} \text{ with } (L+1)(\tilde{X}) &= 0, \quad (B+a)(\tilde{X}) = X \text{ on } \partial M\\ \Rightarrow (L+1)(v) &= 0, \quad (B+a)(v) = Y \text{ on } \partial M\end{aligned}$$

then by applying two times the polarisation formula and this results we have

$$\begin{aligned}- < \dot{\Phi}(a)u, u > &= < v, u > = < v, (L+1)Y >\\ &= < v, Y > + \mathcal{A}(v, Y) - < v, BY >_{L^2(\partial M)}\\ &= < v, Y > + < L(v), Y > + < Bv, Y >_{L^2(\partial M)}\\ &\qquad - < v, BY >_{L^2(\partial M)}\\ &= < Bv, Y >_{L^2(\partial M)} - < v, BY >_{L^2(\partial M)}\\ &= < Y - av, Y >_{L^2(\partial M)} - < v, -aY >_{L^2(\partial M)}\\ &= < Y, Y >_{L^2(\partial M)} > 0\end{aligned}$$

and for any section u, the function of $< \Phi(a)(u), u >$ is strictly decreasing with respect to a and , by the max-min formula, the eigenvalues of $\Phi(a)$ are strictly decreasing functions of a. $\bowtie$

Consequently we can find two numbers $a > 0$ and $\varepsilon > 0$ so that

$$\mathrm{Spec}(L_a) \cap]\alpha_k - \varepsilon, \alpha_k + \varepsilon[= \emptyset$$

Then for every $\alpha \in]\alpha_k - \varepsilon, \alpha_k + \varepsilon[$ the operator $DN(\alpha) + a$ is invertible : in fact $(DN(\alpha) + a)(X) = 0$ means that $u = \mathcal{P}(\alpha)(X)$ satisfies $Bu + au = 0$ on the boundary ; but $(L - \alpha)u = 0$ so $u = 0$ because $\mathrm{Spec}(L_a)$ does not meet $]\alpha_k - \varepsilon, \alpha_k + \varepsilon[$.

Remark 2 . — This previous step is easier for the Laplace operator acting on functions because the operator L_a is obtained by the polarisation of the closed sectorial quadratic form $\int_M |df|^2 + a \int_{\partial M} f^2$ with constant domain $H^1(M)$. The perturbation of L_a is then analytic of type (B) and the remark 4.22 of [K] can be applied : L_a admits a complete analytic family of eigenvectors and eigenvalues, and we have just to differentiate these families to obtain this result.

Step 6 . — *For $X \in C^\infty(\partial M, TM_{|\partial M})$ and $\alpha \in]\alpha_k - \varepsilon, \alpha_k + \varepsilon[$ let $K_a(\alpha)X$ be the restriction on ∂M of the unique solution of the problem*

$$\begin{aligned} (L - \alpha)u &= 0 \\ (B + a)u &= X \text{ on the boundary} \end{aligned}$$

then if $\alpha \neq \alpha_k$ $K_a(\alpha) = (DN(\alpha) + a)^{-1}$.

In fact let $Y = u_{|\partial M}$ then $X = (DN(\alpha) + a)Y$. $\bowtie$

On the other hand $K_a(\alpha)X = \mu X \Leftrightarrow DN(\alpha)X = (1/\mu - a)X$ so

$$n(\alpha) = m_1(\alpha) + m_2(\alpha)$$

where
$$\begin{aligned} m_1(\alpha) &= \sharp \mathrm{Spec} K_a(\alpha) \cap] - \infty, 0[\\ m_2(\alpha) &= \sharp \mathrm{Spec} K_a(\alpha) \cap]\frac{1}{a}, +\infty[\end{aligned}$$

Step 7 . — $\lim_{\alpha \to \alpha_k^+} n(\alpha) - \lim_{\alpha \to \alpha_k -+} n(\alpha) =$***mult***$(\alpha_k, L, B)$-***mult***$(\alpha_k, L, 0)$.

In fact the expression $\lim_{\alpha \to \alpha_k^-} m_1(\alpha) - \lim_{\alpha \to \alpha_k^+} m_1(\alpha)$ equals the multiplicity of $0 \in \mathrm{Spec} K_a(\alpha_k)$ while $\lim_{\alpha \to \alpha_k^+} m_2(\alpha) - \lim_{\alpha \to \alpha_k^-} m_2(\alpha)$ is the multiplicity $1/a \in \mathrm{Spec} K_a(\alpha_k)$.

If $K_a(\alpha)X = 0$ then there exists u such that $(L-\alpha)u = 0$, $(B+a)u = X$ on the boundary and $u_{|\partial M} = 0$. So u is an eigenvector for $(L, 0)$ with eigenvalue α. On the other hand if $K_a(\alpha)X = 1/aX$ there exists u such that $(L-\alpha)u = 0$, $(B+a)u = X$ on the boundary and $u_{|\partial M} = 1/aX \Rightarrow (Bu)_{|\partial M} = 0$ and u is an eigenvector of (L, B) with eigenvalue α. $\bowtie$

This proves the proposition. $\triangleleft$

4 Shift between Dirichlet and Neumann spectrum

Theorem 6 . — *If M is an open bounded domain of a symmetric space of noncompact type and dimension m and if the Lamé functions are constants then, if the symmetric space is of rank bigger than 2 or if the rank is 1 and the dimension of the nilpotent part is less or equal to $\frac{8\mu}{\lambda+2\mu}$:*

$$N(\alpha, L, B) - N(\alpha, L, 0) \geq 1.$$

Remark 3 *In the general case of rank 1 and for general Lamé constants, the proof of the theorem gives*

$$(\lambda + 2\mu)\frac{\dim N}{8} - \mu$$

as upper bound of the first eigenvalue of the Neumann problem (when this number is positive).

Proof. — First in $\mathbb{R}^m$: let v be a vector of $\mathbb{R}^m - \{0\}$ and defines the vector field on M : $X_v(x) = e^{iv.x}v$. It is easy to verify (see the appendix) that $LX_v = (\lambda + 2\mu)|v|^2 X_v$. Then, with $\alpha = (\lambda + 2\mu)|v|^2$, we have

$$X_v = \mathcal{P}(\alpha)(X_{v|\partial M}) \text{ and } \int_{\partial M} (BX_v, X_v)_g = 0.$$

so $n(\alpha) \geq 1$ if we apply the minimax formula to the operator $DN(\alpha)$. $\bowtie$

For the general case we have to make some recall. A Riemannian symmetric space of noncompact type is the quotient G/K of a semisimple non compact (real) Lie group G by a maximal compact subgroup K. Let $0 = [e]$ be the base point, and $\mathfrak{g} = T_eG$ its Lie algebra. The Killing form $B(X, Y) = \mathrm{Tr}(ad(X) \circ ad(Y))$ which is regular and the Cartan involutive automorphism Θ define a metric on $\mathfrak{g}$ by $< X, Y >= -B(\Theta(X), Y)$ and an orthonormal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = \ker(\Theta - 1)$ who is the Lie algebra of K, and $\mathfrak{p} = \ker(\Theta + 1)$ who is not a subalgebra of $\mathfrak{g}$.

This metric is $Ad(K)$ invariant, then it defines a riemannian structure on G by left translation L_g for $g \in G$ and also on G/K so that the action on the left $\tau(g)$ for $g \in G$ is isometric. For this structure the natural projection $\pi : G \to G/K$ is a riemannian submersion ; at the point $g \in G$ the horizontal and vertical are identified with $L_g(\mathfrak{p})$ and $L_g(\mathfrak{k})$. We note $\bar{X}$ the horizontal lift of the tangent vector X of G/K.

Lemma 7 . — *Let $X_0 \in \mathfrak{p}$ and $X = \tau(g)X_0$ a tangent vector of G/K at $m = [g]$, then for any $k \in K$*

$$\bar{X}(gk) = L_{gk}\mathrm{Ad}(k^{-1})X_0.$$

Proof. — From the relation $\pi \circ L_g \circ R_k = \tau(g) \circ \pi$ and $L_g \circ R_k = L_{gk} \circ \mathrm{Ad}(k^{-1})$ one has

$$\pi(L_{gk} \circ \mathrm{Ad}(k^{-1})(X_0) = X$$

and $L_{gk} \circ \mathrm{Ad}(k^{-1})(X_0) \in L_{gk}(\mathfrak{p})$ is horizontal. $\bowtie$

Let ∇, respectively $\bar{\nabla}$, be the Levi-Civita connection of G/K and G. If U and V are left invariant vector fields on G, then $\bar{\nabla}_U V = 1/2 L_g.[U, V]$; and by the O'Neill theorem we have ([CE] p.66) the

Proposition 8 (O'Neill Formula) . — *Let X, Y be two vector fields on G/K then*

$$\overline{\nabla_X Y} = \bar{\nabla}_{\bar{X}}\bar{Y} - \frac{1}{2}[\bar{X}, \bar{Y}]^V = (\bar{\nabla}_{\bar{X}}\bar{Y})^H$$

if subscript V denote the projection on the vertical and subscript H the projection on the horizontal.

The *Iwasawa Decomposition* ([H] p.270) gives a decomposition of the group G. Let $\mathfrak{a}$ be a maximal abelian algebra in $\mathfrak{p}$, then $\mathfrak{g} = \sum_{\rho \in \Sigma} \mathfrak{g}_\rho$ where *the system of (restricted) roots* Σ is a finite generating family of the dual of $\mathfrak{a}$, and $\mathfrak{g}_\rho = \{Y \in \mathfrak{g} ; \forall X \in \mathfrak{a} \quad \mathrm{ad}(X)Y = \rho(X)Y\}$. This eigenspaces satisfies $\Theta\mathfrak{g}_\rho = \mathfrak{g}_{-\rho}$ and $[\mathfrak{g}_\rho, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\rho+\mu}$. Let $m_\rho = \dim \mathfrak{g}_\rho$ be the multiplicity of the root ρ. A base on $\mathfrak{a}^*$ gives a (lexicographic) order on $\mathfrak{a}^*$; Σ^+ is the set of positive roots for this order. Then $\mathfrak{n} = \sum_{\rho \in \Sigma^+} \mathfrak{g}_\rho$ is a subalgebra of $\mathfrak{g}$ which decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. In addition there exist A and N subgroups of G, with corresponding Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$, such that

$$\begin{array}{rcl} \Phi : N \times A \times K & \to & G \\ (n, a, k) & \mapsto & nak \end{array}$$

is an analytic diffeomorphism ; N is nilpotent and A abelian so the exponential is a global card for both. In the sequel we shall omit the Φ.

Finally the restriction of the Killing form, and the restriction of the metric, to $\mathfrak{a}$ equals $2q$ if q is the positive definite symmetric bilinear form

$$q(Y_1, Y_2) = \sum_{\rho \in \Sigma^+} m_\rho \rho(Y_1)\rho(Y_2)$$

defined on $\mathfrak{a}$.

Functions on G/K are identified with functions f on G that satisfy $f(gk) = f(g), \forall g \in G, k \in K$, vector fields on G/K are identified with functions $X : G \to \mathfrak{p}$, $X(gk) = \mathrm{Ad}(k^{-1})X(g)$. Fixing an *orthonormal base* $Y_1, \ldots, Y_r$ of $\mathfrak{a}$ gives then coordinates $(v_j)_{1 \le j \le r}$ on A by inversing the map :

$$\begin{array}{rcl} \mathbb{R}^r & \to & A \\ (v_1, \ldots, v_r) & \mapsto & \exp(v_1 Y_1 + \ldots + v_r Y_r). \end{array}$$

We have then defined functions v_j on G and G/K by $v_j(nak) = v_j(a)$. On the other hand Y_j defines a vector field on G/K by $Y_j(nak) = \mathrm{Ad}(k^{-1})Y_j$.

For certain $s \in \mathbb{C}$ the functions $f_j = \exp(sv_j)$ give eigenfunctions of the Laplace operator on G/K ([M]) :

Lemma 9 . — *The relations between v_j and Y_j are*

$$(\mathrm{d}v_j)^\sharp = Y_j \ ; \quad \Delta v_j = -\mathrm{div} Y_j = -C_j$$

a constant ; $C_j = -\sum_{\rho \in \Sigma^+} \rho(Y_j)$.

Proof. — Let $m = [na] \in G/K$ and $X = \tau(na)(X_0) \in T_m(G/K)$, with $X_0 \in \mathfrak{p}$. For any curve $c(t)$ on G such that $c(0) = e$ and $\dot{c}(0) = X_0$ one has

$$\mathrm{d}_m v_j(X) = \frac{\mathrm{d}}{\mathrm{d}t}(v_j(nac(t))_{|t=0}.$$

We first take an orthonormal base ε_i of $\mathfrak{p}$ such that $(\varepsilon_i)_{1 \le i \le r}$ is an orthonormal base of $\mathfrak{a}$ and the sequel are on the form $X_\rho - \Theta(X_\rho)$ with $X_\rho \in \mathfrak{g}_\rho$ and $\rho \in \Sigma^+$. It defines an orthonormal base of $T_m M$ by $e_i = \tau(na)\varepsilon_i$.

Then we can choose for $\varepsilon_i \in \mathfrak{a}$ the curve $c_i(t) = \exp(t\varepsilon_i)$, while if ε_i is of the second type we write $\varepsilon_i = X_\rho - \Theta X_\rho = 2X_\rho - (X_\rho + \Theta X_\rho) = X_1 + X_2$ with $X_1 = 2X_\rho \in \mathfrak{g}_\rho \subset \mathfrak{n}$ and $X_2 = -(X_\rho + \Theta X_\rho) \in \mathfrak{k}$. We take $c_i(t) = x_1(t)x_2(t)$ with $x_k(t) = \exp(tX_k)$, then $x_1(t) \in N$ and $x_2(t) \in K$.

To prove the first assertion we have to show for any $X = \tau(na)X_0$:
$\mathrm{d}_m v_j(X) = < Y_j, X >_m = < Y_j, X_0 >_0$.

It is sufficient to take X_0 in our base. If $X_0 = \varepsilon_i \in \mathfrak{a}$, then

$$v_j(nac(t)) = v_j(ac(t)) = v(a) + t < Y_j, X_0 > .$$

While if $X_0 = \varepsilon_i = X_\rho - \Theta X_\rho$ remark that $nac_i(t) = \Big(n\mathrm{Ad}(a)(x_1(t))\Big)ax_2(t)$ and N is $\mathrm{Ad}(A)$ invariant, so $v_j(nac(t)) = v_j(a)$ is constant, and

$$\mathrm{d}_m v_j(X) = 0 = < Y_j, X_0 >_0 .$$

To prove the second assertion we shall use the O'Neill formula : If $\varepsilon_i \in \mathfrak{a}$ then $L(nac(t))Y_j$ is the horizontal lift of the vector field Y_j along the curve $nac(t)$, if $c(t) = \exp(t\varepsilon_i)$ as before. It is the restriction of the left invariant vector field issued from Y_j so

$$\bar{\nabla}_{\bar{e_i}}\bar{Y}_j(na) = L(na)[\varepsilon_i, Y_j] = 0, \qquad [\varepsilon_i, Y_j]^V = 0$$

and finally $\overline{\nabla_{e_i}Y_j}(na) = 0$.

While if $\varepsilon_i = X_\rho - \Theta(X_\rho)$, the horizontal lift of Y_j at $nac_i(t)$ is $\bar{Y}_j(nac_i(t)) = L(nac_i(t))\mathrm{Ad}(x_2^{-1}(t))Y_j$. Then by the Leibniz formula

$$\bar{\nabla}_{\bar{e_i}}\bar{Y}_j(na) = L(na)\Big(\frac{1}{2}[\varepsilon_i, Y_j] - \mathrm{ad}(\dot{x}_2(0))(Y_j)\Big) =$$

but $[\varepsilon_i, Y_j]$ is vertical, we obtain then

$$\begin{aligned} \overline{\nabla_{e_i}Y_j}(na) &= L(na)\Big(-\mathrm{ad}(\dot{x}_2(0))(Y_j)\Big) = L(na)[Y_j, X_2] \\ &= -L(na)\rho(Y_j)\Big(X_\rho - \Theta X_\rho\Big) = -\rho(Y_j)\bar{e}_i \end{aligned}$$

We can then calculate the divergence of Y_j :

$$\mathrm{div}(Y_j) = \sum e_i^*(\nabla_{e_i}Y_j) = -\sum_{\rho\in\Sigma+}\rho(Y_j).$$

$\bowtie$

Corollary 10 . — *The endomorphism ∇Y_j is diagonal on the base $e_j = \tau(na)\varepsilon_i$ with coefficient 0 if $\varepsilon_i \in \mathfrak{a}$ and $-\rho(Y_j)$ if $\varepsilon_i = X_\rho - \Theta X_\rho$ with $\rho \in \Sigma^+$. In particular ∇Y_j is symmetric, we shall note $\rho_i =< e_i, \nabla_{e_i}Y_j >$.*

At this stage one sees that if $f_j = \exp(sv_j)$ then

$$\mathrm{d}f_j = sf_j\mathrm{d}v_j \Rightarrow \Delta(f_j) = sf_j\Delta(v_j) - s^2 f_j|\mathrm{d}v_j|^2 = -s(s + C_j)f_j.$$

We can now make the calculations for the elasticity.

Proposition 11 . — *Let $X_j = f_jY_j$ with $f_j = \exp(sv_j)$ for a fixed j. Then*

$$LX_j = \Big(-(\lambda + 2\mu)s(s + C_j) - \mu\Big)X_j$$

Proof. — Recall the expression of LX :

$$LX = -\Big(\mathrm{d}(\lambda\ \mathrm{div}(X))\Big)^\sharp + \nabla^*(\mu\ SX) = -\lambda\Big(\mathrm{d}(\mathrm{div}(X))\Big)^\sharp + \mu\nabla^*(SX)$$

because the Lamé coefficients are supposed constant. For the first term

$$\mathrm{div} X_j = f_j \mathrm{div} Y_j + < (\mathrm{d} f_j)^\sharp, Y_j >= (C_j + s) f_j$$

by the lemma. So

$$(\mathrm{d}(\mathrm{div} X_j))^\sharp = (C_j + s)(s f_j \mathrm{d} v_j)^\sharp = s(C_j + s) X_j.$$

Now the second term : $\nabla X_j = \mathrm{d} f_j \otimes Y_j + f_j \nabla Y_j$ so

$$(\nabla X_j)^t = (Y_j)^\flat \otimes (\mathrm{d} f_j)^\sharp + f_j \nabla Y_j$$

because ∇Y_j is symmetric. We have then by the previous lemma $(\nabla X_j)^t = \mathrm{d} v_j \otimes s f_j Y_j + f_j \nabla Y_j$ so

$$S X_j = 2(\mathrm{d} f_j \otimes Y_j + f_j \nabla Y_j).$$

We apply then the formula of ∇^* with the same orthonormal frame $e_i = \tau(na)\varepsilon_i$ as before.

$$\begin{aligned}
\nabla^*(\mathrm{d} f_j \otimes Y_j) &= -\sum \nabla_{e_i}(\mathrm{d} f_j(e_i) Y_j) + \mathrm{d} f_j(\sum \nabla_{e_i} e_i) Y_j \\
&= -\sum \Big(\mathrm{d}(\mathrm{d} f_j(e_i))(e_i) Y_j + \mathrm{d} f_j(e_i) \nabla_{e_i} Y_j\Big) \ldots \\
&\qquad\qquad\qquad\qquad \ldots + \mathrm{d} f_j(\sum \nabla_{e_i} e_i) Y_j.
\end{aligned}$$

But $\mathrm{d} f_j(e_i) = s f_j < e_i, Y_j >$ so

$$\begin{aligned}
\mathrm{d}(\mathrm{d} f_j(e_i))(e_i) &= s \mathrm{d} f_j(e_i) < e_i, Y_j > + s f_j(< \nabla_{e_i} e_i, Y_j > + < e_i, \nabla_{e_i} Y_j >) \\
&= s^2 f_j < e_i, Y_j >^2 + s f_j(< \nabla_{e_i} e_i, Y_j > + < e_i, \nabla_{e_i} Y_j >)
\end{aligned}$$

and because $|Y_j|^2 = 1$

$$\begin{aligned}
\sum_i \mathrm{d}(\mathrm{d} f_j(e_i))(e_i) &= s^2 f_j + s f_j \mathrm{div} Y_j + s f_j(< \sum_i \nabla_{e_i} e_i, Y_j >) \\
&= s^2 f_j + s f_j \mathrm{div} Y_j + \mathrm{d} f_j(\sum \nabla_{e_i} e_i).
\end{aligned}$$

On the other hand

$$\sum_i \mathrm{d} f_j(e_i) \nabla_{e_i}(Y_j) = s f_j \sum_i < e_i, Y_j > \nabla_{e_i}(Y_j) = s f_j \nabla_{Y_j}(Y_j) = 0.$$

Putting all together gives

$$\nabla^*(\mathrm{d} f_j \otimes Y_j) = -(s^2 f_j + s f_j \mathrm{div} Y_j) Y_j = -s(s + C_j) X_j.$$

Now the calculation of $\nabla^*(f_j \nabla Y_j) = f_j \nabla^* \nabla Y_j + -\nabla_{(\mathrm{d} f_j)^\sharp} Y_j$.

$$\nabla_{(\mathrm{d} f_j)^\sharp} Y_j = s f_j \nabla_{Y_j} Y_j = 0,$$

and $\nabla^*(\nabla Y_j) = -\sum_i \nabla_{e_i}(\nabla_{e_i} Y_j) + \nabla_{\sum_i \nabla_{e_i} e_i} Y_j$ for an orthonormal frame of vector fields.

Lemma 12 . — *The frame of vector fields* $(e_i([na]) = \tau(na)\varepsilon_i)$ *introduced before satisfies* $\nabla_{e_i} e_i = 0$ *if* $\varepsilon_i \in \mathfrak{a}$ *and if* $\varepsilon_i = X_\rho - \Theta(X_\rho)$ *with* $X_\rho \in \mathfrak{g}_\rho$ *and* $\rho \in \Sigma^+$ *then*

$$\nabla_{e_i} e_i = -2\tau(na)[X_\rho, \Theta(X_\rho)].$$

Proof. — If $\varepsilon_i \in \mathfrak{a}$ the case is the same as for Y_j. For the second case we shall use the O'Neill formula, and the notations introduced in lemma 9. The horizontal lift of e_i at $nac_i(t) = \Big(n\mathrm{Ad}(a)x_1(t)\Big)ax_2(t)$ is, by lemma 7,

$$\bar{e}_i(nac_i(t)) = L(nac_i(t)) \circ \mathrm{Ad}(x_2(t))\varepsilon_i$$

then by the Leibniz formula $\bar\nabla_{\bar{e}_i}\bar{e}_i(na) = L(na)\Big(-[\dot{x}_2(0), \varepsilon_i]\Big)$. But

$$-[\dot{x}_2(0), \varepsilon_i] = [X_\rho + \Theta(X_\rho), X_\rho - \Theta(X_\rho)] = -[X_\rho, \Theta(X_\rho)]$$

so $[X_\rho, \Theta(X_\rho)] \in [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ is horizontal, and necessarily

$$\overline{\nabla_{e_i} e_i(na)} = -2L(na)[X_\rho, \Theta(X_\rho)].$$

$\bowtie$

We can now finish the proof of the proposition : we know that $[X_\rho, \Theta(X_\rho)] \in \mathfrak{p}$, on the other and $[X_\rho, \Theta(X_\rho)] \in [\mathfrak{g}_\rho, \mathfrak{g}_{-\rho}] \subset \mathfrak{g}_0$ so $[X_\rho, \Theta(X_\rho)] \in \mathfrak{a}$ can be decomposed on the base Y_k.

We first conclude that $\nabla_{\sum_i \nabla_{e_i} e_i} Y_j = 0$ by the expression of ∇Y_j (see corollary 10).

Secondly for Y_k in our base of $\mathfrak{a}$, we can write, because $X_\rho - \Theta(X_\rho) \in \mathfrak{p}$ and so $\mathrm{ad}(X_\rho - \Theta(X_\rho))$ is symmetric :

$$\begin{aligned}
< [X_\rho, \Theta(X_\rho)], Y_k > &= < [X_\rho - \Theta(X_\rho), \Theta(X_\rho)], Y_k > \\
&= < [X_\rho - \Theta(X_\rho), Y_k], \Theta(X_\rho) > \\
&= -\rho(Y_k) < X_\rho + \Theta(X_\rho), \Theta(X_\rho) > \\
&= -\rho(Y_k)\frac{1}{2}
\end{aligned}$$

which only depends on ρ. Consequently $[X_\rho, \Theta(X_\rho)] = -\frac{1}{2}\sum_k \rho(Y_k)Y_k$ and

$$\begin{aligned}
\sum_i \nabla_{e_i}(\nabla_{e_i} Y_j)(na) &= \sum_i \rho(Y_j)\nabla_{e_i} e_i(na) \\
&= \tau(na)\sum_{k,\rho} \rho(Y_j)\rho(Y_k)Y_k \\
&= \tau(na)\sum_k q(Y_j, Y_k)Y_k = \frac{1}{2}Y_j(na)
\end{aligned}$$

because $B = 2q$ on $\mathfrak{a}$ and the base (Y_j) is orthonormal. Finally

$$\nabla^*(f_j \nabla Y_j) = -f_j\frac{1}{2}Y_j = -\frac{1}{2}X_j$$

which finish the proof of the proposition. $\bowtie$

We conclude then as for the case of $\mathbb{R}^m$: for any $\alpha \geq 0$ there exists s such that $-(\lambda+2\mu)s(s+C_j)-\mu=\alpha$, then $X_j=\mathcal{P}(\alpha)(X_{j|\partial M})$. We still have to verify that $\int_{\partial M}(BX_j,\bar{X}_j)_g \leq 0$. In fact by the formula (2) one has

$$(BX_j,\bar{X}_j)_g=\lambda\ \mathrm{div}(X_j)(\vec{n}_e,\bar{X}_j)_g+\mu\Big((\nabla_{\bar{X}_j}X_j,\vec{n}_e)_g+(\bar{X}_j,\nabla_{\vec{n}_e}X_j)_g\Big).$$

But $\mathrm{div}(X_j)=(s+C_j)f_j$,

$$\nabla_{\bar{X}_j}X_j=\bar{f}_j\mathrm{d}f_j(Y_j)Y_j=s|f_j|^2Y_j$$

while

$$(\bar{X}_j,\nabla_{\vec{n}_e}X_j)_g=\bar{f}_j(Y_j,\nabla_{\vec{n}_e}(f_jY_j))_g=\bar{f}_j\mathrm{d}f_j(\vec{n}_e)=s|f_j|^2(Y_j,\vec{n}_e)_g$$

we can then calculate :

$$\int_{\partial M}(BX_j,\bar{X}_j)_g=(\lambda(s+C_j)+\mu(2s))\int_{\partial M}|f_j|^2(Y_j,\vec{n}_e)_g$$

The Stokes formula gives

$$\int_{\partial M}s|f_j|^2(Y_j,\vec{n}_e)_g=\int_M|\mathrm{d}f_j|^2-\int_M\Delta(f_j)\bar{f}_j.$$

But the previous calculations on v_j show also $|\mathrm{d}f_j|^2=|s|^2|f_j|^2$ and $\Delta(f_j)=-s(s+C_j)f_j$ so

$$\int_{\partial M}s|f_j|^2(Y_j,\vec{n}_e)_g=(|s|^2+s(s+C_j))\int_M|f_j|^2$$

with s solution of $s(s+C_j)=-\alpha'$ if $\alpha'=\frac{\alpha+\mu}{\lambda+2\mu}$ is a positive number. So $s\neq 0$ and

$$\int_{\partial M}|f_j|^2(Y_j,\vec{n}_e)_g=(2\Re(s)+C_j)\int_M|f_j|^2.$$

If the discriminant of the polynome $s(s+C_j)+\alpha'$ is negative or null then $C_j=-2\Re(s)$ and $\int_{\partial M}(BX_j,\bar{X}_j)_g=0$. This is always the case if $C_j=0$ and in general for $\alpha\geq\alpha_0$ if

$$\alpha_0=(\lambda+2\mu)\frac{C_j^2}{4}-\mu.$$

Recall that $C_j=-\sum_{\rho\in\Sigma+}\rho(Y_j)$ is a linear form on Y_j. If $\mathrm{Rank}(G)\geq 2$ one can fix Y_1 in the kernel of $\sum_{\rho\in\Sigma+}\rho$ which is not trivial. With this choice $C_1=0$ and $\int_{\partial M}(BX_1,\bar{X}_1)_g=0$. In rank one we have no choice for Y_1 but only one root. Then $C_1^2=m_\rho q(Y_1,Y_1)=m_\rho\frac{1}{2}$ because Y_1 is of norm 1 and

$$\alpha_0=\frac{(\lambda+2\mu)m_\rho-8\mu}{8}.$$

But $m_\rho=\dim(N)$, this complete the proof of the theorem. $\triangleleft$

5 Appendix

5.1 Expression of ∇^*

In a local orthonormal frame $e_1, \dots e_m$ an endomorphism A defines the matrix $a_{ij} = (Ae_j, e_j)_g$ so that for any vector $AX = \sum a_{ij}(X, e_j)_g e_i$. From the calculation of $(A, \nabla Y)_g \omega_M$ we receive

$$(\nabla^* A, Y)_g \omega_M = -\sum a_{ji}(Y, e_j)_g \mathrm{d}(\imath(e_i)\omega_M) + e_i.a_{ji}(Y, e_j)_g \omega_M + \\ + a_{ji}(Y, \nabla_{e_i} e_j)_g \omega_M$$

The Cartan formula gives $\mathcal{L}_X \omega_M = \imath(X)\mathrm{d}\omega_M + \mathrm{d}(\imath(X)\omega_M) = \mathrm{d}(\imath(X)\omega_M)$then

$$\begin{aligned} \mathrm{d}(\imath(e_i)\omega_M) = \mathcal{L}_{e_i}\omega_M &= \sum_k e_1^* \wedge \dots \wedge \mathcal{L}_{e_i}(e_k^*) e_n^* \\ &= \Big(\sum_k \mathcal{L}_{e_i}(e_k^*)[e_k]\Big)\omega_M \end{aligned}$$

but $\mathcal{L}_{e_i}(e_k^*)[e_k] = -(\mathcal{L}_{e_i}(e_k), e_k)_g$. The Levi-Civita covariant derivative as no torsion, and $(\nabla_{e_i} e_k, e_k)_g = 0$ while $(\nabla_{e_k} e_i, e_k)_g = -(e_i, \nabla_{e_k} e_k)_g$ so

$$\begin{aligned} \mathrm{d}(\imath(e_i)\omega_M) &= = -(e_i, \sum_k \nabla_{e_k} e_k)_g \omega_M \quad \text{and} \\ \sum a_{ji}(Y, e_j)_g \mathrm{d}(\imath(e_i)\omega_M) &= -(A^t Y, \sum_i \nabla_{e_i} e_i)_g. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum e_i.a_{ji}(Y, e_j)_g + a_{ji}(Y, \nabla_{e_i} e_j)_g &= (Y, \sum \nabla_{e_i}(a_{ji} e_j))_g \\ &= (Y, \sum_i \nabla_{e_i}(Ae_i))_g \end{aligned}$$

adding this two results gives the formula (1).

5.2 Calculation of Φ_b.

We have, from the expression of the principal symbol of B

$$\begin{aligned} b(x, \xi)(X) &= -i\bigg(\lambda(x)\xi(X)\vec{n}_i + \mu(x)\Big(\xi(\vec{n}_i)X + (\vec{n}_i, X)_g \xi^\sharp\Big)\bigg) \\ b(x, \xi + D_t \vec{n}_i^\flat)(u) &= -i\bigg(\lambda\Big(\xi(u) + D_t(\vec{n}_i, u)_g\Big)\vec{n}_i + \dots \\ &\quad + \mu\Big(D_t u + (\vec{n}_i, u)_g \xi^\sharp + D_t(\vec{n}_i, u)_g \vec{n}_i\Big)\bigg) \end{aligned}$$

$$= \quad -i\Bigg(\lambda\xi(u)\vec{n}_i + (\lambda+\mu)D_t(\vec{n}_i,u)_g\vec{n}_i + \ldots$$

$$+\mu\Big((\vec{n}_i,u)_g\xi^\sharp + D_t u\Big)\Bigg)$$

and then

$$\begin{aligned}\Phi_b(u) &= b(x,\xi+D_t\vec{n}_i^\flat)(u)[t=0]\\ &= -i\Big(\lambda\xi(U_0)\vec{n}_i + (\lambda+\mu)(\vec{n}_i,U_1)_g\vec{n}_i + \mu((\vec{n}_i,U_0)_g\xi^\sharp + U_1)\Big)\end{aligned}$$

5.3 Calculation of LX_v.

If we write $v = \sum v_j e_j$ we have $\nabla_{e_j} X_v(x) = iv_j e^{iv.x} v$ and then $(\nabla X_v)_{jk} = iv_j v_k e^{iv.x}$ and $\mathrm{div}(X_v) = i\sum v_j^2 e^{iv.x}$. Consequently for any vector Y

$$\nabla_Y X_v = i(Y,v)_g e^{iv.x} v$$

and also $SX_v = 2\nabla X_v$ because the matrix ∇X_v is symmetric, and $\nabla^*(SX_v) = -\sum \nabla_{e_j}(SX_v[e_j]) = 2\sum v_j^2 X_v$. Finally

$$(\mathrm{d}(\mathrm{div}X_v))^\sharp = -\sum v_j^2 X_v \text{ and } LX_v = (\lambda+2\mu)\sum v_j^2 X_v.$$

5.4 Calculation of $\int_{\partial M}(BX_v,X_v)_g$.

From the formula (2) we know that

$$\begin{aligned}(BX_v,v)_g &= i\lambda|v|^2 e^{iv.x}(\vec{n}_e,v)_g + i\mu e^{iv.x}\Big((v,v)_g(v,\vec{n}_e)_g + (v,v)_g(v,\vec{n}_e)_g\Big)\\ &= i(\lambda+2\mu)|v|^2 e^{iv.x}(v,\vec{n}_e)_g\end{aligned}$$

We calculate then

$$\int_{\partial M}(BX_v,X_v)_g = i(\lambda+2\mu)|v|^2\int_{\partial M}(v,\vec{n}_e)_g\omega_{\partial M} = 0$$

because $\int_{\partial M}(v,\vec{n}_e)_g\omega_{\partial M} = \int_{\partial M}\imath(v)\omega = \int_M \mathrm{d}(\imath(v)\omega) = 0$ the form $(\imath(v)\omega)$ being constant.

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