Perturbation of several commuting h-pseudodifferential operators

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June 15, 2006

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Abstract

We consider k pseudodifferential operators $\mathcal{Q}_1(h), \ldots, \mathcal{Q}_k(h)$, acting on $\mathbb{R}^n$ commuting together, and depending on a small parameter h. Under the assumption that the classical Hamiltonian flow of the joint principal symbol q_0 is *periodic with constant period on one given energy level* $q_0^{-1}(E_0)$, in [1] we have shown that the joint spectrum of these operators lying in a h-depending neighborhood $I(h)$ of E_0 is localized near a lattice. This paper followed the now classical method initiated by Helffer and Robert in [11], and in [4] for several operators, which is based on the FIO theory. We investigate in this article the method proposed by Colin de Verdière to Dozias [8] for one operator acting on $\mathbb{R}^n$ and obtain a theorem of perturbation.

Contents

1 Introduction

This article is concerned with the localization, near E_0, a fixed energy level of the joint spectrum of several commuting operators, one of them being a Schrödinger type operator. The joint spectrum of several commuting operators arises naturally when suitable symmetries are present. Let us consider for instance the Schrödinger operator in $\mathbb{R}^3$

$$\mathcal{Q}_1(h) \ = \ -h^2\Delta \ + \ V(x) \ ,$$

and suppose that the potential is real-valued, smooth and symmetric in the variables x_2, x_3, for instance $V(x) = (1+x_1^2)(1+x_2^2+x_3^2)$. Then $\mathcal{Q}_1(h)$ commutes with the kinetic moment

$$\mathcal{Q}_2(h) \ = \ -ih \ (x_2\partial_{x_3} \ - \ x_3\partial_{x_2}) \ ,$$

The eigenvalues of $\mathcal{Q}_1(h)$ correspond to common eigenfunctions of $\mathcal{Q}_1(h)$and $\mathcal{Q}_2(h)$, and the associated joint eigenvalues $\lambda(h) = (\lambda_1(h), \lambda_2(h))$ of the operators $\mathcal{Q}(h) = (\mathcal{Q}_1(h), \mathcal{Q}_2(h))$ have less multiplicity in the spectrum of $\mathcal{Q}(h)$.

More generally, having k commuting operators in $L^2(\mathbb{R}^n)$, $k \le n$, we consider the k-uplet of operators

$$\mathcal{Q}(h) \ = (\mathcal{Q}_1(h), \ldots, \mathcal{Q}_k(h))\,.$$

(H1) *We suppose that the operators $\mathcal{Q}_1(h), \ldots, \mathcal{Q}_k(h)$ are h-admissible pseudodifferential operators, in the sense of Helffer and Robert [12].* Let us recall their definition.

Let p, Φ and Ψ be weight functions on $\mathbb{R}^{2n}$, *i.e.* continuous functions from $\mathbb{R}^{2n}$ into $\mathbb{R}^+$ which are supposed to be σ-temperate in Hörmander's sense (see [14]) and such that the product $\Phi.\Psi$ satisfies the following property : there exist constants δ, $0 < \delta \le 1$, $c_0 > 0$, $c_1 > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2n} \quad c_0 \left(1+|x|^2+|\xi|^2\right)^\delta \le \Phi(x,\xi)\Psi(x,\xi) \le c_1 \left(1+|x|^2+|\xi|^2\right).$$

We denote by $\mathcal{S}_{\Phi,\Psi}(p)$ the space of all the functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for all (α,β) in $\mathbb{N}^{2n}$ there exists $C_{\alpha,\beta} > 0$ verifying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \ \le \ C_{\alpha,\beta}\ p(x,\xi)\ \Phi^{-|\alpha|}\ \Psi^{-|\beta|} \ \ \forall (x,\xi) \in \mathbb{R}^{2n}. \tag{1.1}$$

To such a symbol a corresponds an operator $\mathcal{A}(h) \ = \ Op_h^w(a)$ defined by the Weyl quantization

$$\mathcal{A}\psi(x) \ = \ (2\pi h)^{-n} \int\!\!\int e^{ih^{-1}(x-y\ ,\ \xi)}\ a(h, \frac{x+y}{2}, \xi)\ \psi(y)\ dy\ d\xi$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$.

We suppose that $\mathcal{Q}_j(h) = Op_h^w(q_j)$, where the symbol $q_j(h,x,\xi)$ admits an asymptotic expansion in powers of h : there exists a sequence $\{q_{ij}\}_{i\in\mathbb{N}} \in \mathcal{S}_{\Phi,\Psi}(p\Phi^{-i}\Psi^{-i})$ such that

$$q_j(h,x,\xi) \ \sim \ \sum_{i\ge 0} \ h^i q_{ij}(x,\xi) \tag{1.2}$$

where $\sim$ means that for all $N \in \mathbb{N}$

$$h^{-(N+1)} \left(q_j(h,x,\xi) - \sum_{i=0}^{N} h^i q_{ij}(x,\xi) \right)$$

belongs to a bounded set of $\mathcal{S}_{\Phi,\Psi}(p\Phi^{-(N+1)}\Psi^{-(N+1)})$ for all $h \in]0,h_0[$.

Then $\mathcal{Q}_j(h)$ belongs to $\mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n)\big)$, for each j=1, ..., k. We may assume that each operator $\mathcal{Q}_j(h)$ admits a self-adjoint extension to $L^2(\mathbb{R}^n)$ which will be still denoted by $\mathcal{Q}_j(h)$. It is the case if the principal symbol q_{0j} is itself a weight and if $q_{ij} \in S_{q_{0j}}$ for all i, and more generally if q_{0j} is bounded from below [17].

In our case, we always can reduce to this situation by means of the functional calculus, if the weight is $p = \|q_0\|^2 + 1$, and if the operator $\sum_{j=1}^{k} {Q_j}^2(h)$ is h-admissible : let f be a C^∞ function on $\mathbb{R}^+$ such that $f(x) = x$ for $x \in [0,c]$, (with $0 < c < 1$) f is increasing and asymptotic to 1 at infinity and define $g(x) = f(x)/x$ a C^∞ function on $\mathbb{R}^+$, equal to 1 on the interval $[0,c]$. Then, for $E_0 \in \mathbb{R}^k$, the new operators defined vectorially by

$$P(h) = g(\|Q(h) - E_0\|)(Q(h) - E_0)$$

are h-admissible pseudodifferential operators (remark that the operators $g(\|Q(h) - E_0\|)$ are bounded) with bounded symbols, so every principal symbol is a weight bounded from below. Moreover the $P_j(h)$ commute to each other if the $Q_j(h)$ have this property ; furthermore, on the domain $\{(x,\xi), \|q(h)(x,\xi) - E_0\| \leq c\}$ one has $p(h) = q(h) - E_0$, so $p(h)$ is proper if $q(h)$ is, see hypothesis (H3) below.

Remark. — This method is preferable to the one used in [4] (2.1.3) where the author used instead of g a function with compact support, because in this case the level sets near E_0 are modified and the hypothesis (H3) cannot be kept.

(H2) *We suppose that the operators $\mathcal{Q}_j(h)$ commute to each other, i.e.*

$$\forall i,j = 1,...,k,\ \forall h > 0\ :\ [\mathcal{Q}_i(h), \mathcal{Q}_j(h)] = 0 \text{ on } \mathcal{S}(\mathbb{R}^n). \tag{1.3}$$

Denote by $q_0 = (q_{01}, \ldots, q_{0k})\ :\ T^*\mathbb{R}^n \ \rightarrow\ \mathbb{R}^k$ the principal symbol of the operator

$$\mathcal{Q}(h)\ = (\mathcal{Q}_1(h), \ldots, \mathcal{Q}_k(h))$$

and by $q_1 = (q_{11}, \ldots, q_{1k})$ its subprincipal symbol. As a consequence of the assumption of commutativity, one has $\{q_{0i}, q_{0j}\} = 0$, and the associated symplectic or classical flows also commute.

(H3) Fix $E_0 \in \mathbb{R}^k$ and suppose that E_0 is a *regular value* of the joint principal symbol q_0, and that q_0 is *proper on a neighborhood of* E_0, *i.e.* there exists a compact neighborhood K of E_0 in $\mathbb{R}^k$ such that $q_0^{-1}(K)$ is compact.

The joint spectrum $\Lambda^{\mathcal{Q}}$ for such types of operators has been introduced in [4], and it has been proved that under the condition (H3), the part of the joint spectrum contained in any compact $\tilde{K} \subset \overset{o}{K}$, consists in finitely many joint eigenvalues $\lambda(h) \in \mathbb{R}^k$ of finite multiplicity.

Consequence. — For all $E \in K$ (if K is small enough) the set $\Sigma_E = q_0^{-1}(E)$ has a structure of submanifold of $\mathbb{R}^{2n}$ of codimension k. Σ_E is compact and invariant under the action of the Hamiltonian flow Φ_j^t defined by

$$\Phi_j^0 = \mathrm{Id} \quad ; \quad \frac{\partial}{\partial \mathrm{t}} \Phi_{\mathrm{j}}^{\mathrm{t}} = \mathrm{H}_{\mathrm{j}} \circ \Phi_{\mathrm{j}}^{\mathrm{t}}$$

where H_j is the Hamiltonian field of q_{0j}

$$\mathrm{d}q_{0j} = -H_j \lrcorner\, \omega.$$

Here ω is the canonical symplectic form of $\mathbb{R}^n$

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx_j.$$

Let us remark that $[H_i, H_j] = 0$ in view of the hypothesis (H2), so Σ_E is isotropic ; this implies $k \leq n$.

Each flow Φ_j^t defines a symplectic action on $T^*\mathbb{R}^n$ which preserves any Σ_E, and the joint flow defines an action of $\mathbb{R}^k$ on the level submanifold

$$\begin{array}{rrcl} \rho_E \ : & \mathbb{R}^k \times \Sigma_E & \rightarrow & \Sigma_E \\ & ((t_1, \ldots, t_k), \nu) & \mapsto & \Phi_1^{t_1} \circ \ldots \circ \Phi_k^{t_k}(\nu). \end{array}$$

(H4) *Periodicity with constant period of the joint flow on* $\Sigma_0 = \Sigma_{E_0}$.

This means : we suppose that, on Σ_0, the action ρ_{E_0} of $\mathbb{R}^k$ comes down to an action ρ_0 of a torus $\mathbb{T}^k$; more precisely there exists a base $(e_1, \ldots, e_k)$ of $\mathbb{R}^k$ such that

$$\forall \nu \in \Sigma_0,\ \forall (z_j)_{1 \leq j \leq k} \in \mathbb{Z}^k \qquad \rho_{E_0}\left(2\pi \sum_1^k z_j e_j, \nu\right) = \nu$$

and we will write

$$\forall \gamma = [t] \in \mathbb{T}^k,\ \nu \in \Sigma_0 \quad , \quad \rho_0(\gamma, \nu) = \rho_{E_0}(t, \nu).$$

Remark. — Under this assumption *i.e.* all the points of Σ_0 are periodic with the same period ; but some points may have sub-periods, actually they are isolated : the action ρ_E is locally free (*i.e.* the stabilisator of each point is a discrete subgroup of $\mathbb{R}^k$) because the Hamiltonian fields $(H_1, \ldots, H_k)$ are independent by (H3). So the action ρ_0 of $\mathbb{T}^k$ is locally free, and using the proposition 3.7, p. 94 in [16], we know that the orbits define a foliation with compact leaves, and that the space of the orbits $\Sigma_0/\mathbb{T}^k$ has a structure of *orbifold* : it is a smooth compact manifold with a finite number of conical singularities.

Comparison with the hypothesis of [4] and notations. — Let $(\varepsilon_j)_{1\leq j\leq k}$ be the canonical basis of $\mathbb{R}^k$ and define $\mathbf{a} \in End(\mathbb{R}^k)$ by $e_j = \mathbf{a}(\varepsilon_j) = \sum_i \alpha_{ij}\varepsilon_i$. Then the new symbol :

$$p_0 = (p_{01}, \ldots, p_{0k}) \quad p_0 = \mathbf{a}(q_0)$$

satisfies $dp_{0j} = \sum_i \alpha_{ij} dq_{0i} = -\sum_i \alpha_{ij} H_i \lrcorner\, \omega$ and the corresponding Hamiltonian flow Ψ_j is 2π-periodic on the energy level $\mathbf{a}(E_0)$ in view of (H4).

Consequently the operators

$$\mathcal{P}(h) = \mathbf{a}(Q(h))$$

satisfy the pointwise equivalent of the hypothesis of [4], where it is supposed that there exists a function $f \in C_o^\infty(\mathbb{R}^k, \mathbb{R}^k)$ such that the Hamitonian flows of the principal symbols of the operators $f(\mathcal{Q}(h))$ are periodic in a neighborhood of the energy level.

We will denote by

$$K_j = \sum_i \alpha_{ij} H_i$$

the Hamiltonian field of p_{0j}.

(H5) *The subprincipal symbol q_1 is equal to 0.*

The goal is to construct a new symbol $\tilde{p}(h)$ which approaches $p(h)$ modulo $O(h^2)$ and which follows the assumptions of periodicity of [4].
Example. — If $(\nu, u) \in \Sigma_0 \times]-1, +1[^k$ is a parametrization of a neighborhood of Σ_0 such that the symplectic structure is given by the 2-form ω_1 which is equal to ω on Σ_0 and satisfies

$$du_j = -H_j \lrcorner\, \omega_1 \ ;$$

the vector fields H_j are the Hamiltonian fields of the functions $(\nu, u) \to u_j + E_{0j}$. These functions satisfy the assumption of periodicity in a neighborhood of the energy level, and they coincide with q_{0j}, as well as their derivatives on Σ_0. So they are good candidates for the construction of the principal part of $\tilde{q}(h)$.
The aim of the following section is to show how the general situation can be reduced to this one, using a geometric construction and the theorem of Weinstein.
Afterwards we can apply the Taylor formula (proposition 3) to control the error by h^2. Finally the usual technics of localization for h-pseudo's give the conclusion.

2 Symplectification

We use a connection 1-form on Σ_0 for the construction of a new symplectic form ω_1 which coincides with ω on Σ_0.
Let us recall that each element $A \in \mathbb{R}^k$, the Lie algebra of $\mathbb{T}^k$, defines a vector field $\bar{A}$ on Σ_0 by the locally free action ρ_0

$$\forall \nu \in \Sigma_0 \quad \bar{A}(\nu) = \frac{d}{dt}\big(\rho_0(\exp(tA))(\nu)\big)_{|t=0},$$

where $\exp(A) = (\exp(a_1), \ldots, \exp(a_k))$ for $A = (a_1, \ldots, a_k)$.
In all this section, we will denote the action of $\gamma \in \mathbb{T}^k$ on Σ_0 by $\gamma \cdot \nu$, $\nu \in \Sigma_0$, and by $\gamma_*(X)$ its action on $X \in T\Sigma_0$, the tangent bundle of Σ_0.

Proposition 2.1 . — *There exists a 1-form $\alpha = (\alpha_1, \ldots, \alpha_k)$ on Σ_0, with values in $\mathbb{R}^k$, which satisfies the two properties*
(P1) equivariant property $\forall X \in T\Sigma_0,\ \forall \gamma \in \mathbb{T}^k \quad \alpha(\gamma_*(X)) = \alpha(X)$
(P2) vertical triviality $\forall A \in \mathbb{R}^k \quad \alpha(\bar{A}) = A.$

Comments. — The property (P2) can be written $\alpha_i(K_j) = \delta_{ij}$ (the Kronecker symbol) if $K_j = \sum_i \alpha_{ij} H_i$ is the Hamiltonian field of p_{0j}.
Moreover, when the action ρ_0 of $\mathbb{T}^k$ is free, the quotient $\Sigma_0/\mathbb{T}^k$ has a structure of manifold, and the action

$$\begin{array}{ccc} \mathbb{T}^k & \rightarrow & \Sigma_0 \\ & & \downarrow \\ & & \Sigma_0/\mathbb{T}^k \end{array}$$

defines a structure of principal bundle (see [2] Chap. III, §1, Prop.10, p.104). In this case the 1-form α is a connection 1-form for this principal bundle ; we know that it always exists.
Proof. — First choose a Riemannian metric g on Σ_0. Because the group $\mathbb{T}^k$ is compact, we can suppose that g is invariant by the action of the torus (if it is not the case, take its average by the Haar measure of $\mathbb{T}^k$). For each $\nu \in \Sigma_0$, let V_ν be the subspace of $T_\nu\Sigma_0$ which is generated by the fields $\bar{A}(\nu)$ for all $A \in \mathbb{R}^k$, and H_ν the orthogonal of V_ν for g.

$$\forall \nu \in \Sigma_0, \qquad T_\nu\Sigma_0 = V_\nu \overset{\perp}{\oplus} H_\nu.$$

The action of $\mathbb{T}^k$ is locally free, so the rank of the sub-bundle V is equal to k ($\mathrm{rk}(V) = k$ and $\mathrm{rk}(H) = 2n - k$). We define α by

$$\alpha_{|H} = 0 \quad ; \quad \alpha(\bar{A}) = A \quad \forall A \in \mathbb{R}^k.$$

The property (P2) is fulfilled by construction ; for (P1), we remark that the group $\mathbb{T}^k$ is abelian ; so, for each $\gamma \in \mathbb{T}^k$ and $A \in \mathbb{R}^k$, we have

$$\gamma_*(\bar{A}) = \bar{A}.$$

But γ acts isometrically on Σ_0, so for any $\nu \in \Sigma_0$

$$\gamma_*(H_\nu) = H_{\gamma(\nu)}.$$

Now let $X \in T_\nu\Sigma_0$ and $\gamma \in \mathbb{T}^k$. We decompose X on $V_\nu \overset{\perp}{\oplus} H_\nu$ and calculate

$$\begin{array}{rcl} X & = & \bar{A} + U, \quad A \in \mathbb{R}^k, \quad U \in H_\nu \\ \gamma_*(X) & = & \bar{A} + \gamma_*(U), \quad \gamma_*(U) \in H_{\gamma(\nu)} \\ \alpha(\gamma_*(X)) & = & \alpha(\bar{A}) + 0 = A = \alpha(X). \end{array}$$

■

Corollary 2.1 . — $\forall A \in \mathbb{R}^k \quad \bar{A} \lrcorner\, d\alpha = 0.$

In the case of a principal bundle, this property means that the curvature of any connection is horizontal.
Proof. — For any vector field X, one has

$$d\alpha(X, \bar{A}) = X \cdot \alpha(\bar{A}) - \bar{A} \cdot \alpha(X) - \alpha([X, \bar{A}]).$$

The first term is equal to 0 because $\alpha(\bar{A})$ is constant.
Now suppose that $X = \bar{B}$ is vertical. Then $\alpha(X) = B$ is constant, and we have also $[\bar{B}, \bar{A}] = 0$, because the Lie algebra $\mathbb{R}^k$ is abelian, so the second and the third terms vanish.
Finally if $X \in H$ is horizontal, then we have $\alpha(X) = 0$, and $[X, \bar{A}]$ is also horizontal because $\mathbb{T}^k$ acts isometrically and preserves the horizontal. More precisely

$$[X, \bar{A}] = -\mathcal{L}_{\bar{A}}(X) = \frac{d}{dt}\big(\exp(-tA)_*(X \circ \exp(tA))\big)_{|t=0},$$

so $\alpha([X, \bar{A}]) = 0$. ■

Definition 2.1 . — *Let $\tilde{\alpha}$ be the 1-form defined on $T\mathbb{R}^{2n}{}_{|\Sigma_0}$ in the following way : $\tilde{\alpha}$ is equal to α on $T\Sigma_0$ and equal to 0 on its orthogonal for the canonical metric of $\mathbb{R}^{2n}$; there exist vector fields $\tilde{X}_j \in C^\infty(\Sigma_0, T\mathbb{R}^{2n}{}_{|\Sigma_0})$ such that*

$$\forall j,\ 1 \leq j \leq k \quad \tilde{\alpha}_j = \tilde{X}_j \lrcorner\, \omega,$$

because the form ω is non-degenerated. By definition $\omega(\tilde{X}_j, K_i) = \delta_{ij}$.

Lemma 2.1 . — *There exist vector fields $X_j \in C^\infty(\Sigma_0, T\mathbb{R}^{2n}{}_{|\Sigma_0})$ such that*

$$\forall\ 1 \leq i, j \leq k \quad \omega(X_i, X_j) = 0\ ;\ \omega(X_i, K_j) = \delta_{ij}$$

Proof. — We are looking for X_j of the form

$$X_j = \tilde{X}_j - \sum \beta_{ij} K_i.$$

As a necessary condition, we have $\omega(X_l, X_j) = \omega(\tilde{X}_l, \tilde{X}_j) - \beta_{lj} + \beta_{jl}$, and we obtain the result with $\beta_{lj} = \frac{1}{2}\omega(\tilde{X}_l, \tilde{X}_j) = -\beta_{jl}$. ∎

Moreover, since $K_j \lrcorner\, \omega$ is null on $T\Sigma_0$, we see that $X_j, 1 \leq j \leq k$ form a basis of a complementary space of $T\Sigma_0$ in $T\mathbb{R}^{2n}$.

Actually suppose that $\sum_i \lambda_i X_i = 0$. Then $\omega(\sum_i \lambda_i X_i, K_j) = 0$ which implies $\lambda_j = 0, \forall j$.

Definition 2.2 . — *Let* Exp *be the map*

$$\begin{array}{ccc} \Sigma_0 \times]-\varepsilon, \varepsilon[^k & \overset{\mathrm{Exp}}{\longrightarrow} & \mathbb{R}^{2n} \\ (\nu, u) & \longmapsto & \exp_\nu(\sum_{j=1}^{k} u_j X_j(\nu)) \end{array}$$

where exp *is the exponential map of the canonical Riemannian structure of* $\mathbb{R}^{2n}$.

In particular, one has the formula

$$d_{(\nu,0)}\mathrm{Exp}(\mathrm{X}, \mathrm{U}) = \mathrm{X} + \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{U_j X_j}(\nu) \tag{2.1}$$

for $X \in T_\nu\Sigma_0$ and $U = (U_1, \ldots, U_k) \in \mathbb{R}^k$, and we see that there exists $\varepsilon_0 > 0$ such that Exp is a diffeomorphism on its range for any $0 < \varepsilon < \varepsilon_0$.

Let $\Pi : \Sigma_0 \times]-\varepsilon, \varepsilon[^k \to \Sigma_0$ be the projection on the first factor and $i : \Sigma_0 \to \mathbb{R}^{2n}$ be the canonical imbedding.

Proposition 2.2 . — *If ε is small enough, then the 2-form*

$$\omega_1 = \Pi^*(i^*\omega) + \sum_{j=1}^{k} u_j d(\Pi^*\alpha_j) + \sum_{j=1}^{k} du_j \wedge (\Pi^*\alpha_j)$$

defines a symplectic structure on $\Sigma_0 \times]-\varepsilon, \varepsilon[^k$, which coincides with $\mathrm{Exp}^*\omega$ *on Σ_0.*

Proof. — Here a point ν belonging to Σ_0 is identified with the point $(\nu, 0)$ belonging to $\Sigma_0 \times]-\varepsilon, \varepsilon[^k$ and for such a point we will identify $X \in T_\nu\Sigma_0$ with $(X, 0) \in T_{(\nu,0)}(\Sigma_0 \times]-\varepsilon, \varepsilon[^k)$ and $U \in \mathbb{R}^k$ with $(0, U) \in T_{(\nu,0)}(\Sigma_0 \times]-\varepsilon, \varepsilon[^k)$.

The 2-form ω_1 is closed :

$$\begin{aligned} d\omega_1 &= d\Pi^*(i^*\omega) + \sum_{j=1}^{k} du_j \wedge d(\Pi^*\alpha_j)) - \sum_{j=1}^{k} du_j \wedge d(\Pi^*\alpha_j) \\ &= \Pi^*(i^*d\omega) + 0 = 0 \end{aligned}$$

The 2-form ω_1 coincides with $\mathrm{Exp}^*\omega$ *on* Σ_0 *:* fix $\nu \in \Sigma_0$ and consider first $X \in T_\nu\Sigma_0$; we have

$$X \lrcorner\, \omega_1 = (i \circ \Pi)^*(X \lrcorner\, \omega) - \sum_{j=1}^{k} \alpha_j(X) du_j.$$

Then, for $Y \in T_\nu\Sigma_0$ we can write, since $\mathrm{Exp}_*(\mathrm{X}) = \mathrm{X}$ and $\mathrm{Exp}_*(\mathrm{Y}) = \mathrm{Y}$,

$$\omega_1(X, Y) = \omega(X, Y) = \mathrm{Exp}^*\omega(\mathrm{X}, \mathrm{Y}),$$

while for $U \in \mathbb{R}^k$, $\Pi_*(U) = 0$ and

$$\begin{aligned} \omega_1(X, U) &= -\sum_{j=1}^{k} U_j \alpha_j(X) = -\sum_{j=1}^{k} U_j \omega(X_j, X) \\ &= \omega(X, \sum_{j=1}^{k} U_j X_j) = \omega(X, \mathrm{Exp}_*(\mathrm{U}) = \mathrm{Exp}^*\omega(\mathrm{X}, (0, \mathrm{U})), \end{aligned}$$

following the equation (2.1).
Finally for $U, V \in \mathbb{R}^k$

$$\Pi_*(U) = 0 \text{ and } \Pi_*(V) = 0 \Rightarrow \omega_1(U, V) = 0,$$

while

$$\mathrm{Exp}^*\omega(\mathrm{U}, \mathrm{V}) = \omega(\sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{U_j X_j}, \sum_{\mathrm{j}=1}^{\mathrm{k}} \mathrm{V_j X_j}) = 0$$

in view of Lemma 2.1.
But ω is non-degenerated and this property is open, so ω_1 is also non-degenerated on a neighborhood of Σ_0. ∎
Applying the Weinstein theorem (Lemma 3.14 of [15], or [18]) one obtains

Corollary 2.2 . — *For ε small enough ($\varepsilon > 0$) there exists $\Psi \in$ Diff$(\Sigma_0 \times]-\varepsilon, \varepsilon[^k)$ such that*

$$\Psi^*\omega_1 = \mathrm{Exp}^*(\omega) \quad \text{and} \quad \forall \nu \in \Sigma_0 \quad \Psi(\nu) = \nu,\ \mathrm{d}_\nu\Psi = \mathrm{Id}.$$

Proposition 2.3 . — *Let us define p on $\Sigma_0 \times]-\varepsilon, \varepsilon[^k$ by the formula $p(\nu, u) = u + \mathbf{a}(E_0)$; then $p \circ \Psi$ and $p_0 \circ \mathrm{Exp}$ coincide on Σ_0, as well as their differentials ; while the Hamiltonian flow of each component $p_j \circ \Psi$ is periodic with the same constant period on a neighborhood of $\mathbf{a}(E_0)_j$.*

Proof. — For the first part of the proposition, remark that

$$d_{(\nu,0)}(p_l \circ \Psi) = du_l$$

following Corollary 2.2, while using the expression of $d_{(\nu,0)}\mathrm{Exp}$ given by Formula (2.1), we have

$$\begin{aligned} d_{(\nu,0)}(p_0\circ \mathrm{Exp})(\mathrm{X},\mathrm{U}) &= d_\nu p_0(X+\sum_{j=1}^{j=k} U_j X_j(\nu)) \\ &= \omega(X+\sum_{j=1}^{j=k} U_j X_j(\nu), K_l) = U_l \end{aligned}$$

in view of Lemma 2.1.
For the second part, remark that the Hamiltonian field of p_l for the symplectic structure given by ω_1 is equal to $K_l=(K_l,0)$. Actually $dp_l = du_l$, while

$$\begin{aligned} -K_l\lrcorner\,\omega_1 &= -K_l\lrcorner \Big(\Pi^*(i^*\omega)+\sum_{j=1}^{k} u_j d(\Pi^*\alpha_j)+\sum_{j=1}^{k} du_j\wedge(\Pi^*\alpha_j)\Big) \\ &= du_l. \end{aligned}$$

As a matter of fact remark first that $\Pi_* K_l = K_l$, and that $K_l\lrcorner\, i^*\omega = 0$ (or $K_l\lrcorner\,\omega$ vanishes on tangent vectors of Σ_0), so the first term satisfies

$$K_l\lrcorner\,\Pi^*(i^*\omega) = \Pi^*(K_l\lrcorner\, i^*\omega) = 0.$$

Next, always using $\Pi_* K_l = K_l$, we have

$$K_l\lrcorner\, d(\Pi^*\alpha_j) = \Pi^*(K_l\lrcorner\, d\alpha_j) = \Pi^*(\bar{\varepsilon}_l\lrcorner\, d\alpha_j) = 0$$

by Corollary 2.1.
Finally recall that $\alpha_j(K_l) = \delta_{jl}$ (comments of proposition 2.1), and $du_j(K_l) = 0$; so we have

$$K_l\lrcorner \sum_{j=1}^{k} du_j\wedge(\Pi^*\alpha_j) = -du_l.$$

It is now clear that the Hamiltonian flow of p_l is 2π-periodic for the symplectic structure given by ω_1 ; so the same is true for $p_l\circ\Psi$ with the symplectic structure given by $\Psi^*\omega_1 = \mathrm{Exp}^*(\omega)$. ■

Corollary 2.3 . — *The functions p_0 and $\bar{p} = p\circ\Psi\circ\mathrm{Exp}^{-1}$ defined on $\mathrm{Exp}(\Sigma_0\times]-\varepsilon,\varepsilon[^{\mathrm{k}})$ coincide on Σ_0 as well as their derivatives, while $\bar{p}$ satisfies at $\mathbf{a}(E_0)$ the assumption of [4] : the hamiltonian flow of each component $\bar{p}_j$ is periodic on the level hypersurface $\bar{p}_j^{-1}(y_j)$ for y_j in a neighborhood of $\mathbf{a}(E_0)_j$.*

3 Consequence for the spectrum

First fix ε such that all the previous results are right and write

$$U_0 = \mathrm{Exp}(\Sigma_0 \times] - \varepsilon, \varepsilon[^{\mathrm{k}}).$$

We denote by $\mathrm{Sym}(\mathbb{R}^{\mathrm{k}})$ the space of symmetric matrices on $\mathbb{R}^k$.

Lemma 3.1 *If ε is small enough, there exists a smooth function $C : U_0 \to \mathrm{Sym}(\mathbb{R}^{\mathrm{k}})$ such that the following formula is true for all $\nu \in U_0$*

$$p_0(\nu) - \bar{p}(\nu) = C(\nu)\big[p_0(\nu) - \mathbf{a}(E_0), p_0(\nu) - \mathbf{a}(E_0)\big].$$

Proof. — The function p_0 is a submersion on K, and U_0 is relatively compact. Consequently there exists a finite open covering of U_0 where $p_0 - \mathbf{a}(E_0)$ can be taken as the first coordinates on each open. Take a partition of unit adapted to this covering, and consider the other coordinates as parameters, which actually are coordinates for Σ_0. In each open set of the covering, the announced formula is just the Taylor formula with parameters and an integral remainder, which is applied to the function $p_0 - \bar{p}$ at a point of Σ_0 (remark that at such a point this function as well as its derivative is null). ■

We will write

$$C(\nu)[X, Y] = \left(\sum_{1 \le ij \le k} C^1_{ij}(\nu) X_i Y_j, \ldots, \sum_{ij} C^k_{ij}(\nu) X_i Y_j \right).$$

Now take a cut-off function $\chi \in \mathcal{C}^\infty_o(\mathbb{R}^k)$ with support included in $\overset{o}{K}$ (like in hypothesis (H3)), which is equal to 1 on a open neighborhood W of $\mathbf{a}(E_0)$ and such that $Supp(\chi \circ p_0) \subset U_0$.

Lemma 3.2 *The principal symbol*

$$\tilde{p} = \chi \circ p_0 \, . \, \bar{p} + (1 - \chi \circ p_0) \, . \, p_0$$

defined on $\mathbb{R}^{2n}$ satisfies the hypotheses of [4] .

Proof. — From the formula

$$\tilde{p} - \mathbf{a}(E_0) = p_0 - \mathbf{a}(E_0) + \chi \circ p_0 \, . \, (\bar{p} - p_0)$$

one sees easily that the set $\tilde{p}^{-1}(\mathbf{a}(E_0)) = \Sigma_0$, thus $\mathbf{a}(E_0)$ is a regular value of $\tilde{p}$ which is proper on a neighborhood of Σ_0.

Hint One has, in the coordinates given by Exp, $\|p_0 - \mathbf{a}(E_0)\| \sim \|u\|$ while $\|\chi \circ p_0 \, . \, (\bar{p} - p_0)\| = O(\|u\|^2)$. ■

Proposition 3.1 *Let $\widetilde{\mathcal{P}_j(h)} = Op_h^w(\tilde{p}_j)$ for $1 \leq j \leq k$. For all integer l, $1 \leq l \leq k$, there exist uniformly bounded operators on $L^2(\mathbb{R}^n)$: $D_{ij}^l(h) = Op_h^w(\tilde{C}_{ij}^l)$, $B_j^l(h) = Op_h^w(b_j^l)$, $A^l(h) = Op_h^w(a^l(h))$ such that*

$$\mathcal{P}_l(h) - \widetilde{\mathcal{P}_l(h)} \sim \sum_{ij} D_{ij}^l(h) \circ (\mathcal{P}_i(h) - \mathbf{a}(E_0)_i \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j) + $$

$$+ \quad h \sum_j B_j^l(h) \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j) + h^2 A^l(h).$$

Proof. — With vectorial notations

$$\begin{aligned} \mathcal{P}(h) - \widetilde{\mathcal{P}(h)} &= Op_h^w(p(h) - \tilde{p}) = Op_h^w(p_0 - \tilde{p}) + Op_h^w(p(h) - p_0) \\ &= Op_h^w(\chi \circ p_0 \,.\, (p_0 - \bar{p})) + Op_h^w(p(h) - p_0). \end{aligned}$$

By hypothesis (H5) one has $p(h) - p_0 = h^2 d_1(h)$ and $d_1(h)$ will enter $a(h)$.
By lemma 3.1, $\chi \circ p_0 \,.\, (p_0 - \bar{p}) = \chi \circ p_0 \,.\, C[p_0 - \mathbf{a}(E_0), p_0 - \mathbf{a}(E_0)]$. Let $\tilde{C} = \chi \circ p_0 \,.\, C$; then by hypothesis (H5)

$$Op_h^w(\tilde{C}[p_0 - \mathbf{a}(E_0), p_0 - \mathbf{a}(E_0)]) = Op_h^w(\tilde{C}[p(h) - \mathbf{a}(E_0), p(h) - \mathbf{a}(E_0)) + h^2 Op_h^w(d_2(h))$$

and $d_2(h)$ will enter $a(h)$.
A componant of the first term is equal to $\sum_{ij} Op_h^w(\tilde{C}_{ij}^l(p_i(h) - \mathbf{a}(E_0)_i)(p_j(h) - \mathbf{a}(E_0)_j))$. We will apply here the theorem II-32 in [17], concerning the product of admissible h-pseudo-differential operators to compare $Op_h^w(\tilde{C}_{ij}^l(p_i(h) - \mathbf{a}(E_0)_i)(p_j(h) - \mathbf{a}(E_0)_j))$ and $Op_h^w(\tilde{C}_{ij}^l) \circ (\mathcal{P}_i(h) - \mathbf{a}(E_0)_i) \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j)$. Recall that the product of admissible h-pseudo-differential operators is of the same kind. Moreover, there exists formulae for the principal and the subprincipal symbols : with clear notations, one has for the principal symbol of such operators

$$\sigma_0(A \circ B) = \sigma_0(A)\sigma_0(B)$$

and for the subprincipal symbol

$$\sigma_1(A \circ B) = \sigma_0(A)\sigma_1(B) + \sigma_1(A)\sigma_0(B) + \frac{1}{2i}\{\sigma_0(A), \sigma_0(B)\}.$$

First application

$$\begin{aligned} (\mathcal{P}_i(h) - \mathbf{a}(E_0)_i) \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j) = Op_h^w((p_i(h) - \mathbf{a}(E_0)_i) \quad & (p_j(h) - \mathbf{a}(E_0)_j))) + \\ & h^2 Op_h^w(d_3(h)) \end{aligned}$$

because $\{p_{0i}, p_{0j}\} = 0$ by (H2) ; $d_3(h)$ will enter $a(h)$.
Now take $A = Op_h^w(\tilde{C}_{ij}^l) = D_{ij}^l(h)$ and $B = (\mathcal{P}_i(h) - \mathbf{a}(E_0)_i) \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j)$. The principal symbol of the operator $D_{ij}^l(h)(\mathcal{P}_i(h) - \mathbf{a}(E_0)_i) \circ (\mathcal{P}_j(h) - \mathbf{a}(E_0)_j)$ is equal to

$$\tilde{C}_{ij}^l(p_{0i} - \mathbf{a}(E_0)_i)(p_{0j} - \mathbf{a}(E_0)_j)$$

and for the subprincipal symbol one has

$$\frac{1}{2i}\left\{\tilde{C}^l_{ij},(p_{0i}-\mathbf{a}(E_0)_i)(p_{0j}-\mathbf{a}(E_0)_j)\right\} \;=$$
$$\frac{1}{2i}\left((p_{0i}-\mathbf{a}(E_0)_i)\left\{\tilde{C}^l_{ij},(p_{0j}-\mathbf{a}(E_0)_j)\right\}\right) \;+\; \frac{1}{2i}\left((p_{0j}-\mathbf{a}(E_0)_j)\left\{\tilde{C}^l_{ij},(p_{0i}-\mathbf{a}(E_0)_i)\right\}\right).$$

Define

$$b^l_j=\sum_i\{\tilde{C}^l_{ij},(p_{0i}-\mathbf{a}(E_0)_i)\}+\{\tilde{C}^l_{ji},(p_{0i}-\mathbf{a}(E_0)_i)\}.$$

We have proved that

$$\begin{aligned}\sum_{ij} D^l_{ij}(h)\circ(\mathcal{P}_i(h)-\mathbf{a}(E_0)_i)\circ(\mathcal{P}_j(h)-\mathbf{a}(E_0)_j) \;=&\\ \sum_{ij} Op^w_h(\tilde{C}^l_{ij}(p_{0i}-\mathbf{a}(E_0)_i)(p_{0j}-\mathbf{a}(E_0)_j)) \;&+\; h\sum_j Op^w_h(b^l_j)(\mathcal{P}_j(h)-\mathbf{a}(E_0)_j)\\ &+\; h^2 Op^w_h(d^l_4(h)),\end{aligned}$$

and $d_4(h)$ will enter $a(h)$.

The proof is complete. ■

Now let $I=I_1\times\ldots\times I_k$ be a k-cube around $\mathbf{a}(E_0)$. We suppose $\overline{I}\subset W$ and denote by $\Pi_I=\Pi^1_{I_1}\circ\ldots\circ\Pi^k_{I_k}$ the joint spectral projector of $\mathcal{P}(h)$. One knows, by [4], that the $\Pi^j_{I_j}$ commute to each other and with functions of $\mathcal{P}_j(h)$.

Corollary 3.1 . — *Suppose that I has size h, i.e. $(\mathcal{P}(h)-E_0)\circ\Pi_I=O(h)$ then*

$$(\mathcal{P}(h)-\widetilde{\mathcal{P}(h)})\circ\Pi_I=O(h^2).$$

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