Paracontrolled calculus and singular PDEs through Heat semigroup

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Aim : Let (M, d, μ) be a space of homogeneous type and L a non-negative "second order" operator, self-adjoint and generating a heat semigroup $(e^{-tL})_{t>0}$.

Solve (define what is the meaning of a solution) the following singular PDEs :

Parabolic Anderson Model (PAM)

$$\partial_t u + Lu = u \cdot \xi, \qquad u_{|t=0} = u_0$$

where $u : M \times \mathbb{R}^+ \to \mathbb{R}$ and ξ is a singular distribution on M (mainly a white noise).

Parabolic Anderson Model : $L = -\Delta$ on $\mathbb{T}^d$ or $\mathbb{R}^d$ for $d = 2, 3$.

Difficulty : Define the product $u \cdot \xi$.

First solution : Use Hairer's theory (regularity structure) to transport this equation into another equation on an "abstract structure", where we can perform a fixed point theorem to solve the problem (define the product and solve the heat equation) ... and then come back in the initial space.
(Hairer-Labbé)
(based on a very deep Wavelet analysis, adapted to the Euclidean situation, for the "come back" operation).

Second solution : Use the paracontrolled calculus (introduced by Gubinelli-Imkeller-Perkowski), based on paraproducts to perform a deep analysis of the structure in the product $u\xi$.
(global analysis).

$\Longrightarrow$ Extension of the second approach in a very general setting (in terms of ambiant space and of operator) and resolution of PAM in dimension 2.

1. THE PARACONTROLLED CALCULUS ADAPTED WITH THE OPERATOR L

2. RESOLUTION OF PAM IN 2D

Heat Kernel estimates and Sobolev Algebra property

1. The paracontrolled calculus adapted with the operator L
2. Resolution of PAM in 2D

Assumption : The heat semigroup $(e^{-tL})_t$ has a kernel with pointwise upper Gaussian estimates, as well as its "carré du champ" $(\sqrt{t}\Gamma e^{-tL})_t$, where

$$L(f^2) = 2fL(f) + \Gamma(f)^2.$$

(on the Riemannian setting : $L =$ Laplace-Beltrami operator and Γ the gradient)

Time-frequency analysis adapted to the operator :

$$P_t \simeq e^{-tL} \quad Q_t \simeq (tL)^N e^{-tL}.$$

$Q_t = \phi(tL)$ has to be thought as a frequency truncation operator for frequencies at scale $t^{-1/2}$.

Consider the (inhomogeneous) Hölder space C^s for $s \in \mathbb{R}$

$$\|f\|_{C^s} := \|e^{-L}f\|_\infty + \sup_{0<t<1} t^{-s/2}\|Q_t(f)\|_\infty.$$

(analog of a Besov space, in terms of operator L)

Remark : Under the assumption, C^s corresponds to the usual metric Hölder space for $s \in (0, 1)$.

Calderón reproducing formula : $f = \int_0^1 Q_t(f)\,\frac{dt}{t} + P_1(f)$.

We decompose f, g and the product fg ... due to the “carré du champ” structure, there exist paraproducts :

$$\Pi_g(f) = \int_0^1 Q_t[Q_t(f)P_t(g)]\frac{dt}{t}$$

and a resonant part :

$$\Pi(f,g) = \int_0^1 P_t[Q_t f Q_t g]\frac{dt}{t}$$

such that we have the decomposition

$$fg = \Pi_g(f) + \Pi_f(g) + \Pi(f,g) + \Delta_{-1}(f,g).$$

The “carré du champ” structure allows us to go from one Q_t operator to two Q_t operators !

where

- $\Delta_{-1}(f,g)$ is the low-frequency part (very nice for our study)
- $\Pi(f,g)$ is the resonant part (where f and g have "frequencies at the same scale" and so produce some resonances).
- P_t has to be thought as e^{-tL} and any Q_t operators encode some cancellation
$$Q_t = (tL)e^{-tL}, \qquad \text{or} \qquad Q_t = \sqrt{t}\Gamma e^{-tL}.$$

Paraproduct estimates

Paraproduct estimates

For every $\alpha, \beta \in (-\infty, 1)$, $(f, g) \in C^\alpha \times C^\beta$. If $\beta \geq 0$ then

$$\|\Pi_g(f)\|_{C^\alpha} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta};$$

If $\beta < 0$ then

$$\|\Pi_g(f)\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta};$$

Resonant estimates

For every α, β with $\alpha + \beta > 0$ then

$$\|\Pi(f, g)\|_{C^{\alpha+\beta}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta}.$$

Conclusion : the product fg is only defined if $\alpha + \beta > 0$.
Difficulty : if ξ is a white noise on $\mathbb{R}^d$ then $\xi \in C^{-d/2}$. The solution u is expected to belong in $C^{-d/2+2}$, so the product is not defined for $d \geq 2$!

DEFINITION (PARACONTROLLED FUNCTIONS)

If $X \in C^\alpha$ is "well-known" (is a "model" in terms of Hairer's theory) and $f \in C^\alpha$ we say that f is paracontrolled (or modelled) by X if there exists $g \in C^\alpha$ and $f^\sharp \in C^{2\alpha}$ such that

$$f = \Pi_g(X) + f^\sharp.$$

That means : the $(C^\alpha \setminus C^{2\alpha})$-part of f is "*similar*" to X

Functional rules : product, Schauder estimates on such paracontrolled distributions.

We have to "commute" the paraproduct decomposition with the paraproduct with respect to the model distribution :

Commutator estimates

For $f \in C^\alpha, g \in C^\alpha$ and $h \in C^\gamma$ then

$$\Pi(\Pi_f(g), h) - f \cdot \Pi(g, h) \in C^{2\alpha+\gamma}$$

if $\alpha + \alpha + \gamma > 0$.

Extension of the paracontrolled calculus for time-dependent distributions, resolution of heat equation :
if f is paracontrolled by ξ then the solution of $(\partial_t + L)u = f$ is paracontrolled by X the solution of $(\partial_t + L)X = \xi$ ...

Heat Kernel estimates and Sobolev Algebra property

1. The paracontrolled calculus adapted with the operator L
2. Resolution of PAM in 2D

Then consider (M, d, μ) a doubling space of dimension 2.

We want to solve / give a sense to

$$\partial_t u + Lu = u.\xi \qquad \text{and} \qquad u_{|t=0} = u_0$$

where ξ is a white noise.

Idea : the singularity of u will be given by the white noise !
So it is natural to assume that u will be paracontrolled by X with

$$\partial_t X + LX = \xi \qquad X_{|t=0} = 0.$$

Consider (X, d, μ) of dimension 2 and ξ a white noise on X.
That means that ξ is a centered Gaussian variable such that for every functions f, g then

$$\mathbb{E}_\omega \left[\left(\int f(y)\xi_\omega(y) d\mu \right) \left(\int g(z)\xi_\omega(z) d\mu \right) \right] = \int fg \, d\mu.$$

(ω is the random variable)

Consequence : for all $\alpha \in (1/2, 1)$ we know that a.e. (a.e. ω) then

$$x \to \xi_\omega(x) \in C^{\alpha-2}.$$

And so a.e. we have

$$X \in C_T C^\alpha \cap C_T^{\alpha/2} L^\infty.$$

Then let us work in the class

$$\mathcal{C} := \left\{ u \in C_T C^\alpha, \ u \text{ paracontroled by } X \right\}.$$

We consider the map

$$\gamma : u \in \mathcal{C} \rightarrow v$$

where v is the solution of

$$\partial_t v + Lv = u.\xi \qquad v_{|t=0} = u_0.$$

<u>Problem :</u> How to well-define the product $u.\xi$ in order to use a fix point argument on γ ?
Since $u \in \mathcal{C}$ we write

$$u.\xi = \Pi_\xi(u) + \Pi_u(\xi) + \Pi(u, \xi)$$

and only the last term is non-defined since $\alpha + (\alpha - 2) < 0$.

We use the ansatz given by the class $\mathcal{C}$

$$\begin{aligned} u.\xi &= \Pi_\xi(u) + \Pi_u(\xi) + \Pi(u,\xi) \\ &= \Pi_\xi(u) + \Pi_u(\xi) + \Pi(\Pi_g(X),\xi) + \Pi(u^\sharp,\xi) \\ &= (2\alpha - 2) + \Pi_u(\xi) + g.\Pi(X,\xi) + C(g,X,\xi) + (3\alpha - 2) \end{aligned}$$

if $\alpha > 2/3$. The commutator is also bounded so

$$u.\xi = \Pi_u(\xi) + g.\Pi(X,\xi) + (2\alpha - 2).$$

Condition : If $\Pi(X,\xi)$ is well-defined in $C_T C^{2\alpha-2}$ so that the middle term has a sense in $C_T C^{2\alpha-2}$ and then

$$u.\xi = \Pi_u(\xi) + (2\alpha - 2).$$

u is paracontrolled by ξ !

Then we solve the heat equation (J) : J is the unique operator such that

$$(\partial_t + L)J(f) = f \qquad \text{and} \qquad (Jf)_{|t=0} = 0.$$

Duhamel's formula :

$$J(f)(t) = \int_0^t e^{-(t-\tau)L} f(s)\, ds.$$

We deduce that v takes the form

$$\begin{aligned} v &= J\Pi_u(\xi) + (2\alpha) \\ &= \Pi_u(X) + (2\alpha), \end{aligned}$$

since by definition $X = J\xi$ and use of a commutator estimate (between Π and J).

Conclusion : v belongs also to $\mathcal{C}$ if we could define $\Pi(X, \xi) \in C_T C^{2\alpha-2}$ (which is a condition only on the white noise and independent of u).

In such a case we could then perform a fix point theorem.

Theorem

If $\xi \in C^{\alpha-2}$ with $\alpha \in (2/3, 1)$ and $\Pi(X, \xi)$ is well-defined in $C_T C^{2\alpha-2}$, where $X = J(\xi)$. Then we can solve the heat equation

$$\partial_t u + Lu = u.\xi \qquad u_{|t=0} = u_0 \in C^{2\alpha}.$$

The solution is unique in the class of paracontrolled distributions and depends continuously with respect to the data.

Problem : for ξ the white noise .... $\Pi(X, \xi)$ is a.e. non-defined !

So we have to understand, how to "renormalize" the white noise, to be able to give a sense to this quantity.

Consider $\xi^\epsilon = e^{-\epsilon L}\xi$ an approximation of ξ and $X^\epsilon = e^{-\epsilon L}X$.

PROPOSITION

There exists a deterministic function c_ϵ such that

$$\Pi(X^\epsilon, \xi^\epsilon) - c^\epsilon$$

is a.e. well-defined, uniformly bounded and convergent in $C_T C^{2\alpha-2}$.

Indeed :

$$c^\epsilon \simeq \mathbb{E}_\omega \left[\Pi(X^\epsilon, \xi^\epsilon)\right] (t \to \infty).$$

On the torus or the Euclidean space, invariance by translation of the white noise implies that c_ϵ is a constant

$$c^\epsilon \simeq -\log(\epsilon).$$

Theorem

For $\alpha \in (2/3, 1)$, consider an initial $u_0 \in C^{2\alpha}$. For $\epsilon > 0$, approximate solution

$$\partial_t u^\epsilon + L u^\epsilon = u^\epsilon \cdot (\xi^\epsilon - c^\epsilon), \qquad u^\epsilon_{|t=0} = u_0.$$

Then u^ϵ is a.e. well-defined and uniformly bounded and convergent in $C_T C^{2\alpha-2}$.

Global in time results : compact manifold, weighted white noise or in weighted Hölder space.

Extension

With Dorothee Frey : weaken the pointwise bound on the gradient of the heat kernel.
Indeed, some (G_p) condition is sufficient for a large enough exponent $p > 2$:

$$\sup_{t>0} \|\sqrt{t}\nabla e^{-tL}\|_{p\to p} < \infty.$$

Allows us also to work on Sobolev spaces $W^{s,p} = (1+L)^{-s/2}(L^p)$ for a large enough $p > 2$ (instead of Hölder spaces). Not clear if Hairer's theory could be applied in such a setting.

Project : adapt this approach to the three dimensional situation. Far more difficult because then the white noise is ony $C^{-3/2}$ and $\alpha < 1/2$ (dicted by the renormalization). So $3\alpha - 2 < 0$ and the previous commutator estimate is not sufficient ... Moreover $c_\epsilon \simeq \epsilon^{-1}$ is more divergent so more renormalization has to be done.