An approach of maximal regularity problems with the Hardy spaces.

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Maximal regularity via Hardy spaces

Definition of Maximal Regularity.

Let (Y, d_Y, ν) be a space of homogeneous type and L be the generator of an $L^2(Y)$-analytic semigroup. Consider the Cauchy Problem on $I = [0, T]$

$$\begin{cases} \frac{du}{dt}(t) - Lu(t) = f(t), & t \in I \\ u(0) = 0. \end{cases} \tag{1}$$

The solution u is given by the semigroup

$$u(t) = \int_0^t e^{(t-s)L} f(s, \cdot) ds.$$

Definition

For $p, q \in (1, \infty)$ there is maximal L^q-regularity on $L^p(Y)$ if for every $f \in L^q(I, L^p)$, the solution u satisfies

$$\frac{du}{dt} \in L^q(I, L^p) \quad \textit{and} \quad L(u) \in L^q(I, L^p).$$

Question : Which assumptions on L guarantee us a maximal regularity for some exponents.

Proposition

A positive answer to the maximal regularity in $L^q(I, L^p)$ does not depend on the exponent $q \in (1, \infty)$. So we only speak about maximal regularity in $L^p(Y)$.

Proposition (L. de Simon-1964)

For $p = 2$. If the semigroup $(e^{tL})_{t>0}$ is analytic on $L^2(Y)$ then we have maximal regularity on $L^2(Y)$.

Proposition (L. Weiss-2001)

We have maximal regularity for (p, q) if and only if the complex semigroup satisfies to some R-boundedness

This characterization is not very practicable and we are looking for necessary assumptions on the semigroup $(e^{tL})_{t>0}$, which yields a positive answer. Since we have a saisfying result for $p = 2$, we want to obtain assumptions using L^2-estimates.

Let us define an operator T, for $(t,x) \in I \times Y$

$$T(f)(t,x) := \int_0^t [Le^{(t-s)L} f(s,\cdot)](x)ds = Lu(t,x),$$

where u is solution of (1).

Proposition

For $p \in (1,\infty)$, we have L^p-maximal regularity if and only if T is bounded in $L^p(I \times Y)$.

So we know that T is bounded on $L^2(I \times Y)$ and we are looking for boundedness in $L^p(I \times Y)$.
$\Longrightarrow$ Use of an adapted Hardy spaces H^1_L as follows :

- Construction of an Hardy space H^1_L such that T is bounded from H^1_L into $L^1(I \times Y)$.
- Interpolation between $H^1_L(I \times Y)$ and $L^2(I \times Y)$.

Maximal regularity via Hardy spaces

Results

Let (X, d, μ) be a space of homogeneous type.

Proposition

T is bounded from H^1 into L^1 if for all ball $Q \subset X$ and functions f supported on Q

$$\left(\fint_{C_j(Q)} |T(B_Q(f))|^2 d\mu \right)^{1/2} \leq \alpha_j \left(\fint_Q |f|^2 d\mu \right)^{1/2},$$

with coefficients α_j satisfying

$$\sum_{j \geq 0} \frac{\mu(2^j Q)}{\mu(Q)} \alpha_j < \infty.$$

We define the maximal operator

$$M_\sigma(f)(x) := \sup_{Q \ni x} \left(\fint_Q |A_Q^*(f)|^\sigma d\mu \right)^{1/\sigma}.$$

Theorem

For $p \in (1,2)$ and $\sigma \in (2,\infty)$ with

$$\frac{1}{2} < \frac{1}{p} = (1-\theta) + \frac{\theta}{2} < \frac{1}{\sigma'}.$$

If $M_\sigma \lesssim \mathcal{M}_2$ then $(H^1, L^2)_\theta = L^p$.

Maximal regularity via Hardy spaces

Definition of the adapted Hardy spaces

Assume that L is a second-order differential operator admitting a L^2-analytic semigroup. We consider $X = I \times Y$, equipped with its parabolic metric structure (X, d, μ) :

$$d((t_1, y_2), (t_2, y_2)) = \max\{\sqrt{|t_1 - t_2|}, d_Y(y_1, y_2)\}$$

and

$$d\mu = dt \otimes d\nu.$$

- We have L^2-maximal regularity so T is L^2-bounded
- We are looking to prove that T is bounded in $L^p(X)$ for some $p \in (1, 2)$.

Definition

For $Q = B((t_0, y_0), r)$ a ball of X, we set

$$A_Q(f)(t,x) := A_r(f)(t,x) = \int_0^\infty \phi_r(t-\sigma)[e^{rL} f(\sigma,\cdot)](x) d\sigma,$$

with a smooth function $\phi \in \mathcal{S}(\mathbb{R}^+)$ satisfying

$$\int \phi = 1 \quad and \quad \phi_r(t) := r^{-1}\phi(t/r).$$

Construction of the Hardy space H^1_L on X.

Since ϕ is supported on $\mathcal{S}(\mathbb{R}^+)$, $A_Q(f)(t,x)$ only uses the past information ($\sigma \leq t$). Indeed, we only use fast decay at infinity for ϕ.

Theorem

If $(e^{tL})_{t>0}$, $(tLe^{tL})_{t>0}$ and $(t^2L^2e^{tL})_{t>0}$ satisfy to some L^2-L^2 off-diagonal on Y then T is bounded from H^1_L into $L^1(X)$.

We just require that for $T_t = e^{tL}$ or tLe^{tL} and $t^2L^2e^{tL}$, then for all ball Q of Y

$$\left(\int_{C_j(Q)} |T_u(f)|^2 d\nu\right)^{1/2} \leq \gamma(\frac{2^j r}{u}) \left(\int_Q |f|^2 d\nu\right)^{1/2},$$

with

$$\gamma(x) \leq (1+|x|)^{-4}.$$

Theorem

If for some exponent $p_0 \in (1,2)$, $(e^{tL})_{t>0}$ satisfies to some $L^{p_0} - L^2$ off-diagonal in Y, then

$$M_{p_0'} \lesssim \mathcal{M}_2.$$

Indeed, we ask that for all ball $Q \in Y$ and f supported on Q

$$\left(\fint_{C_j(Q)} |e^{r_Q^2 L}(f)|^2 d\nu \right)^{1/2} \leq \beta_j \left(\fint_Q |f|^{p_0} d\nu \right)^{1/p_0},$$

with coefficients β_j verifying

$$\sum_{j \geq 0} 2^j \beta_j < \infty.$$

Conclusion : Under

- $L^2 - L^2$ off-diagonal decays for $(e^{tL})_{t>0}$, $(tLe^{tL})_{t>0}$ and $(t^2L^2e^{tL})_{t>0}$
- $L^{p_0} - L^2$ off-diagonal decay for $(e^{tL})_{t>0}$,

then T is bounded on $L^p(X)$ for all exponent $p \in (p_0, 2)$. So we have maximal regularity on $L^p(Y)$.

Remarks :

- Good-lambda inequalities permit us to get weighted estimates $\Rightarrow$ Maximal regularity in weighted Lebesgue space.
- For $p \in (2, \infty)$, we can use duality as T^* is very similar to T.
- S. Blunck and P.C. Kunstmann have obtained similar results (via totally different methods) proving that under off-diagonal decays then the semigroup satisfied to the R-boundedness (which is equivalent to the maximal regularity since the work of Weiss). Our estimates are L^2 (excepted for the semigroup), which is weaker that what they assumed.