The bilinear Bochner-Riesz problem

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Let $\Omega = B(0,1)$ be the closed unit ball of $\mathbb{R}^n$ ($n \geq 2$). For $\delta > 0$, the Bochner-Riesz multiplier is then defined by

$$R_\delta(f)(x) := \int_\Omega e^{ix\cdot\xi} \widehat{f}(\xi)(1 - |\xi|^2)_+^\delta d\xi.$$

Idea : [Fefferman,1971] the ball multiplier is not bounded in any L^p-space ($p \neq 2$).
So some extra regularity near the set of singularity (measured by δ) is required to improve the integrability.

Motivation : convergence of Fourier series or truncated Fourier transforms.

Question : Boundedness in L^p of such multipliers ? Fix an exponent $p \in (1, \infty)$, for which exponent δ the multiplier R_δ is bounded on L^p ?

Bochner-Riesz conjecture : For $p \in (1, \infty)$, R_δ is bounded on L^p if and only if

$$n\left|\frac{1}{p} - \frac{1}{2}\right| < \delta + \frac{1}{2} \qquad \text{and} \qquad \delta \geq 0.$$

- Herz and Fefferman have independently proved that the condition is necessary.
- Carleson and Sjölin have solved the conjecture for $n = 2$ (1972).
- For $n \geq 3$, Some progress using restriction estimates, bilinear restriction estimates, ... still open ! The conjecture is proved for a range $p \in (1, q_0] \cup [q_0', \infty)$ with q_0 more and more close to 2.

Naturally, we have the bilinear version :

$$R_\delta(f,g)(x) := \int_{B_{2n}(0,1)} e^{ix\cdot(\xi+\eta)}\widehat{f}(\xi)\widehat{g}(\eta)(1-|\xi|^2-|\eta|^2)^\delta d\xi\, d\eta.$$

It corresponds to the bilinear Fourier multiplier, associated to the symbol $(\xi,\eta)\to(1-|\xi|^2-|\eta|^2)_+^\delta$ having a singularity along the sphere of dimension $2n-1$.

Comparison with other bilinear Fourier multpliers :

- Hörmander class of bilinear pseudodifferential operator, bilinear Calderón-Zygmund operators $\Longrightarrow$ singularity at one point.
- Bilinear Hilbert transform ($n = 1$) $\Longrightarrow$ singularity along a line.
- Disc multiplier ($n = 1$) $\Longrightarrow$ singularity along the circle.
- Twisted paraproducts ($n = 2$) $\Longrightarrow$ singularity along a plane. ....

Here : take advantage of the invariance by rotations (rotations in $\mathbb{R}^{2n}$ and rotations in $\mathbb{R}^n$) to compensate the high-dimension of the singular subset.
Use the non-vanishing curvature to improve some decays.

Question : take $p_1, p_2 \in (1, \infty)$ and p given by Hölder scaling

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

then for wich δ the bilinear Bochner-Riesz multiplier R_δ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$?

Difficulties :

- high dimension of the singular set ;
- bi-radial character of the symbol is not preserved by duality.

For R_δ^{*1}, its first adjoint operator given by

$$\langle R_\delta(f,g),h\rangle = \langle f, R_\delta^{*1}(h,g)\rangle$$

is the bilinear Fourier multiplier given by the symbol

$$(\xi,\eta) \to (1-|\xi+\eta|^2-|\eta|^2)_+^\delta$$

which is not bi-radial.

The case $n = 1$

Theorem

On $\mathbb{R}$, the Bochner-Riesz operator is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^p(\mathbb{R})$ in the following situations :

- *Strict local L^2-case :* $2 < p_1, p_2, p' < \infty$ *and* $\delta \geq 0$.
- *Endpoint cases :* $\{p_1, p_2, p'\} = \{2, 2, \infty\}$ *and* $\delta > 0$.
- *Banach triangle case :* $1 \leq p_1, p_2, p' \leq \infty$ *and* $\delta > 0$.

Proof :
- for the local L^2-case, consequence of [Grafakos - Li] on the disc multiplier.
- the two other cases are a consequence of a spherical decomposition and [B. - Germain] (on bilinear Fourier multipliers with a symbol singular along a curve).
These results are optimal in the strict local L^2 case, in the endpoint cases, and on the boundary of the Banach triangle.

The case $n \geq 2$

Necessary conditions for boundedness :

$$\delta > n(\frac{1}{p} - 1) - \frac{1}{2} \qquad \text{and} \qquad \delta \geq 0.$$

Sufficient condition :

$$\delta > \frac{2n-1}{2} = n - \frac{1}{2}.$$

Proof : Non-Stationnary phase estimates (coming from the curvature of the sphere) to get pointwise estimates (with asymptotic formulas) of the bilinear kernel.

1. The linear Bochner-Riesz multipliers
2. Extension to a bilinear setting
3. Boundedness in the Banach triangle $p_1, p_2, p \in (1, \infty)$
   - $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ boundedness
   - $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ boundedness
   - $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ boundedness
4. Boundedness outside the Banach triangle
5. Interpolation of the different estimates

We split the symbol as

$$m_\delta := (1 - |\xi|^2 - |\eta|^2)_+^\delta = \sum_{j \geq 0} 2^{-j\delta} m_{j,\delta} \qquad (1)$$

with bi-radial symbols $m_{j,\delta}(\xi, \eta) = m_{0,j,\delta}(|\xi|, |\eta|)$, where

$$m_{0,j,\delta}(t, s) = \left[2^j(1 - t^2 - s^2)\right]^\delta \chi\left(2^j(1 - t^2 - s^2)\right),$$

and thus $m_{j,\delta}$ are L^∞-normalized, supported in the annulus $1 - |(\xi, \eta)| \simeq 2^{-j}$ and are regular at the scale 2^{-j}.

Boundedness for R_δ is almost equivalent to a uniform boundedness (with respect to j) of the bilinear multipliers associated to $m_{j,\delta}$.

The elementary symbols $m_{j,\delta}$ still respect the bi-radial property. The corresponding bilinear multiplier as a bilinear kernel $K_{j,\delta}$ satisfying :

$$\left|K_{j,\delta}(y,z)\right| \lesssim 2^{-j}\left(1+2^{-j}(|y|+|z|)\right)^{-M}.$$

Off-diagonal decays $\Longrightarrow$ "localization property" at the scale 2^j in the physical space.

Observation : a factor 2^{-2nj} would imply boundeness in Hölder scaling (and corresponds to a singularity at only one point in the frequency space).

LEMMA

Let $1 \leq p, q \leq \infty$ and $1/r = 1/p + 1/q < \infty$. Let T be a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{q_1}(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ for some s, p_1, q_1 satisfying $0 < r \leq s$, $1 \leq p_1 \leq p$ and $1 \leq q_1 \leq q$ with a ***finite speed propagation property*** *: its bilinear kernel K_T satisfies*

$$\mathrm{supp} K_T \subseteq \{(x,y,z) : \ |x-y| < \rho, |x-z| < \rho\}$$

for some $\rho > 0$. Then

$$\|T\|_{L^p \times L^q \to L^r} \lesssim \rho^{n(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{s})} \|T\|_{L^{p_1} \times L^{q_1} \to L^s}.$$

Up to a small (as we want) loss in the power of ρ, we may allow a fast decreasing kernel as previously for the "localized operators" (and no need to have an exact support condition).

Properties of bi-radial functions :

Proposition

Let $m(\xi,\eta) = m_0(|\xi|,|\eta|)$ be a bi-even symbol m_0. Then if m_0 has a spectrum included into $[-L,L]^2$ then m has a spectrum included into $[-L,L]^{2n}$.

Two different approaches :

- - Using some computations involving Bessel functions.
- - Use the finite speed of propagation property of the Euclidean Laplacian and decompose m with the Wave propagators ($e^{it\sqrt{\Delta}}$ propagates at the speed t).

THE $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ BOUNDEDNESS

THEOREM

The bilinear multiplier R_δ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ if and only if $\delta > 0$.

Proof : The bilinear symbol satisfies for $\alpha \in (0, \delta)$

$$\sup_u \|(1 - u^2 - .^2)^\delta\|_{\dot{W}^{1+\alpha,1}} < \infty.$$

Decomposition of the symbol using bilinear Fourier series
$\Longrightarrow$ Decomposition of R_δ as a tensorial product of L^2-bounded linear Fourier multipliers.

Necessary conditions by using dilations and modulation (the ball may be transformed into any half-space).

Attention : No symmetry (invariance by rotation) under duality so no extension to the local-L^2 case

The $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ boundedness

Theorem

The bilinear multiplier R_δ is bounded from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ as soon as $\delta > \frac{n-1}{2}$.

Proof : Spherical decomposition to reduce the estimate to some "localized" operators.
Use $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ estimates to take advantage of the geometry of the sphere in the frequency space.

Level set estimate :

$$\sup_\xi |\{\eta, \ |(\xi-\eta,\eta)| \simeq 1-\epsilon\}| \lesssim \epsilon$$

because $n \geq 2$.

THEOREM

The bilinear multiplier R_δ is bounded from $L^1(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ as soon as $\delta > \frac{n}{2}$.

Proof : Spherical decomposition to reduce the estimate to some "localized" operators.
Use $L^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ estimates which come from (Bernstein inequality) $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ estimates.
Then use the previous argument based on the regularity of the elementary symbols.

Remarks : All these results can be extended to any bi-radial symbol $m(\xi,\eta) = m_0(|\xi|,|\eta|)$ with the following properties

- m_0 is compactly supported
- Some regularity :

$$\sup_u \|m_0(u,\cdot)\|_{\dot{W}^{1+\delta+,1}} + \sup_v \|m_0(\cdot,v)\|_{\dot{W}^{1+\delta+,1}} < \infty.$$

- A sharp control on the size : for a small parameter $\epsilon > 0$

$$\sup_\xi \left| \left\{ \eta,\ |m(\xi-\eta,\eta)| \simeq \epsilon^\delta \right\} \right| \lesssim \epsilon.$$

Outside the Banach triangle, we use the following

Proposition

Let $1 \leq q < \infty$ *and* $\delta = \sigma + i\tau$. *If* $0 < s < \sigma + \frac{1}{q}$, *then* $(1-|x|^2)_+^\delta \in W^{s,q}(\mathbb{R}^n)$. *Moreover,*

$$\left\| (1-|x|^2)_+^\delta \right\|_{W^{s,q}(\mathbb{R}^n)} \lesssim e^{c|\tau|}.$$

Proof : it relies on sharp estimates involving oscillatory integrals and Bessel functions.
Asymptotic formulas of Bessel functions.
Or can be obtained by using the spherical decomposition and interpolation inequalities.

The linear restriction-extension operator

$$\mathcal{R}_\lambda f(x) := \lambda^{n-1} \int_{\mathbb{S}^{n-1}} e^{i\lambda x\cdot\omega} \widehat{f}(\lambda\omega) d\omega$$
$$:= \int_{\lambda\mathbb{S}^{n-1}} e^{ix\omega} \widehat{f}(\omega) d\omega.$$

Decomposition : Let $m(\xi,\eta) := m_0(|\xi|,|\eta|)$, then

$$T_m(f,g)(x) = \int_0^\infty \int_0^\infty m_0(\lambda_1,\lambda_2) \mathcal{R}_{\lambda_1,\lambda_2}(f,g)(x) d\lambda_1 d\lambda_2,$$

with

$$\mathcal{R}_{\lambda_1,\lambda_2}(f,g)(x) := \mathcal{R}_{\lambda_1} f(x) \mathcal{R}_{\lambda_2} g(x).$$

In the square $[0,1]\times[0,1]$, define

$$A(n)=\Big(\frac{n+1}{2n},0\Big):=(a_n,0),\qquad B(n)=\Big(\frac{n+1}{2n},\frac{n-1}{2n}-\frac{n-1}{n^2+n}\Big),$$

$$A'(n)=\Big(1,\frac{n-1}{2n}\Big),\qquad B'(n)=\Big(\frac{n+1}{2n}+\frac{n-1}{n^2+n},\frac{n-1}{2n}\Big):=(b_n,\frac{n-1}{2n}).$$

and $\Delta(n)$ be the closed pentagon with vertices $A(n), B(n), B'(n), A'(n)$, $(1,0)$ from which closed line segments $[A(n),B(n)],[A'(n),B'(n)]$ are removed. Namely,

$$\Delta(n)=\left\{\begin{array}{l}\Big(\frac{1}{p},\frac{1}{q}\Big)\in[0,1]\times[0,1]:\quad 0\leq\frac{1}{q}\leq\frac{1}{p}\leq 1,\ \frac{1}{p}-\frac{1}{q}\geq\frac{2}{n+1},\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{p}>\frac{n+1}{2n},\ \ \frac{1}{q}<\frac{n-1}{2n}\end{array}\right\}$$

PROPOSITION (BAK)

The linear restriction-extension operator $\mathcal{R}_1$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $(1/p, 1/q) \in \Delta(n)$.

Proof : Closely related to the restriction Fourier estimate (Tomas-Stein for $q = p'$)
Use oscillatory integral estimates of Hörmander.

Can be extended to estimate more general operators, as Bochner-Riesz multipliers of negative order.

Remark : $\mathcal{R}_1$ cannot map into L^2.

Proposition

(i) Let $1/s = 1/q_1 + 1/q_2$, $0 < s \leq \infty$ *and let* $(1/p_1, 1/q_1)$ *and* $(1/p_2, 1/q_2)$ *be both in* $\Delta(n)$.

$$\|\mathcal{R}_{\lambda_1,\lambda_2}(f,g)\|_s \lesssim \lambda_1^{n(\frac{1}{p_1}-\frac{1}{q_1})-1} \lambda_2^{n(\frac{1}{p_2}-\frac{1}{q_2})-1} \|f\|_{p_1} \|g\|_{p_2}.$$

(ii) In the endpoint case $s = 2$ *and* $p_1 = p_2 = 1$ *we have ,*

$$\|\mathcal{R}_{\lambda_1,\lambda_2}(f,g)\|_2 \lesssim \lambda_1^{n-\frac{3}{2}} \lambda_2^{\frac{n-1}{2}} \|f\|_1 \|g\|_1,$$

assuming $\lambda_1 << \lambda_2$.

Proof : (i) comes from Hölder inequality and linear estimates
(ii) can be directly obtained using Plancherel's inequality and some geometric properties of the spheres.

Consequence for bilinear Bochner-Riesz multipliers

$$a_n := \frac{n+1}{2n}, \quad \text{and} \quad b_n := \frac{n+1}{2n} + \frac{n-1}{n^2+n}.$$

Theorem

For $1 \le p_1, p_2 < 2n/(n+1)$ and $1/p = 1/p_1 + 1/p_2$, define

$$\alpha(p_1,p_2) = \begin{cases} \frac{4}{n+1}, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in (a_n,b_n)\times(a_n,b_n) \\ \frac{2}{n+1} - \frac{n-1}{2n} + \frac{1}{p_2}, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in (a_n,b_n)\times[b_n,1] \\ \frac{2}{n+1} - \frac{n-1}{2n} + \frac{1}{p_1}, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in [b_n,1]\times(a_n,b_n) \\ \frac{1}{p_1} + \frac{1}{p_2} - \frac{n-1}{n}, & \text{if } \left(\frac{1}{p_1},\frac{1}{p_2}\right) \in [b_n,1]\times[b_n,1]. \end{cases}$$

If $\delta > n\alpha(p_1,p_2) - 1$, then the bilinear Bochner-Riesz means operator R_δ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Be careful : the operator R_δ depends on δ !

Idea : To interpolate, we get around this technical difficulty by using the spherical decomposition and use the "normalized elementary operators".

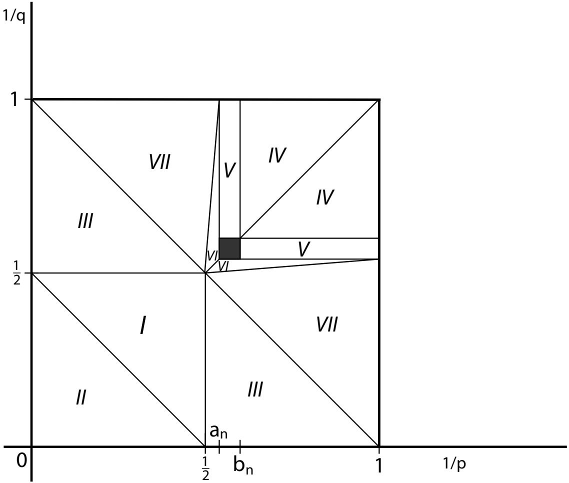

Figure : Exponents $(\frac{1}{p}, \frac{1}{q})$ for $p, q \geq 1$. Here $a_n = \frac{n+1}{2n}$, $b_n = \frac{n+1}{2n} + \frac{n-1}{n^2+n}$

Proposition (Local-L^2 case)

If $p, q, r' \in [2, \infty)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1$ (region I), then R_δ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ if $\delta > \frac{n-1}{r'}$.

Proposition (Banach case)

(A) *If $p, q \in [2, \infty)$ and $r' < 2$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1$ (region II), then R_δ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ if $\delta > \frac{n-1}{2} + n(\frac{1}{r'} - \frac{1}{2})$.*

(B) *If $q, r' \in [2, \infty)$ and $p < 2$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1$ (region III), then R_δ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ if $\delta > n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{r'}$.*

Remaining open questions :

- Improvements near the point $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$? Maybe using "restricted weak-type estimates and interpolation" ...
- Optimality of our results ?