BOUNDEDNESS OF BILINEAR MULTIPLIERS WHOSE SYMBOLS HAVE A NARROW SUPPORT

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Boundedness of bilinear multipliers in sub-Hölder scaling

Definition

On $\mathbb{R}$, a bilinear multiplier is a bilinear operator T which commutes with the simultaneous translations : for all $x \in \mathbb{R}$

$$T(\tau_x f, \tau_x g) = \tau_x T(f, g).$$

Equivalently, there exists a symbol $m \in \mathcal{S}'(\mathbb{R}^2)$ such that

$$T_m(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\xi+\eta)} \widehat{f}(\xi)\widehat{g}(\eta) m(\xi,\eta) d\xi d\eta.$$

Examples :

- If $m = 1$, $T_m(f,g) = fg$. So we expect boundedness under Hölder scaling for bounded symbols m.
- Coifman-Meyer multipliers (paraproducts ...) : $|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|-|\beta|}$.
- Bilinear Hilbert transform : $m(\xi,\eta) = \mathrm{sign}(\xi - \tan(\theta)\eta)$ ...

Consider a curve $\Gamma \subset \mathbb{R}^2$ and its ϵ-neighborhood $\Gamma_\epsilon := \{(\xi,\eta),\ d_\Gamma(\xi,\eta) \leq \epsilon\}$. Chose symbols m_ϵ supported on Γ_ϵ and "regular at the scale ϵ"

$$\left|\partial_\xi^\alpha \partial_\eta^\beta m_\epsilon(\xi,\eta)\right| \lesssim \epsilon^{-\alpha-\beta}.$$

Question : For which Lebesgue exponents $(p,q,r) \in [1,\infty]^3$ and for which functions $\alpha(\epsilon)$ does there hold

$$\|T_{m_\epsilon}\|_{L^p \times L^q \to L^{r'}} \lesssim \alpha(\epsilon) \ \ ? \tag{1}$$

Or, to put it in a more symmetrical fashion,

$$|\langle B_{m_\epsilon}(f,g)\,,\ h\rangle| \lesssim \alpha(\epsilon)\, \|f\|_{L^p} \|f\|_{L^q} \|f\|_{L^r} \ \ ? \tag{2}$$

BOUNDEDNESS OF BILINEAR MULTIPLIERS IN SUB-HÖLDER SCALING

Existing results

Under the Hölder scaling, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ with $\alpha(\epsilon)=1$:

- Γ is a point $\Longrightarrow$ Coifman-Meyer multipliers
- Γ is a line $\Longrightarrow$ Bilinear Hilbert transform (Lacey-Thiele, Muscalu, Tao, Gilbert, Nahmod ...)
- $\Gamma=\mathbb{S}^1$ (or ellipses) $\Longrightarrow$ Disc bilinear multiplier (in the local$-L^2$ case Grafakos and Li)

Assume that (1) holds with $\alpha(\epsilon) = \epsilon^\rho$.

- Singular symbols. If $\rho > 0$ then $T_{\mathbf{1}_K}$ (where $\partial K = \Gamma$) is bounded from $L^p \times L^q$ into $L^{r'}$.
- Bilinear Bochner-Riesz problem. T_m is bounded from $L^p \times L^q$ into $L^{r'}$ where $m(\xi,\eta) = d_\Gamma(\xi,\eta)^\lambda \mathbf{1}_K(\xi,\eta)$ for some $\lambda > \rho$.
- Bilinear restriction-extension inequalities. If $\rho = 1$ then $T_{d\sigma_\Gamma}$ is bounded.

Motivation : Space-time resonance method for dispersive PDEs and more generally bilinear oscillatory integral.

Technical problems

- The geometry of the curve should play a role (at least outside the local-L^2 case)
- Curvature in the frequency space loose integrability in physical space
  the linear ball multiplier result (Fefferman) - bilinear disc multiplier (Grafakos and Li)
- Degeneracy of the curve (when the tangential lines are degenerate).

Necessary conditions

To have

$$|\langle B_{m_\epsilon}(f,g)\,,\,h\rangle| \lesssim \alpha(\epsilon)\,\|f\|_{L^p}\,\|f\|_{L^q}\,\|f\|_{L^r}\,,$$

necessarily

- B_{m_ϵ} bilinear multiplier $\Longrightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r}\geq 1$
- Since $\epsilon^{-1}m_\epsilon$ weakly converges to $d\sigma_\Gamma$, $\epsilon \lesssim \alpha(\epsilon)$
- By scaling, $\epsilon^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \lesssim \alpha(\epsilon)$.

We do not describe all the different situations and just give several results.

Results

The general case : Γ is a continuous line (with assumptions involving uniform rectifiability or Ahlfors-regularity) eventually non bounded.

Theorem

For $p, q, r \geq 2$ then,

$$|\langle T_{m_\epsilon}(f,g), h\rangle| \lesssim \epsilon^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

The decay rate is also optimal.

Proof :

- Split the curve in pieces of diameter ϵ (doable under suitable assumptions on the curve)
- Estimate each pieces by the bilinear fractional integral
- Use Rubio de Francia results on square functions.

Weighted boundedness on bilinear fractional integral $\Longrightarrow$ Weighted boundedness on B_{m_ϵ}.

Theorem

If $p, q, r \leq 2$ then we can take $\alpha(\epsilon) = \epsilon^{\rho}$ with

$$\rho := \min\{p^{-1}, q^{-1}, r^{-1}\}.$$

Proof :

- For $p = q = r = 1$, $\rho = 1$ is easy and $p = q = r = 2$, $\rho = 1/2$ too.
- Study of each points $(2, 2, 1)$ and $(2, 1, 1) \to \rho = 1/2$.
- Interpolation.

If the curve Γ is nowhere characteristic :

THEOREM

If

$$\begin{cases} 1 \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 2 \\ \frac{1}{r} + \frac{1}{q} \leq \frac{3}{2} \\ \frac{1}{p} + \frac{1}{q} \leq \frac{3}{2} \\ \frac{1}{p} + \frac{1}{r} \leq \frac{3}{2}, \end{cases}$$

then

$$\|B_{m_\epsilon}\|_{L^p \times L^q \to L^{r'}} \lesssim \epsilon^{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-1},$$

and this exponent of ϵ is optimal.

Proof :

- Previous proof still hods with at most one exponent p, q, r lower than 2
- Interpolation between the four sub-squares.

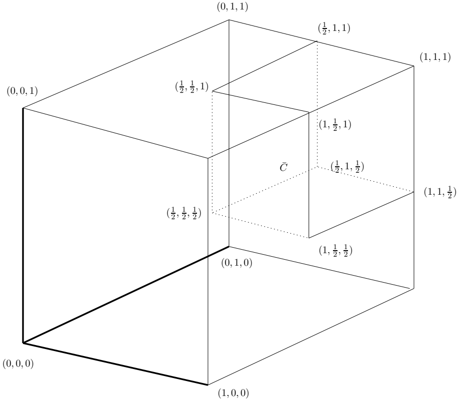

FIG. 1. exponents $(p^{-1},q^{-1},r^{-1}) \in [0,1]^3$

Study of particular points with eventual infinite exponents by a TT^* argument :

- $L^1 \times L^1 \longrightarrow L^2$
- $L^1 \times L^\infty \longrightarrow L^2$
- $L^2 \times L^2 \longrightarrow L^\infty$
- $L^1 \times L^1 \longrightarrow L^1$

General principle : curvature in frequency space is limiting the integrability in physical space.

- The disc isnot a linear multiplier in $L^p(\mathbb{R}^n)$ for $p \neq 2$ as soon as $n \geq 2$
  Fefferman (1971).
- The disc is a bilinear multiplier is the local-L^2 case and is not a bilinear multiplier outside
  Grafakos-Li (2006), Diestel-Grafakos (2007), extension to convex sets.

$\Longrightarrow$ Bocher-Riesz problem : add regularity at the symbol near the boundary to compensate the curvature.

Assume that Γ has a non-vanishing curvature.

THEOREM

- *If* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{5}{2}$ *then*

$$|\langle T_{m_\epsilon}(f,g), h\rangle| \lesssim \epsilon \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}.$$

Proof : Use the non-stationary phase lemma to get decay of the bilinear kernel.

The study becomes more difficult when we tends to the (limit) Hölder scaling $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \to 1$.

THEOREM

- $\alpha(\epsilon) = 1$ *for* $L^2 \times L^2$ *into* L^1
- *"*$\alpha(\epsilon) \to 1$*" for* $L^p \times L^q$ *into* L^∞ *with* $(p, q) \to (\infty, \infty)$*.*

Proof : Use a decomposition of Γ by chords of length $\sqrt{\epsilon}$ and of width ϵ. Use orthogonality between the different pieces.

Closely related to the boundedness of Kakeya maximal operator.

Bilinear Fourier restriction-extension inequalities

We are now interested to symbols carried by the line. Limit point of view when $\epsilon \to 0$. Assume that Γ has still non-vanishing curvature.

Theorem

If

- $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{5}{2}$ *or*
- $\begin{cases} \frac{1}{p}-\frac{3}{q}+\frac{1}{r} \geq -1 \\ \frac{-3}{p}+\frac{1}{q}+\frac{1}{r} \geq -1 \\ \frac{1}{p}+\frac{1}{q}-\frac{3}{r} \geq -1 \\ \frac{1}{p}+\frac{1}{q}+\frac{1}{r} > \frac{7}{3}. \end{cases}$

then $T_{md\sigma_\Gamma}$ is bounded from $L^p \times L^q$ into $L^{r'}$.

Proof : First point as previously.
Second point comes from interpolation between the first point and local-L^2 time frequency analysis.

Question : is there equivalence between bilinear Fourier restriction-extension equality and boundedness with a function decay $\alpha(\epsilon) = \epsilon$?

- Yes, if we consider specific symbols m_ϵ obtained by convolution with $d\sigma_\Gamma$.
- Unknown in general.

To be pursued ....

- Come back to the applications.
- In one-dimensional setting, obtain weighted estimates (with Muckenhoupt-$\mathbb{A}_p$ weights and polynomial weights, useful for applications in dispersive PDEs).
- Generalize in multilinear setting, more difficult geometry since Γ is higher dimensional.
- Muldi-dimensional setting, more difficult due to the projection of Γ on the degenerate planes ....