A $T(1)$-THEOREM RELATED TO A SEMIGROUP

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September 21, 2010

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The classical $T(1)$-theorem

Theorem (David- Journé (1984))

Let T be a linear operator on $\mathbb{R}^n$ such that its kernel $K(x,y)$ satisfies :

$$|K(x,y)| \lesssim |x-y|^{-n}$$

and for some $\epsilon \in (0,1]$ and z with $|z| \leq |x-y|/2$

$$|K(x,y)-K(x,y+z)|+|K(x,y)-K(x+z,y)| \lesssim \frac{|z|^{\epsilon}}{|x-y|^{n+\epsilon}}.$$

Then T is L^2-bounded if and only if $T(\mathbf{1})$, $T^*(\mathbf{1})$ belong to $BMO(\mathbb{R}^n)$ and T satisfies a weak boundedness property.

It requires to give a sense to $T(\mathbf{1})$, $T^*(\mathbf{1})$ (using truncated kernels).
Using paraproducts, we can assume more cancellation : $T(\mathbf{1}) = T^*(\mathbf{1}) = 0$.

Using Calderón-Zygmund theory, such operators are L^p-bounded for all $p \in (1, \infty)$.

Applications :

- Pseudo-differential operators in exotic class $S^0_{1,1}$
- Many extensions, for a non-doubling measure on $\mathbb{R}^n$ (Nazarov, Treil, Volberg (2002) and Tolsa (2001)), with $T(b)$ theorem (where b is an accretive function) and its local variants.
- Extension to ℓ^2-square functions and applications in the Kato conjecture (Auscher, M^{c}Intosh, Hofmann, Lacey and Tchamitchian (2002)) and Cauchy integral ...

Problem : Many operators do not satisfy assumptions on the kernel (for example, Riesz transforms associated to a second order divergence form operator). So we are looking for similar results with other assumptions.

Idea : For the Laplacian, replace the oscillation

$$\fint_Q \left| f - \fint_Q f \right|$$

by

$$\fint_Q \left| f - e^{-r_Q^2 \Delta} f \right|$$

or

$$\fint_Q \left| r_Q^2 \Delta e^{-r_Q^2 \Delta} f \right|.$$

Sometimes, we need more cancellation and so consider $(1 - e^{-r_Q^2\Delta})^M f$ or $(r_Q^2 \Delta)^M e^{-r_Q^2 \Delta} f$.

The framework about Semigroup

Let (M, d, μ) be a doubling Riemannian manifold. We consider a semigroup $(e^{-tL})_{t>0}$ such that L generates a holomorphic semigroup $(e^{-zL})_{z\in S_{\pi/2-\omega}}$.

Assume the following conditions : there exist a positive real $m > 0$ and $\delta > 1$ such that

- For every $z \in S_{\pi/2-\omega}$, the linear operator e^{-zL} is given by a kernel a_z satisfying

$$|a_z(x,y)| \lesssim \frac{1}{\mu(B(x,|z|^{1/m}))}\left(1+\frac{d(x,y)}{|z|^{1/m}}\right)^{-d-2N-\delta}$$

  where d and N are parameter given by the dimension of the space ;

- The operator L has a bounded H_∞-calculus on L^2. That is, there exists c_ν such that for $b \in H_\infty(S_\nu^0)$, we can define $b(L)$ as a L^2-bounded linear operator and

$$\|b(L)\|_{L^2 \to L^2} \leq c_\nu \|b\|_\infty.$$

- We assume that for every integer $k \geq 0$ the square functional

$$f \to \left(\int_0^\infty \left| t^{1/m} \nabla (tL)^k e^{-tL}(f) \right|^2 \frac{dt}{t}\right)^{1/2}$$

is bounded on L^2.

The last condition is satisfied as soon as Riesz transform is bounded in L^2.

Main constraint : the pointwise estimates of the kernel.

Examples :

- $L = -\Delta$ on the Riemannian manifold
- $L = -div(A\nabla\cdot)$ on $\mathbb{R}^n$ with a real-valued matrix A or a complex-valued matrix A for $n \in \{1, 2\}$.

The assumed bounded H_∞-calculus on L^2 implies that for any holomorphic function ψ such that for some $s > 0$, $|\psi(z)| \lesssim \frac{|z|^s}{1+|z|^{2s}}$, the quadratic functional

$$f \to \left(\int_0^\infty |\psi(tL)f|^2 \, \frac{dt}{t} \right)^{1/2}$$

is L^2-bounded.

Then it remains us to define an associated BMO_L space

New BMO spaces

Definition (Duong and Yan (2005))

A function $f \in \mathcal{S}$ belongs to BMO_L if and only if

$$\|f\|_{BMO_L} := \sup_{t>0} \sup_{\substack{B \text{ ball} \\ r_B^m = t}} \frac{1}{\mu(B)} \int_B \left| f - e^{-tL} f \right| d\mu < \infty.$$

Duong and Yan studied these spaces by proving some John-Nirenberg inequalities and have obtained duality results with some Hardy spaces.

As in the classical case, we can build Carleson measures from BMO_L functions.

Proposition (Duong and Yan)

For any $f \in BMO_L$, the measure

$$d\nu(x,t) := \left| t^m L e^{-t^m L} \left(1 - e^{-t^m L}\right) f \right|^2 \frac{d\mu(x)dt}{t}$$

is a Carleson measure. Moreover for all integer $k \geq 1$ the measure

$$d\nu_k(x,t) := \left| (t^m L)^k e^{-t^m L} \left(1 - e^{-t^m L}\right) f \right|^2 \frac{d\mu(x)dt}{t}$$

is a Carleson measure too.

A $T(1)$-theorem related to a semigroup

RESULTS

We have first to define what is "standard kernels" for such semigroup.
Let T be a continuous operator acting from $\mathcal{S}(M)$ into $\mathcal{S}'(M)$. We assume that T and T^* satisfies some $L^2 - L^2$ off diagonal decay as follows : there exists an integer $\kappa \geq 1$ such that for every $s > 0$, every ball Q_1, Q_2 of radius $r := s^{1/m}$ and function $f \in L^2(Q_1)$

- if $d(Q_1, Q_2) \geq 2r$, then we have

$$\left\| (sL)^\kappa e^{-sL} T \right\|_{L^2(Q_1) \to L^2(Q_2)} \lesssim \left(1 + \frac{d(Q_1, Q_2)}{r} \right)^{-d-2N-\delta};$$

  and the dual estimates (replacing L, T by L^*, T^*). This generalizes the regularity assumptions on the kernel in the classical $T(1)$-Theorem.

- if $d(Q_1, Q_2) \leq 2r$, then

$$\left\| (sL)^{\kappa} e^{-sL} T(e^{-sL} \cdot) \right\|_{L^2(Q_1) \to L^2(Q_2)} + \left\| (sL^*)^{\kappa} e^{-sL^*} T(e^{-sL^*} \cdot) \right\|_{L^2(Q_1) \to L^2(Q_2)} \lesssim 1.$$

This last property is the adapted version of the “weak boundedness property” required in the classical $T(1)$-Theorem.

We focus in the case $\kappa = 1$, else we have to consider another BMO_L space, taking into account this extra cancellation.

Theorem

Assume that the Riemannian manifold M satisfies Poincaré inequality (P_2) and has a infinite measure $\mu(M) = \infty$. Suppose the existence of an operator L such that $L(\mathbf{1}) = 0 = L^*(\mathbf{1})$ and such that the corresponding semigroup verifies the above assumptions. Suppose that T is weakly bounded in L^2, then T admits a continuous extension on L^2 if and only if $T(\mathbf{1})$ belong to BMO_L and $T^*(\mathbf{1})$ to BMO_{L^*}.

Moreover $\|T\|_{L^2 \to L^2}$ is only controlled by the implicit constants and $\|T(\mathbf{1})\|_{BMO_L} + \|T^*(\mathbf{1})\|_{BMO_{L^*}}$ (and not by the weak continuity from L^2 to L^2).

Then by using the theory of Hardy spaces (defined by atomic decomposition relatively to the semigroup) and interpolation, we can make varying the exponent :

Theorem

Under the previous assumptions, the operator T is L^p-bounded for all $p \in (1, \infty)$, bounded from L^∞ to BMO_L and from H^1_L to L^1.
Moreover, it is of weak-type $(1,1)$.

Interpolation is based on *Good λ-inequalities* (comparison on the level-sets for the two maximal functions $\mathcal{M}$ and $\mathcal{M}^\sharp$) so we can have weighted estimates with Muckenhoupt weights.

A $T(1)$-theorem related to a semigroup

New paraproducts

We consider

$$\psi_t(L) := (tL)^N e^{-tL}(1-e^{-tL}) \quad \text{and} \quad \phi_t(L) := e^{-tL},$$

with a large enough integer $N > d/m$.
We are interested in "paraproducts" involving these two scales of operators :

$$\pi_b := f \to \int_0^\infty \zeta_t^1(L)(\zeta_t^2(L) f \zeta_t^3(L) b) \frac{dt}{t}$$

where $b \in L^\infty$ and for $i = 1, 2, 3$, ζ_t^i is equal to ψ_t or ϕ_t. By pairing with a third function g,

$$\Lambda(f, b, g) := \langle \pi_b(f), g \rangle = \int_0^\infty \int_M \zeta_t^2(L) f \zeta_t^3(L) b \zeta_t^1(L^*) g d\mu \frac{dt}{t}.$$

The easy case

Proposition

If there are two ψ-functions in $\{\zeta^1, \zeta^2, \zeta^3\}$ and $b \in L^\infty$ then, π_b is bounded in L^p for every $p \in (1, \infty)$.

By duality, we easily have the L^2-boundedness and then by using Hardy space and interpolation we obtain other estimates :

$$\Lambda : L^p \times L^q \times L^{r'} \to \mathbb{R}$$

as soon as $1 < p, q, r \leq \infty$ and $0 < r^{-1} = p^{-1} + q^{-1}$.

The more difficult case

Theorem

If there is one ψ-functions in $\{\zeta^1, \zeta^2, \zeta^3\}$ and $b \in L^\infty$ then, π_b is bounded in L^p for every $p \in (1, \infty)$.

It can be checked that the operator π_b satisfies the properties of our $T(1)$-theorem. Applying it, we get the L^2-boundedness and then the L^p-boundedness.

By trilinear interpolation :

THEOREM

The trilinear form

$$(f,g,h) \to \int_0^\infty \int_M \left[\zeta_t^1(L^*)g\right] \left[\zeta_t^2(L)f\,\zeta_t^3(L)h\right] \frac{dt}{t} d\mu,$$

is bounded on $L^p \times L^q \times L^{r'}$ *for every exponents* $p,q,r \in (1,\infty)$ *satisfying*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = 1.$$

- Moreover we can obtain weighted estimates :

$$\pi_b : L^p(\omega) \to L^r(\omega)$$

  if $b \in L^q(\omega)$ for every weight $\omega \in \mathbb{A}_p \cap \mathbb{A}_q$.
- Consequently, by extrapolation theory for bilinear operators (Martell and Grafakos), the exponent r can be allowed in $(1/2, \infty)$ with $p, q \in (1, \infty)$.
- End-point : with some adapted Hardy space H^1 (relatively to the semigroup), we can prove an estimate from $H^1 \times H^1$ into $L^{1/2}$.

"CLASSICAL" $T(1)$-THEOREM IN A RIEMANNIAN MANIFOLD

We consider the heat semigroup $e^{-t\Delta}$ given by the positive Laplacian. We know that if the heat kernel p_t satisfies the diagonal estimates :

$$\sup_{x\in M} |p_t(x,x)| \leq t^{-d/2}$$

then it self-improves in gaussian upper bounds. Then all the assumptions on the semigroup are satisfied.

Theorem

Let T be a linear operator on M such that its kernel $K(x,y)$ satisfies :

$$|K(x,y)| \lesssim d(x,y)^{-d}$$

and for some $\epsilon \in (0,1]$ and z with $d(y+z,y) \leq d(x,y)/2$

$$|\nabla K(x,y) - \nabla K(x,y+z)| \lesssim \frac{d(y+z,y)^{\epsilon}}{d(x,y)^{d+3N+\epsilon}}$$

and with $d(x+z,x) \leq d(x,y)/2$

$$|\nabla K(x,y) - \nabla K(x+z,y)| \lesssim \frac{d(x+z,x)^{\epsilon}}{d(x,y)^{d+3N+\epsilon}}.$$

Then T is L^2-bounded if and only if $T(\mathbf{1})$, $T^*(\mathbf{1})$ belong to BMO_Δ.

- We require regularity of order $1+\epsilon$ (and not only ϵ as in the classical $T(\mathbf{1})$-theorem).
- $BMO_\Delta \subset BMO$ in general and $BMO_\Delta = BMO$ if we assume extra condition (like pointwise estimates) on the gradient of the heat kernel.
- This result does not use and does not require the differential structure of $\mathbb{R}^n$ and holds in an abstract Riemannian manifold.

SKETCH OF THE PROOF

A) We decompose the operator T at each scale s, with help of the semigroup by composing with $(sL)^\kappa e^{-sL}$. We get

$$T(f) = \int_0^\infty T_s(e^{-sL} f) \frac{ds}{s}.$$

B) At each scale, T_s essentially acts at the scale s and so f can be replaced by a constant at this scale. So we compute

$$T_s(f)(x) = T_s(\mathbf{1})(x) e^{-sL} f(x) + T_s(e^{-sL} f - e^{-sL} f(x))(x).$$

C) This operation makes appear a main term (involving $T(\mathbf{1})$) and another one which can be though as an "error" term.

D) The main term is estimated in L^2 via the Carleson measure theory.

E) The “error” term is bounded because of the “regularity assumption” of the semigroup, which allows us to legitimate the operation b). This more technical study relies on Poincaré inequality and a Sobolev inequality :

Proposition (Sobolev inequality)

For each ball Q of radius r, then

$$\|f\|_{\infty,Q} \lesssim \sum_{i\geq 0} 2^{-iN} \left(\fint_{2^iQ} \left|(1+r^mL)^M f\right|^2 d\mu \right)^{1/2}.$$

Study of the term $T_s(e^{-sL}f - e^{-sL}f(x))(x)$:

- we localize the estimates on a ball of radius $s^{1/m}$
- in this ball, we make the two variables x independent between them via the Sobolev inequality.
- So it remains to study quantities like

$$T_s(e^{-sL}f - e^{-sL}f(y))(x)$$

- Using off-diagonal decay on T (and also on T_s), Poincaré inequality (P_2), that is controlled by the square function involving the gradient of the semigroup.

Some perspectives

Open question : The pointwise estimates on the heat kernel are necessary for step *d*) and the Sobolev inequality. Can we get around and just require some L^2 off-diagonal decay ?

If $L := -\Delta + V$ with a potential. Then under assumptions on V, the semigroup fits into this framework. But $L(\mathbf{1}) \neq 0$!

Open question : Can we consider another function b such that $L(b) = 0$ and obtain a "$T(b)$-Theorem" ?

Open question : Can we do the same with the resolvent operators instead of the semigroup ?