On a bilinear pseudodifferential calculus on $\mathbb{R}$.

Frédéric BERNICOT

Université Paris-Sud

El Escorial 2008

June, 17

The aim :

To define classes of bilinear symbols such that :

- we understand the action of the associated operators on the Sobolev spaces
- we have symbolic rules for the adjoint operation and for the composition with linear pseudodifferential operators.

We look for the largest classes of operators containing the Coifman-Meyer operators, Bilinear Hilbert transforms, Operators satisfying bilinear Marcinkiewicz condition ...

Definition

To a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^3)$, we formally associate the following bilinear operator :

$$T_\sigma(f,g)(x) := \int_{\mathbb{R}^2} e^{ix(\alpha+\beta)}\widehat{f}(\alpha)\widehat{g}(\beta)\sigma(x,\alpha,\beta)d\alpha d\beta.$$

1. Definition of classes
2. Study of classes $BS^{m_1,m_2}_{1,0;\theta}$.
3. The study of the class $BS^{0,0}_{1,1,\theta}$.

   Work joined with R. Torres.

Outline

1. Definition of classes
2. Study of classes $BS^{m_1,m_2}_{1,0;\theta}$.
3. The study of the class $BS^{0,0}_{1,1,\theta}$.

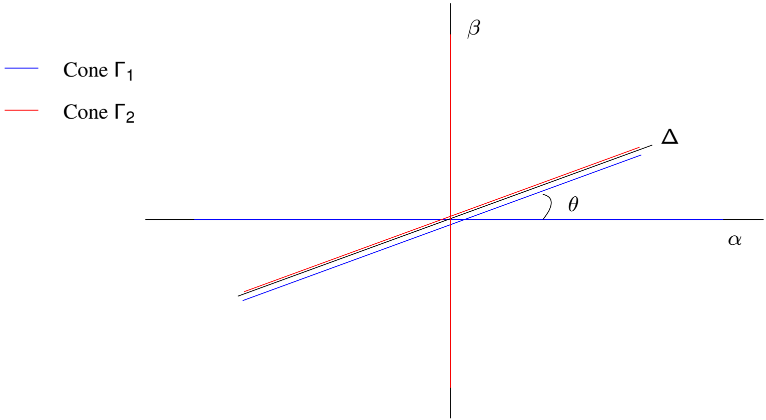
   Work joined with R. Torres.

DEFINITIONS

DEFINITION

For m_1, m_2 two reals, we define the class of bilinear pseudodifferential symbols of order (m_1, m_2). Let θ be an angle and σ be a function in $C^\infty(\mathbb{R}^3)$.
We set that σ belongs to the class $BS^{m_1,m_2}_{\rho,\delta;\theta}$ if and only if for all $a, b, c \geq 0$

$$\begin{aligned}\left|\partial_x^a \partial_\alpha^b \partial_\beta^c \sigma(x,\alpha,\beta)\right| \lesssim & (1+|\alpha|)^{m_1} (1+|\beta|)^{m_2} \\ & (1+\min\{|\alpha|, |\beta-\tan(\theta)\alpha|\})^{-\rho b} \\ & (1+\min\{|\beta|, |\beta-\tan(\theta)\alpha|\})^{-\rho c} \\ & (1+|\beta-\tan(\theta)\alpha|)^{\delta a}.\end{aligned}$$



Differentiation with respect to :

- α then we gain $d((\alpha,\beta),\Gamma_2)$
- β then we gain $d((\alpha,\beta),\Gamma_1)$
- x then we loose $d((\alpha,\beta),\Delta)$.

The dual classes

We exchange the role of the tree variables α, β and $\alpha+\beta$

Definition

We set that σ belongs to the class $BS^{m_1,m_2,1}_{\rho,\delta;\theta}$ if and only if for all $a, b, c \geq 0$

$$\begin{aligned}\left|\partial_x^a \partial_\alpha^b \partial_{\beta-\alpha}^c \sigma(x,\alpha,\beta)\right| \lesssim & (1+|\alpha+\beta|)^{m_1} (1+|\beta|)^{m_2} \\ & (1+\min\{|\alpha+\beta|, |\beta-\tan(\theta)\alpha|\})^{-\rho b} \\ & (1+\min\{|\beta|, |\beta-\tan(\theta)\alpha|\})^{-\rho c} \\ & (1+|\beta-\tan(\theta)\alpha|)^{\delta a}.\end{aligned}$$

We introduce the last class $BS^{m_1,m_2,2}_{\rho,\delta;\theta}$ by exchanging the role of α and β.

Outline

Work joined with R. Torres.

Action on Sobolev spaces

We write $(m)_+ := \frac{m+|m|}{2}$.

Theorem

Let σ be a symbol of order (m_1, m_2) (in one of the three classes $BS^{m_1,m_2}_{1,0;\theta}$, $BS^{m_1,m_2,1}_{1,0;\theta}$ or $BS^{m_1,m_2,2}_{1,0;\theta}$ for $\theta \in]-\pi/2, \pi/2[\setminus\{0, -\pi/4\}$). Let p, q, r be exponents satisfying

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2} \quad \textit{and} \quad 1 < p, q \leq \infty.$$

Then for all real $s \geq 0$ and $\epsilon > 0$, the operator T_σ is continuous from $W^{s+\epsilon+(m_1)_+,p}(\mathbb{R}) \times W^{s+\epsilon+(m_2)_+,q}(\mathbb{R})$ into $W^{s,r}(\mathbb{R})$.

- The $\epsilon > 0$ is necessary to the possibility of bilinear Marcinkiewicz condition (L. Grafakos- N. Kalton, 2001)

Sketch of the proof

- The "global continuities" for the operators associated to such x-indepedent symbols are well-known. (L. Grafakos-N. Kalton (2001) for the bilinear Marcinkiewicz condition, M. Lacey, C. Thiele (1990-2000) for the bilinear Hilbert transform, C. Muscalu, T. Tao, J. Gilbert, A. Nahmod (2000-2006)... for related operators).
- With adding an extra localization argument inside their proofs, we obtain local estimates for such bilinear multiplier.
- With a Sobolev embedding, we are able to assum a smooth spatial dependence for the symbol.

SYMBOLIC RULES.

For T a bilinear operator, we write T^{*1} and T^{*2} for its two adjoints (with respect to the α and to the β variable).

THEOREM (DUALITY RULES)

Let $m_1, m_2 \in \mathbb{R}$ be reals and $\theta \in]-\pi/2, \pi/2[-\{0, -\pi/4\}$ be an angle. If $\sigma \in BS^{m_1,m_2}_{1,0;\theta}$ then $T^{*1}_\sigma = T_{\sigma_1^*}$ with $\sigma_1^* \in BS^{(m_1,m_2),1}_{1,0;\theta^{*1}}$ and $T^{*2}_\sigma = T_{\sigma_2^*}$ with $\sigma_2^* \in BS^{(m_1,m_2),2}_{1,0;\theta^{*2}}$, where the two angles θ^{*1} and θ^{*2} are defined by

$$\cot(\theta) + \cot(\theta^{*1}) = -1 \qquad \textit{and} \qquad \tan(\theta) + \tan(\theta^{*2}) = -1,$$

and so $\theta^{*1}, \theta^{*2} \in]-\pi/2, \pi/2[-\{0, -\pi/4\}$. In addition these two symbols σ_1^* and σ_1^* satisfy an asymptotic formula. We obtain similar results for the two classes $BS^{m_1,m_2,1}_{1,0;\theta}$ and $BS^{m_1,m_2,2}_{1,0;\theta}$.

Theorem (Composition rules)

Let $\theta \in]-\pi/2, \pi/2[$ and $\sigma \in BS^{m_1,m_2}_{1,0;\theta}$ be fixed. Let $\tau_1 \in S^{t_1}_{1,0}$ and $\tau_2 \in S^{t_2}_{1,0}$ be two linear symbols of order t_1 and t_2. Then the operator

$$T(f,g) := T_\sigma\left(\tau_1(x,D)f, \tau_2(x,D)g\right)$$

corresponds to the operator T_m with the symbol $m \in BS^{m_1+t_1,m_2+t_2}_{1,0;\theta}$. This new symbol satisfies an asymptotic formula :

$$m(x,\alpha,\beta) - \sum_{k=0}^{N}\sum_{j=0}^{P} \frac{i^k}{k!}\frac{i^j}{j!}\partial_x^k\tau_1(x,\alpha)\partial_x^j\tau_2(x,\beta)\partial_\alpha^k\partial_\beta^j\sigma(x,\alpha,\beta)$$

is belonging to the class $BS^{m_1+t_1-N,m_2+t_2-P}_{1,0;\theta}$.

$$m(x,\alpha,\beta) \simeq \tau_1(x,\alpha)\tau_2(x,\beta)\sigma(x,\alpha,\beta).$$

Similar results for the dual classes.

Outline

1. Definition of classes
2. Study of classes $BS^{m_1,m_2}_{1,0;\theta}$.
3. The study of the class $BS^{0,0}_{1,1,\theta}$.
   Work joined with R. Torres.

Action on Sobolev spaces

Theorem

Let $\theta \in]-\pi/2, \pi/2[\backslash\{0, -\pi/4\}$ be an angle. If $\sigma \in BS^{0,0}_{1,1;\theta}$, then the bilinear operator T_σ is bounded on $W^{s,p}(\mathbb{R}) \times W^{s,q}(\mathbb{R})$ into $W^{s,t}(\mathbb{R})$ for all exponents $1 < p, q, t < \infty$ satisfying the homogeneous condition

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{q}$$

and $s > 1/2$.

- The result for $s = 0$ is false due to a counter-example.
- The result for all $s > 0$ is true if the symbol only depends on x and on the "singular variable" $\beta - \tan(\theta)\alpha$.

QUESTION

Similarly to results about the linear class $S^0_{1,1}$:

QUESTION

Let $\theta \in]-\pi/2, \pi/2[\setminus\{0, -\pi/4\}$ be an angle. Assume that $\sigma \in BS^{0,0}_{1,1;\theta}$ and if the adjoints σ^{*1} and σ^{*2} belong to the classes $BS^{0,0,i}_{1,1;\theta^{*i}}$ associated to the dual angle. Then does T_σ admit continuous extension on the Lebesgue spaces ?

We have a positive answer if σ only depends on x and on the "singular variable" $\beta - \tan(\theta)\alpha$ (due to the bilinear $T(1)$-Theorem of A. Benyi, C. Demeter, A. Nahmod, C. Thiele, R. Thorres and P. Villarroya - 2008).