On bilinear Littlewood-Paley square functions on $\mathbb{R}$.

Frédéric BERNICOT

Université Paris-Sud
Colloque Orsay 2008

June, 11 2008

The aim :

To get continuities in Lebesgue spaces for bilinear Littlewood-Paley square functions defined with smooth and nonsmooth cutoffs.

1. THE LINEAR RESULTS.

2. THE BILINEAR RESULTS.

DEFINITION

Let I be an interval in $\mathbb{R}$. A smooth function $\phi \in \mathcal{S}(\mathbb{R})$ is said to be frequencialy adapted to I if

- $\forall \xi \in I, \qquad \widehat{\phi}(\xi) = 1$
- $\forall \xi \in (2I)^c, \qquad \widehat{\phi}(\xi) = 0$
- $\forall n \geq 0, \left\| \widehat{\phi}^{(n)} \right\|_\infty \leq |I|^{-n}.$

The nonsmooth cutoff associated to the interval I is the function ϕ_I defined by

$$\widehat{\phi_I} = \mathbf{1}_I.$$

We set π_I, the Fourier multiplier defined by the convolution with ϕ_I.

Outline

1. The linear results.
2. The bilinear results.

The Littlewood-Paley theory

Theorem (J. Littlewood- R. Paley (1935))

For all index $n \in \mathbb{Z}$, let $\psi_n \in \mathcal{S}(\mathbb{R})$ be a smooth function adapted to the interval $[2^n, 2^{n+1}]$. Then for all $p \in (1, \infty)$

$$\left\| \left(\sum_{n \in \mathbb{Z}} |\psi_n * f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

- This square function can be considered as a vector-valued Calderón-Zygmund operator.
- Two extensions : to replace the smooth cutoffs by nonsmooth ones and deal with arbitrary intervals. The main difficulty is the second one.

Extensions :

Extensions for non smooth cutoffs : $\pi_I(f) := \phi_I * f$.

Theorem (L. Carleson (1967))

For all $p \in [2, \infty)$, we have

$$\left\| \left(\sum_{n \in \mathbb{Z}} \left| \pi_{[n,n+1]} * f \right|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}.$$

The restriction $p \geq 2$ is necessary.

Theorem (J.L. Rubio de Francia (1985))

For all $p \in [2, \infty)$, for all collection $\mathcal{I} = (I)_I$ of intervals satisfying $\sum_{I \in \mathcal{I}} \mathbf{1}_I \lesssim 1$, we have

$$\left\| \left(\sum_{I \in \mathcal{I}} |\pi_I(f)|^2 \right)^{1/2} \right\|_p \leq c \|f\|_p.$$

Higher dimensional analogue of this estimate have been obtained by J.L Journé (1985).

OUTLINE

Question : How to define analogues of the linear fourier restriction operator in the bilinear theory.

$\Longrightarrow$ Two methods to define a restiction :

1. the first one

$$\int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\mathbf{1}_I(\xi_1)\mathbf{1}_J(\xi_2)d\xi = \pi_I(f)(x)\pi_J(g)(x)$$

2. or (according to the singularity of the bilinear Hilbert transforms)

$$\int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\mathbf{1}_I(\xi_2-\xi_1)d\xi = \int_{\mathbb{R}} f(x-t)g(x+t)\phi_I(t)dt.$$

THEOREM (G. DIESTEL, (2004))

Let $a, b \in (0,1)$ be two different scales, then for $p, q, r \in (1, \infty)$ satisfying

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

the following square function

$$\left(\sum_{n \in \mathbb{Z}} \left[\int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)} \widehat{f}(\xi_1) \widehat{g}(\xi_2) \mathbf{1}_{[a^n, a^{n-1}]}(\xi_1) \mathbf{1}_{[-b^n, b^n]}(\xi_2) d\xi \right]^2 \right)^{1/2}$$

is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$.

It is a bilinear version of the nonsmooth Littlewood-Paley results.

- We have bilinearized the important role of the point 0 in the frequency plane both in the two variables ξ_1 and ξ_2. These square functions are associated to multiscale paraproducts. The difficulty is due to the fact that the collection $([-b^n, b^n])_n$ is not a bounded covering.
- No restriction for the exponents $p, q \in (1, \infty)$.
- It could be extended for multi-dimensional variables.

New smooth bilinear multipliers

$$\int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\phi_I(\xi_2-\xi_1)d\xi.$$

Theorem (M. Lacey (1996))

Let $p, q \in [2,\infty)$ be exponents satisfying

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{q},$$

and ψ_n be smooth functions frequentialy adapted to the interval $[n, n+1]$. The bilinear square function

$$\left(\sum_{n\in\mathbb{Z}} \left| \int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\psi_{[n,n+1]}(\xi_2-\xi_1)d\xi \right|^2\right)^{1/2}$$

is continuous from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^2(\mathbb{R})$.

The proof is very specific to the exponent 2.

We want to study the following bilinear multiplier :

$$\begin{aligned}\pi_I(f,g)(x) &:= \int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\mathbf{1}_I(\xi_2-\xi_1)d\xi \\ &= \int_{\mathbb{R}} f(x-t)g(x+t)\phi_I(t)dt.\end{aligned}$$

The bilinear Hilbert transform is defined by :

$$\begin{aligned}H(f,g)(x) &:= \frac{1}{\pi}\int_{\mathbb{R}} f(x-t)g(x+t)\frac{dt}{t} \\ &= \pi_{[0,\infty)}(f,g)(x) - \pi_{(-\infty,0]}(f,g)(x).\end{aligned}$$

Using modulations, we have “reverse equalities” : for an interval $I = [a, b]$

$$\pi_{[a,b]}(f, g) = M^a \pi_{[0,\infty)}(f, M^a g) - M^b \pi_{[0,\infty)}(f, M^b g)$$

and

$$\pi_{[0,\infty)}(f, g) = \frac{1}{2} \left[fg + H(f, g) \right].$$

Theorem (M. Lacey and C. Thiele ($\backsim$ 2000))

Let p, q, r be exponents satisfying $1 < p, q \leq \infty$ and

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2}.$$

Then

$$\|H(f,g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

Corollary

For the same exponents, there is a constant c such that for all interval I

$$\|\pi_I(f,g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

Due to a vector valued result for bilinear operators (L.Grafakos and J.M. Martell (2004)), we can deduce

Corollary

For the same exponents, there is a constant c such that for all collection $\mathcal{I} = (I_n)_n$ we have

$$\left\| \left(\sum_n |\pi_{I_n}(f_n, g_n)|^2 \right)^{1/2} \right\|_{L^r} \lesssim \left\| \left(\sum_n |f_n|^2 \right)^{1/2} \right\|_{L^p} \left\| \left(\sum_n |g_n|^2 \right)^{1/2} \right\|_{L^q}.$$

Now we look for square estimates with only one function f or g.

THEOREM (B. (2008))

Let p, q, r be exponents satisfying

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad 2 < p, q, r' < \infty.$$

Then we have

$$\left\| \left(\sum_n \left| \pi_{[n,n+1]}(f,g) \right|^2 \right)^{1/2} \right\|_{L^r} \leq c \left\| f \right\|_{L^p} \left\| g \right\|_{L^q}.$$

The restriction $2 \leq p, q$ is necessary

We have also generalized the result of M. Lacey :

- Boundedness for nonsmooth square function $\Rightarrow$ Boundedness for smooth square function.
- New range for exponents.
- Open question : is the restriction $1 < r < 2$ is necessary ? Probably not ...

- Similar results are expected for an arbitrary collection of intervals satisfying the property of a bounded covering.
- We can use square functions associated to other geometric figures (paralelograms ...).

THEOREM (C.MUSCALU, T.TAO AND C. THIELE, 2002)

Let m be a function on $\mathbb{R}$ satisfying : for all $i \geq 0$

$$\left| m^{(i)}(\xi) \right| \lesssim |\xi|^{-i}$$

and p, q, r be exponents such that $1 < p, q \leq \infty$ and

$$0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \frac{3}{2}.$$

Then the bilinear multiplier

$$T_m(f, g)(x) := \int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)} \widehat{f}(\xi_1) \widehat{g}(\xi_2) m(\xi_2 - \xi_1) d\xi$$

is bounded from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$.

Question : Can we use the results on nonsmooth square functions to obtain continuities for a multiplier T_m under less restrictive regularity-assumptions on m ?