L^p estimates for multilinear operators, generalizing multilinear paraproducts.

FRÉDÉRIC BERNICOT[1]

[1]**Laboratoire de Mathématiques**, Université Paris XI, Orsay

Introduction

We produce here a study report about some multilinear operators continuity in Lebesgue spaces and in Hardy spaces. These operators generalize the concept of multilinear paraproducts. They naturally appear in the decomposition of some multilinear pseudo-differential operators.

Definition of operators

Let's start with some notations : $\mathbf{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$. Let ζ be a regular function on $\mathbb{R}^d$. For all real $t \neq 0$ and vector $q \in \mathbb{R}^d$, we write ζ_t and $\zeta_{t,q}$ the $L^1(\mathbb{R}^d)$-normalized functions defined by :

$$\zeta_t(x) := \frac{1}{|t|^d}\zeta(t^{-1}x), \qquad \zeta_{t,q}(x) := \frac{1}{|t|^d}\zeta(t^{-1}(x-q)).$$

From now on, the notation Ψ is assigned to regular functions whose spectrum is a subset of the corona $\left\{\xi \in \mathbb{R}^d, \quad |\xi| \simeq 1\right\}$. [a] The functions Φ, Φ^i are regular functions whose spectrum is bounded, included in $\left\{\xi \in \mathbb{R}^d, \quad |\xi| \lesssim 1\right\}$. We are going to study the continuity of multilinear operators on the Lebesgue Spaces $L^p(\mathbb{R}^d)$ and on the Hardy Spaces $H^p(\mathbb{R}^d)$.

We define the operators on $\mathbb{R}^d$, which we will study :

Definition : *Let L be a bounded function on $\mathbb{Z}$, $\mu = (\mu_1, .., \mu_n) \in (\mathbb{R}^*)^n$ and $\rho = (\rho_1, .., \rho_n) \in [0,1]^n$, then we define the following operator for functions $f_i \in \mathbf{S}(\mathbb{R}^d)$:*

$$T_{\rho,\mu,L}(f_1, .., f_n)(x) = \sum_{p\in\mathbb{Z}} L(p) \int_{\mathbb{R}^d} \Psi_{2^p}(y) \prod_{i=1}^{n} [\Phi^i_{\mu_i 2^p} * f_i](x - \rho_i\mu_i y)dy.$$

Similarly, if L is a bounded function on $\mathbb{R}^+$, we define the "continuous" version :

$$U_{\rho,\mu,L}(f_1, .., f_n)(x) = \int_0^\infty L(t) \int_{\mathbb{R}^d} \Psi_t(y) \prod_{i=1}^{n} [\Phi^i_{\mu_i t} * f_i](x - \rho_i\mu_i y)dy\frac{dt}{t}.$$

The parameter ρ is useful to make the link with "classical" paraproducts in the following way. Let $\rho_2, .., \rho_n$ tend to 0, keeping $\rho_1 = 1/2$ and $\mu_1 = 1$, then :

$$\forall f_1, .., f_n \in \mathbf{S}(\mathbb{R}^d) \qquad U_{\rho,\mu,L}(f_1, .., f_n)(x) \xrightarrow[\rho_i \to 0]{\rho_1 = 1/2} \int_0^\infty L(t)[\Psi * \Phi^1]_{\mu_1 t} * f_1(x) \prod_{i=2}^{n} [\Phi^i_{\mu_i t} * f_i](x)\frac{dt}{t}. \qquad (1)$$

[a] In [M.T.T.2], some techniques are explained to generalize the resultats with a function Ψ whose only the integral is assumed to be equal to zero. Here we have not consider this problem.

Results

Then we have the following result :

Theorem : *The operators $T_{\rho,\mu,L}$ and $U_{\rho,\mu,L}$ are continuous from $\otimes_{i=1}^n H^{p_i}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ for all exponents $0 < p_i, p < \infty$ verifying :*

$$\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}.$$

In addition the bound of continuity is independent of the parameters $\mu_i \neq 0$ and $\rho_i \in]0,1[$.
Some or all exponents p_i, p can be infinite, in which case H^{p_i} is replaced by L^∞ and L^p by BMO. When some p_i are equal to 1, we can may replace H^1 by L^1 in which case, the target space is $L^{p,\infty}(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d)$. In these limit cases, the bound is uniform in the parameters μ and ρ, if one of the two following situations holds :

$$\max_{1\leq i\leq n} p_i < \infty \qquad or \qquad \gamma := \sum_{\substack{j\in\{1,..,n\} \\ |\mu_j| \simeq \max\{|\mu_k|, \ 1\leq k\leq n\}}} \frac{1}{p_j} \geq \frac{1}{2}.$$

Otherwise the bound of continuity depends of the ratio :

$$\frac{\max\{|\mu_k|, \ 1 \leq k \leq n\}}{\min\{|\mu_k|, \ 1 \leq k \leq n\}}.$$

We don't know if the two conditions are necessary but if $p < \infty$ and $\gamma = 0$, then the previous bounds of continuity cannot be uniform in $\mu \in (\mathbb{R}^*)^n$.
The limit object given by (1) is a multilinear paraproduct and taking limits as indicated above we recover the results in [M.T.T.2] thanks to the uniform bounds.

Motivations

The multilinear paraproducts are a special case of multilinear operators whose symbol is singular at a one point. For example, take a bilinear operator T defined by

$$T(f,g)(x) := \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ix(\alpha+\beta)}\widehat{f}(\alpha)\widehat{g}(\beta)\sigma(\alpha,\beta)d\alpha d\beta, \qquad (2)$$

where the symbol σ is C^∞ on $\mathbb{R}^2 - \{0\}$ and satisfies the following conditions :

$$\forall k, l \in \{1, .., d\}^{\mathbb{N}}, \qquad \left|\partial^k_\alpha \partial^l_\beta \sigma(\alpha,\beta)\right| \lesssim \frac{|\mu_1|^{|k|}|\mu_2|^{|l|}}{(|\mu_1\alpha| + |\mu_2\beta|)^{|k|+|l|}}.$$

Since we know how to decompose T using "classical" paraproducts, we obtain the same continuity (and the same condition for uniform bounds in μ_1, μ_2) for T, than those for $T_{\rho,\mu,L}$.
In the last ten years, some more general operators like the bilinear Hilbert transform have been studied with a time-frequency method, more sophisticated than the Littlewood-Paley analysis that we use here. Now, consider operators T defined by (2) with a symbol σ singular on whole a line V (and not just at a point) of equation $\lambda_1\alpha + \lambda_2\beta = 0$. We assume that the line is non-degenerate : the three real coefficients λ_1, λ_2 and $\lambda_1 - \lambda_2$ are not equal to 0. For the case $d = 1$, if σ verifies

$$\forall k, l \geq 0 \qquad \left|\partial^k_\alpha \partial^l_\beta \sigma(\alpha,\beta)\right| \lesssim |\lambda_1\alpha + \lambda_2\beta|^{-k-l},$$

the operator is showed to be continuous between Lebesgue spaces. This fact is based on the pionnering work of M.Lacey and C.Thiele about the bilinear Hilbert transform, which corresponds to the symbol $\sigma(\alpha,\beta) = sign(\alpha - \beta)$ ([L.T.],[L.T.2]). The strategy to prove this continuity is to decompose such operators as "elementary" operators, which take the following form (for N large enough) :

$$T(f,g)(x) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{2d}} \frac{1}{|\lambda_1\lambda_2|^d(1 + |u/\lambda|^2)^N}$$
$$\left[\sum_{p\in\mathbb{Z}} L(p,j,u) \int \Psi_{\alpha_2,2^p}(y)[(e^{-ij(\lambda_2 2^p)^{-1}\cdot}\Phi^2_{2^p}) * g](x-y)[(e^{ij(\lambda_1 2^p)^{-1}\cdot}\Phi^1_{\alpha_1,\mu 2^p}) * f](x - \mu y)dy\right] du.$$

Here $\mu = \lambda_1\lambda_2^{-1}$, the coefficients L are bounded and are defined by the symbol σ and the vectors $\alpha_i \in \mathbb{R}^d$ are defined by the parameter $u = (u_1, u_2)$:

$$\alpha_2 := \frac{u_2}{\lambda_2} \qquad \alpha_1 := (\frac{u_1}{\lambda_1} - \frac{u_2}{\lambda_2}).$$

The parameter $u \in \mathbb{R}^{2d}$ is not a problem in the summation. We recognize betwen the square brackets, operators precisely in accordance with those we studied before. So the proof of the continuity for T begins by a precise study of "elementary" operators.

In fact, we must understand the projection of these general paraproducts on "trees" in the time-frequency plane (see [M.T.T.] and [B.G.]). To get some uniform continuities of these multilinear operators, it's also important to be able to estimate the multilinear paraproducts with some uniform quantities in μ.
However the study of $T_{\rho,\mu,L}$ and $U_{\rho,\mu,L}$ is not sufficient to study such multilinear operators. One needs some combinatorial arguments, using time-frequency orthogonality, to be able to sum the different estimates and to prove the continuity between Lebesgue spaces for such operators.

Ideas of the proof :

We will deal with the bilinear case : $n = 2$. We can summarize the theorem in the following diagram :

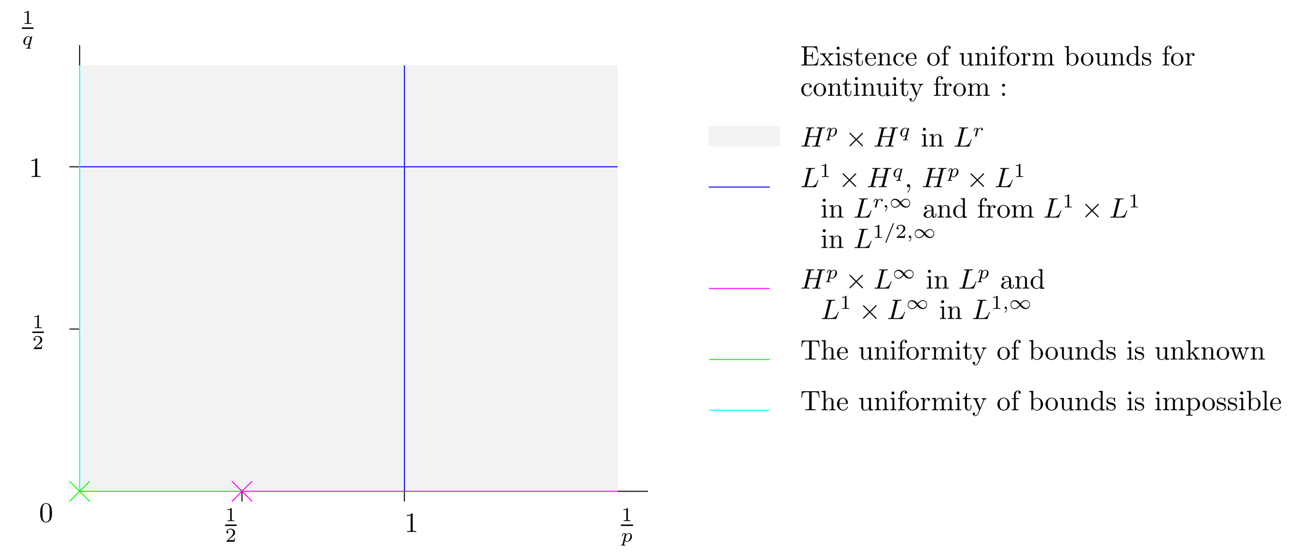


Diagram of the exponents for which the bound of continuity from $H^p \times H^q$ to L^r is uniform in the bilinear case with $|\mu_1| \leq |\mu_2|$.

The case of finite exponents.

We use the characterization of the Hardy spaces by the quadratic functional of Littlewood-Paley. As in the study of paraproducts, the result is also based on the "spectral separation". We decompose the functions f and g by the Littlewood-Paley decomposition. We choose Ψ a regular function, whose spectrum is contained in a corona around 0 and we have

$$T_{\rho,\mu,L}(f,g) := \sum_p \int_0^\infty \int_0^\infty \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle f, \Psi_{u,q}\rangle\langle g, \Psi_{v,s}\rangle F(p,u,v,q,s,x)dqds\,\frac{dv}{v}\frac{du}{u},$$

with

$$F(p,u,v,q,s,x) := L(p)\int_{\mathbb{R}^d} \Psi_{2^p}(y)[\Phi^1_{\mu_1 2^p} * \Psi_{u,q}](x - \rho_1\mu_1 y)[\Phi^2_{\mu_2 2^p} * \Psi_{v,s}](x - \rho_2\mu_2 y)dy.$$

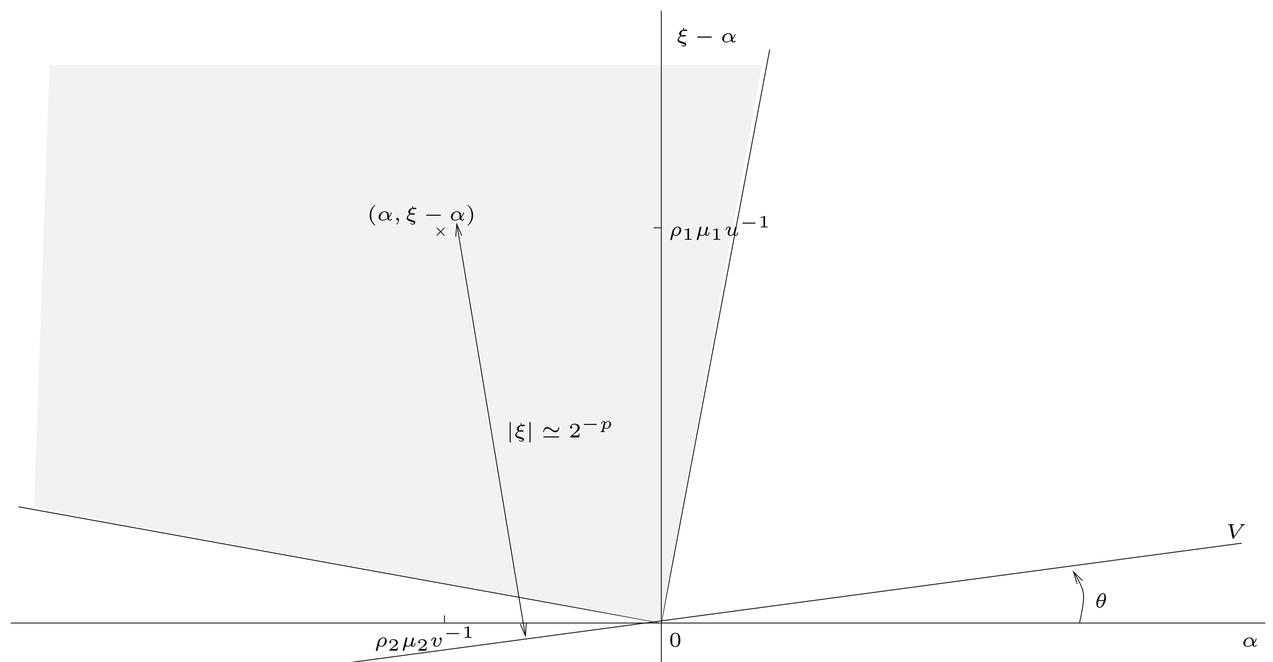
We can write it in the frequency plane, with the Plancherel equality :

$$F(p,u,v,q,s,x) = L(p)\int_{\mathbb{R}^d\times\mathbb{R}^d} \widehat{\Psi_{2^p}}(\xi)\widehat{\Phi^1_{\mu_1 2^p}}((\xi-\alpha)\rho_1^{-1}\mu_1^{-1})\widehat{\Psi_{u,q}}((\xi-\alpha)\rho_1^{-1}\mu_1^{-1})e^{i(\xi-\alpha)x\rho_1^{-1}\mu_1^{-1}}\widehat{\Phi^2_{\mu_2 2^p}}(\mu_2^{-1}\rho_2^{-1}\alpha)$$
$$\widehat{\Psi_{v,s}}(\rho_2^{-1}\mu_2^{-1}\alpha)e^{i\alpha x(\rho_2\mu_2)^{-1}}(\rho_1\rho_2|\mu_1\mu_2|)^{-d}d\alpha d\xi.$$

So the quantity $F(p,u,v,q,s,x)$ will be not null, if the three frequency parameters $u, v, 2^p$ satisfy the condition :

$$\max\{|\mu_1|u^{-1}, |\mu_2|v^{-1}\} \simeq 2^{-p}.$$

So apart from the case $|\mu_1|u^{-1} \simeq |\mu_2|v^{-1}$ (and under this condition the three parameters are also equivalent), we are in the next situation : the integral for $(\alpha,\beta) \in \mathbb{R}^{2d}$ is reduced to an integral over a cone whose vertex is 0.



Region where the symbol of $T_{\rho,\mu,L}$ with $|\mu_2|v^{-1} << |\mu_1|u^{-1}$ is summed.

This cone has a spectral property : it doesn't contain one of the two axes (here the "α"-axis). So we can use a spectral separation with the quadratic functional of Litllewood-Paley at the scale 2^{-p} for f and at the scale v^{-1} for g. As a consequence, we obtain (for M large enough) :

$$\left\|\left(\int_0^\infty |\Psi_t * T_{\rho,\mu,L}(f,g)|^2\frac{dt}{t}\right)^{1/2}\right\|_r \lesssim \left\|\iint_{\mathbb{R}^d\times\mathbb{R}^d} S_{\Psi_{a,1}}(f)S_{\Psi_{b,1}}(g)\,(1+|a|)^{-M}\,(1+|b|)^{-M}\,dadb\right\|_r.$$

We denoted by S_ζ the Littlewood-Paley quadratic functional, associated to a regular function ζ with a null integral on $\mathbb{R}^d$

$$S_\zeta(h) := \left(\int_0^\infty |\zeta_t * h|^2\frac{dt}{t}\right)^{1/2}.$$

Then Hölder's inequality and technical estimates are used to control the dependence of $S_{\Psi_{a,1}}(f)$ with respect to the translation parameter $a \in \mathbb{R}^d$.

The case of infinite exponents.

In the case of infinite exponents, the frequency analysis is not sufficient. Assume by symmetry that $|\mu_2| \geq |\mu_1|$, then we prove first that $T_{\rho,\mu,L}$ is continuous from $L^\infty \times L^2$ to L^2. To handle $T_{\rho,\mu,L}$, we make a development at the first order in $\rho_2 = 0$. Hence $T_{\rho,\mu,L}$ is decomposed in a sum of "classical" paraproducts and terms which can be estimated using Carleson measure arguments by showing that for all $l \in \{1, .., d\}$:

$$\int_0^\infty \int_{\mathbb{R}^d} \left|\int \Psi_t(y)\Phi^1_{\mu_1 t} * f(x - \rho_1\mu_1 y)(\partial_{x_l}\Phi^2)_{\mu_2 t} * g(x + \mu_2 s_2 y)dy\right|^2 \frac{dxdt}{t} \lesssim \|g\|_2^2\|f\|_\infty^2. \qquad (3)$$

By this way, we have the continuity from $L^\infty \times L^2$ into L^2. The same proof can be made under the condition $|\mu_2| \leq |\mu_1|$, but in this case the quadratic estimate (3) is not uniform in μ. Afterwards we generalize the continuity to others spaces, using interpolation arguments, Calderón-Zygmund decomposition (which permits to obtain smaller Lebesgue exponents) and the atomic theory of Hardy spaces. We must be careful with these arguments, because we want to have uniform estimates and only few bounds of the kernel are uniform in ρ and μ. This is why all continuity bounds are not uniform with respect to ρ and μ.

Références

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