Mini-courses

Michela Procesi *Infinite dimensional invariant tori for the 1d NLS*

In the study of close to integrable Hamiltonian PDEs, a fundamental question is to understand the behaviour of "typical" solutions. With this in mind it is natural to study the persistence of almost-periodic solutions and infinite dimensional invariant tori, which are indeed typical in the integrable case. Up to now almost all results in the literature deal with very regular solutions for model PDEs with external parameters giving a large modulation. In this minicourse I shall give an overview of results (particularly those stemming from the seminal work of Bourgain) as well as discussing new developments and perspectives.

Erwan Faou *Splitting methods for Hamiltonian PDEs: A review*

In these lectures, I will review some old and recent results concerning the geometric numerical integration of Hamiltonian PDEs. I will first show the numerical motivations behind splitting methods, and then discuss the notion of convergence, symplecticity, energy preservation, numerical resonances and modulations. The notion of modified Hamiltonian will be extensively studied, in link with the numerical orbital stability of solitary waves over long times.

Talks

Alexander Ostermann *Bourgain techniques for error estimates at low regularity*

Standard numerical integrators such as splitting methods or exponential integrators suffer from order reduction when applied to semi-linear dispersive problems with non-smooth initial data. In this talk, we focus on the cubic nonlinear Schrödinger equation with periodic boundary conditions. For such problems, we present filtered integrators that exhibit superior convergence rates at low regularity. Furthermore, due to the nonexistence of suitable embedding results, the error analysis at very low regularity cannot be carried out in the usual framework of Sobolev spaces. Instead, completely new techniques are required. They are based on Bourgain's seminal work and will be sketched in the talk. Numerical examples illustrating the analytic results will be given.

Louise Gassot *Zero-dispersion limit for the Benjamin-Ono equation on the torus*

We discuss the zero-dispersion limit for the Benjamin-Ono equation on the torus given a single well initial data. We prove that there exist approximate initial data converging to the initial data, such that the corresponding solutions admit a weak limit as the dispersion parameter tends to zero. The weak limit is expressed in terms of the multivalued solution of the inviscid Burgers equation obtained by the method of characteristics. We construct our approximation by using the Birkhoff coordinates of the initial data, introduced by Gérard, Kappeler and Topalov.

Beatrice Langella *Growth of Sobolev norms in quasi integrable quantum systems*

In this talk I will analyze an abstract linear time dependent Schrödinger equation of the form

$$\mathrm{i}\partial_t \psi = (H_0 + V(t))\psi\,, \tag{1}$$

with H_0 a pseudo-differential operator of order $\mathtt{d} > 1$ and $V(t)$ a time dependent family of pseudo-differential operators of order strictly less than $\mathtt{d}$. I will introduce abstract assumptions on H_0, namely steepness and global quantum integrability, under which we can prove a $|t|^\epsilon$ upper bound on the growth of Sobolev norms of all the solutions of (1).
The result I will present applies to several models, as perturbations of the quantum anharmonic oscillator in dimension 2, and perturbations of the Laplacian on a manifold with integrable geodesic flow, and in particular: flat tori, Zoll manifolds, rotation invariant surfaces and Lie groups. The case of several particles on a Zoll manifold, a torus or a Lie group is also covered.

The proof is based a on quantum version of the proof of the classical Nekhoroshev theorem. This is a joint work with Dario Bambusi.

Frédéric Marbach *On some expansions of the solutions to nonlinear ODEs*

Our motivation stems from applications to control theory for nonlinear ODEs. I will start by recalling the main questions, known results and open problems of this research field.
Then, I will give a short comparison of 4 different ways to write an expansion for the solution of such ODEs, driven by analytic vector fields. In particular, we will be interested in error estimates and convergence issues.

Benoît Grébert *Birkhoff normal forms for Hamiltonian PDEs in low regularity*

In the last decades, normal form methods have been very successful in showing the stability in long time of small solutions of nonlinear dispersive equations on bounded domains. However, except in the case of integrable equations, these results only concern very regular solutions. This assumption of regularity seems to be essential from a technical point of view (in order to compensate for the losses due to small divisors) but, surprisingly, numerical simulations strongly suggest that the stability of small solutions should not depend on their regularity. I will present some recent results, obtained in collaboration with Joackim Bernier and Gabriel Rivière, which are more in agreement with these numerical observations.

Patrick Gérard *Multisolitons for the Calogero–Moser Derivative Nonlinear Schrödinger equation*

The CMDNLS equation is a nonlinear Schrödinger equation on the line, displaying a mass critical nonlocal cubic nonlinearity of DNLS type, which conserves the property that the Fourier transform of the solution is supported in the positive half line. We identify a Lax pair for this equation, and this structure allows us to establish an exact normal form for multisoliton solutions, from which we infer long time energy cascades. This a jointwork with Enno Lenzmann (Basel).

Alberto Maspero *Full description of Benjamin-Feir instability of Stokes waves in finite and deep water*

Small-amplitude, traveling, space periodic solutions – called Stokes waves – of the 2 dimensional gravity water waves equations in finite and deep water are linearly unstable with respect to long-wave perturbations, as predicted by Benjamin and Feir in 1967. We completely describe the behavior of the four eigenvalues close to zero of the linearized equations at the Stokes wave, as the Floquet exponent is turned on. We prove in particular the conjecture that a pair of non-purely imaginary eigenvalues depicts a closed figure eight, parameterized by the Floquet exponent, in full agreement with numerical simulations. This is a joint work with M. Berti and P. Ventura.

Massimiliano Berti *Quasi periodic vortex patches*

I will discuss the bifurcation of quasi-periodic vortex patch solutions of the Euler equations close to rotating Kirkhoff ellipses, for a Borel set of eccentricities of asymptotically full Lebesgue measure. This is a joint work with Z. Hassaina and N. Masmoudi.

San Vu Ngoc *Semiclassical normal forms for 1D Hamiltonians and applications to inverse spectral theory*

I will survey recent and old results on symplectic normal forms for Hamiltonians in $\mathbb{R}^2$, and the corresponding microlocal normal norms for pseudodifferential operators. These normal norms have been used with some success for the (semiclassical) inverse spectral problem: can you recover the principal symbol from the spectrum?

Arieh Iserles *On orthogonal systems that recover invariants on the real line*

The design of spectral methods for dispersive equations requires the construction of rapidly-convergent orthonormal bases of the real line. In this talk we present the outlines of recent theory, developed together with Marcus Webb, which mixes harmonic analysis, orthogonal polynomials, asymptotic analysis and tools from numerical linear algebra to sketch a theory of such systems and highlight their implementation in tandem with splitting methods. Our emphasis is on systems that recover invariants, e.g. mass, Sobolev norm and Hamiltonians.

Elena Celledoni *Discrete conservation laws for finite element discretisations of multisymplectic PDEs*

We propose a new, arbitrary order space-time finite element discretisation for Hamiltonian PDEs in multisymplectic formulation. We show that the new method which is obtained by using both continuous and discontinuous discretisations in space, admits a local and global conservation law of energy. We also show existence and uniqueness of solutions of the discrete equations. Further, we illustrate the error behaviour and the conservation properties of the proposed discretisation in extensive numerical experiments on the linear and nonlinear wave equation and the nonlinear Schrödinger equation.

Anxo Biasi *Weak turbulence and integrability in physical models*

I will talk about the rich phenomenology that arises when a resonant equation is under spatial confinement. Under these conditions, the system displays nontrivial dynamics even in the weakly nonlinear limit. In that regime, one may find stationary and time-periodic energy flows, Fermi-Pasta-Ulam recurrences, or even turbulent cascades (the growth of Sobolev norms). I am going to describe how some of these phenomena are related to integrable structures, and how they form a family of resonant equations with physical models for Bose-Einstein condensates, nonlinear waves, or some spacetimes in general relativity.

Paul Alphonse *Smoothing and localizing properties for quadratic evolution equations through the polar decomposition*

In this talk, we will focus on the evolution equations associated with nonselfadjoint quadratic differential operators. The purpose is first to understand how the possible non-commutation phenomena between the selfadjoint and the skew-selfadjoint parts of these operators allow the associated evolution operators to enjoy smoothing and localizing properties in specific directions of the phase space which will be precisely described. These different properties will be deduced from a fine description of the polar decomposition of the evolution operators considered. An application to the generalized Ornstein-Uhlenbeck equations, of which the Kramers-Fokker-Planck equation is a particular case, will be given. We will also explain the local smoothing properties and the gains of integrability for these equations, under a geometric assumptions. These results come from a series of works with J. Bernier (LMJL).

Gigliola Staffilani *On the wave turbulence theory for a stochastic KdV type equation*

This talk is a summary of a recent work completed with Binh Tran. Starting from the stochastic Zakharov-Kuznetsov (ZK) equation, a multidimensional KdV type equation on a hypercubic lattice, we provide a derivation of the 3-wave kinetic equation. We show that the two point correlation function can be asymptotically expressed as the solution of the 3-wave kinetic equation at the kinetic limit under very general assumptions: the initial condition is out of equilibrium, the dimension is $d > 1$, the smallness of the nonlinearity is allowed to be independent of the size of the lattice, the weak noise is chosen not to compete with the weak nonlinearity and not to inject energy into the equation. Unlike the cubic nonlinear Schrödinger equation, for which such a general result is commonly expected without the noise, the kinetic description of the deterministic lattice ZK equation is unlikely to happen. One of the key reasons is that the dispersion relation of the lattice ZK equation leads to a singular manifold, on which not only 3-wave interactions but also all m-wave interactions are allowed to happen. To the best of our knowledge, the work provides the first rigorous derivation of nonlinear 3-wave kinetic equations. Also this is the first derivation for wave kinetic equations in the lattice setting and out-of-equilibrium.

Filippo Giuliani *Transfers of energy in an infinite pendulum lattice*

In this talk we present a recent result (joint work with M. Guardia) about the existence of transfer of energy orbits in a chain of infinitely many weakly coupled pendulums. The model system is posed on an infinite lattice (of any dimension) with formal or convergent Hamiltonian. We develop geometric and functional tools to perform an Arnold diffusion mechanism in infinite dimensional phase spaces. In this way we construct solutions that move the energy of pendulums along any prescribed path. At the best of our knowledge this is the first result of Arnold diffusion for a Hamiltonian system with infinitely many degrees of freedom.

Yvain Bruned *Low regularity integrators via decorated trees*

We will present a general framework of low regularity integrators which allows us to approximate the time dynamics of a large class of equations, including parabolic and hyperbolic problems, as well as dispersive equations. The structure of the local error of the new schemes is driven by nested commutators which require lower regularity assumptions than classical methods do. We use a decorated tree formalism inspired by Regularity Structures in the context of SPDEs. This is a joint work in collaboration with Yvonne Alama Bronsard and Katharina Schratz.

Roberto Feola *Long time NLS approximation for a quasilinear Klein-gordon equation on large compact domains*

We consider a class of Klein–Gordon equations with quasilinear, Hamiltonian and quadratic non-linearities posed on a large box with periodic boundary conditions. We discuss how the cubic NLS equation can be derived to describe, approximately, the evolution of slow modulations in time and space of a spatially and temporarily oscillating wave packet. We show that the approximation is valid over a time scale which goes beyond the natural quadratic lifespan of solutions of cubic equations. We provide error estimates in Sobolev spaces. The proof is based on a combination of normal form techniques and energy methods.

Zaher Hani *The mathematical theory of wave turbulence*

Wave turbulence is the theory of nonequilibrium statistical mechanics for wave systems. Initially formulated in pioneering works of Peierls, Hasselman, and Zakharov early in the past century, wave turbulence is widely used across several areas of physics to describe the statistical behavior of various interacting wave systems. We shall be interested in the mathematical foundation of this theory, which for the longest time had not been established.

The central objects in this theory are: the "wave kinetic equation" (WKE), which stands as the wave analog of Boltzmann's kinetic equation for interacting particle systems, and the "propagation of chaos" hypothesis, which is a fundamental postulate in the field that lacks mathematical justification. Mathematically, the aim is to provide a rigorous justification and derivation of those two central objects; This is Hilbert's Sixth Problem for waves. In this talk, we shall describe some recent results with Yu Deng (University of Southern California) in which we give a full resolution of this problem, namely a rigorous derivation of the wave kinetic equation and a justification of the propagation of chaos, in the context of the nonlinear Schrodinger equation.

Fernando Casas *Applying splitting methods with complex coefficients for solving the Schrödinger equation*

Splitting methods with complex coefficients have been proposed as a way to overcome the well known sign problem when the order is greater than two, especially for parabolic problems. Methods of this class possess properties not shared by standard splitting integrators, both concerning stability and truncation errors. In this talk we present some methods within this class and analyze their applicability for the numerical time integration of Schrödinger equations. Since the problem itself requires working with complex arithmetic, the use of these schemes does not increase the overall computational cost. Nevertheless, the problem of preserving the unitary character of the solution still persists. In this respect, we show how a special symmetry in the compositions leads to numerical integrators that are conjugate to unitary methods for sufficiently small step sizes. As a result, the error in the energy and in the norm of the numerical solution does not possess a secular component over long time intervals. In addition, some of the schemes may lead to better efficiencies than standard splitting methods for this class of problems.
This is a joint work with Sergio Blanes and Alejandro Escorihuela-Tomàs.

Valeria Banica *Growth of energy density for the Schrödinger map*

In this talk we will show the existence of an unbounded growth of the energy density for the 1-D Schrödinger map with values in the 2D sphere, which is the classical continuous Heisenberg model in ferromagnetism. We will give the main ideas of the proof and details on some of the steps, as for instance the construction of a class of critical solutions for the 1D cubic Schrödinger equation. This is a joint work with Luis Vega.