

ASYMPTOTIC PRESERVING HLL SCHEMES

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Abstract. This work concerns the derivation of HLL schemes to approximate the solutions of systems of conservation laws supplemented by source terms. Such a system contains many models such as the Euler equations with high friction or the M1 model for radiative transfer. The main difficulty arising from these models comes from a particular asymptotic behavior. Indeed, in the limit of some suitable parameter, the system tends to a diffusion equation. This paper is devoted to derive HLL methods able to approximate the associated transport regime but also to restore the suitable asymptotic diffusive regime. To access such an issue, a free parameter is introduced into the source term. This free parameter will be a useful correction to satisfy the expected diffusion equation at the discrete level. The derivation of the HLL scheme for hyperbolic systems with source terms comes from a modification of the HLL scheme for the associated homogeneous hyperbolic system. The resulting numerical procedure is robust since the source term discretization preserves the physical admissible states. The scheme is applied to several models of physical interest. The numerical asymptotic behavior is analyzed and an asymptotic preserving property is systematically exhibited. The scheme is illustrated with numerical experiments.

Key words. Hyperbolic system with source term, HLL scheme, asymptotic preserving scheme, Euler equations with high friction, radiative transfer, M1 model, Telegraph equations.

AMS subject classifications. 76M12, 65M12, 35L65

1. Introduction. This work introduces finite volumes methods to approximate the solutions of hyperbolic systems with source terms. Such systems concern many applications and they are essential to model physics involving several regimes governed by relevant parameters.

We derive a class of models issuing from the following partial differential equations:

$$\partial_t w + \partial_x f(w) = \sigma(R(x, w) - w), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_*^+, \quad (1.1)$$

where the vector state w belongs to a convex set $\Omega \subset \mathbb{R}^N$.

We assume that the flux function $f : \Omega \rightarrow \mathbb{R}^N$ admits a Jacobian $\nabla_w f(w)$ which is diagonalizable in $\mathbb{R}$. As a consequence the transport regime:

$$\partial_t w + \partial_x f(w) = 0, \quad (1.2)$$

i.e. the homogeneous system (1.1) with $\sigma = 0$, is hyperbolic.

Concerning the source term, we impose $\sigma \geq 0$. This coefficient σ is eventually a suitable function of x to define $\sigma = \sigma(x)$. Finally $R : \mathbb{R} \times \Omega \rightarrow \Omega$ is a given smooth enough function.

From a numerical point of view, one of the main difficulty comes from a diffusion regime satisfied by the solution of (1.1). Indeed, the associated physics involves a parabolic rescaling parameter (for instance see [12, 7] according to the considered physics) which modifies (1.1) as follows:

$$\varepsilon \partial_t w + \partial_x f(w) = \frac{\sigma}{\varepsilon} (R(w) - w).$$

The derived numerical method must be relevant whenever ε goes to zero. We do not develop here the stiff relaxation theory and the reader is referred to [28] (for instance,

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see also [2, 11, 6, 5, 16, 32] and references therein). In the present work, several examples will illustrate the existence and the interest of a diffusive limit equation satisfied by the solution of (1.1).

As underlined, the model (1.1) proposes a formalism in which enter many applications. In the next section, three of them are given namely the Telegraph model [23, 13], the Euler equations with high friction [12, 34] and the M1 model for radiative transfer [7, 20, 14, 15, 13]. All of them involve their own numerical schemes (see [7, 23, 12] for a brief overview of the numerical procedures, see also [3, 11, 21, 25, 27, 39] for some related works). In fact, from a numerical point of view, the main difficulty comes from the asymptotic regime. Indeed, a numerical method for such a model has to be relevant in both the transport and diffusion regimes. To access such an issue, several authors propose to modify suitable Godunov-type schemes in order to take into account the source terms. Several ideas can be found in the literature. One is based on the work of [25], and proposes the derivation of an hydrostatic reconstruction [3, 12]. This approach can be seen as free from the choice of the approximation technique and for the transport operator (1.2). To satisfy the expected diffusion regime, the usual reconstruction turns out to be very sophisticated and it needs to solve a non-linear equation. As a consequence, the numerical method may not be relevant to approximate some models of the form (1.1).

Another technique, introduced by Gosse-Toscani [23, 24], consists in a structural modification of the Godunov-type scheme considered to approximate the weak solution of (1.2). Roughly speaking, the idea consists in controlling the natural numerical viscosity in order to coincide, in a sense to be prescribed, with the expected diffusion regime (see [7, 20, 14, 15, 13] for further details). The resulting numerical schemes are improved but for very sophisticated derivation. Since a particular attention must be paid on the Godunov-type scheme, several extensions are impossible.

To conclude this brief overview, let us emphasize that all the referred methods deal with a constant σ (see for instance [13]) except for some models where σ may depend on x (see [7, 8]).

Following the above procedure, we propose the derivation of a numerical scheme for (1.1) able to restore any diffusive regime. Put in other words, we modify the HLL approximate Riemann solver for (1.2) to obtain a suitable discretization of (1.1). To exhibit the scheme, we involve a suitable rewriting of the system (1.1). Indeed, let us fix an arbitrary positive parameter $\bar{\sigma}$ to write (1.1) in the following equivalent form:

$$\partial_t w + \partial_x f(w) = (\sigma + \bar{\sigma}) \left(\frac{\sigma}{\sigma + \bar{\sigma}} R(w) + \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} w - w \right).$$

The resulting scheme will involve the arbitrary parameter $\bar{\sigma}$ which can be fixed to preserve the relevant diffusion regime. As a consequence, we preserve the robustness results satisfied by the transport method associated with (1.2).

This paper is organized as follows: in the next section, several models entering the formalism (1.1) are introduced. We describe the diffusion equation satisfied by the model in the limit of a rescaling parameter. Next, a third section concerns the numerical approximation of the solution of (1.1). First, we recall the HLL approximate Riemann solver to approximate the solutions of (1.2). Then we modify this standard procedure to take into account the source terms. The resulting numerical scheme is proved to be robust. The last part of this section concerns the correction to obtain an asymptotic-preserving scheme and thus to satisfy the expected diffusion equation as soon as the asymptotic regime is reached. The following section is dedicated to the application models described in section 2. The derived scheme is applied to each

of them. A specific attention is paid on the asymptotic behavior in the diffusion limit. Finally, in the last section, numerical experiments are performed to highlight the relevance of the scheme.

2. Some mathematical models. In this section, we give several applications which can take the form of the system (1.1). The interest of these applications comes from the existence of a diffusion regime in the limit of some relevant parabolic rescaling parameter. The limit equation is here systematically exhibited since it plays a central role in the numerical simulations.

2.1. The Telegraph equations. The first model we propose to consider is given by the Telegraph (or Goldstein-Taylor) equations as follows:

$$\begin{aligned}\partial_t u + a \partial_x u &= \sigma(v - u), \\ \partial_t v - a \partial_x v &= \sigma(u - v),\end{aligned}\tag{2.1}$$

where a and σ are given positive constants and $(u, v) \in \mathbb{R}^2$. This system clearly writes into the formalism (1.1) by choosing:

$$w = (u, v)^T, \quad f(w) = (au, -av)^T \quad \text{and} \quad R(w) = (v, u)^T.\tag{2.2}$$

Now, let us introduce a parametric rescaling to write:

$$\begin{aligned}\varepsilon \partial_t u^\varepsilon + a \partial_x u^\varepsilon &= \frac{\sigma}{\varepsilon}(v^\varepsilon - u^\varepsilon), \\ \varepsilon \partial_t v^\varepsilon - a \partial_x v^\varepsilon &= \frac{\sigma}{\varepsilon}(u^\varepsilon - v^\varepsilon),\end{aligned}\tag{2.3}$$

where ε is a (positive) parameter devoted to tend to zero. Let us then consider the asymptotic regime when ε is small and exhibit the diffusion equation satisfied by the solution of (2.3). Here we skip the rigorous analysis of the asymptotic expansion and the reader is referred to [34] for further details. First, we set:

$$U^\varepsilon = u^\varepsilon - v^\varepsilon \quad \text{and} \quad V^\varepsilon = u^\varepsilon + v^\varepsilon,$$

to write (2.3) as follows:

$$\begin{aligned}\varepsilon \partial_t U^\varepsilon + a \partial_x V^\varepsilon &= -\frac{2\sigma}{\varepsilon}U^\varepsilon, \\ \varepsilon \partial_t V^\varepsilon + a \partial_x U^\varepsilon &= 0.\end{aligned}$$

Then, we formally adopt a Chapman-Enskog expansion (see for instance [9] and references therein) for U^ε of the form:

$$U^\varepsilon = \varepsilon U^1 + O(\varepsilon^2).$$

Taking the leading term from the evolution law of U^ε , we deduce:

$$U^1 = -\frac{a}{2\sigma} \partial_x V^\varepsilon.$$

Using this expression, we get the following first-order asymptotic diffusive equation satisfied by the limit of V^ε (called V):

$$\partial_t V - \partial_x \left(\frac{a^2}{2\sigma} \partial_x V \right) = 0,$$

which reads:

$$\partial_t(u+v) - \partial_x\left(\frac{a^2}{2\sigma}\partial_x(u+v)\right) = 0. \quad (2.4)$$

One of the main numerical challenge concerning the approximation of the solution of (2.1) is the derivation of a scheme able to behave well in both the transport regime (where $\sigma = 0$) and the diffusion regime governed by (2.4).

2.2. Euler equation with high friction. The second application we propose to consider is devoted to the simulation of an isentropic gas flow in a porous media [33, 44]. Such a model is governed by the isentropic Euler equations supplemented by a source term (namely the friction term):

$$\partial_t\rho + \partial_x\rho u = 0, \quad (2.5)$$

$$\partial_t\rho u + \partial_x(\rho u^2 + p(\rho)) = -\sigma\rho u, \quad (2.6)$$

where ρ is the density, u is the velocity and σ is the friction coefficient. Usually a polytropic gas with a pressure law given by:

$$p(\rho) = \kappa\rho^\gamma, \quad \gamma \in (1, 3], \quad \kappa > 0,$$

is used (see for instance [34]). In the present work, we do not impose such a restriction. As a consequence, the pressure p may be any positive increasing smooth function. Moreover, we emphasize that most works on the numerical approximation of this system consider only a given positive constant friction coefficient. Unlike them, we consider the friction coefficient to be a function of x .

To put (2.5)-(2.6) into the formalism (1.1), let us rewrite (2.5) as follows:

$$\partial_t\rho + \partial_x\rho u = \sigma(\rho - \rho).$$

Hence, (2.5)-(2.6) writes into the formalism (1.1) by setting:

$$w = (\rho, \rho u)^T, \quad f(w) = (\rho u, \rho u^2 + p)^T \quad \text{and} \quad R(w) = (\rho, 0)^T. \quad (2.7)$$

The admissible space of state is defined by:

$$\Omega = \left\{ w \in \mathbb{R}^2, \quad \rho > 0, \quad u \in \mathbb{R} \right\}. \quad (2.8)$$

Let us note that $R(w) \in \Omega$ for all $w \in \Omega$.

To conclude this brief description of the model, we consider a high friction term to reach the diffusion regime. Indeed, let us introduce a rescaling parameter ε to rewrite the model in the following form:

$$\begin{aligned} \varepsilon\partial_t\rho^\varepsilon + \partial_x\rho^\varepsilon u^\varepsilon &= 0, \\ \varepsilon\partial_t\rho^\varepsilon u^\varepsilon + \partial_x\left(\rho^\varepsilon(u^\varepsilon)^2 + p(\rho^\varepsilon)\right) &= -\frac{\sigma}{\varepsilon}\rho^\varepsilon u^\varepsilon. \end{aligned}$$

When ε tends to zero, [34, 33] proved the existence of the following diffusion equation satisfied by ρ (the limit of ρ^ε):

$$\partial_t\rho - \partial_x\left(\frac{1}{\sigma}\partial_x p(\rho)\right) = 0. \quad (2.9)$$

We underline that the proof in [34] was only given for a polytropic gas and a constant friction coefficient. Formally, a Chapman-Enskog expansion yields to the diffusion equation (2.9) as soon as the pressure law and the positive friction coefficient are smooth enough functions.

2.3. The M1 model for radiative transfer. This last example concerns a moment model for radiative transfer. It was introduced by Dubroca and Feugeas in [20] and have been later studied or extended in [14, 15, 18, 38, 43] (for instance, see also [29]). It reads as follows:

$$\partial_t E + \partial_x F = c\sigma^e a T^4 - c\sigma^a E, \quad (2.10)$$

$$\partial_t F + c^2 \partial_x P = -c\sigma^f F, \quad (2.11)$$

$$\rho C_v \partial_t T = c\sigma^a E - c\sigma^e a T^4. \quad (2.12)$$

where E denotes the radiative energy, F the radiative flux and T the material temperature. The unknown vector state is therefore $w = (E, F, T)^T$ and the admissible space of state is defined by:

$$\Omega = \left\{ w \in \mathbb{R}^3, E > 0, |F/cE| < 1, T > 0 \right\}. \quad (2.13)$$

Moreover, the radiative pressure $P = P(E, F)$ is a given function defined by:

$$P(E, F) = \chi\left(\frac{F}{cE}\right)E, \text{ where } \chi(z) = \frac{3 + 4z^2}{5 + 2\sqrt{4 - 3z^2}}.$$

The reader is referred to [1, 30, 36] for pionner derivations of the radiative pressure law $P(E, F)$.

Finally, c is the speed of the light, $a > 0$ is a given constant, $\rho = \rho(T)$ is the density which is supposed to be a given function and σ^e , σ^a and σ^f denote the opacities mean values. These mean values are critical physical functions of the unknowns (see [42] and also [35, 37] for their expression). Let us emphasize that the discrepancy between the present work and classical numerical methods that can be found in the literature (see [7, 20, 14, 15, 13] for instance) concerns the choice of these opacities mean values. Indeed, most of the numerical schemes impose $\sigma^e = \sigma^a = \sigma^f$. Such a restrictive assumption is rejected here since the opacity may depend on the space.

Let us now write the M1 model using the formalism (1.1). We have clearly $f(w) = (F, c^2 P(E, F), 0)^T$. The main difficulty comes from the source term that has to be given in the form $\sigma(R(w) - w)$. To access such an issue, two assumptions must be made on the opacities as stated in the following result:

LEMMA 1. *If we set $\sigma^m = \max(\sigma^a, \sigma^f, \frac{\sigma^e a T^3}{\rho C_v})$, then there exist three positive coefficients σ_1 , σ_2 and σ_3 such that:*

$$\sigma^m = \sigma^a + \sigma_1 = \sigma^f + \sigma_2 = \frac{\sigma^e a T^3}{\rho C_v} + \sigma_3. \quad (2.14)$$

With these notations, the system (2.10)-(2.12) can be written in the form (1.1) if we set $\sigma = c\sigma^m$ and

$$R(w) = \left(\frac{\sigma^e a T^4 + \sigma_1 E}{\sigma^m}, \frac{\sigma_2 F}{\sigma^m}, \frac{\frac{\sigma^a E}{\rho C_v} + \sigma_3 T}{\sigma^f} \right)^T. \quad (2.15)$$

Proof. Involving the definition of σ_1 , σ_2 and σ_3 , the components of the source

term in (2.10)-(2.12) write:

$$\begin{aligned}\sigma^e aT^4 - \sigma^a E &= \sigma^e aT^4 - (\sigma^m - \sigma_1)E = \sigma^m \left(\frac{\sigma^e aT^4 + \sigma_1 E}{\sigma^f} - E \right), \\ -\sigma^f F &= \sigma^m \left(\frac{\sigma_2 F}{\sigma^m} - F \right), \\ \frac{\sigma^a E - \sigma^e aT^4}{\rho C_v} &= \sigma^m \left(\frac{\frac{\sigma^a E}{\rho C_v} + \sigma_3 T}{\sigma^m} - T \right).\end{aligned}$$

The system (2.10)-(2.12) thus writes in the following form:

$$\begin{aligned}\partial_t E + \partial_x F &= c\sigma^m \left(\frac{\sigma^e aT^4 + \sigma_1 E}{\sigma^m} - E \right), \\ \partial_t F + c^2 \partial_x P &= c\sigma^m \left(\frac{\sigma_2 F}{\sigma^m} - F \right), \\ \partial_t T &= c\sigma^m \left(\frac{\frac{\sigma^a E}{\rho C_v} + \sigma_3 T}{\sigma^m} - T \right).\end{aligned}$$

The definition of $R(w)$ is easily deduced from the last result and the proof is therefore achieved. $\square$

The M1 model has to degenerate into the diffusion regime reached by introducing a rescaling parameter ε as follows:

$$\begin{aligned}\varepsilon \partial_t E^\varepsilon + \partial_x F^\varepsilon &= \frac{1}{\varepsilon} (c\sigma^e a(T^\varepsilon)^4 - c\sigma^a E^\varepsilon), \\ \varepsilon \partial_t F^\varepsilon + c^2 \partial_x P(E^\varepsilon, F^\varepsilon) &= -\frac{c\sigma^f}{\varepsilon} F^\varepsilon, \\ \varepsilon \rho C_v \partial_t T^\varepsilon &= \frac{1}{\varepsilon} (c\sigma^a E^\varepsilon - c\sigma^e a(T^\varepsilon)^4).\end{aligned}$$

As ε tends to zero, the M1 model degenerates into the so-called equilibrium diffusion equation given by:

$$\partial_t (\rho C_v T + aT^4) - \partial_x \left(\frac{c}{3\sigma^f} \partial_x (aT^4) \right) = 0, \quad (2.16)$$

where σ^f stands for the Rosseland's mean value of the opacity.

3. A HLL scheme with source terms. We now consider the numerical approximation of the system (1.1). To access such an issue, we propose to modify any suitable HLL scheme associated with the transport limit (1.2). In this sense, we first introduce a finite volumes scheme to approximate (1.2) according to the formalism proposed by Harten, Lax and van Leer [26]. Then, we suggest a relevant modification of the involved approximate solver to produce a discretization of the source terms. The obtained scheme will be shown to satisfy robustness properties. We conclude the section with the introduction of an appropriate correction to be applied in order to get the required asymptotic preserving property. Indeed, such a property makes the scheme suitable to approximate the diffusion regimes as described in the previous section.

3.1. The initial approximate Riemann solver. We assume to be given a finite volumes method to approximate the weak solution of (1.2). Several strategies

can be adopted and the reader is referred to Godlewski and Raviart [22] (see also [31, 40]). In the present work, we propose to consider the formalism of Harten, Lax and van Leer as detailed in [26]. For the sake of completeness, we briefly recall this approach here.

Let us consider a uniform mesh of constant size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $i \in \mathbb{Z}$ and we denote Δt the time increment with $t^{n+1} = t^n + \Delta t$ for all $n \in \mathbb{N}$.

We assume known a piecewise constant approximate solution at time t^n , denoted $w^h(x, t^n) \in \Omega$ and defined by:

$$w^h(x, t^n) = w_i^n \text{ if } x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}).$$

To evolve this approximation in time, we consider an approximate HLL Riemann solver stated at each interface $x_{i+\frac{1}{2}}$. This approximate Riemann solver, denoted by $w_{\mathcal{R}}(\frac{x}{t}; w_L, w_R)$ is built as follows:

$$w_{\mathcal{R}}\left(\frac{x}{t}; w_L, w_R\right) = \begin{cases} w_L, & \text{if } \frac{x}{t} < b^-, \\ w^*(w_L, w_R), & \text{if } b^- < \frac{x}{t} < b^+, \\ w_R, & \text{if } \frac{x}{t} > b^+, \end{cases} \quad (3.1)$$

where b^+ and b^- denote the minimum and maximum velocity waves involved by the approximate Riemann solver. Concerning the constant state w^* , it describes the approximate solution inside the dependence cone characterized by $x = b^-t$ and $x = b^+t$. Following [26], we recall that the approximate Riemann solver (3.1) must satisfy a consistency condition given by:

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} w_{\mathcal{R}}\left(\frac{x}{\Delta t}; w_L, w_R\right) dx = \frac{1}{2}(w_L + w_R) - \frac{\Delta t}{\Delta x} (f(w_R) - f(w_L)), \quad (3.2)$$

where b^- and b^+ are large enough (see [26]).

The intermediate state, $w^*(w_L, w_R)$ is thus given by the following constant:

$$w^*(w_L, w_R) = \frac{b^+ w_R - b^- w_L}{b^+ - b^-} - \frac{1}{b^+ - b^-} (f(w_R) - f(w_L)). \quad (3.3)$$

Next, the approximate HLL Riemann solver $w_{\mathcal{R}}(\frac{x - x_{i+\frac{1}{2}}}{t + t^n}; w_i^n, w_{i+1}^n)$ is stated at each cell interface $x_{i+\frac{1}{2}}$. We thus obtain a non-interacting juxtaposition of Riemann solvers as soon as the following CFL-like condition is assumed:

$$\frac{\Delta t}{\Delta x} \max(|b_{i+\frac{1}{2}}^-|, |b_{i+\frac{1}{2}}^+|) \leq \frac{1}{2}. \quad (3.4)$$

Then we define an approximate solution at time $t^n + t$ for all $t \in (0, \Delta t)$ as follows:

$$w^h(x, t^n + t) = w_{\mathcal{R}}\left(\frac{x - x_{i+\frac{1}{2}}}{t + t^n}; w_i^n, w_{i+1}^n\right), \text{ if } x \in (x_i, x_{i+1}).$$

The projection of this solution on the piecewise constant functions gives the expected updated approximation:

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w^h(x, t^n + \Delta t) dx. \quad (3.5)$$

Now let us involve the condition (3.2) to rewrite the scheme in a standard conservation form:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} \right), \quad (3.6)$$

where we have:

$$\mathcal{F}_{i+\frac{1}{2}} = \frac{b_{i+\frac{1}{2}}^+ f(w_i^n) - b_{i+\frac{1}{2}}^- f(w_{i+1}^n)}{b_{i+\frac{1}{2}}^+ - b_{i+\frac{1}{2}}^-} + \frac{b_{i+\frac{1}{2}}^+ b_{i+\frac{1}{2}}^-}{b_{i+\frac{1}{2}}^+ - b_{i+\frac{1}{2}}^-} (w_{i+1}^n - w_i^n). \quad (3.7)$$

To conclude this short description of the numerical approximation of the weak solution of (1.2), we recall the robustness of the scheme. This property is stated in the following result:

LEMMA 2. *Let w_L and w_R be two constant states in Ω . Let us assume that $w^*(\frac{x}{t}; w_L, w_R)$, defined in (3.1), belongs to Ω . Then, as soon as $w_i^n \in \Omega$ for all $i \in \mathbb{Z}$, we have $w_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$.*

We skip the proof of this result (see for instance [4, 11, 26] for further details).

3.2. Extension to include source terms. We correct the above approximate Riemann solver to take into account the source terms involved in (1.1). We suggest to modify the intermediate state function as follows:

$$\tilde{w}^* \left(\frac{x}{t}; w_L, w_R \right) = \begin{cases} \alpha w^*(w_L, w_R) + (1 - \alpha) R(0^-, w_L), & \text{if } \frac{x}{t} < 0, \\ \alpha w^*(w_L, w_R) + (1 - \alpha) R(0^+, w_R), & \text{if } \frac{x}{t} > 0, \end{cases}$$

where we have set:

$$\alpha = \frac{b^+ - b^-}{b^+ - b^- + \sigma \Delta x} > 0, \quad (3.8)$$

and where $0^\pm$ denotes the left and right limit as x goes to zero. Such a definition of the parameter α is not unique and some comments will be made of this section.

Let us recall that the parameter σ may depend on x . With some abuse in the notations, we write $\sigma = \sigma(x = 0)$. According to recent work devoted to well-balanced schemes for source terms, the considered approximate Riemann solver involved a stationary discontinuity to deal with the source terms. Similar techniques can be found in [25] or [11].

We note that the corrected intermediate states are nothing but a convex combination involving the approximate intermediate state for the system (1.2), w^* , and the source term. As a consequence, if $\sigma = 0$, then $\alpha = 1$ and $\tilde{w}^* = w^*$ hence we get the expected transport regime. Later on, we will see that as soon as σ tends to infinity, and thus α tends to zero, the intermediate state tends to R and the diffusion regime is reached.

The approximate Riemann solver is now given by (see Figure 3.1):

$$\tilde{w}_{\mathcal{R}} \left(\frac{x}{t}; w_L, w_R \right) = \begin{cases} w_L, & \text{if } \frac{x}{t} < b^-, \\ \alpha w^*(w_L, w_R) + (1 - \alpha) R(0^-, w_L), & \text{if } \min(0, b^-) < \frac{x}{t} < \min(0, b^+), \\ \alpha w^*(w_L, w_R) + (1 - \alpha) R(0^+, w_R), & \text{if } \max(0, b^-) < \frac{x}{t} < \max(0, b^+), \\ w_R, & \text{if } \frac{x}{t} > b^+, \end{cases} \quad (3.9)$$

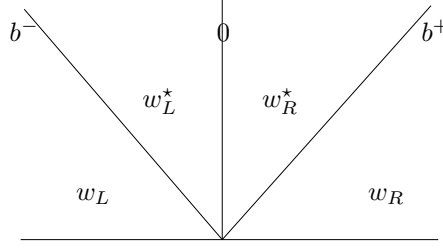


FIG. 3.1. The Riemann wave structure modified by the source term where $w_L^* = \alpha w^*(w_L, w_R) + (1 - \alpha)R(0^-, w_L)$ and $w_R^* = \alpha w^*(w_L, w_R) + (1 - \alpha)R(0^+, w_R)$.

where b^- and b^+ were introduced in (3.1). Once again, we consider the juxtaposition of the Riemann solvers stated at each interface. Such a juxtaposition is non-interacting as long as Δt satisfies the CFL-like condition (3.4).

Now, at time $t^n + t$ for all $t \in (0, \Delta t)$, the juxtaposition of non-interacting approximate Riemann solutions reads as follows:

$$\tilde{w}^h(x, t^n + t) = \tilde{w}_{\mathcal{R}}\left(\frac{x - x_{i+\frac{1}{2}}}{t^n + t}; w_i^n, w_{i+1}^n\right), \text{ if } x \in (x_i, x_{i+1}).$$

As usual, the updated states are mean values of the solution at time $t + \Delta t$ inside each cell:

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{w}^h(x, t + \Delta t) dx. \quad (3.10)$$

This above integral formulation can be developed to write the scheme under a conservation form supplemented by a relevant discretization of the source term. To address such an issue, let us consider clear notations according to the definition of the approximate Riemann solver at the interface $x_{i+\frac{1}{2}}$. In addition, for the sake of simplicity, we set:

$$\begin{aligned} \bar{x}_{i-\frac{1}{2}}^L &= x_{i-\frac{1}{2}} + \Delta t \max(0, b_{i-\frac{1}{2}}^-), \\ \bar{x}_{i-\frac{1}{2}}^R &= x_{i-\frac{1}{2}} + \Delta t \max(0, b_{i-\frac{1}{2}}^+), \\ \underline{x}_{i+\frac{1}{2}}^L &= x_{i+\frac{1}{2}} + \Delta t \min(0, b_{i+\frac{1}{2}}^-), \\ \underline{x}_{i+\frac{1}{2}}^R &= x_{i+\frac{1}{2}} + \Delta t \min(0, b_{i+\frac{1}{2}}^+). \end{aligned} \quad (3.11)$$

Then we have:

$$\begin{aligned} \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} \tilde{w}^h(x, t + \Delta t) dx &= \frac{\alpha_{i-\frac{1}{2}}}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} w^h(x, t + \Delta t) dx \\ &+ \frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} \left(\int_{x_{i-\frac{1}{2}}}^{\bar{x}_{i-\frac{1}{2}}^L} w_{i-1}^n dx + \int_{\bar{x}_{i-\frac{1}{2}}^L}^{\bar{x}_{i-\frac{1}{2}}^R} R(x_{i-\frac{1}{2}}^+, w_i^n) dx + \int_{\bar{x}_{i-\frac{1}{2}}^R}^{x_i} w_i^n dx \right). \end{aligned}$$

After a straightforward computation, we obtain:

$$\begin{aligned} \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} \tilde{w}^h(x, t + \Delta t) dx &= \frac{1}{2} w_i^n + \frac{\Delta t}{\Delta x} \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}} - \frac{\Delta t}{\Delta x} f(w_i^n) \\ &\quad + \frac{\Delta t}{\Delta x} (1 - \alpha_{i-\frac{1}{2}}) S_{i-\frac{1}{2}}^+, \end{aligned}$$

where:

$$\begin{aligned} S_{i-\frac{1}{2}}^+ &= \max(0, b_{i-\frac{1}{2}}^-) \left(w_{i-1}^n - R(x_{i-\frac{1}{2}}^+, w_i^n) \right) \\ &\quad + \max(0, b_{i-\frac{1}{2}}^+) \left(R(x_{i-\frac{1}{2}}^+, w_i^n) - w_i^n \right) + f(w_i^n). \end{aligned}$$

A similar computation leads to:

$$\begin{aligned} \frac{1}{\Delta x} \int_{x_i}^{x_{i+\frac{1}{2}}} \tilde{w}^h(x, t + \Delta t) dx &= \frac{1}{2} w_i^n - \frac{\Delta t}{\Delta x} \alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}} + \frac{\Delta t}{\Delta x} f(w_i^n) \\ &\quad + \frac{\Delta t}{\Delta x} (1 - \alpha_{i+\frac{1}{2}}) S_{i+\frac{1}{2}}^-, \end{aligned}$$

where:

$$\begin{aligned} S_{i+\frac{1}{2}}^- &= \min(0, b_{i+\frac{1}{2}}^-) \left(w_i^n - R(x_{i+\frac{1}{2}}^-, w_i^n) \right) \\ &\quad + \min(0, b_{i+\frac{1}{2}}^+) \left(R(x_{i+\frac{1}{2}}^-, w_i^n) - w_{i+1}^n \right) - f(w_i^n). \end{aligned}$$

The scheme hence reads as follows:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} \left(\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}} \right) \\ &\quad + \Delta t \left(\frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} S_{i-\frac{1}{2}}^+ + \frac{1 - \alpha_{i+\frac{1}{2}}}{\Delta x} S_{i+\frac{1}{2}}^- \right). \end{aligned} \quad (3.12)$$

First of all, we note that if we impose $\alpha_{i+\frac{1}{2}} = 1$ for all $i \in \mathbb{Z}$ we recover the initial finite volumes scheme (3.6). Then, by fixing $\alpha_{i+\frac{1}{2}} = 0$, we obtain a source term discretization:

LEMMA 3. *If $w_{i-1}^n = w_i^n = w_{i+1}^n = w \in \Omega$ a constant state, and assume σ is constant. Then the source term discretization:*

$$S_i^n = \frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} S_{i-\frac{1}{2}}^+ + \frac{1 - \alpha_{i+\frac{1}{2}}}{\Delta x} S_{i+\frac{1}{2}}^- \quad (3.13)$$

is a first-order approximation of the exact source term $\sigma(R(x_i, w) - w)$.

Proof. First, we use the usual assumption $b_{i-\frac{1}{2}}^\pm = b^\pm(w_{i-1}^n, w_i^n)$. Since $w_{i-1}^n = w_i^n = w$ then we have:

$$S_{i-\frac{1}{2}}^+ = \left(\max(0, b^+) - \max(0, b^-) \right) \left(R(x_{i-\frac{1}{2}}^+, w) - w \right) + f(w),$$

and using $w_{i+1}^n = w_i^n = w$ we also have:

$$S_{i+\frac{1}{2}}^- = \left(\min(0, b^+) - \min(0, b^-) \right) \left(R(x_{i+\frac{1}{2}}^-, w) - w \right) - f(w).$$

Moreover, from the definition of $\alpha_{i+\frac{1}{2}}$, we have:

$$1 - \alpha_{i+\frac{1}{2}} = \frac{\sigma \Delta x}{b^+ - b^- + \sigma \Delta x} \text{ and } 1 - \alpha_{i-\frac{1}{2}} = \frac{\sigma \Delta x}{b^+ - b^- + \sigma \Delta x}.$$

As a consequence, we immediately obtain:

$$S_i^n = \sigma(R(x_i, w) - w) + \mathcal{O}(\Delta x).$$

The proof is therefore completed. $\square$

To conclude the description of the numerical procedure, let us note that the proposed scheme involves a numerical flux function given by $\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}$ instead of only $\mathcal{F}_{i+\frac{1}{2}}$. Such a modified flux function is known in the literature to derive asymptotic preserving methods. The reader is referred to [8, 7, 13, 15, 23] where the weight $\alpha_{i+\frac{1}{2}}$ is introduced.

The proposed modified HLL scheme for source terms is easily proved to preserve the invariant region Ω as soon as the HLL scheme for (1.2) does it. Indeed, we establish the following statement:

THEOREM 3.1. *Assume that the CFL condition (3.4) holds, and assume that the scheme (3.6) preserves Ω . Moreover, assume that $w_i^n \in \Omega$ for all $i \in \mathbb{Z}$. Then, for all $i \in \mathbb{Z}$, the updated state w_i^{n+1} , defined by (3.12), belongs to Ω .*

Proof. Since Ω is a convex space, from the definition of w_i^{n+1} given by (3.10), the result is established as soon as we have $\tilde{w}^h(x, t^n + \Delta t) \in \Omega$ for all $x \in \mathbb{R}$. Now $\tilde{w}^h$ turns out to be the juxtaposition of functions of the form $\tilde{w}_{\mathcal{R}}$ given by (3.9) composed of states $w_i^n \in \Omega$ and convex combinations of states in Ω . We immediately deduce that $\tilde{w}^h(x, t^n + \Delta t) \in \Omega$ for all $x \in \mathbb{R}$, which concludes the proof. $\square$

To achieve the derivation of the extension of source term discretization, let us note that several choices can be suggested to define the parameters $(\alpha_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$. According to (3.12), we require the following consistency conditions:

$$\alpha_{i+\frac{1}{2}} \in (0, 1), \quad (3.14)$$

$$\alpha_{i+\frac{1}{2}} = 1 + \mathcal{O}(\Delta x), \quad (3.15)$$

$$\frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} S_{i-\frac{1}{2}}^+ + \frac{1 - \alpha_{i+\frac{1}{2}}}{\Delta x} S_{i+\frac{1}{2}}^- = \sigma(R(w_i^n) - w_i^n) + \mathcal{O}(\Delta x), \quad (3.16)$$

which are obviously satisfied by the definition (3.8)

3.3. An asymptotic preserving correction. In the present section, we do not exhibit the asymptotic preserving property satisfied by the scheme. Such an analysis will be done for each considered application. In fact, we establish here the existence of a rewriting of the system (1.1) and hence of the scheme (3.10) in order to introduce an additional parameter. This new parameter can be seen as a new degree of freedom which is essential to ensure the expected diffusive regime.

To access such an issue, let us introduce $\bar{\sigma} \geq 0$ arbitrarily chosen to write (1.1) as follows:

$$\partial_t w + \partial_x f(w) = \sigma R(x, w) - \sigma w + (\bar{\sigma} - \sigma)w, \quad (3.17)$$

$$= (\sigma + \bar{\sigma}) \left(\left(\frac{\sigma}{\sigma + \bar{\sigma}} R(x, w) + \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} w \right) - w \right). \quad (3.18)$$

For all $w \in \Omega$ we set:

$$\bar{R}(x, w) = \frac{\sigma}{\sigma + \bar{\sigma}} R(x, w) + \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} w \quad (3.19)$$

in order to have:

$$\partial_t w + \partial_x f(w) = (\sigma + \bar{\sigma})(\bar{R}(w) - w), \quad (3.20)$$

which exactly enters the formalism of (1.1). Indeed, since Ω is a convex set and the parameter σ and $\bar{\sigma}$ are positive, we have $\bar{R}(w) \in \Omega$ for all $w \in \Omega$.

As expected, this new formulation is free from the definition of $\bar{\sigma}$ which may depend on x . As a consequence, the numerical scheme (3.10) can be considered to approximate the solution of (1.1) but using the following modified definition of the discrete source term:

$$\begin{aligned} S_{i+\frac{1}{2}}^- &= \min(0, b_{i+\frac{1}{2}}^-) \left(w_i^n - \bar{R}(x_{i+\frac{1}{2}}^-, w_i^n) \right) \\ &\quad + \min(0, b_{i+\frac{1}{2}}^+) \left(\bar{R}(x_{i+\frac{1}{2}}^-, w_i^n) - w_{i+1}^n \right) - f(w_i^n), \\ S_{i-\frac{1}{2}}^+ &= \max(0, b_{i-\frac{1}{2}}^-) \left(w_{i-1}^n - \bar{R}(x_{i-\frac{1}{2}}^+, w_i^n) \right) \\ &\quad + \max(0, b_{i-\frac{1}{2}}^+) \left(\bar{R}(x_{i-\frac{1}{2}}^+, w_i^n) - w_i^n \right) + f(w_i^n). \end{aligned} \quad (3.21)$$

From a practical point of view, we have the following discretization of the modified nonlinear function $\bar{R}$:

$$\bar{R}(x_{i+\frac{1}{2}}^\pm, w_i^n) = \frac{\sigma_{i+\frac{1}{2}}}{\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}} R(x_{i+\frac{1}{2}}^\pm, w_i^n) + \frac{\bar{\sigma}_{i+\frac{1}{2}}}{\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}} w_i^n,$$

where the parameter $\bar{\sigma}_{i+\frac{1}{2}}$ will be defined later on according to the required asymptotic property.

To summarize, we will consider the following scheme with free parameter $\bar{\sigma}_{i+\frac{1}{2}}$:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}) + \Delta t S_i^n, \\ \alpha_{i+\frac{1}{2}} &= \frac{b_{i+\frac{1}{2}}^+ - b_{i+\frac{1}{2}}^-}{b_{i+\frac{1}{2}}^+ - b_{i+\frac{1}{2}}^- + (\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}) \Delta x}, \\ S_i^n &= \frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} S_{i-\frac{1}{2}}^+ + \frac{1 - \alpha_{i+\frac{1}{2}}}{\Delta x} S_{i+\frac{1}{2}}^-, \end{aligned} \quad (3.22)$$

where $S_{i+\frac{1}{2}}^\pm$ is defined by (3.21). Here, the positive parameters $(\bar{\sigma}_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$ stay free and will be defined later on.

To conclude the description of the correction, we note that the schemes (3.12) and (3.22) exactly coincide as soon as $\bar{\sigma}_{i+\frac{1}{2}} = 0$ for all i in $\mathbb{Z}$.

3.4. Extension to Godunov type solvers. The above numerical method has been derived in the framework of the HLL scheme. Arguing the work by Harten, Lax and van Leer [26], we now propose to consider more general approximate Riemann solvers. Instead of (3.1) we suggest to adopt the following Riemann solver:

$$w_{\mathcal{R}}\left(\frac{x}{t}; w_L, w_R\right) = \begin{cases} w_L, & \text{if } \frac{x}{t} < b^-, \\ w^*\left(\frac{x}{t}; w_L, w_R\right), & \text{if } b^- < \frac{x}{t} < b^+, \\ w_R, & \text{if } \frac{x}{t} > b^+, \end{cases} \quad (3.23)$$

where the intermediate state w^* now depends on $\frac{x}{t}$. Arguing [26], let us note that the consistency condition (3.2) must be satisfied. A choice of intermediate state given

by (3.3) straightforwardly enters the formalism (3.23) though other possibilities may be proposed. For instance, let us adopt an exact Riemann solver to fix b^- and b^+ given by the exact minimum and maximum characteristic speeds while $w^*(x/t)$ turns out to be the exact Riemann solution into the dependence cone. The HLLC scheme, proposed by [41] (see [40, 4] for several examples and extensions), also enters such a framework.

Next, we state an approximate Riemann solver $w_{\mathcal{R}}(\frac{x-x_{i+\frac{1}{2}}}{t+t^n}; w_i^n, w_{i+1}^n)$ at each cell interface $x_{i+\frac{1}{2}}$, to define, under the CFL restriction (3.4), the following updated approximation:

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_i} w_{\mathcal{R}}\left(\frac{x-x_{i-\frac{1}{2}}}{t^n+\Delta t}; w_{i-1}^n, w_i^n\right) dx + \frac{1}{\Delta x} \int_{x_i}^{x_{i+\frac{1}{2}}} w_{\mathcal{R}}\left(\frac{x-x_{i+\frac{1}{2}}}{t^n+\Delta t}; w_i^n, w_{i+1}^n\right) dx. \quad (3.24)$$

Involving (3.2), we obtain the conservative scheme:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} \right), \quad (3.25)$$

where:

$$\mathcal{F}_{i+\frac{1}{2}} = f(w_{i+1}^n) - \frac{\Delta x}{2\Delta t} w_{i+1}^n + \frac{1}{\Delta t} \int_{x_{i+\frac{1}{2}}}^{x_i} w_{\mathcal{R}}\left(\frac{x-x_{i+\frac{1}{2}}}{t^n+\Delta t}; w_i^n, w_{i+1}^n\right) dx. \quad (3.26)$$

After [26], this numerical method preserves the admissible states. Indeed, as soon as $w_i^n \in \Omega$ for all $i \in \mathbb{Z}$, we have $w_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$.

Now, let us correct the intermediate state function w^* involved in (3.23), as follows:

$$\tilde{w}^*(x/t; w_L, w_R) = \begin{cases} \alpha w^*(x/t; w_L, w_R) + (1-\alpha)R(0^-; w_L), & \text{if } x/t < 0, \\ \alpha w^*(x/t; w_L, w_R) + (1-\alpha)R(0^+; w_R), & \text{if } x/t > 0, \end{cases}$$

where the parameter α is once again defined by (3.8). The resulting approximate Riemann solver is therefore given by:

$$\tilde{w}_{\mathcal{R}}\left(\frac{x}{t}; w_L, w_R\right) = \begin{cases} w_L, & \text{if } \frac{x}{t} < b^-, \\ \alpha w^*\left(\frac{x}{t}; w_L, w_R\right) + (1-\alpha)R(0^-; w_L), & \text{if } \min(0, b^-) < \frac{x}{t} < \min(0, b^+), \\ \alpha w^*\left(\frac{x}{t}; w_L, w_R\right) + (1-\alpha)R(0^+; w_R), & \text{if } \max(0, b^-) < \frac{x}{t} < \max(0, b^+), \\ w_R, & \text{if } x/t > b^+. \end{cases} \quad (3.27)$$

Under the CFL-like condition (3.4), at time $t^n + t$ for all $t \in (0, \Delta t)$, the juxtaposition of non-interacting approximate Riemann solutions reads as follows:

$$\tilde{w}^h(x, t^n + t) = \tilde{w}_{\mathcal{R}}\left(\frac{x-x_{i+\frac{1}{2}}}{t^n+t}; w_i^n, w_{i+1}^n\right), \text{ if } x \in (x_i, x_{i+1}).$$

Hence, the updated states are given by:

$$w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{w}^h(x, t + \Delta t) dx. \quad (3.28)$$

We skip the detail of the computations, to obtain the following expanded formula:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}} \right) + \Delta t \left(\frac{1 - \alpha_{i-\frac{1}{2}}}{\Delta x} S_{i-\frac{1}{2}}^+ + \frac{1 - \alpha_{i+\frac{1}{2}}}{\Delta x} S_{i+\frac{1}{2}}^- \right),$$

where we have set:

$$\begin{aligned} S_{i-\frac{1}{2}}^+ &= \max(0, b_{i-\frac{1}{2}}^-) \left(w_{i-1}^n - R(x_{i-\frac{1}{2}}^+; w_i^n) \right) \\ &\quad + \max(0, b_{i-\frac{1}{2}}^+) \left(R(x_{i-\frac{1}{2}}^+; w_i^n) - w_i^n \right) + f(w_i^n). \\ S_{i+\frac{1}{2}}^- &= \min(0, b_{i+\frac{1}{2}}^-) \left(w_i^n - R(x_{i+\frac{1}{2}}^-; w_i^n) \right) \\ &\quad + \min(0, b_{i+\frac{1}{2}}^+) \left(R(x_{i+\frac{1}{2}}^-; w_i^n) - w_{i+1}^n \right) - f(w_i^n). \end{aligned}$$

This scheme obviously generalizes (3.12). It also preserves the invariant region Ω as soon as the finite volumes scheme for (3.25) does it. Indeed, we establish the following statement:

THEOREM 3.2. *Assume that the CFL condition (3.4) holds, and assume that the scheme (3.25) preserves Ω . Moreover, assume that $w_i^n \in \Omega$ for all $i \in \mathbb{Z}$. Then, $\forall i \in \mathbb{Z}$ the updated state w_i^{n+1} , defined by (3.28), belongs to Ω .*

Proof. Since Ω is a convex space, from the definition of w_i^{n+1} given by (3.28), the result is established as soon as we have $\tilde{w}^h(x, t^n + \Delta t) \in \Omega$ for all $x \in \mathbb{R}$. Now $\tilde{w}^h$ turns out to be the juxtaposition of functions of the form $\tilde{w}_{\mathcal{R}}$ given by (3.27) composed of states $w_i^n \in \Omega$ and convex combinations of states in Ω . We immediately deduce that $\tilde{w}^h(x, t^n + \Delta t) \in \Omega$ for all $x \in \mathbb{R}$, which concludes the proof. $\square$

To achieve the extension of the numerical procedure to adapt all Godunov type solvers, let us emphasize that the asymptotic preserving correction, given by (3.22), can be adopted.

4. Several applications. We now consider the mathematical models introduced in section 2 for numerical approximation. The modified HLL scheme with a correction parameter (3.22) is now applied to exhibit a robust numerical method that is able to deal with both the transport and diffusive regimes. We note that most of the work devoted to asymptotic preserving schemes pay a particular attention on the derivation of the initial HLL scheme associated to the transport regime (1.2). One of the main interest of the numerical procedure introduced in this paper stays in the lack of assumption on the initial scheme to obtain an asymptotic preserving method.

We emphasize that usual asymptotic preserving schemes (see for instance [7, 12, 13]) need very sophisticated control of the velocities $b^\pm$. Here, such a control is not necessary and we can enforce $b^+ = -b^- = b$ a given positive constant, large enough to ensure the robustness and the stability (see [10, 11, 7]). Hence, the following applications will be done using an approximate Riemann solver (3.1) defined by:

$$w^*(w_R, w_L) = \frac{1}{2}(w_L + w_R) - \frac{1}{2b}(f(w_R) - f(w_L)).$$

As a consequence, from (3.7) the numerical flux function is given by:

$$\mathcal{F}_{i+\frac{1}{2}} = \frac{1}{2}(f(w_i^n) + f(w_{i+1}^n)) - \frac{b_{i+\frac{1}{2}}}{2}(w_{i+1}^n - w_i^n). \quad (4.1)$$

Before considering applications introduced in section 2, we underline that robustness Theorem 3.1 easily holds. Hence, the invariant domain Ω is preserved by the scheme (3.21)-(3.22)-(4.1).

4.1. The Telegraph equations. The first considered application concerns the Telegraph equations (2.1). We will here use the notations (2.2). Before constructing a relevant asymptotic preserving scheme, we suggest to use this academic model with benefit to address the main difficulties which come from the preservation of the asymptotic regime, here defined by (2.4). To access such an issue, and thus to exhibit a wrong asymptotic regime, let us first omit the asymptotic preserving correction i.e. set $\bar{\sigma}_{i+\frac{1}{2}} = 0$ in the scheme (3.22).

In order to simplify our purpose, we propose to fix σ as a positive constant. Moreover, after [26, 11] (see also [13]), the parameter b in (4.1) may be fixed to the wave velocity a . Such a choice preserves the required robustness of the method. Involving these restrictive assumptions, the scheme (3.22)-(4.1) reads as follows:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a + \sigma \Delta x} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) + \Delta t S_i^n, \\ \mathcal{F}_{i+\frac{1}{2}} &= \frac{1}{2} (f(w_i^n) + f(w_{i+1}^n)) - \frac{a}{2} (w_{i+1}^n - w_i^n), \\ S_i^n &= \frac{2a\sigma}{2a + \sigma \Delta x} (R(w_i^n) - w_i^n). \end{aligned}$$

Let us now study the asymptotic behavior of this numerical method as soon as a parabolic rescaling is introduced. Denoting ε the rescaling parameter, the rescaled scheme is given by:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\varepsilon \Delta x} \frac{2a}{2a + \sigma \Delta x / \varepsilon} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) + \frac{\Delta t}{\varepsilon} S_i^n, \\ S_i^n &= \frac{2a\sigma / \varepsilon}{2a + \sigma \Delta x / \varepsilon} (R(w_i^n) - w_i^n). \end{aligned} \quad (4.2)$$

The following result is now obtained when ε tends to zero:

THEOREM 4.1 (Bad result). *When ε tends to zero, the unknown vector $w_i^{n+1} = (u_i^{n+1}, v_i^{n+1})^T$, defined by (4.2), satisfies the following discrete diffusive equation:*

$$(u + v)_i^{n+1} = (u + v)_i^n - \frac{\Delta t}{\Delta x^2} \frac{a^2}{\sigma} \left((u + v)_{i+1}^n - 2(u + v)_i^n + (u + v)_{i-1}^n \right). \quad (4.3)$$

We remark that we obtain a wrong diffusive equation since the required diffusion coefficient is $a^2/(2\sigma)$ while (4.3) gives a^2/σ .

Proof. We set $\mathcal{F}_{i+\frac{1}{2}} = (\mathcal{F}_{i+\frac{1}{2}}^u, \mathcal{F}_{i+\frac{1}{2}}^v)^T$ to write (4.2) as:

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a\varepsilon + \sigma \Delta x} (\mathcal{F}_{i+\frac{1}{2}}^u - \mathcal{F}_{i-\frac{1}{2}}^u) + \frac{\Delta t}{\varepsilon} \frac{2a}{2a\varepsilon + \sigma \Delta x} (v_i^n - u_i^n), \\ v_i^{n+1} &= v_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a\varepsilon + \sigma \Delta x} (\mathcal{F}_{i+\frac{1}{2}}^v - \mathcal{F}_{i-\frac{1}{2}}^v) + \frac{\Delta t}{\varepsilon} \frac{2a}{2a\varepsilon + \sigma \Delta x} (u_i^n - v_i^n). \end{aligned}$$

Let us next consider the updated formula for $u - v$ to get:

$$\begin{aligned} (u - v)_i^{n+1} &= (u - v)_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a\varepsilon + \sigma \Delta x} \left((\mathcal{F}_{i+\frac{1}{2}}^u - \mathcal{F}_{i+\frac{1}{2}}^v) - (\mathcal{F}_{i-\frac{1}{2}}^u - \mathcal{F}_{i-\frac{1}{2}}^v) \right), \\ &\quad + \frac{\Delta t}{\varepsilon} \frac{4a}{2a\varepsilon + \sigma \Delta x} (v - u)_i^n. \end{aligned}$$

As soon as ε tends to zero, we obtain for all $i \in \mathbb{Z}$, $(u - v)_i^n = 0$.

Then we consider the scheme satisfied by $u + v$:

$$(u + v)_i^{n+1} = (u + v)_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a\varepsilon + \sigma\Delta x} \left((\mathcal{F}_{i+\frac{1}{2}}^u + \mathcal{F}_{i+\frac{1}{2}}^v) - (\mathcal{F}_{i-\frac{1}{2}}^u + \mathcal{F}_{i-\frac{1}{2}}^v) \right).$$

As ε tends to 0 we have:

$$(u + v)_i^{n+1} = (u + v)_i^n - \frac{\Delta t}{\Delta x^2} \frac{2a}{\sigma} \left((\mathcal{F}_{i+\frac{1}{2}}^u + \mathcal{F}_{i+\frac{1}{2}}^v) - (\mathcal{F}_{i-\frac{1}{2}}^u + \mathcal{F}_{i-\frac{1}{2}}^v) \right)_{|(u-v)_i^n=0}.$$

By definition of the numerical flux, we have:

$$\begin{aligned} \mathcal{F}_{i+\frac{1}{2}}^u &= \frac{av_i^n + av_{i+1}^n}{2} - \frac{a}{2}(u_{i+1}^n - u_i^n), \\ \mathcal{F}_{i+\frac{1}{2}}^v &= \frac{-au_i^n - av_{i+1}^n}{2} - \frac{a}{2}(v_{i+1}^n - v_i^n). \end{aligned}$$

and therefore we have:

$$\left((\mathcal{F}_{i+\frac{1}{2}}^u + \mathcal{F}_{i+\frac{1}{2}}^v) - (\mathcal{F}_{i-\frac{1}{2}}^u + \mathcal{F}_{i-\frac{1}{2}}^v) \right)_{|(u-v)_i^n=0} = a \left((u + v)_{i+1}^n - 2(u + v)_i^n + (u + v)_{i-1}^n \right).$$

The proof is then achieved. $\square$

To recover the required diffusion coefficient, we have to consider the asymptotic preserving correction. We therefore consider $\bar{\sigma} \geq 0$, a constant to be defined, to write (3.22) as:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\Delta x} \frac{2a}{2a + (\sigma + \bar{\sigma})\Delta x} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) + \Delta t S_i^n, \\ \mathcal{F}_{i+\frac{1}{2}} &= \frac{1}{2}(f(w_i^n) + f(w_{i+1}^n)) - \frac{a}{2}(w_{i+1}^n - w_i^n), \\ S_i^n &= \frac{2a(\sigma + \bar{\sigma})}{2a + (\sigma + \bar{\sigma})\Delta x} (\bar{R}(w_i^n) - w_i^n) = \frac{2a\sigma}{2a + (\sigma + \bar{\sigma})\Delta x} (R(w_i^n) - w_i^n). \end{aligned}$$

Now we introduce the rescaling parameter ε to obtain the following rescaled scheme:

$$\begin{aligned} w_i^{n+1} &= w_i^n - \frac{\Delta t}{\varepsilon \Delta x} \frac{2a}{2a + (\sigma + \bar{\sigma})\Delta x / \varepsilon} (\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}) + \frac{\Delta t}{\varepsilon} S_i^n, \\ S_i^n &= \frac{2a\sigma / \varepsilon}{2a + (\sigma + \bar{\sigma})\Delta x / \varepsilon} (R(w_i^n) - w_i^n). \end{aligned} \tag{4.4}$$

The asymptotic regime satisfied by this numerical method is stated in the following result:

THEOREM 4.2 (Good result). *Assume $\bar{\sigma} = \sigma$. When ε tends to zero, the unknown vector $w_i^{n+1} = (u_i^{n+1}, v_i^{n+1})^T$, defined by (4.4), satisfies the following discrete diffusive equation:*

$$(u + v)_i^{n+1} = (u + v)_i^n - \frac{\Delta t}{\Delta x^2} \frac{a^2}{2\sigma} \left((u + v)_{i+1}^n - 2(u + v)_i^n + (u + v)_{i-1}^n \right).$$

Enforcing a relevant $\bar{\sigma}$ therefore allows us to obtain the correct diffusive regime.

Proof. The computations are similar to those used to establish (4.3). We finally obtain:

$$(u + v)_i^{n+1} = (u + v)_i^n - \frac{\Delta t}{\Delta x^2} \frac{a^2}{\sigma + \bar{\sigma}} \left((u + v)_{i+1}^n - 2(u + v)_i^n + (u + v)_{i-1}^n \right).$$

Using $\bar{\sigma} = \sigma$ completes the proof. $\square$

To conclude this application, let us emphasize that extensions to non-constant σ does not yield any difficulty. Indeed, we have:

THEOREM 4.3. *Consider the scheme (3.22)-(4.1) for the Telegraph equations (2.1). Assume that $\forall i \in \mathbb{Z}$, $b_{i+\frac{1}{2}}^+ = -b_{i+\frac{1}{2}}^- = a$ and $\bar{\sigma}_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}$. Then, the asymptotic behavior of the scheme is given by:*

$$(u+v)_i^{n+1} = (u+v)_i^n - \frac{a^2 \Delta t}{2\Delta x^2} \left(\frac{(u+v)_{i+1}^n - (u+v)_i^n}{\sigma_{i+\frac{1}{2}}} - \frac{(u+v)_i^n - (u+v)_{i-1}^n}{\sigma_{i-\frac{1}{2}}} \right).$$

4.2. Euler with high friction. We here look for the numerical approximation for the Euler equation with friction given by (2.5)-(2.6). We use the notation (2.7) to consider the scheme (3.22)-(4.1). To complete this scheme, we suggest after [26, 11, 12] to define b as:

$$b_{i+\frac{1}{2}} = \max(|u_i^n \pm c_i^n|, |u_{i+\frac{1}{2}}^* \pm c_{i+\frac{1}{2}}^*|, |u_{i+1}^n \pm c_{i+1}^n|), \quad (4.5)$$

where $c = \sqrt{p'(\rho)}$. We denote $\mathcal{F}_{i+\frac{1}{2}}^\rho$ and $\mathcal{F}_{i+\frac{1}{2}}^{\rho u}$ the components of $\mathcal{F}_{i+\frac{1}{2}}$ and we set:

$$\alpha_{i+\frac{1}{2}} = \frac{2b_{i+\frac{1}{2}}}{2b_{i+\frac{1}{2}} + (\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}})\Delta x},$$

where the positive sequence $(\bar{\sigma}_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$ has to be specified. After straightforward computations, the scheme reads:

$$\begin{aligned} \rho_i^{n+1} &= \rho_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}^\rho - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}^\rho) + \Delta t \frac{\alpha_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}}{\Delta x} (\rho u)_i^n, \\ (\rho u)_i^{n+1} &= (\rho u)_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}^{\rho u} - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}^{\rho u}) + \Delta t S_i^{\rho u}, \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} S_i^{\rho u} &= \left(\frac{b_{i-\frac{1}{2}} \sigma_{i-\frac{1}{2}}}{2b_{i-\frac{1}{2}} + (\sigma_{i-\frac{1}{2}} + \bar{\sigma}_{i-\frac{1}{2}})\Delta x} + \frac{b_{i+\frac{1}{2}} \sigma_{i+\frac{1}{2}}}{2b_{i+\frac{1}{2}} + (\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}})\Delta x} \right), \\ &+ \frac{\alpha_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}}{\Delta x} (\rho_i^n (u_i^n)^2 + p(\rho_i^n)). \end{aligned} \quad (4.7)$$

Now, let us focus on the asymptotic preserving property satisfied by the scheme and exhibit the relevant choice of $\bar{\sigma}_{i+\frac{1}{2}}$. In this sense, we consider the rescaled scheme obtained by substituting Δt by $\Delta t/\varepsilon$ and $\sigma_{i+1/2}, \bar{\sigma}_{i+1/2}$ by $\sigma_{i+1/2}/\varepsilon, \bar{\sigma}_{i+1/2}/\varepsilon$. As ε tends to zero, from a straightforward computation, we obtain $(\rho u)_i^n = 0$. The asymptotic behavior of the updated formula for the density is given by:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x^2} \left(\frac{2b_{i+\frac{1}{2}}}{\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}} \mathcal{F}_{i+\frac{1}{2}}^\rho - \frac{2b_{i-\frac{1}{2}}}{\sigma_{i-\frac{1}{2}} + \bar{\sigma}_{i-\frac{1}{2}}} \mathcal{F}_{i-\frac{1}{2}}^\rho \right) \Big|_{(\rho u)_i^n = 0}.$$

Since our choice of $\mathcal{F}^\rho$ yields:

$$\mathcal{F}_{i+\frac{1}{2}}^\rho = \frac{1}{2} ((\rho u)_i^n + (\rho u)_{i+1}^n) - \frac{b_{i+\frac{1}{2}}}{2} (\rho_{i+1}^n - \rho_i^n),$$

we get:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x^2} \left(\frac{b_{i+\frac{1}{2}}^2}{\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}} (\rho_{i+1}^n - \rho_i^n) - \frac{b_{i-\frac{1}{2}}^2}{\sigma_{i-\frac{1}{2}} + \bar{\sigma}_{i-\frac{1}{2}}} (\rho_i^n - \rho_{i-1}^n) \right).$$

In the diffusive limit equation (2.9), the diffusion coefficient is given by $p'(\rho)/\sigma$. As a consequence, an immediate choice would be to fix $\bar{\sigma}_{i+\frac{1}{2}}$ to:

$$\bar{\sigma}_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} \left(\frac{2b_{i+\frac{1}{2}}^2}{p'(\rho_i^n) + p'(\rho_{i+1}^n)} - 1 \right). \quad (4.8)$$

Let us note that $\bar{\sigma}$ may become negative. To overcome this, we may choose larger values of b , and thus we have the following statement:

THEOREM 4.4. *Let $(\rho_i^{n+1}, (\rho u)_i^{n+1})^T$ be defined by the numerical scheme (4.6)-(4.7)-(4.8). Consider the rescaled scheme obtained by substituting Δt by $\Delta t/\varepsilon$, $\sigma_{i+\frac{1}{2}}$ by $\sigma_{i+\frac{1}{2}}/\varepsilon$ and $\bar{\sigma}_{i+\frac{1}{2}}$ by $\bar{\sigma}_{i+\frac{1}{2}}/\varepsilon$. Then, when the rescaling parameter tends to zero, ρ_i^n satisfies the following diffusive equation for all $i \in \mathbb{Z}$:*

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x^2} \left(\frac{p'(\rho_i^n) + p'(\rho_{i+1}^n)}{2\sigma_{i+\frac{1}{2}}} (\rho_{i+1}^n - \rho_i^n) - \frac{p'(\rho_i^n) + p'(\rho_{i-1}^n)}{2\sigma_{i-\frac{1}{2}}} (\rho_i^n - \rho_{i-1}^n) \right).$$

To conclude this application, let us illustrate the behavior of the scheme when a more accurate method is used. Indeed, the numerical flux function given by (4.1) turns out to be too viscous for most applications of interest.

We now consider the numerical flux (3.7) where, after [11] (see also [5, 16]) we propose to set $b_{i+\frac{1}{2}}^+ = u_{i+1} + a_{i+\frac{1}{2}}$, $b_{i+\frac{1}{2}}^- = u_i^n - a_{i+\frac{1}{2}}$ with $a_{i+\frac{1}{2}} = \max(c_i^n, c_{i+1}^n)$. Skipping the details of the now classic computations, we exhibit the limit scheme satisfied by the density in the limit to zero of the rescaling parameter:

$$\rho_i^{n+1} = \rho_i^n + \frac{\Delta t}{\Delta x^2} \left(\frac{-b_{i+\frac{1}{2}}^+ b_{i+\frac{1}{2}}^-}{\sigma_{i+\frac{1}{2}} + \bar{\sigma}_{i+\frac{1}{2}}} (\rho_{i+1}^n - \rho_i^n) - \frac{-b_{i-\frac{1}{2}}^+ b_{i-\frac{1}{2}}^-}{\sigma_{i-\frac{1}{2}} + \bar{\sigma}_{i-\frac{1}{2}}} (\rho_i^n - \rho_{i-1}^n) \right). \quad (4.9)$$

Let us set $p'_{i+\frac{1}{2}} = (p'(\rho_i^n) + p'(\rho_{i+1}^n))/2$. With $-b_{i+\frac{1}{2}}^+ b_{i+\frac{1}{2}}^- > p'_{i+\frac{1}{2}}$, we choose

$$\bar{\sigma}_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} \left(\frac{-b_{i+\frac{1}{2}}^+ b_{i+\frac{1}{2}}^-}{p'_{i+\frac{1}{2}}} - 1 \right),$$

then we get the following expected diffusion regime from (4.9):

$$\rho_i^{n+1} = \rho_i^n + \frac{\Delta t}{\Delta x^2} \left(\frac{p'_{i+\frac{1}{2}}}{\sigma_{i+\frac{1}{2}}} (\rho_{i+1}^n - \rho_i^n) - \frac{p'_{i-\frac{1}{2}}}{\sigma_{i-\frac{1}{2}}} (\rho_i^n - \rho_{i-1}^n) \right),$$

which is nothing but the discrete form of (2.9).

4.3. The M1 model for radiative transfer. Let us apply the scheme (3.22)-(4.1) to the M1 model for radiative transfer (2.10)-(2.12). According to [7, 26, 13], the parameter b is assumed to be large enough to ensure the robustness and stability requirements. We here choose $b_{i+\frac{1}{2}} = c$ where c is the speed of the light. As before,

we note $\mathcal{F}_{i+\frac{1}{2}}^E$, $\mathcal{F}_{i+\frac{1}{2}}^F$ and $\mathcal{F}_{i+\frac{1}{2}}^T$ the components of the numerical flux function. We also set:

$$\alpha_{i+\frac{1}{2}} = \frac{2c}{2c + (\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}})\Delta x},$$

where $(\bar{\sigma}_{i+\frac{1}{2}})_{i \in \mathbb{Z}}$ is the free parameter that has to be specified. The numerical schemes becomes:

$$\begin{aligned} E_i^{n+1} &= E_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}^E - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}^E) + \Delta t S_i^E, \\ F_i^{n+1} &= F_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}^F - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}^F) + \Delta t S_i^F, \\ \rho C_v T_i^{n+1} &= \rho C_v T_i^n - \frac{\Delta t}{\Delta x} (\alpha_{i+\frac{1}{2}} \mathcal{F}_{i+\frac{1}{2}}^T - \alpha_{i-\frac{1}{2}} \mathcal{F}_{i-\frac{1}{2}}^T) + \Delta t S_i^T. \end{aligned}$$

Concerning the definition of the source term, let us recall that the non-linear function $R(x, w)$ defined by (2.15) depends on the parameter $\sigma(x)$. Then we have

$$R(x_{i+\frac{1}{2}}^\pm, w_i^n) = \left(\frac{\sigma_{i+\frac{1}{2}}^e a(T_i^n)^4 + \sigma_{i+\frac{1}{2}}^1 E_i^n}{\sigma_{i+\frac{1}{2}}^m}, \frac{\sigma_{i+\frac{1}{2}}^2 F_i^n}{\sigma_{i+\frac{1}{2}}^m}, \frac{\frac{\sigma_{i+\frac{1}{2}}^a E_i^n}{\rho C_v} + \sigma_{i+\frac{1}{2}}^{3,\pm} T_i^n}{\sigma_{i+\frac{1}{2}}^m} \right)^T, \quad (4.10)$$

where we have set, according to (2.14):

$$\begin{aligned} \sigma_{i+\frac{1}{2}}^1 &= \sigma_{i+\frac{1}{2}}^m - \sigma_{i+\frac{1}{2}}^a, \\ \sigma_{i+\frac{1}{2}}^2 &= \sigma_{i+\frac{1}{2}}^m - \sigma_{i+\frac{1}{2}}^f, \\ \sigma_{i-\frac{1}{2}}^{3,+} &= \sigma_{i-\frac{1}{2}}^m - \frac{\sigma_{i-\frac{1}{2}}^e a(T_i^n)^3}{\rho C_v}, \\ \sigma_{i+\frac{1}{2}}^{3,-} &= \sigma_{i+\frac{1}{2}}^m - \frac{\sigma_{i+\frac{1}{2}}^e a(T_i^n)^3}{\rho C_v}. \end{aligned}$$

After a straightforward computation, the source term given by (3.21)-(3.22)-(4.1)-(4.10) reads as follows:

$$\begin{aligned} S_i^E &= \frac{c}{2 + (\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}})\Delta x} (\sigma_{i+\frac{1}{2}}^e a(T_i^n)^4 - \sigma_{i+\frac{1}{2}}^a E_i^n) \\ &\quad + \frac{c}{2 + (\sigma_{i-\frac{1}{2}}^m + \bar{\sigma}_{i-\frac{1}{2}})\Delta x} (\sigma_{i-\frac{1}{2}}^e a(T_i^n)^4 - \sigma_{i-\frac{1}{2}}^a E_i^n) + \frac{\alpha_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}}{\Delta x} F_i^n, \\ S_i^F &= \frac{c}{2 + (\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}})\Delta x} \sigma_{i+\frac{1}{2}}^m F_i^n \\ &\quad + \frac{c}{2 + (\sigma_{i-\frac{1}{2}}^m + \bar{\sigma}_{i-\frac{1}{2}})\Delta x} \sigma_{i-\frac{1}{2}}^m F_i^n + \frac{\alpha_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}}{\Delta x} c^2 P(E_i^n, F_i^n), \\ S_i^T &= \frac{c}{2 + (\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}})\Delta x} (\sigma_{i+\frac{1}{2}}^a E_i^n - \sigma_{i+\frac{1}{2}}^e a(T_i^n)^4) \\ &\quad + \frac{c}{2 + (\sigma_{i-\frac{1}{2}}^m + \bar{\sigma}_{i-\frac{1}{2}})\Delta x} (\sigma_{i-\frac{1}{2}}^a E_i^n - \sigma_{i-\frac{1}{2}}^e a(T_i^n)^4). \end{aligned}$$

Now let us consider the asymptotic regime obtained by substituting Δt by $\Delta t/\varepsilon$, the opacity σ by σ/ε , and $\bar{\sigma}$ by $\bar{\sigma}/\varepsilon$. The details of the computations, which are easy but arduous, are not given here. As soon as ε tends to zero, we obtain $\forall i \in \mathbb{Z}$:

$$F_i^n = 0 \quad \text{and} \quad E_i^n = a(T_i^n)^4.$$

Moreover, the following discrete diffusive equation is reached for the temperature:

$$\begin{aligned} (\rho C_v T + a T^4)_i^{n+1} &= (\rho C_v T + a T^4)_i^n \\ &- \frac{\Delta t}{\Delta x^2} \left(\frac{2(\mathcal{F}_{i+\frac{1}{2}}^E + \mathcal{F}_{i+\frac{1}{2}}^T)}{\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}}} - \frac{2(\mathcal{F}_{i-\frac{1}{2}}^E + \mathcal{F}_{i-\frac{1}{2}}^T)}{\sigma_{i-\frac{1}{2}}^m + \bar{\sigma}_{i-\frac{1}{2}}} \right) \Big|_{F_i^n=0, E_i^n=a(T_i^n)^4}. \end{aligned}$$

Since the numerical flux function is given by (4.1), we get:

$$\begin{aligned} (\rho C_v T + a T^4)_i^{n+1} &= (\rho C_v T + a T^4)_i^n \\ &+ c \frac{\Delta t}{\Delta x^2} \left(\frac{((\rho C_v T + a T^4)_{i+1}^n - (\rho C_v T + a T^4)_i^n)}{\sigma_{i+\frac{1}{2}}^m + \bar{\sigma}_{i+\frac{1}{2}}} - \right. \\ &\quad \left. \frac{((\rho C_v T + a T^4)_i^n - (\rho C_v T + a T^4)_{i-1}^n)}{\sigma_{i-\frac{1}{2}}^m + \bar{\sigma}_{i-\frac{1}{2}}} \right). \end{aligned}$$

In order to ensure that this scheme approximates the equilibrium diffusion equation (5.3), we fix:

$$\bar{\sigma}_{i+\frac{1}{2}} = 3\sigma_{i+\frac{1}{2}}^f \left(1 + \rho C_v \frac{T_{i+1}^n - T_i^n}{a(T_{i+1}^n)^4 - a(T_i^n)^4} \right) - \sigma_{i+\frac{1}{2}}^m. \quad (4.11)$$

Which is positive as soon as $3\sigma_{i+\frac{1}{2}}^f \geq \sigma_{i+\frac{1}{2}}^m$.

5. Numerical Examples.

5.1. Telegraph equations. We consider the Telegraph equations to validate the proposed asymptotic preserving numerical correction. To address such an issue, we approximate the solution of the Cauchy problem made of (2.1) and the following initial data:

$$u(x, 0) = v(x, 0) = \begin{cases} 1 & \text{if } x < 0, \\ 0.005 & \text{if } x > 0. \end{cases}$$

Concerning the model constants, we have fixed:

$$a = 2 \quad \text{and} \quad \sigma = 1000.$$

In Figure 5.1, we compare the approximated solutions, at time $t = 10$, obtained with the modified HLL scheme (3.12) without asymptotic preserving correction, and with the corrected asymptotic preserving HLL scheme (3.22). As expected, we note the crucial impact of the correction.

5.2. Euler equations with high friction. In this paragraph, we apply our scheme to the resolution of Euler equations with high friction (2.5)-(2.6). The objective is to illustrate its behavior when considering σ that is a function of x . To access such an issue, we consider the following expression for σ :

$$\sigma = \sigma(x) = e^x [\kappa \gamma e^{-(\gamma+1)x} - (\rho u)_L],$$

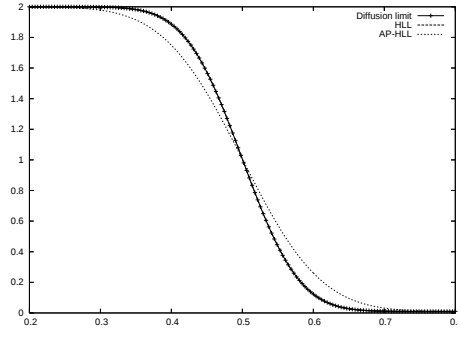


FIG. 5.1. *Telegraph equation approximations: comparison of the diffusion solution, the approximation obtained by the HLL scheme with source term discretization but without correction, and the asymptotic preserving HLL approximation.*

where $(\rho u)_L$ is the left boundary value imposed on ρu . With this form of σ , the steady solution has the following form:

$$\begin{aligned}\rho &= e^{-x}, \\ \rho u &= (\rho u)_L.\end{aligned}\tag{5.1}$$

Figure (5.2) shows the results of two computations performed with this expression of σ and two different space discretizations. We have chosen $\kappa = 50$, $\gamma = 2$, $\rho_L = 1$, $(\rho u)_L = 1$ and $\bar{\sigma} = 0$ and compared the steady states with the exact solution (5.1). We can see that even with a coarse grid the behavior of the solution is perfectly

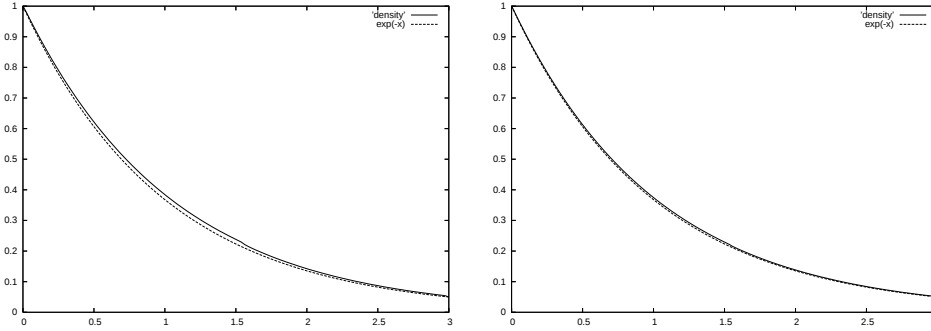


FIG. 5.2. *Density - Euler with friction $\sigma(x)$, $\Delta x = 2.5 \cdot 10^{-2}$ (left) and $\Delta x = 1. \cdot 10^{-2}$ (right).*

captured.

Since we are free to take σ as a function of x in our definition of the numerical scheme, it is able to deal even with highly non-linear choices. This is a very important edge compared to other schemes found in the literature.

5.3. M1 model for radiative transfer. We consider here the case of a left entering Marshak wave in an highly absorbing medium. In such an application, the M1 model (2.10)-(2.12) has to degenerate into the equilibrium diffusion equation (2.16) which describes the behavior of the temperature.

For the computations, we set $\sigma^e = 1500$ and $\rho C_v = 10^{-4}$. The initial and left boundary conditions are respectively given by $E_0 = aT_0^4$, $F_0 = 0$ and $E_L = aT_L^4$, $F_L = 0$.

Figure (5.3) shows the comparison between the solutions computed by four different numerical schemes. The first one approximates the equilibrium diffusion equation (2.16) and serves as a reference here. The other schemes consist in approximations of the M1 model for radiative transfer (2.10)-(2.12) given by our formalism (3.22)-(4.1) for different choices of numerical fluxes associated with the transport limit (1.2). We have used a variable speed HLL scheme but also the classical HLL scheme with $b^+ = -b^- = c$ and eventually an asymptotic preserving correction parameter $\bar{\sigma}$. Without the correction parameter ($\bar{\sigma} = 0$), the HLL scheme $b^+ = -b^- = c$ is not

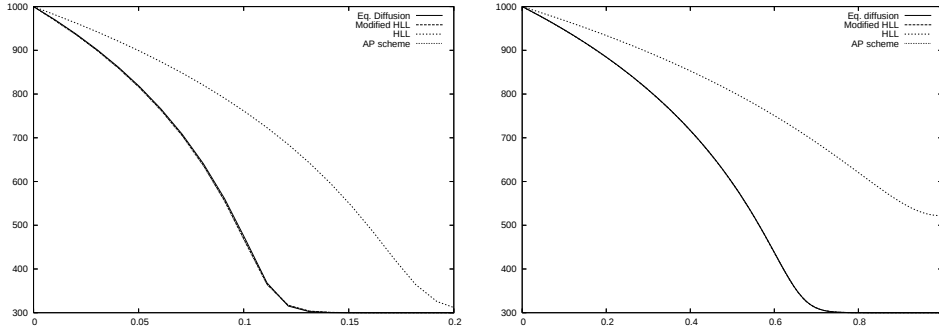


FIG. 5.3. Temperature predicted by the M1-model compared with the equilibrium diffusion equation at time $t = 3.10^{-6}s$ and $t = 1.10^{-4}s$.

asymptotic preserving and we clearly see on figure (5.3) that it does not predict the correct behavior of the solution. However, if we introduce the choice of $\bar{\sigma}$ according to (4.11), we can see that the resulting numerical scheme is asymptotic preserving. Indeed, its results are very close to those of the variable speed HLL (which is asymptotic preserving in this particular case) and of the equilibrium diffusion. Moreover, it behaves well even for small times.

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