# TOWARDS WEAKLY HYPERBOLIC MODELS FOR INACCESSIBLE PORE VOLUME MODEL IN POLYMER FLOODING

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**AMS subject classifications.** 35L65, 76S05

**Key words.** polymer flooding, inaccessible pore volume, adsorption, system of conservation laws, weak hyperbolicity

**Abstract.** In the simulations of enhanced oil recovery by polymer flooding, it is widely acknowledged that the inaccessible pore volume (IPV) must be properly accounted for. The effect of IPV is to make the interstitial velocity of polymer molecules higher than that of water molecules. This acceleration is traditionally modeled by a velocity enhancement factor. The difficulty, however, lies in finding a relevant closure law for this factor that preserves well-posedness of the overall flow model. Previous works by Bartelds et al. (*Transp. Porous Med.* **26**, 75–88, 1997) and Hilden et al. (*Transp. Porous Med.* **114**, 65–86, 2016) have demonstrated that ill-posedness occurs for the popular constant enhancement factor and they proposed some alternative physical IPV laws that partially addressed the question of stability.

This paper is aimed at bringing new mathematical insights into the design of IPV laws that ensure weak hyperbolicity for a Buckley-Leverett type two-phase polymer flow model. To begin with, we show that by switching to Lagrangian coordinates, it is possible to derive several practical and meaningful sufficient conditions for weak hyperbolicity. Next, special solutions to some of these sufficient conditions are shown to coincide with well-known IPV laws in the literature and then investigated more thoroughly. Finally, by patching together these piecewise sufficient conditions, we are in a position to develop original IPV laws that guarantee weak hyperbolicity for the flow model.

## 1. Introduction.

**1.1. Context and objectives.** Polymer flooding is an enhanced oil recovery (EOR) process by which a mixture of water and polymer is injected into an oil field [13, 31] in order to increase production. The viscosity of polymer reduces the mobility of the water, thus avoiding hydrodynamic instabilities of water-oil fronts [25, 28] and enabling oil, especially a very viscous one, to be better pushed, which results in a higher extraction rate. Numerous experiments have shown that in such a mixture, the polymer molecules tend to travel faster than the water molecules. This effect, called *velocity enhancement* is essentially due to the *inaccessible pore volume* (IPV): in the porous medium, big polymer molecules cannot penetrate all the pores, unlike small water molecules. Having less volume to flood, the polymer molecules flow at an increased velocity [7, 20].

In most physical models, this acceleration effect is represented by an enhancement or IPV factor that measures the ratio between the polymer velocity and the water velocity. This factor $\gamma \geq 1$ is to be expressed as an algebraic function of other state variables, such as the water saturation $s \in [0, 1]$ and the polymer concentration $c \geq 0$ for a two-phase flow. The simplest instance of such a closure law $\gamma(s, c)$ is a constant IPV factor $\gamma \equiv 1 + \eta$, $\eta \geq 0$, which is commonly used in industrial codes. Since we have to model wave propagation phenomena with bounded velocities, we here adopt a nonlinear transport equation of Buckley-Leverett type. Unfortunately, it was shown [4, 15] that, when combined with a Buckley-Leverett type one-dimensional two-phase

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polymer flow model [5, 26]

$$\partial_t(s) \qquad + \partial_x(f) \quad = 0, \tag{1.1a}$$
$$\partial_t(sc + a) + \partial_x(fc\gamma) = 0, \tag{1.1b}$$

where the fractional flux $f(s, c)$ and the adsorption $a(c)$ are given functions, a constant IPV factor $\gamma > 1$ yields an ill-posed system: there exist states $(s, c)$ at which the characteristic speeds of (1.1) are conjugate complex numbers. The loss of weak hyperbolicity for the system (1.1) at these states is a major impediment to the stability of linear perturbations [12, 22], which in turn prevents numerical simulations to be carried out without a blow-up in the polymer concentration at the water front [15].

The main difficulty comes from the the saturation-dependent *percolation* law $\gamma(s)$. For instance, in [4] the authors introduced $\gamma(s) = s/(s - s_\bullet)$ in order to recover real characteristic speeds for $s > s_\bullet$, where $s_\bullet$ is the inaccessible pore volume saturation. The shortcoming of the percolation law is that it requires $s_\bullet < s_\flat$, where $s_\flat$ is the irreducible water saturation. For $s_\bullet > s_\flat$, [15] advocated two models called *uniform polymer diffusion* and *non-uniform polymer diffusion* based on a less stringent notion of "acceptable" shocks instead of weak hyperbolicity. As shown by [15] themselves and further analyzed by [27], these polymer diffusion models may violate hyperbolicity and end up giving nonphysical polymer concentration values. A detailed review of the stability issues associated with various IPV factors can be found in [14, 33].

On the grounds of this state of the art, it seems that weak hyperbolicity of (1.1) must be the primary concern in the design of IPV laws. Unlike [4, 15] where microscopic models are adopted, here we study the macroscopic model (1.1). Note that we are focused only on weak hyperbolicity, i.e., the existence of two real eigenvalues of the Jacobian matrix of (1.1). We do not ask anything about any stronger kind of hyperbolicity. This leaves open the possibility for resonance, but to successfully rule out ellipticity.

**1.2. Main results and outline.** We first provide a precise mathematical statement of the problem in section 2, where the basic concepts are recalled. Emphasis is laid on the properties of the fractional flux, in relation with its specific dependence on the phase mobilities and the mobility reduction. Classical and more recent IPV laws are reviewed and discussed in from the standpoint of weak hyperbolicity.

Section 3 is devoted to the derivation of several practically sufficient conditions for real eigenvalue existence. The key ingredient to this endeavor is the Lagrangian framework, which greatly simplifies calculations. Sufficient conditions can then be obtained by various splittings of the discriminant of the characteristic polynomial into two parts, one of which is a perfect square. Aside from this usual path already followed by [4, 15], we expose another methodology inspired from traffic flow modeling, which amounts to specifying a velocity value that must lie in-between the two real eigenvalues. This viewpoint makes more physical sense and gives rise to even more sufficient conditions. Besides, we point out a crucial connection between the model without adsorption ($a \equiv 0$) and the model with adsorption ($a \geq 0$). Sufficient conditions for the latter can then be inferred from those for the former by means of a change of variables.

A sufficient condition for weak hyperbolicity is a partial differential inequality on $\gamma$ or a related function. Sometimes, an explicit solution can be determined. In section 4, we study such particular solutions and we show that classical IPV laws, including percolation, can be recovered in this way. For the percolation law, we exhibit a change of variable that transforms the model without adsorption into that without IPV. This testifies to the fact that the mathematical behavior of the model without adsorption, equipped with the percolation law, is similar to that of a system introduced by [19], the Riemann problem of which was solved by [17]. For the constant IPV law, we revisit the existence of elliptic states with a rigorous characterization of the frontier between the weak hyperbolic region and the elliptic one for the model without adsorption.

From the observation that a single sufficient condition can secure weak hyperbolicity only on a subdomain of $(s, c) \in [0, 1] \times \mathbb{R}_+$, it is natural to combine different sufficient conditions on different subdomains in the hope of building up an IPV law that is weakly hyperbolic, over the whole domain $(s, c) \in [0, 1] \times \mathbb{R}_+$. This is the program that we undertake in section 5. The main obstacle to the feasibility of this construction is that, due to the matching conditions, the maximal amplitude of the IPV factor may be more or less limited. Nevertheless, there are ways to improve this construction.

**2. Setting of the problem.** We start by formulating the question of weak hyperbolicity for a prototype model in §2.1. Analytical results in the trivial no-IPV case ($\gamma \equiv 1$) are recalled in §2.2, together with some notions that will be helpful in the sequel. The definition and the limit of existing non-trivial IPV laws are highlighted in §2.3.

**2.1. Flow model with IPV in polymer flooding.** We consider a two-phase (water-oil) immiscible incompressible polymer flow in a one-dimensional porous media. The derivation of its equations from a full three-dimensional model, detailed in [4, 9, 26], results in the system of conservation laws

$$\partial_t(s) + \partial_x f(s, c) = 0, \tag{2.1a}$$
$$\partial_t(sc + a(c)) + \partial_x(f(s, c)c\gamma(s, c)) = 0, \tag{2.1b}$$

where the unknowns $(s, c)$ depend on $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Here, $s \in [0, 1]$ represents the water saturation, so that $1 - s$ is the oil saturation, and $c \in \mathbb{R}_+$ the polymer concentration in water. In (2.1), the adsorption $a(\cdot)$ is a $\mathcal{C}^1$-function of $c \in \mathbb{R}_+$ such that

$$a(0) = 0, \qquad a'(c) \geq 0, \tag{2.2}$$

hence $a(c) \geq 0$ for all $c \geq 0$.

The fractional flux $f(\cdot, \cdot)$ is a $\mathcal{C}^1$-function of $(s, c) \in [0, 1] \times \mathbb{R}_+ =: \Omega$ given by

$$f(s, c) = \frac{\Lambda_w(s)/R(c)}{\Lambda_o(1 - s) + \Lambda_w(s)/R(c)}, \tag{2.3}$$

where the mobility reduction $R(\cdot)$ is a $\mathcal{C}^1$-function of $c \in \mathbb{R}_+$, which satisfies

$$R(0) = 1, \qquad R'(c) > 0, \qquad \lim_{c \to +\infty} R(c) = +\infty. \tag{2.4}$$

Hence, $R(c) > 1$ for $c > 0$.

From the physical point of view, the polymer concentration must naturally be bounded from above by a maximal value representing the limit of dissolution in water. Likewise, there is a maximal limit in the mobility reduction. However, from a mathematical point of view, it is necessary to allow the polymer concentration to go to infinity in order to cope with a possible model singularity.

The quantities $\Lambda_w(s)$ and $\Lambda_o(1-s)$ are respectively the water and oil mobilities. Their definitions involve two parameters $s_\flat$ and $s_\sharp$ such that

$$0 \le s_\flat < s_\sharp \le 1. \tag{2.5}$$

These two quantities characterize the porous medium where $s_\flat \in [0,1)$ denotes the irreducible water saturation, while $s_{\text{or}} := 1 - s_\sharp \in [0,1)$ stand for the residual oil saturation. It is then assumed that $\Lambda_w(\cdot)$ and $\Lambda_o(\cdot)$ are $\mathcal{C}^1$-functions defined over $[0,1]$ such that

$$\Lambda_w(s) = 0 \ \text{ on } [0, s_\flat], \qquad \Lambda_w(s) > 0 \ \text{ on } (s_\flat, 1], \qquad \Lambda_w'(s) \ge 0, \tag{2.6a}$$
$$\Lambda_o(s) = 0 \ \text{ on } [0, 1-s_\sharp], \qquad \Lambda_o(s) > 0 \ \text{ on } (1-s_\sharp, 1] \qquad \Lambda_o'(s) \ge 0. \tag{2.6b}$$

Thus, $\Lambda_w(s)$ vanishes for $s \in [0, s_\flat]$ and increases with respect to $s$, while $\Lambda_o(1-s)$ vanishes for $s \in [s_\sharp, 1]$ and decreases with respect to $s$. A prominent example of mobility functions comes from laws detailed in [6] for relative permeabilties which reads

$$\Lambda_w(s) = \Lambda_w^{\text{m}} \left( \frac{s - s_\flat}{s_\sharp - s_\flat} \right)_+^{\alpha_w} \qquad \text{and} \qquad \Lambda_o(1-s) = \Lambda_o^{\text{m}} \left( \frac{s_\sharp - s}{s_\sharp - s_\flat} \right)_+^{\alpha_o}, \tag{2.7}$$

where $r_+ = \max(r, 0)$ stands for the positive part of $r \in \mathbb{R}$ and $(\Lambda_w^{\text{m}}, \Lambda_o^{\text{m}}) \in (\mathbb{R}_+^*)^2$ are two positive constants. The use of the positive part function enables us to extend the definition of the mobilities over $s \in [0,1]$ without any impact on the fractional flux. Note that to ascertain differentiability of $\Lambda_w(\cdot)$ at $s_\flat$ when $s_\flat > 0$ and of $\Lambda_o(\cdot)$ at $s_\sharp$ when $s_\sharp < 1$, in (2.7) we must require

$$\alpha_w > 1 \qquad \text{and} \qquad \alpha_o > 1. \tag{2.8}$$

Thanks to the assumptions made on $R$, $\Lambda_w$ and $\Lambda_o$, the fractional flux $f$ satisfies

$$f(s,c) = 0 \ \text{ on } [0, s_\flat] \times \mathbb{R}_+ \qquad \text{and} \qquad f(s,c) = 1 \ \text{ on } [s_\sharp, 1] \times \mathbb{R}_+. \tag{2.9}$$

Furthermore, on $(s_\flat, s_\sharp) \times \mathbb{R}_+$,

$$0 < f(s,c) < 1, \qquad f_s(s,c) > 0, \qquad f_c(s,c) < 0, \tag{2.10}$$

where $f_s$ is the partial derivative with respect to $s$ at fixed $c$ and $f_c$ is the partial derivative with respect to $c$ at fixed $s$.

From the most general viewpoint, the velocity enhancement or IPV function $\gamma(s,c)$ is continuous and piecewise differentiable function of $(s,c) \in \Omega$. The fact that its derivatives $\gamma_s$ and $\gamma_c$ are authorized to have jumps will make it easier to define suitable IPV laws over the whole domain $\Omega$. However, it is fundamental to demand that the product $fc\gamma$ be a $\mathcal{C}^1$ function of $(s,c) \in \Omega$. From physical considerations, it is assumed that

$$\gamma(s,c) \ge 1 \qquad \text{and} \qquad \gamma_s(s,c) \le 0. \tag{2.11}$$

The last inequality, which asserts that $\gamma$ is a non-increasing function of $s$ at fixed $c$, is a reasonable physical behavior.

Once the closure laws $a$, $R$, $\Lambda_w$, $\Lambda_o$, $\gamma$ have been selected according to conditions (2.2), (2.4), (2.6), (2.11) and their regularity assumptions, the system (2.1) can be cast under the abstract form

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0, \tag{2.12}$$

where $\mathbf{w} = (s, sc + a)$ is the vector of conservative variables and $\mathbf{f} = (f, fc\gamma)$ is the smooth enough flux function such that the Jacobian matrix $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$ exists.

The system (2.1) is said to be

- *hyperbolic* if the $2 \times 2$ matrix $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$ is diagonalizable over $\mathbb{R}$;
- *weakly hyperbolic* if the $2 \times 2$ matrix $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$ possesses two real eigenvalues, the eigenspace is not required to be of dimension two.

Hyperbolicity is a desirable feature of the model, insofar as it guarantees stability of linear perturbation around a steady state [12, 22]. We clarify conditions on the IPV law $\gamma$ so that the system (2.1) is weakly hyperbolic. In other words, we accept defective eigenvalues and the associated *resonance* phenomenon, which is known to appear in many celebrated models: pressureless gas [23], nozzle flow [1], Baer-Nunziato [3]. The situation we actually want to forbid is *ellipticity*, that is, the existence of states at which $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$ possesses two conjugate complex eigenvalues with nonzero imaginary parts, since this gives rise to nonphysical ill-posedness.

**2.2. Preliminary notions and results.** The no-IPV case, which corresponds to a constant $\gamma = 1$, has been extensively analyzed for $[s_\flat, s_\sharp] = [0, 1]$. In particular, the Riemann problem associated with

$$\partial_t(s) \qquad + \partial_x(f) \;\; = 0, \tag{2.13a}$$
$$\partial_t(sc + a) + \partial_x(fc) = 0, \tag{2.13b}$$

was solved by [18] for $a'(c) > 0$, $a''(c) < 0$, and by [24] for $a'(c) > 0$, $a''(c) > 0$. A straightforward calculation shows that the characteristic speeds of system (2.13), namely the eigenvalues of the Jacobian matrix $\nabla_{\mathbf{w}}\mathbf{f}(\mathbf{w})$, at any state $\mathbf{w} = (s, sc + a)$ are always real and equal to

$$f_s(s, c) \qquad \text{and} \qquad \frac{f(s, c)}{s + a'(c)}. \tag{2.14}$$

This testifies to the weak hyperbolicity of (2.13). Hyperbolicity is in fact *strict* (two distinct real eigenvalues) except on the set

$$f_s = f/(s + a'). \tag{2.15}$$

The no-adsorption subcase ($a = 0$) of the no-IPV case has also received a lot of attention. Indeed, the system

$$\partial_t(s) \;\; + \partial_x(f) \;\; = 0, \tag{2.16a}$$
$$\partial_t(sc) + \partial_x(fc) = 0, \tag{2.16b}$$

belongs to a family of systems introduced by [19] in elasticity. System (2.16) is weakly hyperbolic with real characteristic speeds

$$f_s(s, c) \qquad \text{and} \qquad \frac{f(s, c)}{s}. \tag{2.17}$$

Resonance occurs when $f_s(s,c) = f(s,c)/s$. The Riemann problem associated with system (2.16) was solved by [17] for $[s_\flat, s_\sharp] = [0,1]$.

Let us notice that the quantity

$$u(s,c) = \frac{f(s,c)}{s}, \tag{2.18}$$

referred to as the water *interstitial* velocity, and it is of main importance in the general case. It can be verified that

$$u(s,c) = 0 \ \text{ on } [0, s_\flat] \times \mathbb{R}_+ \qquad \text{and} \qquad u(s,c) = 1/s \ \text{ on } [s_\sharp, 1] \times \mathbb{R}_+. \tag{2.19}$$

Note that $u$ is well-defined at $s = 0$ by continuity of $u(s,c) = 0$ for $s \in (s_\flat, s_\sharp)$. Furthermore,

$$0 \le u(s,c) \le 1/s \qquad \text{and} \qquad u_c(s,c) \le 0, \tag{2.20}$$

with strict inequalities on $(s_\flat, s_\sharp) \times \mathbb{R}_+$.

The domain characterized by $f_s(s,c) = f(s,c)/s$, which can then be equivalently characterized as $u_s(s,c) = 0$, will turn out to play a very special role in the general case as detailed in section 5. The following result gives a precise description of this set.

Proposition 2.1. *Assume that the function*

$$\Upsilon(s) = -\frac{s\Lambda_o(1-s)}{\Lambda_w(s)}, \tag{2.21}$$

*is increasing on $(s_\flat, s_\sharp)$, while its derivative $\Upsilon'(s)$ is decreasing on the same interval, with*

$$\lim_{s \downarrow s_\flat} \Upsilon'(s) = +\infty, \qquad \lim_{s \uparrow s_\sharp} \Upsilon'(s) = 0. \tag{2.22}$$

*Then, the set of states $(s,c) \in \Omega$ such that $f_s(s,c) = f(s,c)/s$, or equivalently $u_s(s,c) = 0$, namely the resonant region, consists of the vertical strip $[0, s_\flat] \times \mathbb{R}_+$ and the* transition *curve*

$$s = (\Upsilon')^{-1}(1/R(c)) =: s_*(c). \tag{2.23}$$

*The above transition curve lies entirely in the region $(s_\flat, s_\sharp) \times \mathbb{R}_+$, is increasing with respect to $s \ge (\Upsilon')^{-1}(1) = s_*(0)$ and admits the vertical asymptote $s = s_\sharp$ when $c \to +\infty$. Moreover, we have*

$$u_s(s,c) > 0 \ \text{ for } s_\flat < s < s_*(c) \qquad \text{and} \qquad u_s(s,c) < 0 \ \text{ for } s_*(c) < s < s_\sharp. \tag{2.24}$$

*Proof.* As long as $s$ stays in $[0, s_\flat]$, we have $f(s,c) = 0$ and $u(s,c) = 0$, so that the strip $[0, s_\flat] \times \mathbb{R}_+$ belongs to the searched resonant region. On the other hand, for $s \in [s_\sharp, 1]$, since $f(s,c) = 1$ and $u(s,c) = 1/s$, then we have $u_s(s,c) = -1/s^2 \neq 0$. Thus, no state in the strip $[s_\sharp, 1] \times \mathbb{R}_+$ can be in the resonant domain.

We now focus on $s \in (s_\flat, s_\sharp)$. From definition (2.3), it follows that

$$\frac{sf_s(s,c)}{f(s,c)} = \frac{s\left(\Lambda_w'(s)\Lambda_o(1-s) + \Lambda_o'(1-s)\Lambda_w(s)\right)}{\Lambda_w(s)\Lambda_o(1-s) + \Lambda_w^2(s)/R(c)}. \tag{2.25}$$

With the resonant region defined by the equation $sf_s(s,c)/f(s,c) = 1$, the above identity now reads locally over the resonant region

$$\begin{aligned}\frac{1}{R(c)} &= \frac{s\left(\Lambda_w'(s)\Lambda_o(1-s)+\Lambda_o'(1-s)\Lambda_w(s)\right)-\Lambda_w(s)\Lambda_o(1-s)}{\Lambda_w^2(s)},\\ &= \Upsilon'(s), \qquad (2.26)\end{aligned}$$

where $\Upsilon(s)$ is the function defined by (2.21). Since $\Upsilon'$ is imposed to be a decreasing function of $s$, therefore, the reciprocal function $(\Upsilon')^{-1}$ is well-defined as a decreasing function from $(0,+\infty)$ to $(s_\flat, s_\sharp)$. Applying $(\Upsilon')^{-1}$ to both sides of (2.26) yields (2.23). Since $R$ is an increasing function, $1/R$ is a decreasing function and its composition with the decreasing function $(\Upsilon')^{-1}$ results in an increasing function $s_* = (\Upsilon')^{-1}\circ(1/R)$. When $c\to+\infty$, the last assumption of (2.4) implies $1/R(c)\to 0$, thus $s_*(c)\to s_\sharp$. Statement (2.24) is an easy consequence and the proof is achieved. ∎

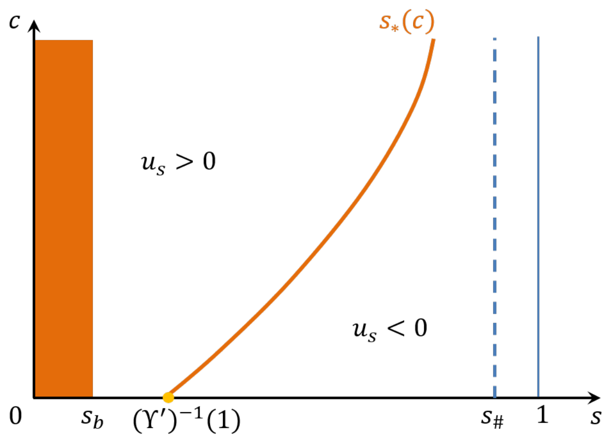


FIG. 2.1. *States for which $u_s = 0$ (in orange) the $(s,c)$-plane.*

The behavior of $s_*(\cdot)$ is depicted in Figure 2.1. In the literature, the existence and uniqueness of $s_*(c)$ have to be postulated over a finite interval $c\in[0,c^{\mathrm{m}}]$, and nothing can be said about its increasing or decreasing behavior. Here, Proposition 2.1 connects these properties to the function $\Upsilon$. It turns out that the requirements made of $\Upsilon$ are met by the Brooks-Corey laws.

PROPOSITION 2.2. *For the Brooks-Corey laws* (2.7)*, the conditions on $\Upsilon$ hold true if*

$$\alpha_o \geq \alpha_w > 1. \qquad (2.27)$$

*Proof.* Plugging (2.7) into (2.21), we get the explicit expression

$$\Upsilon(s) = -(s_\sharp - s_\flat)^{\alpha_w-\alpha_o}\frac{\Lambda_o^{\mathrm{m}}}{\Lambda_w^{\mathrm{m}}}\frac{s(s_\sharp - s)^{\alpha_o}}{(s-s_\flat)^{\alpha_w}} \qquad (2.28)$$

in the Brooks-Corey case. It is convenient to put

$$\sigma = \frac{s-s_\flat}{s_\sharp - s_\flat}\in[0,1] \qquad \text{and} \qquad S = s_\sharp - s_\flat > 0,$$

so that $s = S\sigma + s_\flat$ and

$$\Upsilon(s) = -\frac{\Lambda_o^{\mathrm{m}}}{\Lambda_w^{\mathrm{m}}}(S\sigma + s_\flat)\frac{(1-\sigma)^{\alpha_o}}{\sigma^{\alpha_w}} = -\frac{\Lambda_o^{\mathrm{m}}}{\Lambda_w^{\mathrm{m}}}(1-\sigma)^{\alpha_o}(S\sigma^{-\alpha_w+1} + s_\flat\sigma^{-\alpha_w}). \tag{2.29}$$

A straightforward calculation shows that

$$\Upsilon'(s) = \frac{\Lambda_o^{\mathrm{m}}}{\Lambda_w^{\mathrm{m}}} S\,(1-\sigma)^{\alpha_o-1}\sigma^{-\alpha_w-1}\Big\{s_\flat\alpha_w + [s_\flat(\alpha_o-\alpha_w) + S(\alpha_w-1)]\sigma s + S(\alpha_o-\alpha_w+1)\sigma^2\Big\}, \tag{2.30}$$

so that $\Upsilon'(s) > 0$ for all $s \in (s_\flat, s_\sharp)$ if $\alpha_o \geq \alpha_w \geq 1$. Moreover, it is easily seen that

$$\lim_{s\downarrow s_\flat}\Upsilon'(s) = +\infty, \qquad \lim_{s\uparrow s_\sharp}\Upsilon'(s) = 0.$$

Concerning the monotonicity of $\Upsilon'$, a further differentiation leads to

$$\Upsilon''(s) = -\frac{\Lambda_o^{\mathrm{m}}}{\Lambda_w^{\mathrm{m}}} S^2\,(1-\sigma)^{\alpha_o-2}\sigma^{-\alpha_w-2}\Big\{(\alpha_o-1)\sigma P(\sigma) + (1-\sigma)Q(\sigma)\Big\} \tag{2.31}$$

with

$$P(\sigma) = s_\flat\alpha_w + [s_\flat(\alpha_o-\alpha_w) + S(\alpha_w-1)]\sigma + S(\alpha_o-\alpha_w+1)\sigma^2, \tag{2.32a}$$

$$Q(\sigma) = s_\flat\alpha_w(\alpha_w+1) + \alpha_w[s_\flat(\alpha_o-\alpha_w) + S(\alpha_w-1)]\sigma + S(\alpha_w-1)(\alpha_o-\alpha_w+1)\sigma^2. \tag{2.32b}$$

If $\alpha_o \geq \alpha_w \geq 1$, then $P > 0$ and $Q > 0$ for all $s \in (s_\flat, s_\sharp)$. Hence, $\Upsilon'' < 0$, which completes the proof. ∎

**2.3. Standard IPV laws and their limits.** The mathematically pleasant IPV law $\gamma = 1$ is not in agreement with physical observations, because this means that the large size of the polymer molecules does not influence their path in a porous medium. For realistic reservoir simulations, an IPV factor $\gamma > 1$ is mandatory. Below we enumerate the IPV laws available in the literature.

1. By far, the most frequently used IPV law is the constant IPV

$$\gamma(s,c) = 1 + \eta, \qquad \eta > 0, \tag{2.33}$$

   due to its simplicity. Many physical arguments have been raised [4, 15] against (2.33). From a mathematical standpoint, the main result to date [4, 15] is that when $\eta > 0$ is small enough, then the states $(s_*(c), c)$ on the transition curve of the no-IPV law become elliptic. In §4.1, we will shed more light on this question.

2. In [4], a saturation-dependent percolation law is introduced as follows:

$$\gamma(s,c) = \frac{s}{s - s_\bullet}, \qquad s > s_\bullet, \tag{2.34}$$

   where $s_\bullet \in (0, s_\sharp)$ stand for a new parameter called *IPV saturation*. The advantage of (2.34) is that it ensures weak hyperbolicity for the system (2.1). In §4.3, we will show that for the no-adsorption case, system (2.1) equipped with (2.34) is in fact equivalent, in a sense to be prescribed, to a no-adsorption no-IPV model (2.16). This brings a new understanding to the percolation law. Now, the shortcoming of (2.34) is that it holds only for $s > s_\bullet$. For the percolation law to be valid and remain bounded on $(s_\flat, s_\sharp)$, it is necessary that $s_\bullet < s_\flat$. But such a condition is not always physically observed.

3. The uniform diffusion law

$$\gamma(s,c) = \begin{cases} \dfrac{s}{\Lambda_w(s)} \dfrac{\Lambda_w(s) - \Lambda_w(s_\bullet)}{s - s_\bullet} & \text{if } s_\bullet > s_\flat \text{ and } s > s_\bullet, \\ \dfrac{s}{s - s_\bullet} & \text{if } s_\bullet \le s_\flat \text{ and } s > s_\bullet, \\ 0 & \text{if } s \le s_\bullet, \end{cases} \tag{2.35}$$

and the non-uniform diffusion law

$$\gamma(s,c) = \begin{cases} \left[1 - \dfrac{s_\flat}{s_\bullet} \dfrac{\Lambda_w(s_\bullet)}{\Lambda_w(s)}\right] \dfrac{s}{s - s_\flat} & \text{if } s_\bullet > s_\flat \text{ and } s > s_\bullet, \\ \dfrac{s}{s - s_\bullet} & \text{if } s_\bullet \le s_\flat \text{ and } s > s_\bullet, \\ 1 & \text{if } s \le s_\bullet, \end{cases} \tag{2.36}$$

proposed by [15] in order to tackle the difficult case $s_\bullet > s_\flat$, as well as to extend $\gamma$ to the whole interval $s \in [0,1]$. In place of weak hyperbolicity, the authors opted for a less demanding but more handy notion of stability for shock fronts. While numerical simulations demonstrate an usually better behavior, the remaining elliptic regions may still cause trouble, as pointed out by the authors themselves and confirmed by [27]. Besides, note that the condition $\gamma_s \le 0$ is not always satisfied.

Our main goal is to design a physically acceptable IPV law that ensures weakly hyperbolic for (2.1) over $\Omega$.

**3. Sufficient conditions for weak hyperbolicity.** We first set up in §3.1 a framework which mimics compressible hydrodynamics and where the calculation of characteristic polynomials becomes simpler. Next, in §3.2 we follow the usual technique of splitting the discriminant in order to derive some first sufficient conditions. These conditions give an interpretation in terms of control of eigenvalues, which then appears as a more productive technique.

**3.1. Transformation to Lagrange coordinates.** The velocity $u$ introduced in (2.18) enables to reformulate the system (2.1) under a form with more resemblance to usual hydrodynamics, namely,

$$\partial_t(s) \qquad + \partial_x(su) \qquad = 0, \tag{3.1a}$$

$$\partial_t(sc + a) + \partial_x(scu + q) = 0, \tag{3.1b}$$

in which we propose to call

$$q = (\gamma - 1)scu \tag{3.2}$$

the *IPV deviation*. Instead of working with the conservative variables $(s, sc + a)$, which is not convenient because an inversion of $sc + a$ is needed to compute $c$, it is more judicious to take $(s, c)$ as primary variables. The system (3.1) can then be put under the quasilinear form

$$\begin{pmatrix} 1 & 0 \\ c & s + a' \end{pmatrix} \partial_t \begin{pmatrix} s \\ c \end{pmatrix} + \begin{pmatrix} u + su_s & su_c \\ c(u + su_s) + q_s & s(u + cu_c) + q_c \end{pmatrix} \partial_x \begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.3}$$

This leads to consider the characteristic polynomial

$$\mathcal{P}^{\mathrm{Eul}}(\zeta) = \begin{vmatrix} u + su_s - \zeta & su_c \\ c(u + su_s) + q_s - \zeta c & s(u + cu_c) + q_c - \zeta(s + a') \end{vmatrix}, \tag{3.4}$$

where the superscript "Eul" reminds that we are in the Eulerian coordinates $(x, t)$. Multiplying the first row of (3.4) by $c$ and subtracting the product from the second row, we obtain

$$\mathcal{P}^{\mathrm{Eul}}(\zeta) = \begin{vmatrix} u + su_s - \zeta & su_c \\ q_s & su + q_c - \zeta(s + a') \end{vmatrix}. \tag{3.5}$$

The Eulerian eigenvalues are the roots of $\mathcal{P}^{\mathrm{Eul}}$. To answer the question of how to make them real, we suggest to adopt a Lagrangian type formulation.

The equation (3.1a), rearranged as $\partial_t(s) = \partial_x(-su)$, can be seen as the Schwarz condition for cross partial derivatives. As a consequence, there exists a function $m$ of $(x, t)$ such that $\mathrm{d}m = s\,\mathrm{d}x - su\,\mathrm{d}t$. In other words,

$$\partial_x m = s, \qquad \partial_t m = -su. \tag{3.6}$$

We refer to $m$ as the Lagrangian abscissa. By a change of referential from $(x, t)$ to $(m, t)$, the system (3.1) becomes

$$\partial_t(\tau) \qquad - \partial_m(u) \qquad = 0, \tag{3.7a}$$

$$\partial_t(c + a\tau) + \partial_m(q - au) = 0, \tag{3.7b}$$

with

$$\tau = \frac{1}{s}. \tag{3.8}$$

The full procedure is detailed in [9, 29, 32]. In compressible hydrodynamics, $\tau$ is the specific volume, while $s$ (often denoted by $\rho$) is the density. The interest of Lagrangian coordinates is to suppress the nonlinearities $(su, scu)$ associated with the Eulerian representation and to keep only the genuine nonlinearities $(u, q)$ intrinsic to the problem.

From the quasilinear form

$$\begin{pmatrix} 1 & 0 \\ a & 1 + a'\tau \end{pmatrix} \partial_t \begin{pmatrix} \tau \\ c \end{pmatrix} + \begin{pmatrix} -u_\tau & -u_c \\ q_\tau - au_\tau & q_c - (au)_c \end{pmatrix} \partial_m \begin{pmatrix} \tau \\ c \end{pmatrix} = 0, \tag{3.9}$$

of (3.7), we define the Lagrangian characteristic polynomial

$$\mathcal{P}^{\mathrm{Lag}}(\lambda) = \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau - au_\tau - \lambda a & q_c - (au)_c - \lambda(1 + a'\tau) \end{vmatrix}. \tag{3.10}$$

Multiplying the first row of (3.10) by $a$ and subtracting the product from the second row, we obtain

$$\mathcal{P}^{\mathrm{Lag}}(\lambda) = \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}. \tag{3.11}$$

After expansion and simplification, we get

$$\mathcal{P}^{\mathrm{Lag}}(\lambda) = (1 + a'\tau)\lambda^2 + [(1 + a'\tau)u_\tau - (q_c - a'u)]\lambda + q_\tau u_c - (q_c - a'u)u_\tau. \tag{3.12}$$

The roots of $\mathcal{P}^{\mathrm{Lag}}$ are related to those of $\mathcal{P}^{\mathrm{Eul}}$ as followins.

LEMMA 3.1. *For all $\lambda \in \mathbb{C}$, we have*

$$\mathcal{P}^{\mathrm{Lag}}(\lambda) = s\mathcal{P}^{\mathrm{Eul}}(u + \tau\lambda). \tag{3.13}$$

*Proof.* Inserting $\zeta = u + \tau\lambda$ into (3.5) and since $\tau s = 1$, we get

$$\mathcal{P}^{\mathrm{Eul}}(u + \tau\lambda) = \begin{vmatrix} su_s - \tau\lambda & su_c \\ q_s & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}. \tag{3.14}$$

Using now $su_s = -\tau u_\tau$ and $q_s = -\tau^2 q_\tau$, we successively have

$$\begin{aligned}
\mathcal{P}^{\mathrm{Eul}}(u + \tau\lambda) &= \begin{vmatrix} -\tau u_\tau - \tau\lambda & su_c \\ -\tau^2 q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}, \\
&= (-s) \begin{vmatrix} \tau^2 u_\tau + \tau^2\lambda & -u_c \\ -\tau^2 q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}, \\
&= (-s)(-\tau^2) \begin{vmatrix} -u_\tau - \lambda & -u_c \\ q_\tau & q_c - a'u - \lambda(1 + a'\tau) \end{vmatrix}, \\
&= \tau\mathcal{P}^{\mathrm{Lag}}(\lambda).
\end{aligned}$$

Multiplying both sides by $s$, we get the expected relation (3.13). □

Since the relation between the Lagrangian eigenvalue $\lambda$ and the Eulerian eigenvalue $\zeta = u + \tau\lambda$ involves real numbers $u$ and $\tau$, requiring that $\mathcal{P}^{\mathrm{Eul}}$ has real roots is nothing but requiring that $\mathcal{P}^{\mathrm{Lag}}$ has real roots.

**3.2. Splitting of the discriminant.** The most elementary way to guarantee that the quadratic polynomial $\mathcal{P}^{\mathrm{Lag}}$ has two real roots is to make sure that its discriminant is non-negative, which leads to a partial differential inequality. The idea is to split the discriminant as the sum of a perfect square and a residual term with a sign to be controlled.

THEOREM 3.2. *Let $g = (\gamma - 1)sc$ so that $q = gu$. If any of the following Splitting Conditions (SC) is satisfied at state $(s, c) \in \Omega$:*

$$\text{[SC0]} \qquad (g_c - a')u_\tau - g_\tau u_c \geq 0, \tag{3.15}$$

$$\text{[SC1]} \qquad q_\tau \geq 0, \tag{3.16}$$

$$\text{[SC2]} \qquad (1 + a'\tau)g_\tau - g(g_c - a') \geq 0, \tag{3.17}$$

*then the Lagrangian characteristic polynomial $\mathcal{P}^{Lag}$ admits two real roots.*

*Proof.* Involving (3.12), the discriminant of $\mathcal{P}^{\mathrm{Lag}}$ reads

$$\Delta = \left((1 + a'\tau)u_\tau - (q_c - a'u)\right)^2 + 4(1 + a'\tau)\left((q_c - a'u)u_\tau - q_\tau u_c\right). \tag{3.18}$$

Then, the splitting condition [SC0] imposes that the second summand is non-negative. Indeed, because of (2.2) and $\tau > 0$, we get $s \geq 0$ as soon as we have

$$(q_c - a'u)u_\tau - q_\tau u_c \geq 0. \tag{3.19}$$

Substituting $gu$ for $q$, the above inequality becomes

$$(g_c u - a'u)u_\tau - g_\tau u u_c \geq 0. \tag{3.20}$$

Involving (2.20), we get $u \geq 0$, and we immediately notice that (3.15) implies (3.19).

Next, applying the identity $(A-B)^2 + 4AB = (A+B)^2$ with $A = (1+a'\tau)u_\tau$ and $B = q_c - a'u$ to (3.18), we may rewrite the discriminant as follows:

$$\Delta = \left((1+a'\tau)u_\tau + (q_c - a'u)\right)^2 - 4(1+a'\tau)q_\tau u_c, \tag{3.21}$$

to get another splitting reformulation of the discriminant. Then, we obtain $\Delta \geq 0$ as long as

$$-u_c q_\tau \geq 0. \tag{3.22}$$

With (2.20), we have $u_c \leq 0$, and it is thus obvious that (3.16) implies (3.22).

An even more subtle splitting of the discriminant can be worked out by first substituting $gu$ for $q$ in (3.21) to have

$$\Delta = \left((1+a'\tau)u_\tau - a'u + g_c u + g u_c\right)^2 - 4(1+a'\tau)(g_\tau u + g u_\tau)u_c, \tag{3.23}$$

and next by applying the identity $(A+B)^2 = (A-B)^2 + 4AB$ with $A = (1+a'\tau)u_\tau - a'u + g_c u$ and $B = g u_c$ to obtain

$$\Delta = \left((1+a'\tau)u_\tau - a'u + g_c u - g u_c\right)^2 - 4\left((1+a'\tau)g_\tau - g(g_c - a')\right) u u_c, \tag{3.24}$$

after some algebraic cancellations. Then, [SC2] results from imposing that the second summand is non-negative, that is,

$$-u u_c \left((1+a'\tau)g_\tau - g(g_c - a')\right) \geq 0. \tag{3.25}$$

Since $u \geq 0$ and $u_c \leq 0$, we note that (3.17) implies (3.25), and the proof is completed. $\square$

From now on, we stress out that the three sufficient splitting conditions of Theorem 3.2 are not equivalent, since they are associated with three different splittings of the discriminant. It is also worth mentioning that similar calculations were carried out in [4, 15] using Eulerian coordinates, which made them a little more cumbersome.

The splitting condition [SC1] may be more convenient since it involves the adsorption $a$. However, it does involve the velocity $u$. In contrast, [SC2] is attractive since it does not involve the velocity $u$, but for involving the adsorption rate $a'$. The dependence with respect to $a'$ can be removed in the following way. Rewriting (3.17) as

$$g_\tau - g g_c + a'(\tau g)_\tau \geq 0, \tag{3.26}$$

we observe that $(\tau g)_\tau = (\,(\gamma-1)c\,)_\tau = \gamma_\tau = -s^2\gamma_s$. Therefore, if there can be found an IPV law $\gamma$ such that

$$[\mathsf{SC2'}] \qquad g_\tau - g g_c \geq 0, \qquad \gamma_s \leq 0, \tag{3.27}$$

then (3.26) will be automatically satisfied. We recall that the condition $\gamma_s \leq 0$ is physically natural and was already postulated in (2.11).

Now, the sufficient conditions [SC0], [SC1] and [SC2] are completed by other possibilities and by information about the location of booth real roots of $\mathcal{P}^{\mathrm{Lag}}$, or equivalently $\mathcal{P}^{\mathrm{Eul}}$.

Lemma 3.3. *If any of the conditions in Table 3.1 is satisfied at state $(s,c) \in \Omega$, then $\mathcal{P}^{Lag}$, or equivalently $\mathcal{P}^{Eul}$, admits two real roots. Moreover, the two Lagrangian roots enclose $\lambda_*$ while the two Eulerian roots enclose $\zeta_* = u + \tau\lambda_*$.*

| # | practical expression | $\lambda_*$ | $\zeta_*$ |
|---|---|---|---|
| [SC0] | $(g_c - a')u_\tau - g_\tau u_c \geq 0$ | $0$ | $u$ |
| [SC1] | $q_\tau \geq 0$ | $-u_\tau$ | $f_s$ |
| [SC2] | $(1 + a'\tau)g_\tau - g(g_c - a') \geq 0$ | $\dfrac{(g_c - a')u}{1 + a'\tau}$ | $\dfrac{(\gamma c)_c f}{s + a'}$ |
| [SC3] | $(1 + a'\tau)(q - au)_\tau - a(q - au)_c \geq 0$ | $-u_\tau + \dfrac{au_c}{1 + a'\tau}$ | $f_s + \dfrac{au_c}{s + a'}$ |
| [SC4] | $(1 + a'\tau)(g_c u_\tau - g_\tau u_c) - a'(gu)_c \geq 0$ | $-\dfrac{a'u}{1 + a'\tau}$ | $\dfrac{f}{s + a'}$ |

TABLE 3.1
*Sufficient conditions for weak hyperbolicity.*

*Proof.* First, we notice that $\mathcal{P}^{\mathrm{Lag}}$, defined by (3.12), is a strictly convex quadratic polynomial, with $\lim_{\lambda\to\pm\infty} \mathcal{P}^{\mathrm{Lag}}(\lambda) = +\infty$. If $\mathcal{P}^{\mathrm{Lag}}(\lambda_*) < 0$ for some $\lambda_* \in \mathbb{R}$, then according to the intermediate value Theorem, there exist $\lambda_- \in (-\infty, \lambda_*)$ and $\lambda_+ \in (\lambda_*, \infty)$ such that $\mathcal{P}^{\mathrm{Lag}}(\lambda_-) = \mathcal{P}^{\mathrm{Lag}}(\lambda_+) = 0$, and $\lambda_- < \lambda_* < \lambda_+$.

If $\mathcal{P}^{\mathrm{Lag}}(\lambda_*) = 0$ for some $\lambda_* \in \mathbb{R}$, then $\lambda_*$ is a real root. Since $\mathcal{P}^{\mathrm{Lag}}$ has real coefficients, the other root is also real and we still have $\lambda_- \leq \lambda_* \leq \lambda_+$.

For each row of Table 3.1, insert the value of $\lambda_*$ in the third column into (3.12) to compute $\mathcal{P}^{\mathrm{Lag}}(\lambda_*)$. After algebraic cancellations, we get

$$
\begin{aligned}
\mathcal{P}^{\mathrm{Lag}}(0) &= u[u_c g_\tau - u_\tau(g_c - a')],\\
\mathcal{P}^{\mathrm{Lag}}(-u_\tau) &= u_c q_\tau,\\
\mathcal{P}^{\mathrm{Lag}}\left(\frac{(g_c - a')u}{1 + a'\tau}\right) &= uu_c\left[g_\tau - g\frac{g_c - a'}{1 + a'\tau}\right]\\
\mathcal{P}^{\mathrm{Lag}}\left(-u_\tau + \frac{au_c}{1 + a'\tau}\right) &= u_c\left[(q - au)_\tau - \frac{a(q - au)_c}{1 + a'\tau}\right],\\
\mathcal{P}^{\mathrm{Lag}}\left(-\frac{a'u}{1 + a'\tau}\right) &= u\left[g_\tau u_c - g_c u_\tau + \frac{a'}{1 + a'\tau}(gu)_c\right].
\end{aligned}
$$

Since $1+a'\tau$, $u$ and $-u_c$ are non-negative, the expression in each row implies $\mathcal{P}^{\mathrm{Lag}}(\lambda_*) \leq 0$. The proof is thus achieved. ◻

As a matter of fact, this approach is inspired from traffic flow modeling [2, 21, 34] where it is customary to prescribe some physically reasonable value $\lambda_* \in \mathbb{R}$ as characteristic speed of the model, namely, $\mathcal{P}^{\mathrm{Lag}}(\lambda_*) = 0$. In our problem, most values of $\zeta_*$ in the last column are taken to be the eigenvalues of (2.1) in special cases such as (2.14) or (2.17). Now, let us motivate the prescribed velocities $\lambda_*$ for [SC2] and [SC3].

**3.3. From the case without adsorption to the case with adsorption.** The influence of the adsorption $a$ on some sufficient conditions of Table 3.1 can be better understood by means of a change of variables that turns system (3.7) into a system that formally does not contain any adsorption. Let us introduce the new variables

$$\widetilde{\tau} = \tau, \qquad \widetilde{c} = c + a(c)\tau. \tag{3.28}$$

Since the function $c \mapsto c + a(c)\tau$ is increasing at fixed $\tau > 0$ and has range in $[0, +\infty)$, it can be inverted so as to recover $c = c(\widetilde{\tau}, \widetilde{c})$. Upon defining the new IPV deviation

$$\widetilde{q}(\widetilde{\tau}, \widetilde{c}) = q(\tau, c) - a(c)u(\tau, c), \qquad \widetilde{g}(\widetilde{\tau}, \widetilde{c}) = g(\tau, c) - a(c), \tag{3.29}$$

we can rewrite the Lagrangian model (3.7) as

$$\partial_t \widetilde{\tau} - \partial_m \widetilde{u} = 0, \tag{3.30a}$$

$$\partial_t \widetilde{c} + \partial_m \widetilde{q} = 0, \tag{3.30b}$$

where the velocity

$$\widetilde{u}(\widetilde{\tau}, \widetilde{c}) = u(\tau, c)$$

is the same as earlier but is now seen as a function of the new variables. System (3.30) has the same structure as the old one (3.7) in which the adsorption $a$ has vanished. This facilitates the expression of some sufficient conditions for weak hyperbolicity of (3.30).

THEOREM 3.4. *The sufficient conditions* [SC0], [SC2] *and* [SC3] *of Table 3.1 are equivalent to their counterparts in Table 3.2 in the new variables* $(\widetilde{\tau}, \widetilde{c})$.

| # | practical expression | $\lambda_*$ | $\zeta_*$ |
|---|---|---|---|
| [SC0] | $\widetilde{g}_{\widetilde{c}}\, \widetilde{u}_{\widetilde{\tau}} - \widetilde{g}_{\widetilde{\tau}}\, \widetilde{u}_{\widetilde{c}} \geq 0$ | $0$ | $\widetilde{u}$ |
| [SC2] | $\widetilde{g}_{\widetilde{\tau}} - \widetilde{g}\, \widetilde{g}_{\widetilde{c}} \geq 0$ | $\widetilde{g}_{\widetilde{c}}\, \widetilde{u}$ | $(\gamma c)_{\widetilde{c}}\, \widetilde{u}$ |
| [SC3] | $\widetilde{q}_{\widetilde{\tau}} \geq 0$ | $-\widetilde{u}_{\widetilde{\tau}}$ | $f_{\widetilde{s}}$ |

TABLE 3.2
*Sufficient conditions [SC2] and [SC3] in the new variables.*

*Proof.* For each row at issue in Table 3.1, we apply the chain rule

$$\partial_\tau = \partial_{\widetilde{\tau}} + a\, \partial_{\widetilde{c}} \qquad \text{and} \qquad \partial_c = (1 + a'\tau)\, \partial_{\widetilde{c}}, \tag{3.31}$$

and use (3.29) to get the corresponding new expression in Table 3.2. Let us do the calculations for [SC2], the other conditions being similar. Starting from Table 3.1, we have

$$0 \leq (1 + a'\tau)g_\tau - g(g_c - a') = (1 + a'\tau)\widetilde{g}_\tau - (\widetilde{g} + a)\widetilde{g}_c,$$

$$= (1 + a'\tau)\,(\widetilde{g}_{\widetilde{\tau}} + a\widetilde{g}_{\widetilde{c}} - (\widetilde{g} + a)\widetilde{g}_{\widetilde{c}})\,,$$
$$= (1 + a'\tau)(\widetilde{g}_{\widetilde{\tau}} - \widetilde{g}\,\widetilde{g}_{\widetilde{c}}).$$

It follows that $\widetilde{g}_{\widetilde{\tau}} - \widetilde{g}\,\widetilde{g}_{\widetilde{c}} \geq 0$. The other relations are similarly obtained. □

In Table 3.2, the partial derivatives $-\widetilde{u}_{\widetilde{\tau}}$ and $f_{\widetilde{s}}$ are taken at fixed $\widetilde{c}$. Everything happens as if the effect of adsorption can be hidden behind the modified composition $\widetilde{c}$ and the modified IPV deviation $\widetilde{q}$. The connection between the case with adsorption and the case without adsorption can be further exploited for the design of IPV laws. Indeed, assume that we have an IPV deviation $q_0(s,c)$ that ensures weak hyperbolicity for

$$\partial_t(s) \;\; + \partial_x(su) \qquad = 0, \tag{3.32a}$$
$$\partial_t(sc) + \partial_x(scu + q) = 0, \tag{3.32b}$$

which embodies (3.1) with $a = 0$. Then,

$$q_1(s,c) = a(c)u(s,c) + q_0(s, c + \tau a(c)) \tag{3.33}$$

is an IPV deviation that ensures weak hyperbolicity for the system (2.1). If $q_0$ is derived from an IPV factor, namely,

$$q_0(\widetilde{s},\widetilde{c}) = (\gamma_0(\widetilde{s},\widetilde{c}) - 1)\,\widetilde{s}\,\widetilde{c}\,\widetilde{u}(\widetilde{s},\widetilde{c}),$$

then $q_1$ can be put under the form

$$q_1(s,c) = (\gamma_1(s,c) - 1)\,sc\,u(s,c),$$

with

$$\gamma_1(s,c) = \left(1 + \frac{a(c)}{c}\tau\right)\gamma_0(s, c + \tau a(c)). \tag{3.34}$$

Unfortunately, $\gamma_1$ may become unbounded for $s \downarrow 0$. Moreover, the requirement $(\gamma_1)_s \leq 0$ also seems difficult to comply with. This is why we shall not proceed any further in this direction.

**4. Special solutions to some sufficient conditions and their importance.** We now illustrate the interest of the above sufficient conditions by linking them to some well-known IPV laws. The constant IPV (2.33) appears to be a special solution of [SC0] on a region of the state space (§4.1) for the model without adsorption. The concentration-dependent IPV deviation $q = q(c)$ emerges (§4.2) as a special solution [SC1] and will play a role in the global construction (section 5). The percolation law (2.34) reveals itself (§4.3) to be a special solution of [SC2] on a region of the state space.

**4.1. The splitting condition [SC0] and the constant IPV factor.** First, let us present a statement useful in the sequel.

Proposition 4.1. *Over the strip* $[0, s_\flat] \times \mathbb{R}_+$, *the system* (2.1)*, equipped with the constant IPV law* $\gamma(s,c) = 1 + \eta$*, admits two real eigenvalues for all* $\eta > 0$.

*Proof.* In the region $[0, s_\flat] \times \mathbb{R}_+$, $u = 0$ identically so that the equality $(g_c - a')u_\tau - g_\tau u_c = 0$ obviously holds. As a consequence, $\mathcal{P}^{\mathrm{Lag}}$ admits two real roots and the proof is completed. □

Concerning the combination of a constant IPV factor with the model without adsorption (3.32), we have the following result.

Proposition 4.2. *Over the region*

$$\mathcal{U} = \big\{(s,c) \in \Omega \;\mid\; su_s \le cu_c\big\}, \tag{4.1}$$

*the system* (3.32)*, equipped with the constant IPV law* $\gamma(s,c) = 1+\eta$*, admits two real eigenvalues for all* $\eta > 0$*.*

*Proof.* When $a = 0$, [SC0] boils down to

$$g_c u_\tau - g_\tau u_c \ge 0. \tag{4.2}$$

Next, plugging $g = \eta c/\tau$ into (4.2) gives

$$\eta \frac{\tau u_\tau + cu_c}{\tau^2} \ge 0, \tag{4.3}$$

which is equivalent to

$$\tau u_\tau + cu_c = -su_s + cu_c \ge 0, \tag{4.4}$$

to recover the expected definition of $\mathcal{U}$ given by (4.1). □

The region $\mathcal{U}$ always contains the strip $[0, s_\flat] \times \mathbb{R}_+$, where $u_s = u_c = 0$. It also includes the strip $[s_\sharp, 1] \times \mathbb{R}_+$, where $u = 1/s$ and $su_s = s(-1/s^2) = -1/s < 0 = cu_c$. A state $(s,c)$ such that $s_\flat < s < s_*(c)$ cannot possibly belong to $\mathcal{U}$, since $u_s > 0 > u_c$. The following Proposition delivers a more accurate characterization of $\mathcal{U}$ for $s \in (s_\flat, s_\sharp)$.

Proposition 4.3. *Consider a function* $\Upsilon(s) = s\Lambda_o(1-s)/\Lambda_w(s)$ *which satisfies the assumptions stated in Proposition 2.1. In addition, consider a* $\mathcal{C}^2$*-function* $R$ *such that*

$$R'(c) + cR''(c) \ge 0, \qquad \textit{for all } \; c \ge 0. \tag{4.5}$$

*The set of states* $(s,c) \in (s_\flat, s_\sharp) \times \mathbb{R}_+$*, such that* $su_s = cu_c$ *is satisfied, is the* frontier *curve* $s = s_{**}(c)$ *where, for all* $c \ge 0$*,* $s_{**}(c)$ *is the unique solution in* $(s_\flat, s_\sharp)$ *of the equation*

$$R(c)\Upsilon'(s) + cR'(c)\frac{\Upsilon(s)}{s} = 1. \tag{4.6}$$

*The curve* $s_{**}$ *(see Figure 4.1) lies entirely in the region* $(s_\flat, s_\sharp) \times \mathbb{R}_+$*, on the right of* $s_*(c)$*, that is,* $s_{**}(c) > s_*(c)$ *except for* $s_{**}(0) = s_*(0)$*. Moreover, the function* $s_{**}$ *is increasing as long as* $s \ge (\Upsilon')^{-1}(1) = s_{**}(0)$ *and admits the vertical asymptote* $s = s_\sharp$ *when* $c \to +\infty$*. Then,*

$$\mathcal{U} = \big\{(s,c) \in \Omega \;\mid\; 0 \le s \le s_\flat \;\; \textit{or} \;\; s_{**}(c) \le s \le 1\big\}. \tag{4.7}$$

*Proof.* By definition of $u$ given by (2.18), the equality $su_s = cu_c$ is equivalent to

$$\frac{sf_s}{f} = \frac{cf_c}{f} + 1. \tag{4.8}$$

In view of the expression (2.25) for $sf_s/f$ and of

$$\frac{cf_c}{f}(s,c) = -\frac{\Lambda_w(s)\Lambda_o(1-s)}{\Lambda_w(s)\Lambda_o(1-s)+\Lambda_w^2(s)/R(c)}\frac{cR'(c)}{R(c)}, \tag{4.9}$$

which is derived from (2.3), the equation (4.8) becomes

$$s\left(\Lambda_w'(s)\Lambda_o(1-s)+\Lambda_o'(1-s)\Lambda_w(s)\right) = \Lambda_w(s)\Lambda_o(1-s)\left(1-\frac{cR'(c)}{R(c)}\right)+\frac{\Lambda_w^2(s)}{R(c)} \tag{4.10}$$

after multiplication by $\Lambda_w(s)\Lambda_o(1-s)+\Lambda_w^2/R(c)$. Next, multiplying again by $R(c)/\Lambda_w^2(s)$ and after some rearrangement, we end up with

$$R(c)\Upsilon'(s)+cR'(c)\frac{\Lambda_o(1-s)}{\Lambda_w(s)}-1=0, \tag{4.11}$$

which is nothing but (4.6), since we have $\Lambda_o(1-s)/\Lambda_w(s)=\Upsilon(s)/s$.

Now, let us set

$$\Phi(s,c) = R(c)\Upsilon'(s)+cR'(c)\frac{\Lambda_o(1-s)}{\Lambda_w(s)}-1 \tag{4.12}$$

to denote the left-hand side of (4.11). At a fixed $c \geq 0$, the function $\Phi(\cdot,c)$ is continuous and monotonically decreasing with

$$\lim_{s\downarrow s_\flat}\Phi(s) = +\infty, \qquad \lim_{s\uparrow s_\sharp}\Phi(s) = -1, \tag{4.13}$$

thanks to the assumptions imposed to $\Upsilon$, $\Lambda_w$ and $\Lambda_o$. Therefore, there exists a unique $s_{**}(c)\in(s_\flat,s_\sharp)$ such that $\Phi(s_{**}(c),c)=0$. Since we have

$$R(c)\Upsilon'(s_{**}(c))+cR'(c)\frac{\Lambda_o(1-s_{**}(c))}{\Lambda_w(s_{**}(c))}=1$$

with

$$R(c)\Upsilon'(s_{**}(c))\geq 0 \qquad \text{and} \qquad cR'(c)\frac{\Lambda_o(1-s_{**}(c))}{\Lambda_w(s_{**}(c))}\geq 0,$$

necessarily, we get $R(c)\Upsilon'(s_{**}(c))\leq 1$. Moreover, when $c\to+\infty$, we have $R(c)\to+\infty$. For the product $R(c)\Upsilon'(s_{**}(c))$ to remain bounded, it is necessary that $\Upsilon'(s_{**}(c))\downarrow 0$, which implies $s_{**}(c)\uparrow s_\sharp$. In other words, the line $s=s_\sharp$ is an asymptote for the curve $s_{**}$.

To establish that $s_{**}$ is increasing with respect to $c$, we assume differentiability for $cR'(c)$ and invoke the implicit function theorem, by virtue of which

$$\begin{aligned}\frac{\mathrm{d}s_{**}}{\mathrm{d}c}(c) &= -\frac{\Phi_c(s_{**}(c),c)}{\Phi_s(s_{**}(c),c)}\\ &= -\frac{R'(c)\Upsilon'(s_{**}(c))+(R'(c)+cR''(c))(\Lambda_o(1-\cdot)/\Lambda_w)(s_{**}(c))}{R(c)\Upsilon''(s_{**}(c))+cR'(c)(\Lambda_o(1-\cdot)/\Lambda_w)'(s_{**}(c))}.\end{aligned} \tag{4.14}$$

The assumptions $R'>0$ and $R'+cR''\geq 0$ enforce $\Phi_c>0$. Since $\Phi$ has been shown decreasing with respect to $s$, then $\Phi_s<0$. Thus, $\mathrm{d}s_{**}/\mathrm{d}c>0$. By monotonicity, it can be checked that for $s_{**}(c)<s<s_\sharp$, we have $su_s<cu_c$. Because $R'>0$, it is

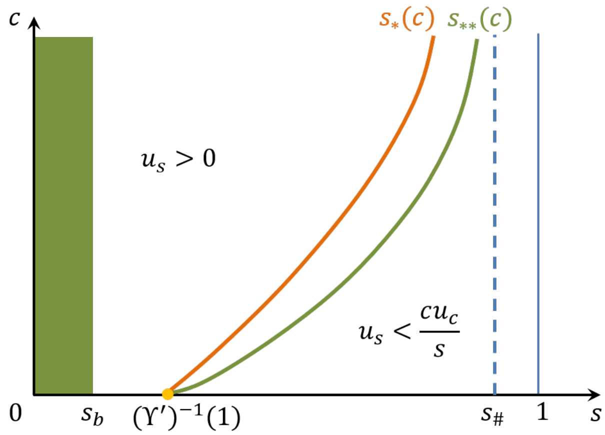


FIG. 4.1. *States for which $su_s = cu_c$ (in green) the $(s,c)$-plane.*

readily proved that $s_{**}(c) > s_*(c)$ for all $c > 0$. Equality occurs only at $c = 0$, where $s_{**}(0) = s_*(0) = (\Upsilon')^{-1}(1)$. The proof is thus completed. □

The behavior of $s_{**}(\cdot)$ is depicted in Figure 4.1. Note that the additional requirement (4.5) on $R$ is quite reasonable. In practice, most mobility reduction functions are of the type [11]

$$R(c) = 1 + \sum_{i \in \mathcal{I}} \varsigma_i c^{m_i}, \qquad \varsigma_i > 0, \tag{4.15}$$

where $\mathcal{I}$ is a finite subset of $\mathbb{N}$ and $1 \leq m_1 < m_2 < \dots$ is an increasing sequence of exponents. Any $R$ of this form fulfills (2.4) and (4.5). The family (4.15) encapsulates other classical mobility reduction laws such as [16] and [8].

So far, we have shown that the no-adsorption system (3.32) equipped with the constant IPV law (2.33) is weakly hyperbolic over $\mathcal{U}$ uniformly in the strength $\eta > 0$.

Now, we focus our attention to the converse. The following Theorem asserts that the only states in $\Omega$ that preserve real eigenvalues for the model (3.32) uniformly in the strength of the constant IPV are those defined by $\mathcal{U}$. It also describes the change in behavior of system (3.32) with respect to this strength $\eta$ for the states in $\Omega \setminus \mathcal{U}$. To this end, [SC0] is not enough and we will have to study the exact discriminant.

THEOREM 4.4. *If for a state $(s,c) \in \Omega$, the model* (3.32) *admits two real eigenvalues uniformly in the strength $\eta > 0$ of the constant IPV law, then $(s,c)$ necessarily belongs to region $\mathcal{U}$. For any other state $(s,c) \in \Omega \setminus \mathcal{U}$, there are two amplitudes $\eta_m(\omega) \geq 0$ and $\eta_M(\omega) > 0$ that depend on the ratio $\omega = -su_s/u$ such that the Lagrangian characteristic polynomial $\mathcal{P}^{Lag}$, defined by* (3.12)*, admits*

- *two real roots if $\eta \in [0, \eta_m(\omega)] \cup [\eta_M(\omega), +\infty)$;*
- *two conjugate complex roots with nonzero imaginary parts if $\eta \in (\eta_m(\omega), \eta_M(\omega))$.*

*Proof.* For the system under consideration, the discriminant (3.21) of the Lagrangian characteristic polynomial can be successively reformulates as follows:

$$\begin{aligned}
\Delta &= (u_\tau + q_c)^2 - 4u_c q_\tau,\\
&= (-s^2 u_s + q_c)^2 + 4u_c s^2 q_s,\\
&= \left(-s^2 u_s + \eta s (cu)_c\right)^2 + 4u_c s^2 \eta c (su)_s,
\end{aligned}$$

$$= s^2\big( (-su_s + \eta(u + cu_c))^2 + 4\eta cu_c(u + su_s)\big).$$

Factorizing by $u^2$, we obtain

$$\Delta = (su)^2\left(\left(-\frac{su_s}{u} + \eta\left(1 + \frac{cu_c}{u}\right)\right)^2 + 4\eta\frac{cu_c}{u}\left(1 + \frac{su_s}{u}\right)\right). \tag{4.16}$$

To write the above relation, we have assume that $u \neq 0$, which holds true for $s \notin [0, s_\flat]$. Since the behavior of the system is trivially known in the strip $[0, s_\flat] \times \mathbb{R}_+$, we may consider $u \neq 0$. Next, we introduce the reduced quantities

$$\omega = -\frac{su_s}{u} \in \mathbb{R}, \qquad \vartheta = -\frac{cu_c}{u} > 0, \tag{4.17}$$

associated with a state $(s, c)$. The first ratio $\omega$ can be thought of as an algebraic distance from the curve $u_s = 0$. If $\omega < 0$, we are on the left; if $\omega > 0$, we are on the right. Then,

$$\begin{aligned} \Delta/(su)^2 &= [\omega + \eta(1 - \vartheta)]^2 - 4\eta\vartheta(1 - \omega), \\ &= \omega^2 + 2(1 + \vartheta)\omega\eta + (1 - \vartheta)^2\eta^2 - 4\vartheta\eta, \\ &=: \mathcal{Q}_\vartheta(\omega, \eta), \end{aligned} \tag{4.18}$$

is a quadratic form in the variables $(\omega, \eta)$ at a fixed $\vartheta$. In the $(\omega, \eta)$-plane, the conic $\mathcal{Q}_\vartheta(\omega, \eta) = 0$ can be shown to be a hyperbola of center

$$(\omega_c, \eta_c) = \left(\frac{1 + \vartheta}{2}, -\frac{1}{2}\right), \tag{4.19}$$

and to have asymptotic directions

$$\frac{\eta}{\omega} = -\frac{1}{(1 \pm \sqrt{\vartheta})^2}. \tag{4.20}$$

Only one of its branches lies in the half-plane $\eta \geq 0$. This branch, sketched out in Figure 4.2, is tangent to the horizontal line $\eta = 0$ at $(\omega_h, \eta_h) = (0, 0)$. On the other hand, it is tangent to the vertical line $\omega = \eta$ at $(\omega_v, \eta_v) = (\vartheta, \vartheta/(1 - \vartheta))$.

The elliptic region $\mathcal{Q}_\vartheta(\omega, \eta) < 0$ matches with the upper part of the half-plane bordered by this branch of the hyperbola. As illustrated in Figure 4.2, this elliptic region contains the vertical segments $(\eta_{\min}(\omega), \eta_{\max}(\omega))$. Each of these segments comes from the intersection of the vertical line at abscissa $\omega < \vartheta$ with this branch of the hyperbola. But $\omega < \vartheta$ is just equivalent to $-su_s < -cu_c$, that is, $su_s > cu_c$. In other words, we are on the left side of the curve $s = s_{**}(c)$ in the $(s, c)$-plane.

The region $\mathcal{Q}_\vartheta(\omega, \eta) \geq 0$, where the Lagrangian characteristic polynomial admits two real roots, is the complementary part of the region in the half-plane where the roots are conjugate complex with nonzero imaginary parts. For $\omega < \vartheta$, real roots hold only for $\eta \leq \eta_m(\omega)$ or $\eta \geq \eta_M(\omega)$. In such conditions, we do not have real roots uniformly with respect to $\eta$. For real roots to be uniform in $\eta$, we must be able to freely move along the vertical direction without crossing the interior region bounded by the branch of hyperbola. This happens only for $\omega \geq \vartheta$, in other words, for $su_s \leq cu_c$. □

It is worth mentioning that when $\omega = 0$, the interval where roots of $\mathcal{P}^{\mathrm{Lag}}$become complex, is given by

$$\eta_m(0) = 0, \qquad \eta_M(0) = \frac{4\vartheta}{(1 - \vartheta)^2},$$

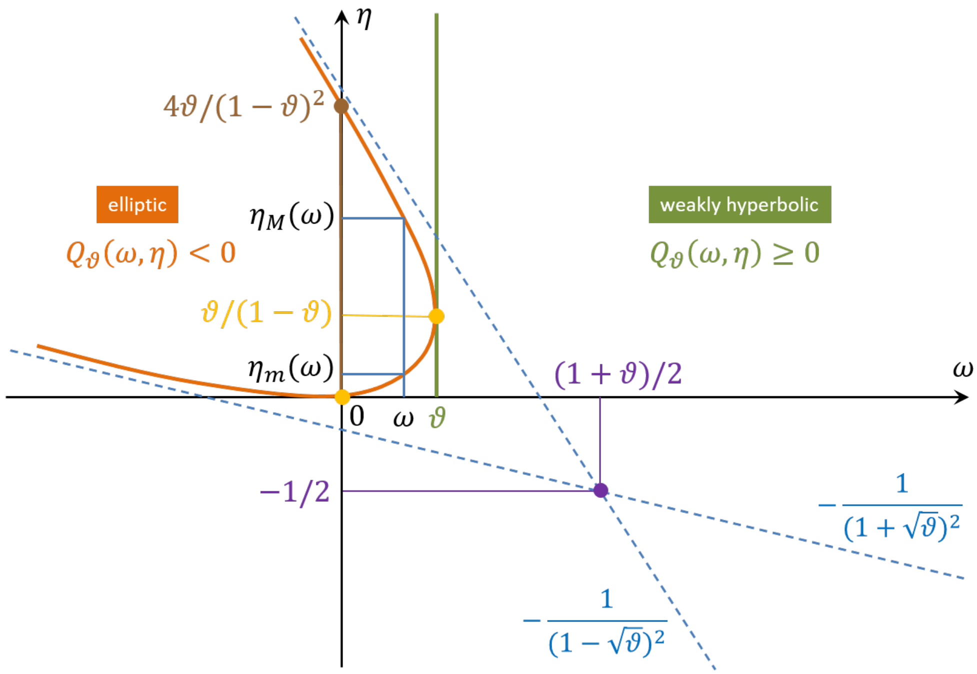


FIG. 4.2. *Elliptic and weakly hyperbolic regions for the model without adsorption equipped with the constant IPV law.*

which results from solving $\mathcal{Q}_\vartheta(0,\eta)=0$ for $\eta$. This corroborates observations detailed in [4] where, $\eta$ is small enough, the states $(s,c)$ such that $u_s = 0$ involve complex roots for $\mathcal{P}^{\mathrm{Lag}}$.

**4.2. The splitting condition [SC1] and the concentration-dependent IPV deviation.** As pointed out, the sufficient condition $q_\tau \geq 0$ does not involve the adsorption $a$ and is therefore highly relevant to the full system (2.1). A special solution of [SC1] corresponds to the equality

$$q_\tau = 0, \tag{4.21}$$

which amounts to saying that the IPV deviation $q = (\gamma - 1)fc$ depends only on $c$.

THEOREM 4.5. *The IPV factor*

$$\gamma(s,c) = 1 + \frac{\Xi(c)}{f(s,c)}, \tag{4.22}$$

*where* $\Xi(c) \geq 0$ *is an arbitrary function, solves* [SC1] *for* $s \in (s_\flat, 1]$. *The associated IPV deviation reads*

$$q(c) = c\,\Xi(c). \tag{4.23}$$

*Proof.* By imposing $q$, defined by (4.23), to satisfy (4.21), we get (4.22) whenever division by $f$ is legitimate, i.e., for $s \in (s_\flat, 1]$. ∎

The choice $\Xi = 0$ leads to $\gamma = 1$ and has no physical interest. When $\Xi > 0$, the right-hand side of (4.22) makes sense only for $s \in (s_\flat, 1]$. Due to the blow-up

$$\lim_{s \downarrow s_\flat} \gamma(s,c) = +\infty, \tag{4.24}$$

the IPV law (4.22) must be restricted to an interval of the form $[s_\bullet, 1]$ with $s_\bullet > s_\flat$. We shall make use of (4.22) in section 5 to build an IPV law that ensures global weak hyperbolicity for (2.1). The model without adsorption (3.32) equipped with an IPV deviation of the type $q = q(c)$ was studied by [30]. It was shown in particular that the Riemann problem may have structurally stable Riemann solutions without the standard constant state.

Another solution of [SC1] corresponding to the strict inequality $q_\tau > 0$ deserves our attention, even though it may not be relevant from the physics point of view.

Proposition 4.6. *The IPV factor*

$$\gamma(s,c) = 1 + \Xi(s,c)\frac{\Lambda_o(1-s)}{\Lambda_w(s)}, \tag{4.25}$$

*where $\Xi(s,c) \geq 0$ is an arbitrary function that is decreasing with respect to $s$, satisfies* [SC1] *for $s \in (s_\flat, 1]$. Strict inequality $q_\tau > 0$ holds in $(s_\flat, s_\sharp) \times \mathbb{R}_+^*$.*

The interest of (4.25) lies in the possibility to consider a positive constant for $\Xi$, which yields an IPV factor that does not depend on $c$, similarly to the percolation law. The shortcoming of (4.25) is that $\gamma(s_\sharp, c) = 1$ for all $c \geq 0$. However, this loss of the IPV effect does not make sense from a practical point of view.

*Proof.* First, with $\tau = 1/s$, enforcing $q_\tau > 0$ is equivalent to get $q_s < 0$. Since $q_s = c[\gamma_s f + (\gamma - 1) f_s]$ and $c \geq 0$, the inequality $q_s \leq 0$ is obtained as soon as

$$\gamma_s f + (\gamma - 1) f \leq 0. \tag{4.26}$$

Assuming $\gamma > 1$ and $f > 0$, this above inequality turns out to be equivalent to

$$\frac{\gamma_s}{\gamma - 1} + \frac{f_s}{f} \leq 0, \tag{4.27}$$

where

$$\frac{f_s}{f} = \left(\frac{\Lambda_w'(s)}{\Lambda_w(s)} + \frac{\Lambda_o'(1-s)}{\Lambda_o(1-s)}\right)\left(1 + \frac{\Lambda_w(s)}{\Lambda_o(1-s)R(c)}\right)^{-1}. \tag{4.28}$$

With $R(c) \geq 1$, the above quantity can be controlled, with strict inequality for $(s,c) \in (s_\flat, s_\sharp) \times \mathbb{R}_+^*$, by a $c$-independent upper-bound as follows:

$$\frac{f_s}{f} \leq \frac{\Lambda_w'(s)}{\Lambda_w(s)} + \frac{\Lambda_o'(1-s)}{\Lambda_o(1-s)} \tag{4.29}$$

As a consequence, we can enforce (4.27) by demanding

$$\frac{\gamma_s(s,c)}{\gamma(s,c) - 1} + \frac{\Lambda_w'(s)}{\Lambda_w(s)} + \frac{\Lambda_o'(1-s)}{\Lambda_o(1-s)} \leq 0, \tag{4.30}$$

which is equivalent to impose to the following function:

$$(s,c) \mapsto \log\left((\gamma(s,c) - 1)\,\frac{\Lambda_w(s)}{\Lambda_o(1-s)}\right)$$

to be decreasing with respect to $s$. Such a condition is immediately satisfied by adopting (4.25) with $\Xi(s,c)$ a decreasing function with respect to $s$. The proof is thus completed. ∎

**4.3. The splitting condition [SC2] and the percolation law.** We recall that [SC2], given by (3.17), can be equivalently reformulated by (3.26) which does not involve the velocity $u$ and is physically interesting. The stronger version (3.27) is also free from the adsorption $a$. Let us look for a solution of [SC2'] such that

$$g_\tau - g g_c = 0 \qquad \text{and} \qquad \gamma_s \leq 0. \tag{4.31}$$

First, we consider the equality $g_\tau - g g_c = 0$.

LEMMA 4.7. *All smooth solutions of $g_\tau - g g_c = 0$ are given by*

$$g(\tau, c) = g_1(c_1(\tau, c)), \tag{4.32}$$

*where $g_1$ is an arbitrary $\mathcal{C}^1$-function over $\mathbb{R}_+$ representing the initial condition at $\tau = 1$, i.e.,*

$$g_1(c) = g(1, c), \qquad c \geq 0,$$

*and $c_1(\tau, c)$ solves the implicit equation*

$$c = c_1(\tau, c) - g_1(c_1(\tau, c))(\tau - 1). \tag{4.33}$$

*This solution (4.32)–(4.33) is well-defined for*

$$1 \leq \tau < 1 + \min_{c \geq 0} \frac{1}{\max(g_1'(c), 0)} =: \tau_\bullet. \tag{4.34}$$

*Proof.* Up to a sign, the equation in (4.31) recasts as follows:

$$g_\tau - (\frac{1}{2} g^2)_c = 0.$$

We immediately recognize a Burgers equation, in which $\tau$ plays the role of time with initial date $\tau_0 = 1$ so that $s = 1/\tau$ remains below 1, and $c$ the role of space. Smooth solutions can then be found by the method of characteristics [22].

In the $(c, \tau)$-plane, we introduce the characteristic curves $\tau \mapsto C(\tau; c_1)$ defined by

$$\frac{\mathrm{d}C}{\mathrm{d}\tau}(\tau; c_1) = -g(\tau, C(\tau; c_1)), \tag{4.35a}$$

$$C(1; c_1) = c_1, \tag{4.35b}$$

for $c_1 \geq 0$. Since we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g(\tau, C(\tau; c_1)) = (g_\tau - g g_c)(\tau, C(\tau; c_1)) = 0. \tag{4.36}$$

the unknown $g$ remains constant along a characteristic curve. Therefore, $g(\tau, C(\tau; c_1) = g(1, C(1; c_1)) = g_1(c_1)$ and the characteristic curve (4.35) is a straight line with a Cartesian representation given by

$$c = c_1 - g_1(c_1)(\tau - 1). \tag{4.37}$$

Equation (4.33) thus gives the foot $c_1(\tau, c)$ of the characteristic line going through $(\tau, c)$. Inversion can be carried out as long as the right-hand side is a monotone function of $c_1$, namely, $\tau < \tau_\bullet$ as supplied by (4.34). □

Next, by selecting an appropriate function $g_1 = g(1,\cdot)$, we can make sure that $\gamma_s \le 0$.

THEOREM 4.8. *The percolation law*

$$\gamma(s,c) = \frac{s}{s - s_\bullet}, \tag{4.38}$$

*where* $s_\bullet \in (0,1)$ *is a parameter, solves* (4.31) *for alls* $\in (s_\bullet, 1]$. *It corresponds to the linear initial function*

$$g_1(c) = \frac{s_\bullet\, c}{1 - s_\bullet} \tag{4.39}$$

*and the maximal time* $\tau_\bullet = 1/s_\bullet$.

*Proof.* By the choice (4.39), equation (4.33) becomes linear in $c_1$, that is,

$$c = c_1 - \frac{s_\bullet\, c_1}{1 - s_\bullet}(\tau - 1) = \frac{1 - s_\bullet \tau}{1 - s_\bullet} c_1. \tag{4.40}$$

The solution

$$c_1 = \frac{1 - s_\bullet}{1 - s_\bullet \tau} c \tag{4.41}$$

makes sense only for $1 - s_\bullet \tau > 0$, namely, $\tau < 1/s_\bullet$. It turns out that $\tau_\bullet = 1/s_\bullet$ also coincides with the value imposed by (4.34). In this case,

$$g(\tau, c) = g_1(c_1) = \frac{s_\bullet\, c}{1 - s_\bullet} \frac{1 - s_\bullet}{1 - s_\bullet \tau} c = \frac{s_\bullet c}{1 - s_\bullet \tau} = \frac{s_\bullet s c}{s - s_\bullet}. \tag{4.42}$$

Comparing this with the definition $g = (\gamma - 1)sc$, we deduce the value (4.38) for $\gamma$ and notice that $\gamma_s = -s_\bullet/(s - s_\bullet)^2 \le 0$. □

At this stage, it is natural to wonder about those solutions of (3.27) with, possibly strict, inequality $g_\tau - g g_c \ge 0$ instead of equality $g_\tau - g g_c = 0$. Although we do not have a full characterization (as in Lemma 4.7 for equality), the following Proposition singles out a class of such solutions.

PROPOSITION 4.9. *Whenever it is well-defined, the generalized percolation law*

$$\gamma(s,c) = \frac{s}{s - \sigma(s,c)}, \tag{4.43}$$

*where the* $\mathcal{C}^1$*-function* $(s,c) \mapsto \sigma(s,c) \in (0,s)$ *is non-increasing with respect to* $s$ *and* $c$*, satisfies the splitting condition [SC2'] given by* (3.27).

*Proof.* Let us plug $g = (\gamma - 1)c/\tau$ into $g_\tau - g g_c \ge 0$ and since we have

$$g_\tau = \gamma_\tau c/\tau - (\gamma - 1)c/\tau^2, \qquad g_c = \gamma_c c/\tau + (\gamma - 1)/\tau$$

then we get

$$\tau\gamma_\tau - \gamma(\gamma - 1) - c(\gamma - 1)\gamma_c \ge 0, \tag{4.44}$$

which is equivalent to

$$s\gamma_s + \gamma(\gamma - 1) + c(\gamma - 1)\gamma_c \le 0. \tag{4.45}$$

A special class of solutions can be obtained by enforcing

$$s\gamma_s + \gamma(\gamma - 1) \leq 0 \qquad \text{and} \qquad (\gamma - 1)\gamma_c \leq 0. \tag{4.46}$$

To solve the first inequality, let us consider the function $\psi = s(1-1/\gamma)$ with derivative with respect to $s$ given by

$$\psi_s = 1 - \frac{1}{\gamma} + s\frac{\gamma_s}{\gamma^2} = \frac{s\gamma_s + \gamma(\gamma-1)}{\gamma^2} \tag{4.47}$$

must be non-positive, i.e., $\psi_s \leq 0$. Let $\sigma(s,c)$ be arbitrary function non-increasing in $s$. Setting $\psi = \sigma$ and solving for $\gamma$, we get (4.43). The condition $0 < \sigma(s,c) < s$ is required for $\gamma$ to be well-defined and greater than 1. Since $\gamma > 1$, the second inequality $(\gamma-1)\gamma_c \leq 0$ reduces to

$$\gamma_c = \frac{s\sigma_c}{\left(s - \sigma(s,c)\right)^2} \leq 0. \tag{4.48}$$

Therefore, $\sigma_c \leq 0$ and $\sigma$ must also be non-increasing with respect to $c$. Finally, we directly get $\gamma_s \leq -\tau\gamma(\gamma-1) \leq 0$. □

To conclude this section, we report a most remarkable property of the percolation law (4.38) that seems to have been little emphasized in the literature (for instance, see [10]).

THEOREM 4.10. *The change of variables*

$$\mathtt{S} = s/\gamma, \qquad \mathtt{C} = \gamma c, \qquad \mathtt{F}(\mathtt{S},\mathtt{C}) = f(s,c) \tag{4.49}$$

*turns the model without adsorption* (3.32)*, considered for* $(s,c) \in (s_\bullet, 1] \times \mathbb{R}_+$ *and equipped with the percolation law* (4.38)*, into the well-known Keyfitz-Kranzer no-IPV model*

$$\partial_t(\mathtt{S}) \;\; + \partial_x(\mathtt{F}) \;\; = 0, \tag{4.50a}$$
$$\partial_t(\mathtt{SC}) + \partial_x(\mathtt{FC}) = 0, \tag{4.50b}$$

*considered for* $(\mathtt{S},\mathtt{C}) \in (0, 1-s_\bullet] \times \mathbb{R}_+$*, and conversely. The equivalence between the two systems holds for smooth as well as discontinuous solutions.*

*Proof.* From the identity

$$\frac{s}{\gamma} = s - s_\bullet, \tag{4.51}$$

we have for smooth solutions

$$\partial_t(\mathtt{S}) = \partial_t\left(\frac{s}{\gamma}\right) = \partial_t(s - s_\bullet) = \partial_t(s) = -\partial_x(f) = -\partial_x(\mathtt{F}), \tag{4.52}$$

which proves (4.50a). We also have

$$\partial_t(\mathtt{SC}) = \partial_t\left(\frac{s}{\gamma}\gamma c\right) = \partial_t(sc) = -\partial_x(\gamma f c) = -\partial_x(\mathtt{FC}), \tag{4.53}$$

which proves (4.50b). Thus, the two systems are equivalent for smooth solutions.

Next, for discontinuous solutions, let us consider a shock between two states $(s_L, c_L)$ and $(s_R, c_R)$ for the system with the percolation law. These states satisfy the Rankine-Hugoniot relations

$$[\![f]\!] = \sigma[\![s]\!], \tag{4.54a}$$
$$[\![\gamma fc]\!] = \sigma[\![sc]\!], \tag{4.54b}$$

where $[\![\cdot]\!] = \cdot_R - \cdot_L$ denotes the jump and $\sigma$ the velocity of the discontinuity. Once again, arguing (4.51) we get

$$\left[\!\!\left[\frac{s}{\gamma}\right]\!\!\right] = [\![s - s_\bullet]\!] = [\![s]\!], \tag{4.55}$$

so that

$$[\![\mathtt{F}]\!] = [\![f]\!] = \sigma[\![s]\!] = \sigma\left[\!\!\left[\frac{s}{\gamma}\right]\!\!\right] = [\![\mathtt{S}]\!] \tag{4.56a}$$

and

$$[\![\mathtt{FC}]\!] = [\![f\gamma c]\!] = \sigma[\![sc]\!] = \sigma\left[\!\!\left[\frac{s}{\gamma}\gamma c\right]\!\!\right] = \sigma[\![\mathtt{SC}]\!]. \tag{4.56b}$$

The equalities (4.56) are nothing but the Rankine-Hugoniot relations for the system (4.50). The equivalence for weak solutions of both systems (3.32) and (4.50) is thus established. □

Owing to this previous equivalence, the model (3.32) equipped with the percolation law (4.38) is weakly hyperbolic on $(s_\bullet, 1] \times \mathbb{R}_+$. Its characteristic speeds are deduced from (2.17) as

$$\mathtt{F}/\mathtt{S} = f/(s/\gamma) = \gamma u,$$
$$\mathtt{F}_\mathtt{S} = f_s s_\mathtt{S} + f_c c_\mathtt{S} = (su)_s + (\gamma - 1)cu_c,$$

the first one $\gamma u$ being linearly degenerate.

**5. Construction of IPV laws ensuring global weak hyperbolicity.** We now proceed with the more ambitious task of building IPV laws that make (2.1) weakly hyperbolic on $\Omega$. This is achieved by patching up various sufficient conditions on different strips of $\Omega$. For the easier case $s_\bullet < s_\flat$, we expose in §5.1 a rather straightforward extension of the percolation law. For the more delicate case $s_\bullet > s_\flat$, we exhibit in §5.2 a mathematically consistent construction that appears to be a significant advance despite some minor physical limitations.

**5.1. For $s_\bullet < s_\flat$.** The difficulty with the percolation law (4.38) is that $\gamma$ blows up as $s \downarrow s_\bullet$. But if $s_\flat > s_\bullet$, it is advisable to stop the percolation law at $s_\flat$ and to extend $\gamma$ continuously by a constant to the left. The idea is sketched out in Figure 5.1.

THEOREM 5.1. *When $s_\bullet < s_\flat$, the IPV law*

$$\gamma(s, c) = \begin{cases} \dfrac{s}{s - s_\bullet} & \text{if } \ s_\flat \le s \le 1, \\ \dfrac{s_\flat}{s_\flat - s_\bullet} & \text{if } \ 0 \le s \le s_\flat, \end{cases} \tag{5.1}$$

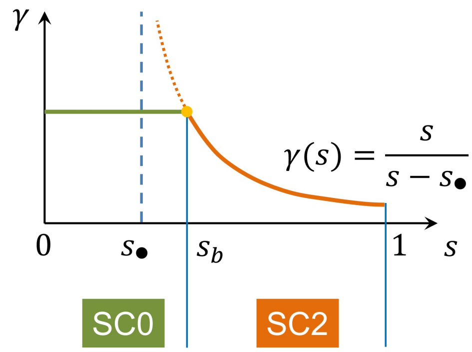


FIG. 5.1. *Construction principle for $s_\bullet < s_\flat$.*

*gives rise to a $\mathcal{C}^1$ flux $fc\gamma$ and guarantees weak hyperbolicity for the system* (2.1) *over* $\Omega$.

*Proof.* For $s \in [s_\flat, 1]$, weak hyperbolicity is due to [SC2'], as explained in §4.3. For $s \in [0, s_\flat]$, weak hyperbolicity is due to [SC0], as shown in Proposition 4.1. The only thing that remains to be verified is that the flux component $fc\gamma = scu + q$ is continuously differentiable with respect to $s$.

Since $q = 0$ for $s \in [0, s_\flat]$, we have $\lim_{s \uparrow s_\flat} q_s(s, c) = 0$. For $s > s_\flat$,

$$q_s = \gamma_s fc + (\gamma - 1) f_s c.$$

When $s \downarrow s_\flat$, the limit of $\gamma_s fc$ is 0, since $f(s_\flat, c) = 0$ and $\gamma_s = -s/(s - s_\bullet)^2$ remains bounded. The limit of $(\gamma - 1) f_s c$ is also 0, thanks to

$$f_s(s, c) = \frac{\Lambda_w'(s)\Lambda_o(1-s) + \Lambda_w(s)\Lambda_o'(1-s)}{R(c)\left(\Lambda_w(s)/R(c) + \Lambda_o(1-s)\right)^2}, \tag{5.2}$$

and more specifically to $\Lambda_w'(s_\flat) = 0$. Therefore, $\lim_{s \downarrow s_\flat} q_s(s, c) = 0$ and finally $q_s(s_\flat, c) = 0$. $\square$

In the previous proof, what really matters for differentiability of $fc\gamma$ at $s_\flat$ is not the values of $\gamma$ and $\gamma_s$, but the equality $\Lambda_w'(s_\flat) = 0$ which comes from the differentiability assumption on $\Lambda_w$ and from (2.8)–(2.5). The same argument will be reiterated in §5.2.

### 5.2. For $s_\bullet > s_\flat$.

**Theoretical derivation.** Because of the blow-up at $s_\bullet$, the percolation law is no longer applicable. Instead, we deploy the [SC1]-type IPV law (4.25) on $[s_\bullet, 1]$, that we extrapolate on $[s_\flat, s_\bullet]$ by an IPV law subject to [SC2'] with strict inequality. On $[0, s_\flat]$, the IPV factor $\gamma$ is again continuously extended by a constant. The strategy is summarized in Figure 5.2.

The form imposed to $\gamma$ on $[s_\flat, s_\bullet]$ is meant to make

$$h = (\gamma - 1)s,$$

affine with respect to $\tau$, namely, $h(\tau, c) = A(c)\tau + B(c)$. The justification of this ansatz, along with that of the maximal amplitude $\Xi(c) = \eta/R(c)$ for $\gamma - 1$ on $[s_\sharp, 1]$,

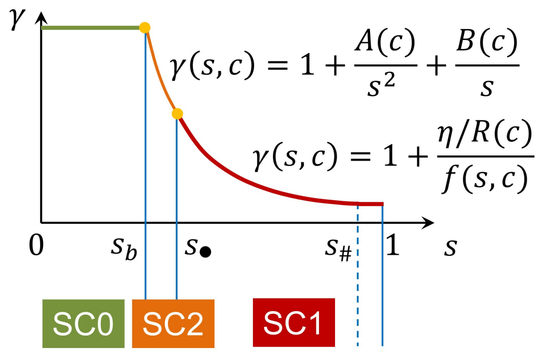


FIG. 5.2. *Construction principle for* $s_\bullet > s_\flat$.

is postponed until the proof of Theorem 5.2 is completed.

THEOREM 5.2. *When* $s_\bullet > s_\flat$*, the IPV law*

$$\gamma(s,c) = \begin{cases} 1 + \dfrac{\eta}{R(c) f(s,c)} & \text{if } s \in [s_\bullet, 1], \\ 1 + \dfrac{A(c)}{s^2} + \dfrac{B(c)}{s} & \text{if } s \in [s_\flat, s_\bullet], \\ 1 + \dfrac{A(c)}{s_\flat^2} + \dfrac{B(c)}{s_\flat} & \text{if } s \in [0, s_\flat], \end{cases} \tag{5.3}$$

*where* $\eta > 0$ *is a given constant and*

$$A(c) = -\frac{\eta}{R(c)} \frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)}, \qquad B(c) = \frac{\eta}{R(c)} \frac{u(\tau_\bullet, c) + \tau_\bullet u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)}, \tag{5.4}$$

*gives rise to a* $\mathcal{C}^1$ *flux* $fc\gamma$ *and guarantees weak hyperbolicity for the system* (2.1) *over* $\Omega$*, provided that the assumptions of Proposition 2.1 on* $\Upsilon(s) = -s\Lambda_o(1-s)/\Lambda_w(s)$ *are met and that*

$$s_\bullet < \min\{2s_\flat,\, s_*(0)\}, \qquad \eta < \frac{-u_\tau(\tau_\bullet, 0)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, 0)(\tau_\flat - \tau_\bullet)\right)^2}, \tag{5.5}$$

*with* $\tau_\bullet = 1/s_\bullet$*,* $\tau_\flat = 1/s_\flat$*.* To prove this Theorem, we need a few technical lemmas. The first one clarifies the significance of the coefficients $A(c)$ and $B(c)$.

LEMMA 5.3. *An alternative expression for* $A(c)$ *is*

$$A(c) = \eta s_\bullet^2 \left( \Upsilon'(s_\bullet) - \frac{1}{R(c)} \right), \tag{5.6}$$

*where* $\Upsilon$ *is the function defined in* (2.21)*. The coefficients* $A(c)$ *and* $B(c)$ *defined in* (5.4) *are adjusted so that* $h$ *(and* $q$*,* $fc\gamma$*) is continuously differentiable at* $s_\bullet$*, that is,*

$$h(\tau_\bullet, c) = \frac{\eta}{R(c)} \frac{1}{u(\tau_\bullet, c)}, \qquad h_\tau(\tau_\bullet, c) = -\frac{\eta}{R(c)} \frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)}. \tag{5.7}$$

*Proof.* Taking the limit of $h = (\gamma - 1)s$ on the right of $s_\bullet$, i.e., on the left of $\tau_\bullet$, we have

$$\lim_{\tau\uparrow\tau_\bullet} h(\tau, c) = \frac{\eta}{R(c)}\,\frac{1}{u(\tau_\bullet, c)}, \qquad \lim_{\tau\uparrow\tau_\bullet} h_\tau(\tau, c) = -\frac{\eta}{R(c)}\,\frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)}. \tag{5.8}$$

Taking the limit of $h = A\tau + B$ on the left of $s_\bullet$, i.e., on the right of $\tau_\bullet$, we have

$$\lim_{\tau\uparrow\tau_\bullet} h(\tau, c) = A(c)\tau_\bullet + B(c), \qquad \lim_{\tau\uparrow\tau_\bullet} h_\tau(\tau, c) = A(c). \tag{5.9}$$

Requiring that the quantities in (5.9) are equal to their counterparts in (5.8), we obtain a linear system in $(A(c), B(c))$ whose solution is given by (5.4). The value of $A(c)$ can still be written as

$$A(c) = \frac{\eta}{R(c)}\left(\frac{1}{u}\right)_\tau (\tau_\bullet, c) = -\frac{\eta s_\bullet^2}{R(c)}\left(\frac{s}{f}\right)_s (s_\bullet, c). \tag{5.10}$$

But from the definition (2.3) of the fractional flux, it is readily checked that

$$\frac{s}{f(s,c)} = s - R(c)\Upsilon(s). \tag{5.11}$$

Hence,

$$\left(\frac{s}{f}\right)_s (s, c) = 1 - R(c)\Upsilon'(s). \tag{5.12}$$

Inserting this into (5.10) yields the expression (5.6). ◻

The next lemma shows that extrapolation of $h$ by an affine function in $\tau$ on $[\tau_\bullet, \tau_\flat]$ induces favorable behaviors on $\gamma$.

Lemma 5.4. *Under the assumptions of Proposition 2.1 on $\Upsilon$ and if $s_\bullet < s_*(0)$, we have*

$$A(c) > 0, \qquad A'(c) > 0, \qquad h > 0, \qquad \gamma > 1, \qquad \gamma_s < 0, \tag{5.13}$$

*for all $c \geq 0$ and for all $\tau \in [\tau_\bullet, \tau_\flat]$.*

*Proof.* According to the study of the curve $s_*$ in Proposition (2.1), the assumption $s_\bullet < s_*(0)$ implies $s_\bullet < s_*(c)$, which in turn entails $\Upsilon'(s_\bullet) > 1/R(c)$, that is, $A(c) > 0$. Its derivative

$$A'(c) = \eta s_\bullet^2 \frac{R'(c)}{R^2(c)}, \tag{5.14}$$

is positive by virtue of conditions (2.4) on $R$.

For $\tau \in [\tau_\bullet, \tau_\flat]$,

$$h(\tau, c) = h(\tau_\bullet, c) + A(c)(\tau - \tau_\bullet) > 0, \tag{5.15}$$

since $h(\tau_\bullet, c) > 0$ after (5.7). Consequently, $\gamma > 1$. Taking the derivative of $\gamma = 1 + A(c)\tau^2 + B(c)\tau$ with respect to $\tau$ gives

$$\gamma_\tau = 2A(c)\tau + B(c) = A(c)\tau + h(\tau, c). \tag{5.16}$$

Thus, $\gamma_\tau > 0$ for $\tau \in [\tau_\bullet, \tau_\flat]$, which means $\gamma_s < 0$ for $s \in [s_\flat, s_\bullet]$. ◻

The last lemma will be needed at the very end of the proof of Theorem 5.2.

Lemma 5.5. *If $s_\flat < s_\bullet < \min\{s_*(0), 2s_\flat\}$, then*

$$\min_{c\geq 0} \frac{-R(c)u_\tau(\tau_\bullet, c)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, c)(\tau_\flat - \tau_\bullet)\right)^2} = \frac{-u_\tau(\tau_\bullet, 0)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, 0)(\tau_\flat - \tau_\bullet)\right)^2}. \tag{5.17}$$

*Proof.* The function

$$\Theta(c) = \frac{-R(c)u_\tau(\tau_\bullet, c)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, c)(\tau_\flat - \tau_\bullet)\right)^2}, \tag{5.18}$$

is well-defined for all $c \geq 0$, since the denominator in the right-hand side is always greater than 1, thanks to $u(\tau_\bullet, c) > 0$, $-u_\tau(\tau_\bullet, c) > 0$ [due to $s_\bullet < s_*(0)$] and $\tau_\flat > \tau_\bullet$. A more convenient form of (5.18) can be obtained by noting on the one hand that from

$$-u_\tau = s^2 u_s = sf_s - f = f\left(\frac{sf_s}{f} - 1\right)$$

and from the expression (2.25) for $sf_s/f$, we have

$$-u_\tau(\tau_\bullet, c) = \frac{\Lambda_w(s_\bullet)/R(c)}{\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)/R(c)}\left(s_\bullet \frac{\{\Lambda_w'/\Lambda_w\}(s_\bullet) + \{\Lambda_o'/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/(R(c)\Lambda_o(1-s_\bullet))} - 1\right). \tag{5.19}$$

On the other hand,

$$u(\tau_\bullet, c) = \tau_\bullet \frac{\Lambda_w(s_\bullet)/R(c)}{\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)/R(c)}. \tag{5.20}$$

Plugging (5.19)–(5.20) into (5.18), we end up with

$$\Theta(c) = \frac{\Lambda_w^2(s_\bullet)}{s_\bullet\left(\Lambda_w'(s_\bullet)\Lambda_o(1-s_\bullet) + \Lambda_w(s_\bullet)\Lambda_o'(1-s_\bullet)\right)} \frac{z(c)(z(c)+1)}{\left(1 + z(c)(s_\bullet\tau_\flat - 1)\right)^2}, \tag{5.21}$$

where

$$z(c) = s_\bullet \frac{\{\Lambda_w'/\Lambda_w\}(s_\bullet) + \{\Lambda_o'/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/(R(c)\Lambda_o(1-s_\bullet))} - 1 \tag{5.22}$$

is an obviously increasing function of $c \geq 0$. When $c$ ranges from 0 to $+\infty$, $z(c)$ ranges from

$$z_m = s_\bullet \frac{\{\Lambda_w'/\Lambda_w\}(s_\bullet) + \{\Lambda_o'/\Lambda_o\}(1-s_\bullet)}{1 + \Lambda_w(s_\bullet)/\Lambda_o(1-s_\bullet)} - 1 = \frac{1}{1 + \Lambda_o(1-s_\bullet)/\Lambda_w(s_\bullet)}\left(\Upsilon'(s_\bullet) - 1\right),$$

to

$$z_M = s_\bullet\left(\frac{\Lambda_w'}{\Lambda_w}(s_\bullet) + \frac{\Lambda_o'}{\Lambda_o}(1-s_\bullet)\right) - 1 = \frac{\Lambda_w(s_\bullet)}{\Lambda_o(1-s_\bullet)}\Upsilon'(s_\bullet).$$

Note that $z_m > 0$ because $s_\bullet < s_*(0)$. This leads us to study the function

$$\Psi(z) = \frac{z(z+1)}{\left(1 + z(s_\bullet\tau_\flat - 1)\right)^2}, \tag{5.23}$$

which coincides with $\Theta(c)$ up to a positive factor, for $z \in [z_m, z_M]$. The derivative of $\Psi$ reads

$$\Psi'(z) = \frac{(3 - s_\bullet \tau_\flat)z + 1}{\left(1 + z(s_\bullet \tau_\flat - 1)\right)^3}. \tag{5.24}$$

If $s_\bullet < 2s_\flat$, then $s_\bullet \tau_\flat < 2$ and $3 - s_\bullet \tau_\flat > 1 > 0$. Therefore $\Psi'(z) > 0$ for all $z$ of interest and the minimum of $\Psi$ on $[z_m, z_M]$ is reached at $z = z_m$. Going back $\Theta$, we can now state that the minimum of $\Theta$ on $\mathbb{R}_+$ is reached at $c = 0$, which proves (5.17). $\square$

At last, we are in a position to prove Theorem 5.2.

*Proof of Theorem 5.2.* Weak hyperbolicty on $[s_\bullet, 1] \times \mathbb{R}_+$ comes from applying Theorem 4.5 with $\Xi(c) = \eta/R(c)$. By construction, the flux $fc\gamma$ is differentiable at $s_\bullet$. Weak hyperbolicity on $[0, s_\flat] \times \mathbb{R}_+$ results from Proposition 4.1. By an argument similar to Theorem 5.1, $fc\gamma$ is differentiable at $s_\flat$. The only thing that remains to be established is weak hyperbolicity on $(s_\flat, s_\bullet)$.

To this end, we are going to use (3.27). The second part $\gamma_s \leq 0$ is already settled in Lemma 5.4. To prove that $g_\tau - gg_c \geq 0$ on $(\tau_\bullet, \tau_\flat)$, we note that since $g = hc$, this is equivalent to

$$h_\tau - h^2 - chh_c \geq 0, \tag{5.25}$$

after simplification by $c \geq 0$. Plugging $h = A(c)\tau + B(c)$ into (5.25), the inequality to be proven becomes

$$A(c) - \left(A(c)\tau + B(c)\right)^2 - c\left(A(c)\tau + B(c)\right)\left(A'(c)\tau + B'(c)\right) \geq 0. \tag{5.26}$$

The left-hand side of (5.26) is a quadratic polynomial in $\tau$, whose leading term in $\tau^2$ has a negative coefficient, namely,

$$-A^2(c) - cA(c)A'(c) < 0, \tag{5.27}$$

by Lemma 5.4. Put another way, the left-hand side of (5.26) is a concave function. It is known, however, that on a compact interval, a concave function reaches its minimum value on the boundary. Here, this minimum is achieved at $\tau_\bullet$ or $\tau_\flat$. Therefore, condition (5.26) for all $\tau \in (\tau_\bullet, \tau_\flat)$ is equivalent to

$$A(c) - \left(A(c)\tau_\bullet + B(c)\right)^2 - c\left(A(c)\tau_\bullet + B(c)\right)\left(A'(c)\tau_\bullet + B'(c)\right) \geq 0, \tag{5.28a}$$

$$A(c) - \left(A(c)\tau_\flat + B(c)\right)^2 - c\left(A(c)\tau_\flat + B(c)\right)\left(A'(c)\tau_\flat + B'(c)\right) \geq 0. \tag{5.28b}$$

Let us first try to comply with (5.28a). By taking the derivative of

$$A(c)\tau_\bullet + B(c) = \frac{\eta}{R(c)} \frac{1}{u(\tau_\bullet, c)} = \frac{\eta}{R(c)} \frac{s_\bullet}{f(s_\bullet, c)} = \eta s_\bullet \left( \frac{1}{R(c)} + \frac{\Lambda_o(1 - s_\bullet)}{\Lambda_w(s_\bullet)} \right), \tag{5.29}$$

with respect to $c$, we have

$$A'(c)\tau_\bullet + B'(c) = -\eta s_\bullet \frac{R'(c)}{R^2(c)} < 0. \tag{5.30}$$

As a consequence, the third summand of (5.28a) is positive, that is,

$$-c\left(A(c)\tau_\bullet + B(c)\right)\left(A'(c)\tau_\bullet + B'(c)\right) > 0. \tag{5.31}$$

It is thus sufficient to impose $A(c) - [A(c)\tau_\bullet + B(c)]^2 \geq 0$, namely,

$$-\frac{\eta}{R(c)}\frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)} - \frac{\eta^2}{R^2(c)}\frac{1}{u^2(\tau_\bullet, c)} \geq 0. \tag{5.32}$$

This boils down to

$$\eta \leq -R(c)\, u_\tau(\tau_\bullet, c). \tag{5.33}$$

Next, let us investigate the feasibility of (5.28b). We observe that

$$A(c)\tau_\flat + B(c) = (A(c)\tau_\bullet + B(c)) + A(c)(\tau_\flat - \tau_\bullet) > 0. \tag{5.34}$$

By taking the derivative of the above equality with respect to $c$,

$$A'(c)\tau_\flat + B'(c) = -\eta s_\bullet \frac{R'(c)}{R^2(c)} + \eta s_\bullet^2 \frac{R'(c)}{R^2(c)}(\tau_\flat - \tau_\bullet) = \eta s_\bullet \frac{R'(c)}{R(c)}(s_\bullet \tau_\flat - 2). \tag{5.35}$$

If $s_\bullet < 2s_\flat$, as supposed by the first condition of (5.5), then $s_\bullet \tau_\flat < 2$ and $A'(c)\tau_\flat + B'(c) < 0$. In this case, the third summand of (5.28b) is positive, that is,

$$-c\,(A(c)\tau_\flat + B(c))\,(A'(c)\tau_\flat + B'(c)) > 0. \tag{5.36}$$

It is thus sufficient to prescribe $A(c) - (A(c)\tau_\flat + B(c))^2 \geq 0$, namely,

$$-\frac{\eta}{R(c)}\frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)} - \left(\frac{\eta}{R(c)}\frac{1}{u(\tau_\bullet, c)} - \frac{\eta}{R(c)}\frac{u_\tau(\tau_\bullet, c)}{u^2(\tau_\bullet, c)}(\tau_\flat - \tau_\bullet)\right)^2 \geq 0. \tag{5.37}$$

After some algebra, this boils down to

$$\eta \leq \frac{-R(c)\, u_\tau(\tau_\bullet, c)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, c)(\tau_\flat - \tau_\bullet)\right)^2}. \tag{5.38}$$

Owing to $-u_\tau(\tau_\bullet, c) > 0$, $u(\tau_\bullet, c) > 0$ and $\tau_\flat > \tau_\bullet$, the upper-bound of (5.38) is more restrictive than that of (5.33). In other words, it is enough to require (5.38) for all $c \geq 0$. According to Lemma 5.5,

$$\eta \leq \min_{c\geq 0} \frac{-R(c)\, u_\tau(\tau_\bullet, c)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, c)(\tau_\flat - \tau_\bullet)\right)^2} = \frac{-u_\tau(\tau_\bullet, 0)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, 0)(\tau_\flat - \tau_\bullet)\right)^2}, \tag{5.39}$$

which is the second condition of (5.5).

The existence of a $(s_\flat, s_\sharp, s_\bullet)$-dependent maximal value

$$\eta_M(s_\flat, s_\sharp, s_\bullet) = \frac{-u_\tau(\tau_\bullet, 0)}{\left(1 - \dfrac{u_\tau}{u}(\tau_\bullet, 0)(\tau_\flat - \tau_\bullet)\right)^2} \tag{5.40}$$

allowed for the amplitude of $\gamma(1, 0) = \eta$ should not come as a surprise. After all, such a limitation was already in place for the percolation law (4.38). Indeed,

$$\eta_M(s_\bullet) := \gamma(1) - 1 = \frac{1}{1 - s_\bullet} - 1 = \frac{s_\bullet}{1 - s_\bullet}. \tag{5.41}$$

**Numerical illustration.** To gain insight into the order of magnitude of the upper-bound (5.40) for the amplitude of $\eta$, we undertake a short numerical study by drawing a series of curves. First, in Figure 5.3, we plot $s_\bullet \mapsto \eta_M(s_\flat, s_\sharp, s_\bullet)$ for $s_\bullet \in (0.2, 0.4)$ corresponding to

$$(s_\flat, s_\sharp) = (0.2,\ 0.7), \qquad (\alpha_w, \alpha_o) = (2,\ 3),$$

and

$$(\Lambda_w^{\mathrm{m}}, \Lambda_o^{\mathrm{m}}) = (1.4804 \cdot 10^{-11},\ 2.4673 \cdot 10^{-12}) \ \ [\mathrm{m}^2 \cdot \mathrm{Pa}^{-1} \cdot \mathrm{s}^{-1}].$$

The mobility reduction law is taken to be

$$R(c) = 1 + a_1 c + a_2 c^2 + a_4 c^{15/4}$$

with

$$a_1 = 2 \qquad a_2 = 2.8, \qquad a_4 = 0.16005.$$

For this configuration, we found the resonant saturation for $c = 0$ takes the value

$$s_*(0) = 0.42588.$$

Depending on the value of $s_\bullet$ selected, the maximal amplitude $\eta_M$ can take values from $\eta_M^{\min} = 0$ to $\eta_M^{\max} = 0.20799$, reached at $s_\bullet = 0.37255$.

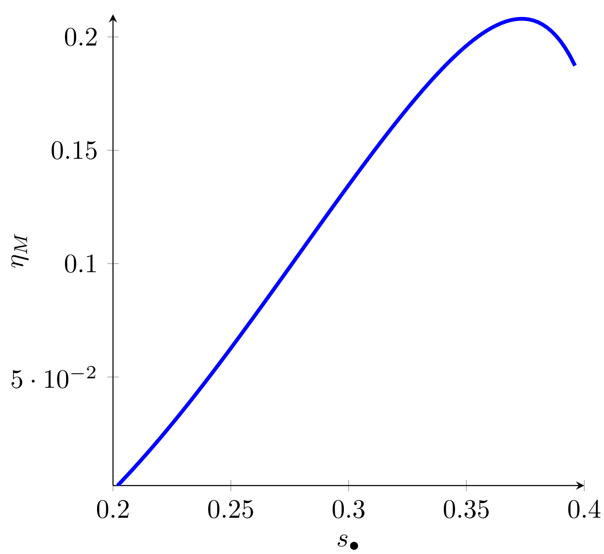


FIG. 5.3. *Upper-bound $\eta_M(s_\flat, s_\sharp, s_\bullet)$ as a function of $s_\bullet \in [s_\flat, \min\{2s_\flat, s_*(0)\}]$.*

Next, in Figure 5.4, the IPV factor $\gamma$ is displayed as a function of $s \in [0, 1]$ for different values of concentration $c \in \{0,\ 0.5,\ 1,\ 2\}$ $[\mathrm{g} \cdot \mathrm{L}^{-1}]$, using the same parameters as above and setting

$$s_\bullet = 0.37255$$

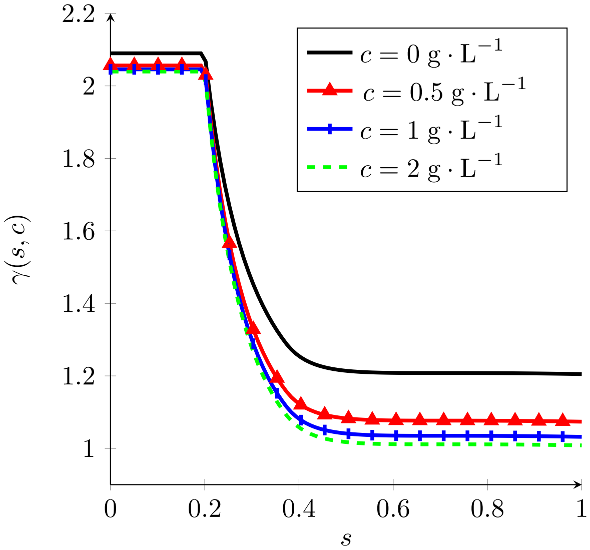


FIG. 5.4. *IPV factor $\gamma$ as a function of $s$ for different concentration values $c \in \{0, 0.5, 1, 2\}$ $[g \cdot L^{-1}]$.*

so as to benefit from the largest possible amplitude in $\eta_M$. We see that the range of $\gamma$ matches physical observations. As expected by construction, $\gamma$ decreases when $c$ increases. The decay is more and more substantial when $s$ approaches 1.

Finally, in Figure 5.5, we investigate the maximal amplitude with respect to $s_\bullet$, that is,

$$\eta_M^{\max}(s_\flat, s_\sharp) = \max_{s_\bullet \in [s_\flat, \min\{2s_\flat, s_*(0)\}} \eta_M(s_\flat, s_\sharp, s_\bullet) \tag{5.42}$$

as a function of $s_\sharp$ for different values of $s_\flat$. It can be seen that this quantity decreases with respect to $s_\sharp$. Moreover, it can range from unrealistically small values (for $s_\sharp$ approaching 1) to physically acceptable values (for small $s_\sharp$).

**Further discussion.** The previous study seems to show that it is a delicate task to choose the characteristic parameters $(s_\flat, s_\sharp, s_\bullet)$ of the medium in order to have a physically correct magnitude for $\eta$. However, it is capital to underline the fact that the upper-bound (5.40) is very far from being optimal. Indeed, in the proof of Theorem 5.2, we have neglected the favorable positive terms (5.31), (5.36) for the sake of simplicity. The proposed conditions $A(c) - [A(c)\tau + B(c)]^2 \geq 0$ for $\tau \in \{\tau_\bullet, \tau_\flat]$ are therefore quite abrupt. To put it another way, the actual maximal value for $\eta$ that is compatible with weak hyperbolicity may be much larger, but the challenge of estimating it in a pragmatic and efficient way seems out of our reach for the time being.

We now comment on two fundamental aspects of the construction (5.3). The first one is concerned with the choice of the form $h(\tau, c) = A(c)\tau + B(c)$ for the IPV law on $[s_\flat, s_\bullet]$. Broadly speaking, the most natural extrapolation is the affine one. But the question remains as to which quantity should be linearly extrapolated and with respect to which variable. Several pairs of (quantity, variable) have been attempted. The difficulty is that, except for the pair $(h, \tau)$, they do not supply with an acceptable

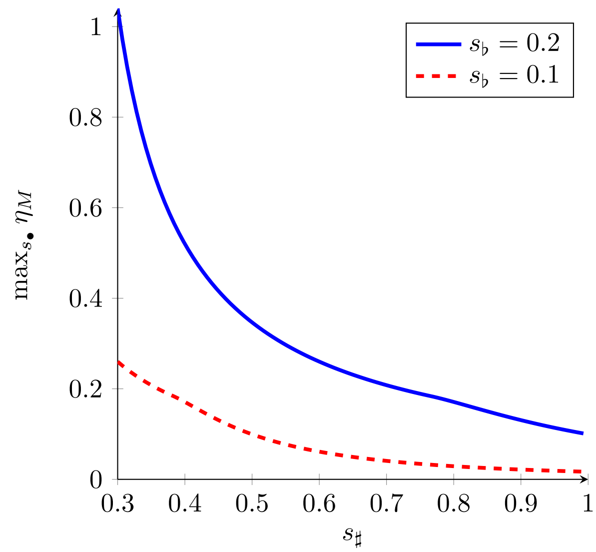


FIG. 5.5. *Maximal upper-bound $\eta_M^{\max}(s_\flat, s_\sharp)$ as a function of $s_\sharp \in [0.3, 1)$ for $s_\flat \in \{0.1, 0.2\}$.*

maximal value on the IPV amplitude $\eta$ for [SC2'] on $(s_\flat, s_\bullet)$. For instance, the pair $(\gamma - 1, s)$ leads to the condition

$$\eta < \min_{c \geq 0} \min \left\{ -R(c) u_\tau(\tau_\bullet, c),\ -\frac{R(c)[u_\tau(\tau_\bullet, c) + 2(s_\bullet - s_\flat) f_s(s_\bullet, c)]}{\left[1 + \dfrac{f_s}{f}(s_\bullet, c)(s_\bullet - s_\flat)\right]^2} \right\}. \tag{5.43}$$

The sign of the second argument in the minimum function of the right-hand side cannot be controlled. It is not easy either to study the minimum over $c \geq 0$ of this upper-bound, which makes it highly impractical.

The second aspect to be discussed is the division of $\eta$ by $R(c)$ on $[s_\bullet, 1]$, which seems objectionable at first sight. The purpose of this operation is to have a non-zero upper-bound $\eta_M(s_\flat, s_\sharp, s_\bullet)$ that guarantees weak hyperbolicity for all $c \geq 0$. Had we merely searched for weak hyperbolicity on a finite range $c \in [0, c_M]$, it would not have been necessary to divide $\eta$ by $R(c)$. Nevertheless, the corresponding upper-bound would have depended on $c_M$ in a decreasing manner. This purely mathematical argument can be supplemented by a more physical one: the more polymer molecules there are, the less likely they are accelerated, but the more likely they collide with each other or with the pores of the medium, thus lowering its overall speed. A rough but solid analogy is perhaps to be found in traffic flow. If some exceptional vehicles (police, emergency) are authorized to move faster than the mainstream cars, the speed-up they gain must be necessarily tempered when there are too many of them.

**6. Conclusion.** In this work, we laid down some foundational principles for the design of IPV laws that guarantee weak hyperbolicity of a prototype polymer flow model. Derived from a passage to Lagrangian coordinates, the sufficient conditions for local weak hyperbolicity were given a clearer physical meaning and appeared to be easier to deal with. Special solutions to these sufficient conditions were shown to coincide with classical IPV laws and brought more insight into them. A strategy using these sufficient conditions piecewise was proposed for the construction of an IPV law that ensured global weak hyperbolicity.

Further to these encouraging theoretical results, there is still a long way to go in order to expand this idea of solution into a fully satisfactory one for the weak hyperbolicity problem. First, it is important to improve the practical upper-bound $\eta_M(s_\flat, s_\sharp, s_\bullet)$ for the maximal IPV amplitude $\eta = \gamma(1,0) - 1$. Regardless of the success of this theoretical effort, it is also essential to perform numerical simulations with the new global IPV law using physically "sensible" values for $\eta$, $s_\flat$, $s_\sharp$ and $s_\bullet$. The observation of numerical results on typical test cases will ultimately validate the IPV laws. This part requires appropriate numerical schemes to be developed and will be the subject of an upcoming work.

**Declarations.**

**Funding and/or Conflicts of interests/Competing interests.** The authors declare that no funds, grants or any financial support (other than the salaries from their respective institutions) were received during the preparation of this work. Furthermore, they have no financial or non-financial competing interests to disclose.

**Data transparency.** The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

**Credit for contribution.** All authors contributed equally to this work. They read and approved the final manuscript.

## REFERENCES

[1] N. Andrianov and G. Warnecke, *On the solution to the Riemann problem for the compressible duct flow*, SIAM J. Appl. Math., 64 (2004), pp. 878–901.
[2] A. Aw and M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math., 60 (2000), pp. 916–938.
[3] M. R. Baer and J. W. Nunziato, *Mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials*, Int. J. Multiphase Flow, 12 (1986), pp. 861–889.
[4] G. A. Bartelds, J. Bruining, and J. Molenaar, *The modeling of velocity enhancement in polymer flooding*, Transp. Porous Med., 26 (1997), pp. 75–88.
[5] B. Braconnier, C. Preux, É. Flauraud, Q.-H. Tran, and C. Berthon, *An analysis of physical models and numerical schemes for polymer flooding simulations*, Comput. Geosci., 21 (2017), pp. 1267–1279.
[6] R. H. Brooks and A. T. Corey, *Hydraulic properties of porous media and their relation to drainage design*, Trans. ASAE, 7 (1964), pp. 26–28.
[7] R. Dawson and R. B. Lantz, *Inaccessible pore volume in polymer flooding*, SPE Journal, 12 (1972), pp. 448–452.
[8] P.-G. de Gennes, *Scaling Concepts in Polymer Physics*, Cornell University Press, Ithaca, 1979.
[9] G. L. Dongmo Nguepi, *Modèles mathématiques et numériques avancés pour la simulation du polymère dans les réservoirs pétroliers*, thèse de doctorat, Université Paris-Saclay, 2021.
[10] G. L. Dongmo Nguepi, B. Braconnier, C. Preux, Q.-H. Tran, and C. Berthon, *A relaxation method for the simulation of possibly non-hyperbolic polymer flooding models with inaccessible pore volume effect*, in Finite Volumes for Complex Applications IX - Methods, Theoretical Aspects, Examples, R. Klöfkorn, E. Keilegavlen, F. A. Radu, and J. Fuhrmann, eds., Cham, 2020, Springer International Publishing, pp. 445–453.
[11] P. J. Flory, *Principles of Polymer Chemistry*, Cornell University Press, Ithaca, New York, 1953.
[12] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, vol. 118 of Applied Mathematical Sciences, Springer-Verlag, New York, 1996.
[13] D. W. Green and G. P. Willhite, *Enhanced Oil Recovery*, vol. 6 of SPE Textbook Series, Society of Petroleum Engineers, Richardson, Texas, 1998.
[14] E. Guarnerio, *Velocity enhancement models for polymer flooding in reservoir simulations*, Master's thesis, Delft University of Technlogy, 2018.

[15] S. Hilden, H. Nilsen, and X. Raynaud, *Study of the well-posedness of models for the inaccessible pore volume in polymer flooding*, Transp. Porous Med., 114 (2016), pp. 65–86.
[16] M. L. Huggins, *Solutions of long chain compounds*, J. Chem. Phys., 9 (1941), p. 440.
[17] E. L. Isaacson and J. B. Temple, *Analysis of a singular hyperbolic system of conservation laws*, J. Diff. Eqs, 65 (1986), pp. 250–268.
[18] T. Johansen and R. Winther, *The solution of the Riemann problem for a hyperbolic system of conservation laws modeling polymer flooding*, SIAM J. Math. Anal., 19 (1988), pp. 541–566.
[19] B. L. Keyfitz and H. C. Kranzer, *A system of non-strictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rat. Mech. Anal., 72 (1980), pp. 219–241.
[20] L. W. Lake, *Enhanced Oil Recovery*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.
[21] J.-P. Lebacque, S. Mammar, and H. Haj-Salem, *The Aw-Rascle and Zhang's model: Vacuum problems, existence and regularity of the solutions of the Riemann problem*, Transp. Res. Part B, 41 (2007), pp. 710–721.
[22] R. J. LeVeque, *Numerical Methods for Conservation Laws*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992.
[23] ———, *The dynamics of pressureless dust clouds and delta waves*, J. Hyperb. Differ. Eq., 01 (2004), pp. 315–327.
[24] V. Matos, J. D. Silva, and D. Marchesin, *Loss of hyperbolicity changes the number of wave groups in Riemann problems*, Bull. Braz. Math. Soc., New Series, 47 (2016), pp. 545–559.
[25] B. Noetinger, V. Artus, and L. Ricard, *Dynamics of the water-oil front for two-phase, immiscible flow in heterogeneous porous media. 2- Isotropic media*, Transp. Porous Med., 56 (2004), pp. 305–328.
[26] G. A. Pope, *The application of fractional flow theory to enhanced oil recovery*, SPE Journal, 20 (1980), pp. 191–205.
[27] J. Romate and E. Guarnerio, *Velocity enhancement models for polymer flooding in reservoir simulation*, in Proceedings of 20th European Symposium on Improved Oil Recovery, April 8-11, Pau, France, European Association of Geoscientists & Engineers, 2019, pp. 1–16.
[28] P. Saffman and G. Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. R. Soc. A, 245 (1958), pp. 312–329.
[29] D. Serre, *Systèmes de Lois de Conservation I. Hyperbolicité, Entropies, Ondes de Choc*, Collection Fondations, Diderot Éditeur Arts et Sciences, Paris, 1996.
[30] J. D. Silva and D. Marchesin, *Riemann solutions without an intermediate constant state for a system of two conservation laws*, J. Diff. Eqs, 256 (2014), pp. 1295–1316.
[31] K. S. Sorbie, *Polymer-Improved Oil Recovery*, Springer Science, New York, 1991.
[32] D. H. Wagner, *Equivalence of the Euler and Lagrangian equations of gas dynamics for weak solutions*, J. Diff. Eqs, 68 (1987), pp. 118–136.
[33] L. Wiegman, *Numerical aspects of transport modelling in enhanced oil recovery*, Master's thesis, Delft University of Technlogy, 2017.
[34] H. M. Zhang, *A non-equilibrium traffic model devoid of gas-like behavior*, Transp. Res. Part B, 36 (2002), pp. 275–290.