Mathematical Models and Methods in Applied Sciences

Shock layers for turbulence models

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Received (Day Month Year)
Revised (Day Month Year)
Communicated by (xxxxxxxxxx)

The present work is devoted to an extension of the Navier-Stokes equations where the fluid governed by two independent pressure laws. Several turbulence models typically enter this framework. The striking novelty over the usual Navier-Stokes equations stems from the impossibility to recast equivalently the system of interest in full conservation form. Opposing to systems of conservation laws, where the end states of the viscous shock are completely characterized by *jump relations*, the lack of conservation implies the absence of *jump relations*. We analyze the traveling wave behaviors according to the ratio of viscosities, and we show that the traveling wave solutions of our system tend to the traveling wave solutions of a fully conservative system. This result is used to exhibit asymptotic expansions of the end states. Such an asymptotic behavior achieves a deep physical interpretation when illustrated in the case of compressible turbulent flows.

Keywords: Turbulence models, Traveling wave solutions, Asymptotic behaviors.

AMS Subject Classification: 35L67, 35Q35, 76F50

1. Introduction

The present work considers the Navier-Stokes equations for compressible fluid dynamics modeled by two independent pressures. *Two independent pressures* mean more precisely that each of them is characterized with its own specific entropy. The smooth solutions of the system undergo simultaneously two independent entropy balance equations.

However, we will show latter that such fluid models exhibit several close relationships with the usual Navier-Stokes system. The fundamental discrepancy stays in the lack of an admissible change of variables that recasts the governing equations in full conservation form. As a consequence, we have to deal with a convection-diffusion system in non-conservation form.

Such systems are a natural extension of the classical Navier-Stokes equations and they actually occur in several distinct physical settings. They occur in plasma physics where the electronic pressure must be distinguished from the mixture pressure of the other heavy species (see Coquel and Marmignon[5]). They can also be recognized within the frame of the so-called *two transport equation* models for turbulent compressible flows where the average thermodynamic pressure should be distinguished from the specific turbulent kinetic energy (for instance, see Mohammadi and Pironneau[15]).

The convection-diffusion systems with nonconservative products may not be defined in the usual distributional sense (see Dal Maso, LeFloch and Murat[6] or Colombeau, Leroux, Noussair and Perrot[4]) if discontinuous shock wave solutions appear. Since the system admits a non zero diffusion operator, we can expect a regularization of shock waves to obtain viscous shock layers. Unfortunately, some convection-diffusion system which admit discontinuous shock wave solutions have already been observed in the literature (see Zel'dovich and Raizer[26]). It is therefore necessary to prove the existence and uniqueness of smooth traveling wave solutions (*i.e.* viscous shock layers).

The shock layers are formally described by the triples $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ where σ denotes the propagation speed of the wave while $(\mathbf{v}_L, \mathbf{v}_R)$ denotes the pair of the end states. When convection-diffusion systems in conservation form, like the usual Navier-Stokes equations, are studied, the triples $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ are solutions of the classical Rankine-Hugoniot relations. Hence, we have $\mathbf{v}_R = \mathbf{v}_R(\sigma; \mathbf{v}_L)$ or reversely $\mathbf{v}_L = \mathbf{v}_L(\sigma; \mathbf{v}_R)$. The end state $\mathbf{v}_R(\sigma; \mathbf{v}_L)$, or reversely $\mathbf{v}_L(\sigma; \mathbf{v}_R)$, stays completely free from the choice of the diffusion when considering a system of conservation laws. The characterization of the end states turns out to be completely different from the classical Navier-Stokes equations in the setting of our extended system and more generally in the setting of non-conservative systems. Indeed, its non-conservation form makes the triples $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ to depend on the precise shape of the diffusion tensor.

To be more precise, the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ will be shown to depend on a ratio of viscosities in the form: μ_1/μ_2. A natural question thus arises: What is the asymptotic behaviors as one viscosity goes to zero? In the limit of μ_1, respectively μ_2, to zero, the traveling wave solutions are shown to be solutions of a convection-diffusion system in full conservation form. In addition, the limit systems find a physical interpretation detailed below. Since the limit triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ is solution of a Rankine-Hugoniot relations associated to the limit conservative system, we propose an asymptotic expansion of the end states in a small parameter (the ratio of the viscosities).

The present work is organized as follows. In the next section, we introduce the set of equation and we state some algebraic properties. In the third section, we prove the existence and uniqueness of the traveling wave solutions and we give a characterization of the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$. The next section is devoted to the asymptotic behaviors of the viscous shock layers according to the ratio of viscosities. We state our main result of convergence and we propose asymptotic expansions of the end states. Finally in the last section, we prove the stated convergence result.

2. Mathematical model

The present work aims at studying the existence and the behavior of the traveling wave solutions of the following system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + \frac{2}{3}\rho k) = \partial_x \left((\mu + \mu_\tau) \partial_x u \right), \\ \partial_t \rho E + \partial_x (\rho E + p + \frac{2}{3}\rho k) u = \partial_x ((\mu + \mu_\tau) u \partial_x u), \\ \partial_t \rho k + \partial_x \rho k u + \frac{2}{3}\rho k \partial_x u = \mu_\tau \left(\partial_x u \right)^2, \\ \partial_t \rho \epsilon + \partial_x \rho \epsilon u + C_{\epsilon 1} \frac{2}{3} \rho \epsilon \partial_x u = C_{\epsilon 1} \frac{\epsilon}{k} \mu_\tau \left(\partial_x u \right)^2, \\ \rho E = \rho \frac{u^2}{2} + \frac{p}{\gamma - 1} + \rho k. \end{cases} \tag{2.1}$$

The gas is characterized by its density $\rho > 0$, its velocity $u \in \mathbb{R}$ and its total energy E. We denote by k the specific kinetic turbulent energy and ϵ the dissipation rate of k. This system coincides with the celebrate (k, ϵ) model which governs compressible turbulent flows (see Mohammadi and Pironneau[15], Hirsh[11] or Vandromme and Minh[22] to further details).

In the model, $C_{\epsilon 1} \in (1, 2)$ denotes a given constant. In addition, the physics prescribes the laminar viscosity μ while the turbulent viscosity reads as follows:

$$\mu_\tau = C_\mu \rho \frac{k^2}{\epsilon}, \tag{2.2}$$

with $C_\mu > 0$ a modeling constant.

First, let us note that the smooth solutions of (2.1) obey the following additional conservation law (see Smith[21] or Mohammadi and Pironneau[15]):

$$\partial_t \rho \frac{(\frac{2}{3}k)^{C_{\epsilon 1}}}{\epsilon} + \partial_x \rho \frac{(\frac{2}{3}k)^{C_{\epsilon 1}}}{\epsilon} u = 0. \tag{2.3}$$

As a consequence, the quantity $k^{C_{\epsilon 1}}/\epsilon$ stays constant through a viscous shock layer. With no restriction, we eliminate ϵ from the unknowns to reduce our initial system.

For the sake of simplicity in the notations, we set $p_2 = (\gamma_2 - 1)\rho k$ where $\gamma_2 = 5/3$, to write the ρk equation as follows:

$$\partial_t p_2 + u \partial_x p_2 + \gamma_2 p_2 \partial_x u = (\gamma_2 - 1)\mu_2 (\partial_x u)^2,$$

where $\mu_2 := \mu_\tau$. Let us recall that the evolution law satisfied by the pressure p is given by:

$$\partial_t p + u \partial_x p + \gamma p \partial_x u = (\gamma - 1)\mu (\partial_x u)^2.$$

The initial model (2.1) can be summarized as follows. Let us consider a gas with density $\rho > 0$ and velocity u, which is modeled by two independent pressures $p_1 > 0$

and $p_2 > 0$, associated to the adiabatic exponents $\gamma_1 > 1$ and $\gamma_2 > 1$. The system of PDE's that motivates our analysis thus reads:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \qquad x \in \mathbb{R},\ t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = \partial_x \left((\mu_1 + \mu_2) \partial_x u \right), \\ \partial_t p_1 + \partial_x p_1 u + (\gamma_1 - 1) p_1 \partial_x u = (\gamma_1 - 1) \mu_1 \left(\partial_x u \right)^2, \\ \partial_t p_2 + \partial_x p_2 u + (\gamma_2 - 1) p_2 \partial_x u = (\gamma_2 - 1) \mu_2 \left(\partial_x u \right)^2. \end{cases} \tag{2.4}$$

The convection-diffusion system (2.4) can be understood as an extension of the standard Navier-Stokes equations when considering an additional PDE for governing an additional pressure. Similarly to the classical Navier-Stokes equations, the smooth solutions of system (2.4) obey additional governing equations as we state in the following result:

Lemma 2.1. *Smooth solutions of (2.4) satisfy the following conservation law:*

$$\partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = \partial_x ((\mu_1 + \mu_2) u \partial_x u), \tag{2.5}$$

where the total energy ρE is defined by:

$$\rho E = \frac{(\rho u)^2}{2\rho} + \frac{p_1}{\gamma_1 - 1} + \frac{p_2}{\gamma_2 - 1}. \tag{2.6}$$

Smooth solutions satisfy in addition the following balance equations:

$$\partial_t \rho \log(s_1) + \partial_x \rho \log(s_1) u = \frac{\gamma_1 - 1}{T_1} \mu_1 (\partial_x u)^2, \tag{2.7}$$

$$\partial_t \rho \log(s_2) + \partial_x \rho \log(s_2) u = \frac{\gamma_2 - 1}{T_2} \mu_2 (\partial_x u)^2, \tag{2.8}$$

where the specific entropies are respectively given by

$$s_1 := \frac{p_1}{\rho^{\gamma_1}}, \quad s_2 := \frac{p_2}{\rho^{\gamma_2}} \tag{2.9}$$

and the involved temperatures respectively read $T_1 = p_1/\rho$ and $T_2 = p_2/\rho$. As a consequence, smooth solutions of (2.4) obey:

$$\frac{\mu_2}{\mu_1 + \mu_2} \frac{T_1}{\gamma_1 - 1} \left(\partial_t \rho \log(s_1) + \partial_x \rho \log(s_1) u \right) - \frac{\mu_1}{\mu_1 + \mu_2} \frac{T_2}{\gamma_2 - 1} \left(\partial_t \rho \log(s_2) + \partial_x \rho \log(s_2) u \right) = 0. \tag{2.10}$$

More details and the proof concerning this result are given in Berthon and Coquel[2,3]. Let us emphasize that the three balance equations (2.5), (2.7) and (2.8) can be proved to be the only non trivial additional equations for smooth solutions (up to some standard nonlinear transforms in s_1 and s_2). As a consequence and besides several close relationships with the usual Navier-Stokes system, the very discrepancy stays in the lack of four non trivial conservation laws. Indeed and without restrictive modeling assumptions (see below), none of the equations (2.7), (2.8) and (2.10) boils down to conservation laws. As a consequence, (2.4) cannot be recast, generally speaking, in full conservation form. Thus, we adopt the following abstract form of (2.4):

$$\partial_t \mathbf{v} + \mathbf{A}(\mathbf{v})\partial_x \mathbf{v} = \mathcal{E}(\mathbf{v}, \partial_{xx}\mathbf{v}), \quad x \in \mathbb{R},\ t > 0, \tag{2.11}$$

where the right hand side involves diffusion terms in conservation form as well as dissipation rates in non-conservation form. The vector state

$$\mathbf{v} = (\rho, \rho u, p_1, p_2) \tag{2.12}$$

is assumed to belong to the phase space

$$\Omega = \left\{\mathbf{v} \in \mathbb{R}^4 :\ \rho > 0,\ \rho u \in \mathbb{R},\ p_1 > 0,\ p_2 > 0\right\}.$$

After the pioneering works by LeFloch[14], Dal Maso, LeFloch and Murat[6], Raviart and Sainsaulieu[17] and Sainsaulieu[19] let us highlight that the non conservation form met by (2.4) makes the end states of viscous shock layers to intrinsically depend on the closure relations for the transport coefficients μ_1 and μ_2. More precisely, such a dependence occurs in term of the ratio of μ_1 and μ_2. To exemplify the role played by this ratio, let us note that the straightforward derivation of the non standard balance equation (2.10) reflects a cancellation property. Namely, the entropy balance equations (2.7) and (2.8) are not independent but actually evolve proportionally to the ratio of the two viscosities μ_1 and μ_2. Indeed and at least formally, once integrated with respect to the space variable, the equation (2.10) reads:

$$\begin{aligned} &\frac{\mu_2}{\mu_1+\mu_2}\left\{\int_{\mathbb{R}} \frac{T_1}{\gamma_1 - 1}(\partial_t \rho \log(s_1) + \partial_x \rho \log(s_1) u) dx\right\} - \\ &\frac{\mu_1}{\mu_1+\mu_2}\left\{\int_{\mathbb{R}} \frac{T_2}{\gamma_2 - 1}(\partial_t \rho \log(s_2) + \partial_x \rho \log(s_1) u) dx\right\} = 0, \end{aligned} \tag{2.13}$$

so that the evolution in time of the two entropies must be kept in balance according to the ratio of the two viscosities. Let us underline that by contrast with (2.7) and (2.8) where entropy dissipation rates are actually independently imposed, the weighted equation (2.10) exhibits a compared rate of both the entropy dissipations.

Involving several restrictive assumptions, the following result exhibits the role played by the ratio of viscosities:

Lemma 2.2. *For smooth solutions of (2.4), the equation (2.10) reads as follows:*

$$\frac{\mu_2}{\mu_1+\mu_2}\frac{\rho^{\gamma_1-1}}{\gamma_1-1}\left(\partial_t \rho s_1 + \partial_x \rho s_1 u\right) - \frac{\mu_1}{\mu_1+\mu_2}\frac{\rho^{\gamma_2-1}}{\gamma_2-1}\left(\partial_t \rho s_2 + \partial_x \rho s_2 u\right) = 0. \tag{2.14}$$

(i) Assume that $\mu_i > 0$ and $\mu_j = 0$ with $(i,j) = (1,2)$ or $(i,j) = (2,1)$. Then (2.14) reduces to the following conservation law:

$$\partial_t \rho s_i + \partial_x \rho s_i u = 0. \tag{2.15}$$

(ii) Assume that μ_1 and μ_2 are two given positive constants. If moreover $\gamma_1 = \gamma_2$, then (2.14) reduces to the following conservation law:

$$\partial_t \rho\left(\mu_2 s_1 - \mu_1 s_2\right) + \partial_x \rho\left(\mu_2 s_1 - \mu_1 s_2\right) u = 0. \tag{2.16}$$

(iii) Let μ_1^0 and μ_2^0 denote two positive constants and assume that $\mu_1 = \mu_1^0 T_1$, $\mu_2 = \mu_2^0 T_2$. Then (2.10) coincides with the following conservation law:

$$\partial_t \rho \left(\frac{\mu_2^0}{\gamma_1 - 1} \log(s_1) - \frac{\mu_1^0}{\gamma_2 - 1} \log(s_2) \right) + \partial_x \rho \left(\frac{\mu_2^0}{\gamma_1 - 1} \log(s_1) - \frac{\mu_1^0}{\gamma_2 - 1} \log(s_2) \right) u = 0. \tag{2.17}$$

These above restrictive assumptions yield for additional non trivial conservation laws that are encoded in the non standard balance equation (2.10) or its equivalent form (2.14). When considering their associated Rankine-Hugoniot condition, they clearly indicate that the end states of the viscous shock layers under consideration actually depend on the ratio of both viscosities. The above restrictive assumptions will no longer be adopted in the sequel. The dependence we have just pointed out is illustrated in Figure 1 in a more general setting in which the system (2.4) does not admit an equivalent full conservation form. For a given left end state $\mathbf{v}_L$ and a given

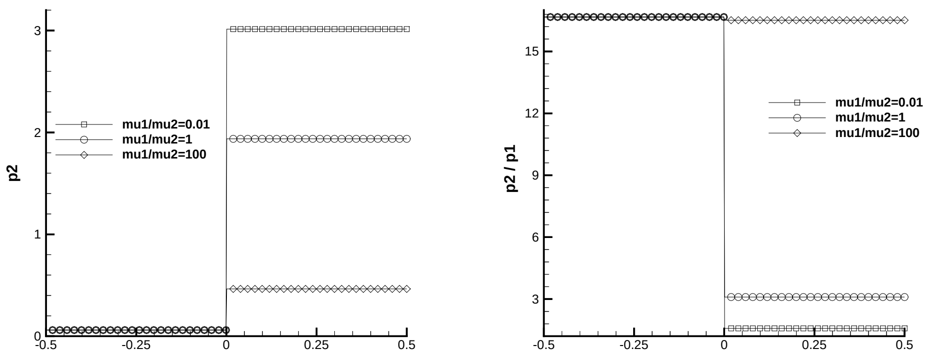

Fig. 1. Exact stationary traveling wave solutions of (2.4) for various ratios μ_1/μ_2.

velocity σ, the required right end states $\mathbf{v}_R$ are defined when solving numerically the nonlinear ODE's system governing traveling wave solutions (see below) for various ratios of the viscosities. In a sense described below, (2.10) or (2.14) continue to play a major role in a general setting since they encode a *generalized jump condition* which turns out to play a central role for our purpose.

3. Traveling wave solutions and jump relations

In this section, we derive generalized jump relations that are needed to characterize the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ associated with a given viscous shock layer. In that aim, let us first recall that a traveling wave solution of (2.4) is a smooth solution $\mathrm{C}^1(\mathbb{R}\times\mathbb{R}_+,\Omega)$ in the form $\mathbf{v}(x,t)=\widehat{\mathbf{v}}(x-\sigma t)$ with

$$\lim_{\xi\to-\infty}\widehat{\mathbf{v}}(\xi)=\mathbf{v}_L,\quad \lim_{\xi\to+\infty}\widehat{\mathbf{v}}(\xi)=\mathbf{v}_R,\quad \xi=x-\sigma t, \tag{3.1}$$

where the triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$ is prescribed. Hence, the function $\widehat{\mathbf{v}}$ thus satisfies the following nonlinear ODE's system:

$$-\sigma d_\xi\widehat{\mathbf{v}}+\mathbf{A}(\widehat{\mathbf{v}})d_\xi\widehat{\mathbf{v}}=\mathcal{E}(\widehat{\mathbf{v}},d^2_{\xi\xi}\widehat{\mathbf{v}}),\quad \xi\in\mathbb{R}, \tag{3.2}$$

where we have used the abstract form (2.11).

Let us recall that traveling wave solutions can only be associated with genuinely nonlinear fields in the underlying first order system extracted from (2.4) (see Serre[20] for instance). This one is seen to be hyperbolic over the phase space Ω with the following distinct eigenvalues

$$u-c(\mathbf{v}),\quad u,\quad u+c(\mathbf{v}),\quad c(\mathbf{v})=\sqrt{\frac{\gamma_1p_1+\gamma_2p_2}{\rho}},\quad \mathbf{v}\in\Omega,$$

where u has two order of multiplicity. The extreme fields are genuinely nonlinear while the intermediate ones are linearly degenerate. For frame invariance properties satisfied by the PDE model (2.4), note that it suffices to study the traveling wave solutions associated with the first extreme field[20].

Assume the following hypothesis satisfied by the viscosity functions:

$$\begin{aligned}&\text{The smooth functions } \mu_1,\mu_2\in\mathrm{C}^1(\Omega,\mathbb{R}) \text{ satisfy } \mu_1(\mathbf{v})\geq 0,\\ &\mu_2(\mathbf{v})\geq 0 \text{ and } \mu_1(\mathbf{v})+\mu_2(\mathbf{v})>0 \text{ for all } \mathbf{v}\in\Omega,\end{aligned} \tag{3.3}$$

to state existence of viscous shock layers.

Theorem 3.1. *Let the viscosities satisfy (3.3). Let $\mathbf{v}_L\in\Omega$ and $\sigma\in\mathbb{R}$ be given such that the following Lax condition is satisfied:*

$$u_L-c_L>\sigma,\quad c_L^2=\frac{\gamma_1(p_1)_L}{\rho_L}+\frac{\gamma_2(p_2)_L}{\rho_L}. \tag{3.4}$$

Then there exists a unique state $\mathbf{v}_R\in\Omega$ and a traveling wave solution of (2.4), unique up to a translation (i.e. a parameterization), which connects $\mathbf{v}_L$ and $\mathbf{v}_R$.

The proof of this result will be given at the end of the present section.

Now, after LeFloch[14], Raviart-Sainsaulieu[17], our purpose is to highlight that the end states of the traveling wave solutions of (3.1)-(3.2) does not depend on the amplitude of the diffusion tensor modeled in (3.2) but just on its shape. Indeed let us introduce the function

$$\mathbf{v}_\delta(x,t)=\widehat{\mathbf{v}}\left(\frac{x-\sigma t}{\delta}\right),$$

where $\delta>0$ denotes a positive rescaling parameter. For all $\delta>0$, $\mathbf{v}_\delta(x,t)$ turns out to be a traveling wave solution of (2.4) but for the viscosity functions μ_1^δ, μ_2^δ defined by

$$\mu_1^\delta=\delta\mu_1,\quad \mu_2^\delta=\delta\mu_2.$$

The function $\mathbf{v}_\delta(x,t)$ is obviously associated with the same triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$ for all values of the rescaling parameter $\delta>0$. The expected independence property with respect to the diffusion amplitude thus follows. Hence, the characterization of the triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$ follows from the next result.

Theorem 3.2. *Assume that the viscosity functions μ_1,μ_2 satisfy (3.3). The triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$ associated with the resulting dissipative tensor necessarily obeys the jump relations:*

$$-\sigma[\rho]+[\rho u]=0, \tag{3.5a}$$
$$-\sigma[\rho u]+[\rho u^2+p_1+p_2]=0, \tag{3.5b}$$
$$-\sigma[\rho E]+[(\rho E+p_1+p_2)u]=0, \tag{3.5c}$$

and necessarily satisfies the two entropy inequalities:

$$-\sigma[\rho s_1]+[\rho s_1 u]>0, \tag{3.6a}$$
$$-\sigma[\rho s_2]+[\rho s_2 u]>0. \tag{3.6b}$$

In order to specify (3.6), let us consider for all $\delta>0$, $\mathbf{v}_\delta$ a rescaled traveling wave for the same triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$. Let us then define

$$\int_{\mathbb{R}}\frac{\mu_2(\widehat{\mathbf{v}}_\delta)}{\mu_1(\widehat{\mathbf{v}}_\delta)+\mu_2(\widehat{\mathbf{v}}_\delta)}\frac{\widehat{\rho_\delta}^{\gamma_1-1}}{\gamma_1-1}\left\{\partial_t(\widehat{\rho s_1})_\delta+\partial_x(\widehat{\rho s_1 u})_\delta\right\}(\xi)\,d\xi=\zeta_1(\sigma;\mathbf{v}_L,\mathbf{v}_R,\frac{\mu_2}{\mu_1}), \tag{3.7}$$
$$\int_{\mathbb{R}}\frac{\mu_1(\widehat{\mathbf{v}}_\delta)}{\mu_1(\widehat{\mathbf{v}}_\delta)+\mu_2(\widehat{\mathbf{v}}_\delta)}\frac{\widehat{\rho_\delta}^{\gamma_2-1}}{\gamma_2-1}\left\{\partial_t(\widehat{\rho s_2})_\delta+\partial_x(\widehat{\rho s_2 u})_\delta\right\}(\xi)\,d\xi=\zeta_2(\sigma;\mathbf{v}_L,\mathbf{v}_R,\frac{\mu_2}{\mu_1}). \tag{3.8}$$

Then the total masses ζ_1 and ζ_2 of the two entropy inequalities are bounded and do not depend on the rescaling parameter $\delta>0$ but only depend on the ratio of the viscosities. The masses are linked through the following identity:

$$\zeta_1(\sigma;\mathbf{v}_L,\mathbf{v}_R,\frac{\mu_2}{\mu_1})-\zeta_2(\sigma;\mathbf{v}_L,\mathbf{v}_R,\frac{\mu_2}{\mu_1})=0. \tag{3.9}$$

We omit the proof of the above result since it is straightforward when (2.4), (2.5), (2.7) and (2.8) are integrated with respect of traveling waves.

Rephrasing Theorem 3.2, the triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$ is entirely characterized by the classical jump relations (3.5a), (3.5b), (3.5c) and the *generalized jump condition* (3.9). As a consequence, for fixed $\mathbf{v}_L$ and σ, the end state $\mathbf{v}_R$ depends on $(\sigma,\mathbf{v}_L)$ but also on the ratio of the viscosities (see Figure 1):

$$\mathbf{v}_R:=\mathbf{v}_R(\sigma,\mathbf{v}_L,\frac{\mu_2}{\mu_1}). \tag{3.10}$$

Let us emphasize that the average balance equation (2.13) with respect of traveling waves, coincides with the generalized jump relation (3.9). This non-standard dependence arises from the lack of four conservation laws but it holds false when the classical conservative Navier-Stokes equations are considered where $(\mathbf{v}_R:=\mathbf{v}_R(\sigma,\mathbf{v}_L))$.

After the work of LeFloch[14] or Raviart-Sainsaulieu[17], a family of traveling wave solutions $\{\hat{\mathbf{v}}_\delta\}_{\delta>0}$, associated with a given triple $(\sigma;\mathbf{v}_L,\mathbf{v}_R)$, converges to, in the L^1_{loc} strong topology, to the following step function as δ goes to zero:

$$\mathbf{v}_0(x,t) = \begin{cases} \mathbf{v}_L, & x<\sigma t, \\ \mathbf{v}_R, & x>\sigma t. \end{cases} \tag{3.11}$$

This so-called shock-solution satisfies by construction the (generalized) jump conditions (3.5), (3.9).

To conclude the present section, Theorem 3.1 is established and we give useful properties of traveling wave solutions. First, we give the precise form of the autonomous system which governs the viscous profile we study for existence. For the sake of clarity in the notations, the notation $\hat{v}$ associated with traveling wave is omitted in the sequel.

Lemma 3.1. *The function* $\mathbf{v}$ *is an heterocline solution of (3.2) if and only if* $\mathbf{v}$ *is an heterocline of the following algebraic-differential system:*

$$\begin{cases} \rho(u-\sigma)=M, \\ \frac{p_1}{\gamma_1-1}+\frac{p_2}{\gamma_2-1}=\frac{\rho}{\rho_L}\left(\frac{(p_1)_L}{\gamma_1-1}+\frac{(p_2)_L}{\gamma_2-1}\right)+\left(1-\frac{\rho}{\rho_L}\right)\left(\frac{M^2}{2}-((p_1)_L+(p_2)_L)\right), \end{cases} \tag{3.12}$$

$$\begin{cases} (\mu_1+\mu_2)M\tau d_\xi\tau = \mathcal{F}(\tau)+\dfrac{\gamma_2-\gamma_1}{\gamma_2-1}s_2\tau^{1-\gamma_2}, \\ d_\xi s_2 = (\gamma_2-1)\mu_2 M\tau^{\gamma_2-1}(d_\xi\tau)^2, \end{cases} \tag{3.13}$$

where $\tau=1/\rho$, $s_2=p_2\tau^{\gamma_2}$ *and*

$$\mathcal{F}(\tau)=M^2\left(\frac{\gamma_1+1}{2}\right)(\tau-\tau_L)(\tau-\tau^\star)-\frac{\gamma_2-\gamma_1}{\gamma_2-1}\tau_L(p_2)_L, \tag{3.14}$$

with the constants

$$M=\rho_L(u_L-\sigma), \quad \tau^\star=\frac{\gamma_1-1}{\gamma_1+1}\frac{1}{\rho_L}+\frac{2\gamma_1}{\gamma_1+1}\frac{(p_1)_L+(p_2)_L}{M^2}. \tag{3.15}$$

Note that (3.13) has a symmetric form when involving $s_1=p_1\tau^{\gamma_1}$. For the sake of simplicity, we adopt (3.13) where s_2 plays the role of the *main* specific entropy, and we enforce $\gamma_1\le\gamma_2$.

Since (3.13) is an autonomous system independent of (3.12), the existence of a heteroclinic solution of (3.2) is governed by the existence of a heteroclinic solution of (3.13). This dynamical system is endowed with the following open subset of $\mathbb{R}^2$:

$$\mathcal{E}=\left\{\omega=(\tau,s_2)\in\mathbb{R}^2:\ \tau>0, s_2>0\right\}.$$

The smoothness assumptions (3.3) made on the viscosity functions make the vector field of (3.13) $X:\mathcal{E}\to\mathbb{R}^2$ to be continuously differentiable. Well-known results then assert that prescribing at $\xi=0$ an initial data ω_0 in (3.13) gives rise to a non-extensible solution of Σ^3 under the generic assumption of a continuous vector field X (see Reinhard[18] or Walter[25] for instance), uniqueness being achieved under the strengthened condition of a Lipschitz continuous field (see again Reinhard[18] or Walter[25] for the Picard-Lindelöf Theorem and for counterexamples).

Lemma 3.1 is now established.

Proof of Lemma 3.1. For convenience, we set $\xi = x - \sigma t$. A traveling wave $\mathbf{v}$ with speed σ is a solution of (2.4) if and only if it is solution of the following ODE's system:

$$\begin{cases} -\sigma d_\xi \rho + d_\xi \rho u = 0, \\ -\sigma d_\xi \rho u + d_\xi \left(\rho u^2 + p_1 + p_2\right) = d_\xi \left((\mu_1 + \mu_2) d_\xi u\right), \\ -\sigma d_\xi p_i + d_\xi p_i u + (\gamma_i - 1) p_i d_\xi u = (\gamma_1 - 1)\mu_i (d_\xi u)^2, \quad i = 1, 2. \end{cases} \tag{3.16}$$

First, let us notice that $-\sigma d_\xi \rho + d_\xi \rho u = 0$ reads for all $\xi \in \mathbb{R}$

$$\rho(u - \sigma) = M,$$

when integrating over $(-\infty, \xi)$. For the sake of simplicity, let us set $\tau = 1/\rho$ to write $M\tau = u - \sigma$. Hence, a traveling wave $\mathbf{v}$ with velocity σ is a solution of (2.4) if and only if it is solution of the following system:

$$\begin{cases} \rho(u - \sigma) = M, & (3.17a) \\ M^2 d_\xi \tau + d_\xi p_1 + d_\xi p_2 = d_\xi \left(M(\mu_1 + \mu_2) d_\xi \tau\right), & (3.17b) \\ \tau d_\xi p_i + \gamma_i p_i d_\xi \tau = M(\gamma_i - 1)\mu_i (d_\xi \tau)^2, \quad i = 1, 2. & (3.17c) \end{cases}$$

A second algebraic relation can be exhibited:

$$\frac{p_1}{\gamma_1 - 1} + \frac{p_2}{\gamma_2 - 1} = \frac{\rho}{\rho_L}\left(\frac{(p_1)_L}{\gamma_1 - 1} + \frac{(p_2)_L}{\gamma_2 - 1}\right) + \left(1 - \frac{\rho}{\rho_L}\right)\left(\frac{M^2}{2} - ((p_1)_L + (p_2)_L)\right). \tag{3.18}$$

Indeed, when integrating over $(-\infty, \xi)$ for all $\xi \in \mathbb{R}$, the equation (3.17b) reads:

$$(\mu_1 + \mu_2) M d_\xi \tau = H - H_L, \tag{3.19}$$

where we set $H = M^2\tau + p_1 + p_2$ and $H_L = M^2\tau_L + (p_1)_L + (p_2)_L$. The sum of $(3.17c)_{i=1}$ and $(3.17c)_{i=2}$ gives

$$d_\xi\Big(\tau\big(\frac{p_1}{\gamma_1 - 1} + \frac{p_2}{\gamma_2 - 1}\big) - \frac{M^2}{2}\tau^2\Big) + H d_\xi \tau = \Big\{(\mu_1 + \mu_2) M d_\xi \tau\Big\} d_\xi \tau. \tag{3.20}$$

Invoking (3.19), we have

$$d_\xi\left(\tau\left(\frac{p_1}{\gamma_1 - 1} + \frac{p_2}{\gamma_2 - 1}\right) + H_L \tau - \frac{M^2}{2}\tau^2\right) + H d_\xi \tau = (H - H_L) d_\xi \tau,$$

to write

$$d_\xi\Big(\tau\big(\frac{p_1}{\gamma_1 - 1} + \frac{p_2}{\gamma_2 - 1}\big) + H_L \tau - \frac{M^2}{2}\tau^2\Big) = 0. \tag{3.21}$$

The expected relation (3.18) arises when (3.21) is integrated on $(-\infty, \xi)$ for all $\xi \in \mathbb{R}$. Hence, the algebraic system (3.12) is established. Now, we turn exhibiting the system (3.13). To access such an issue, let us integrate (3.17b to write:

$$(\mu_1 + \mu_2)M d_\xi \tau = M^2\tau + p_1 + p_2 - \left(M^2\tau_L + (p_1)_L + (p_2)_L\right).$$

Invoking (3.18), we eliminate p_1 from the above relation to obtain:

$$(\mu_1 + \mu_2)M\tau d_\xi \tau = \mathcal{F}(\tau) + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} s_2 \tau^{1-\gamma_2},$$

where $s_2 = p_2\tau^{\gamma_2}$ and the smooth function $\mathcal{F}$ is defined by (3.14).

The last equation of (3.13) is easily obtained from $(3.17\mathrm{c})_{\mathrm{i}=2}$, and the proof is achieved. □

Now, we turn proving the existence of a unique heteroclinic solution of (3.13) which connects $\omega_L = (\tau_L, (s_2)_L)$ in the past (namely $\xi \to -\infty$). Let us note from now on that the heteroclines of interest connect two end points, ω_L and ω_R, which necessarily belong to the following manifold:

$$G^{-1}(0) = \{\omega \in \mathcal{E} : \; G(\omega) = 0\}, \tag{3.22}$$

where we have set

$$G(\omega) = \mathcal{F}(\tau) + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} s_2 \tau^{1-\gamma_2}. \tag{3.23}$$

First we have:

Lemma 3.2. *Assume (3.3) and the Lax condition (3.4). There exists two orbit solutions of (3.13) which connect ω_L in the past. At most one of them can be heteroclinic.*

We denote $\mathcal{L}$ the orbit possibly heteroclinic. This orbit will be seen below emanate from the region $\{\tau \leq \tau_L, s_2 \geq (s_2)_L\}$ and made of states satisfying $d_\xi\tau < 0$. Put in other words, the viscous shock we are proving for existence, necessarily is a compressive shock.

Proof. The local existence of two solutions of (3.13) which connect ω_L as ξ goes to $-\infty$ is a direct consequence of the Central Manifold Theorem[1,12] applied to the rest point ω_L. Indeed, straightforward computations ensure that ω_L is a non hyperbolic point of the dynamical system (3.13) since the linearization admits $\lambda_1 = (M^2\tau - \gamma_1 p_1 - \gamma_2 p_2)/((\mu_1 + \mu_2)M\tau) > 0$ and $\lambda_2 = 0$ as eigenvalues. The associated eigenspaces are given by $T^u(\omega_L) = \{e_1\}$ and $T^c(\omega_L) = \{e_2\}$ where $\{e_1, e_2\}$ stands for the canonical orthonormal basis of $\mathbb{R}^2$. From the Central Manifold Theorem, there exists two locally invariant manifolds $W^u(\omega_L)$ and $W^c(\omega_L)$ tangent to $T^u(\omega_L)$ and $T^c(\omega_L)$ respectively. From the Lax condition (3.4), the nontrivial eigenvalue λ_1 is positive. Hence, $W^u(\omega_L)$ is referred to as the unstable manifold and is made of the totality of the orbits which tend exponentially fast as $\xi \to -\infty$. Since $W^u(\omega_L)$ is one dimensional and tangent to e_1, there exists exactly two horizontal orbits which approach ω_L as $\xi \to -\infty$ from the two opposite direction $\tau \geq \tau_L$ and $\tau \leq \tau_L$.

To conclude the proof, we establish that the orbit emanating from the region $\tau \geq \tau_L$ cannot be heteroclinic. Indeed, an easy study of the dynamical system's vector field ensures the nonexistence of a rest point for (3.13) in the region $\{\tau > \tau_L, s_2 \geq (s_2)_L\}$ where $d_\xi \tau > 0$. The proof is thus completed. □

Therefore, the orbit $\mathcal{L}$ is shown to be trapped in the region

$$I = \{\omega \in \mathcal{E} :\ \tau < \tau_L, s_2 > (s_2)_L, G(\omega) \leq 0\},$$

where G is defined by (3.23). Indeed, we have

Lemma 3.3. *Assume$\gamma_1 \leq \gamma_2$. The set I is a positively invariant set of the dynamical system (3.13). In addition, the orbit $\mathcal{L}$ stays within a compact subset $\mathcal{K}$ of I:*

$$\mathcal{K} = \left\{\omega \in \mathcal{E} :\ \tau_m < \tau < \tau_L, (s_2)_L < s_2 < s_2^M, G(\omega) \leq 0\right\},$$

where τ_m and s_2^M solely depend of the pair $(\sigma, \mathbf{v}_L)$.

Proof. We denote $\omega_0 \cdot \xi$ the positive semi-flow solution associated with the initial data ω_0 defined for all $\xi \in [0, \xi^+(\omega_0))$, where $\xi^+(\omega_0)$ stands for the maximal positive time of existence. First, we establish that $\omega_0 \cdot \xi \in I$ for all ω_0 in I. This statement is obvious as soon as ω_0 satisfies $G(\omega_0) = 0$, since ω_0 thus become a rest point. Now, for $\omega_0 \in I$ but for $G(\omega_0) < 0$, we prove $G(\omega_0 \cdot \xi) < 0$ for all ξ in $[0, \xi^+(\omega_0))$. Assume the existence of a finite ξ_c such that $G(\omega_0 \cdot \xi_c) = 0$, *i.e.* $\omega_0 \cdot \xi_c$ defines an end point of (3.13). Since the vector field is Lipschitz-continuous over $\mathcal{E}$, such a rest point cannot be reached for finite ξ. Then I is a positive invariant set for (3.13).

Now, we turn proving the estimates for both τ and s_2. First, let us assume $\gamma_1 < \gamma_2$ to emphasize that the manifold $G^{-1}(0)$ is parameterized by τ:

$$G^{-1}(0) = \left\{\omega \in \mathcal{E} :\ s_2 = -\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1}\mathcal{F}(\tau)\tau^{\gamma_2 - 1}\right\}. \tag{3.24}$$

A straightforward study ensures the required estimates to define the compact subset $\mathcal{K}$.

Now, let us assume $\gamma_1 = \gamma_2$ to obtain $G(\omega) = M\frac{\gamma_1+1}{2}(\tau - \tau_L)(\tau - \tau^\star)$ where $\tau^\star$ is defined by (3.15). Hence, τ satisfies the following required estimate:

$$\tau^\star \leq \tau < \tau_L.$$

We conclude bounding $s_2(\xi)$ for all $\xi \in [0, \xi^+(\omega_L))$. We have

$$d_\xi s_2 = \frac{\mu_2}{\mu_1 + \mu_2}(\gamma_2 - 1)\tau^{\gamma_2 - 2}\mathcal{F}(\tau)d_\xi \tau,$$

where

$$-\frac{\mu_2}{\mu_1 + \mu_2}(\gamma_2 - 1)\tau^{\gamma_2 - 2}\mathcal{F}(\tau) \leq -(\gamma_2 - 1)\mathcal{F}\left(\frac{\tau^\star + \tau_L}{2}\right) \max_{\tau \in \{\tau^\star, \tau_L\}} (\tau^{\gamma_2 - 2}), \quad \forall \omega \in I.$$

Let us set $C(\sigma, \mathbf{v}_L)$ the right hand side of the above inequality to write:

$$0 \leq d_\xi s_2 \leq C(\sigma, \mathbf{v}_L) d_\xi \tau.$$

The above relation is integrated with respect of ξ to obtain:

$$0 \leq s_2(\xi) - (s_2)_L \leq C(\sigma, \mathbf{v}_L)(\tau_L - \tau^\star) \quad \forall \xi \in [0, \xi^+(\omega_L))$$

and the proof is achieved. □

From the above result, the orbit $\mathcal{L}$ is relatively compact and well-known consideration (see Reinhard[18] for instance) ensure that the ϖ-limit set is nonempty, compact and connected. Moreover, the function $\omega \to \tau(\omega)$, defined for all $\omega \in \mathcal{E}$, is a Lyapounov function over I since $d_\xi \tau \leq 0$ on I. When applying the LaSalle Theorem (see Reinhard[18]), the ϖ-limit set is included in $\{\omega \in I : \ G(\omega) = 0\}$. Hence, there exists $\omega_R \in I$ which is connected by $\mathcal{L}$ in the future. As a consequence, Theorem 3.1 is proved as soon as $\gamma_1 < \gamma_2$. The same result holds with $\gamma_2 < \gamma_1$ when reversing the role played by p_1 and p_2. The proof of Theorem 3.1 is thus achieved.

The present section is achieved when establishing the following property satisfied by the heteroclinic solutions of (3.13):

Lemma 3.4. *The orbit $\mathcal{L}$ can be parameterized by τ and thus defines a function $S_2 \in C^1([\tau_R, \tau_L], \mathbb{R}_+)$ such that $S_2(\tau) = s_2$ for all $\omega \in \mathcal{L}$. The function S_2 is solution of*

$$\begin{cases} \dfrac{dS_2}{d\tau} = \dfrac{\mu_2}{\mu_1 + \mu_2}(\gamma_2 - 1)\tau^{\gamma_2 - 1} \left\{ \mathcal{F}(\tau) + \dfrac{\gamma_2 - \gamma_1}{\gamma_2 - 1} S_2 \tau^{1-\gamma_2} \right\}, \quad \tau \in [\tau_R, \tau_L], \\ S_2(\tau_L) = (s_2)_L. \end{cases} \tag{3.25}$$

Proof. Let $\xi \to (\tau(\xi), s_2(\xi))$ be solution of (3.13) such that the orbit $\mathcal{L}$ is defined as $\{(\tau(\xi), s_2(\xi)) : \ \xi \in \mathbb{R}\}$. Since $\xi \to \tau(\xi)$ is a strictly decreasing function then $\mathcal{L}$ can be parameterized by τ to obtain a function S_2 with:

$$\frac{dS_2}{d\tau} = \frac{d_\xi s_2(\xi)}{d_\xi \tau(\xi)}.$$

As a consequence, $\tau \to S_2(\tau)$ is solution of the following Cauchy problem:

$$\begin{cases} \dfrac{dS_2}{d\tau} = \dfrac{\mu_2}{\mu_1 + \mu_2}(\gamma_2 - 1)\tau^{\gamma_2 - 1} \left\{ \mathcal{F}(\tau) + \dfrac{\gamma_2 - \gamma_1}{\gamma_2 - 1} S_2 \tau^{1-\gamma_2} \right\}, \quad \tau \in (\tau_R, \tau_L), \\ S_2(\tau(0)) = s_2(0). \end{cases} \tag{3.26}$$

To conclude, let us underline that the field function of (3.26) is $\mathrm{C}^1(\mathbb{R}_+)$ and the proof is completed when applying the Picard-Lindelöf Theorem. □

4. Limit behaviors of shock layers as $\mu_2 \to 0$

The generalized jump relation (3.9) clearly suggests that the system (2.4) admits a limit system in full conservation form as soon as the ratio of viscosities μ_2/μ_1 goes to zero. In the sequel, to avoid a zero μ_1 viscosity we assume:

There exists a constant $C_{\mu_1} > 0$ such that $\mu_1(\mathbf{v}) > C_{\mu_1}$ for all $\mathbf{v} \in \Omega$. (4.1)

To illustrate our purpose, momentarily let us assume that the viscosity functions are constant with μ_2 understood as a parameter arbitrarily small. The generalized jump relation (3.9) clearly tends to $\zeta_2(\sigma; \mathbf{v}_L, \mathbf{v}_R, 0) = 0$ while μ_2/μ_1 goes to zero. Such identity will be shown to coincide with the following classical jump relation:

$$-\sigma[\rho s_2] + [\rho s_2 u] = 0. \tag{4.2}$$

Hence, in the limit of μ_2/μ_1 to zero, the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ is entirely characterized by the classical jump relations (3.5a), (3.5b), (3.5c) and (4.2). This triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ thus coincides with one which arises when viscous shock layers are considered for the following conservation system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p_1 + p_2) = \partial_x(\mu_1 \partial_x u), \\ \partial_t \rho E + \partial_x(\rho E + p_1 + p_2)u = \partial_x(\mu_1 u \partial_x u), \\ \partial_t \rho s_2 + \partial_x \rho s_2 u = 0. \end{cases} \tag{4.3}$$

First, let us underline that (4.3) is nothing but the conservation form of (2.4) but for the following specific choice of viscosities: $\mu_1 > 0$ and $\mu_2 = 0$ (see Lemma 2.2). Since the assumption (3.3) is satisfied by such a choice of viscosity functions, Theorem 3.1 can be applied. Hence, the global existence and uniqueness (up to a translation) of traveling wave solutions of (4.3) is once again ensured. To be more precise, for all $\mathbf{v}_L \in \Omega$ and $\sigma \in \mathbb{R}$ fixed so that (3.4) is satisfied, there exists a unique $\mathbf{v}_R^0 \in \Omega$ and a unique shock layer $\mathbf{v}^0(x,t) = \widehat{\mathbf{v}}^0(x - \sigma t)$ smooth solution of (4.3) which connects the end states $\mathbf{v}_L$ and $\mathbf{v}_R^0$.

A second remark is devoted to the symmetry played by the pressures. Indeed, our initial system (2.4) involved symmetric pressures while such a symmetry is lost considering the limit system (4.3). In fact, the limit behaviors does not preserve the symmetry since one of the viscosities is preferred. All the stated results hold true reversing the role played by the pressures and their associated viscosities.

No longer we adopt the restrictive assumptions of two constant viscosities. As prescribed by physics, the viscosities are assumed to be nonlinear functions on the unknowns. Concerning μ_2, we assume that this function depends on a parameter $\eta > 0$ to read:

$$\mu_2(\mathbf{v}) = \eta \nu_2(\mathbf{v}), \quad \forall \mathbf{v} \in \Omega. \tag{4.4}$$

This dependence on the parameter η implies that traveling wave solutions of (2.4) define a sequence indexed to η. These solutions are thus denoted $\mathbf{v}^\eta(x,t) = \widehat{\mathbf{v}}^\eta(x - \sigma t)$. Involving (3.10), the right end state also depends on η to write: $\mathbf{v}_R^\eta$.

Now, we state the main result of this section:

Theorem 4.1. *Assume that the viscosity functions satisfy (3.3)-(4.1) while* μ_2 *satisfies (4.4). Let* $\mathbf{v}_L \in \Omega$ *and* $\sigma \in \mathbb{R}$ *be fixed so that (3.4) is satisfied. Let* $\widehat{\mathbf{v}}^0 \in \mathrm{C}^1(\mathbb{R}, \Omega)$ *be a traveling wave solution of (4.3) associated with the triple* $(\sigma; \mathbf{v}_L, \mathbf{v}_R^0)$ *where the parameterization is assumed to be fixed. Assuming a suitable parameterization, the traveling wave solutions of (2.4),* $\widehat{\mathbf{v}}^\eta \in \mathrm{C}^1(\mathbb{R}, \Omega)$*, associated with the triple* $(\sigma; \mathbf{v}_L, \mathbf{v}_R^\eta)$*, tend to* $\widehat{\mathbf{v}}^0$ *as* η *goes to* 0*. The convergence is uniform in* $\mathbb{R}$*.*

The next section will be devoted to the proof of this statement.

Now, we turn considering the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R^\eta)$ to propose a characterization of the end state $\mathbf{v}_R^\eta$. Indeed, by opposition with the usual Navier-Stokes equations, $\mathbf{v}_R^\eta$ remains implicitly known as solution of the non-standard generalized jump relation (3.9). With $\mu_1 > 0$ and $\nu_2 \in \mathrm{C}^1(\Omega, \mathbb{R}_+)$ fixed, Theorem 4.1 ensures the following limit:

$$\lim_{\eta \to 0} \mathbf{v}_R^\eta(\sigma; \mathbf{v}_L, \eta \frac{\nu_2}{\mu_1}) = \mathbf{v}_R^0(\sigma; \mathbf{v}_L). \tag{4.5}$$

Now, we exhibit an asymptotic expansion of $\mathbf{v}_R^\eta$ involving the ratio of the viscosities. Such an expansion turns out to be useful in applications of practical interest.

Theorem 4.2. *Let* $\mathbf{v}_L \in \Omega$ *and* $\sigma \in \mathbb{R}$ *so that (3.4) is satisfied. Assume that the viscosity* $\mu_1 > 0$ *is constant while* μ_2 *is given by (4.4) with the function* ν_2 *defined as follows:*

$$\nu_2(\mathbf{v}) = p_2^{2-C_{\epsilon 1}} \rho^{C_{\epsilon 1}}.$$

Then for all $n \in \mathbb{N}$*, there exists a vector sequence* $(\mathbf{v}_i)_{1 \le i \le n} \in \mathbb{R}^4$ *which only dependents on the pair* $(\sigma, \mathbf{v}_L)$ *such that:*

$$\mathbf{v}_R^\eta(\sigma; \mathbf{v}_L, \eta \frac{\nu_2}{\mu_1}) = \mathbf{v}_R^0(\sigma; \mathbf{v}_L) + \sum_{1 \le i \le n} \mathbf{v}_i(\sigma; \mathbf{v}_L) \delta^i + O(\delta^{n+1}), \tag{4.6}$$

where $\delta = \eta / \mu_1$*.*

Let us note that the viscosity function μ_2 prescribed by the physics is given by (2.2). Since the quantity $k^{C_{\epsilon 1}}/\epsilon$ stays constant across a viscous shock layer, the function μ_2 recast as follows:

$$\mu_2(\mathbf{v}) = \eta \nu_2(\mathbf{v}), \quad \eta = \frac{C_\mu (\frac{2}{3} k_L)^{C_{\epsilon_1}} / \epsilon_L}{(\gamma_\tau - 1)^2}, \quad \nu_2(\mathbf{v}) = p_2^{2-C_{\epsilon 1}} \rho^{C_{\epsilon 1}}. \tag{4.7}$$

Hence, Theorem 4.2 can be applied with viscosities of physical interest.

We return proving Theorem 4.2. The proof is directly obtained from the three following technical results. Let us just emphasize that these lemmas specified the detailed form of the asymptotic expansion (4.6). For the sake of simplicity in the notations, the index η is omitted.

First, we will prove that the end state $\mathbf{v}_R$ necessarily satisfies three algebraic relations independent on δ. These relations, associated with the conservation of density, momentum and total energy, will be shown to link the unknowns ρ_R, $(\rho u)_R$ and $(p_1)_R$ to the unknown $(s_2)_R = (p_2)_R/\rho_R^{\gamma_2}$ but for a globally non-invertible identity between ρ_R and $(s_2)_R$ (except for the specific case $\gamma_1 = \gamma_2$). To consider the unknown s_2 to express the other unknowns is deliberate according to Lemma 3.4. Indeed, we will propose an asymptotic expansion of the integral curve $S_2(\tau)$ to obtain an expansion of $(s_2)_R$ since we have $S_2(\tau_R) = (s_2)_R$. It will appear that the coefficients of this expansion depend on $(\sigma, \mathbf{v}_L)$ but also on $\tau_R = 1/\rho_R$. The expansions of $(s_2)_R$ and ρ_R will be exhibited when involving the limit relation (4.5).

Lemma 4.1. *The end state $\mathbf{v}_R$ necessarily satisfies the following identities:*

$$\rho_R(u_R - \sigma) = M, \tag{4.8}$$

$$\frac{(p_1)_R}{\gamma_1 - 1} + \frac{(p_2)_R}{\gamma_2 - 1} = \frac{\rho_R}{\rho_L}\left(\frac{(p_1)_L}{\gamma_1 - 1} + \frac{(p_2)_L}{\gamma_2 - 1}\right) + \left(1 - \frac{\rho_R}{\rho_L}\right)\left(\frac{M^2}{2} - ((p_1)_L + (p_2)_L)\right). \tag{4.9}$$

In addition, if $\gamma_1 = \gamma_2$, we have $\rho_R = \frac{1}{\tau^\star}$. In the case of two distinct adiabatic coefficients, $\gamma_1 \neq \gamma_2$, $(s_2)_R$ is defined as follows:

$$(s_2)_R = -\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1}(\tau_R)^{\gamma_2 - 1}\mathcal{F}(\tau_R). \tag{4.10}$$

We omit the proof of this result which is a direct consequence of identities (3.12) and (3.22) in the limit of ξ to $+\infty$.

Let us introduce the notation $\Theta(\tau)$ to denote some generic smooth function defined by:

$$\Theta(\tau) = \sum_{l=0}^{L} a_l \tau^{b_l}, \tag{4.11}$$

where $L \in \mathbb{N}$ is given, and (a_l, b_l) denote real constant pairs which only depend on $(\sigma, \mathbf{v}_L)$. Of course, primitives of such a function is known and once again reads under the generic form Θ.

Lemma 4.2. *There exists a function sequence $(\Theta_i)_{1\leq i\leq n}$ so that*

$$(s_2)_R = (s_2)_L + \sum_{1\leq i\leq n} \Theta_i(\tau_R)\delta^i + O(\delta^{n+1}). \tag{4.12}$$

Proof. First, an asymptotic expansion in the parameter δ of the integral curve $S_2 \in \mathrm{C}^1([\tau_R, \tau_L], \mathbb{R}_+)$, solution of (3.25), is proposed. For the sake of simplicity, let us set $\theta = 2 - C_{\varepsilon 1}$. Since we have

$$\frac{\mu_2}{\mu_1} = \delta \frac{S_2(\tau)^\theta}{\tau^{\theta(\gamma_2 - 1)+1}},$$

a straightforward computation gives the following ODE satisfied by S_2:

$$\frac{1}{1-\theta}\ \tau^{(\gamma_2-1)(\theta-1)+\gamma_1}\ \frac{d}{d\tau}\left(S_2(\tau)^{1-\theta}\right)+\delta\ \frac{d}{d\tau}\left(\frac{S_2(\tau)}{\tau^{\gamma_2-\gamma_1}}\right)= \delta\ \tau^{\gamma_1-2}(\gamma_2-1)\mathcal{F}(\tau), \quad (4.13)$$

with $S_2(\tau_L)=(s_2)_L$ as initial data. In order to exhibit an expansion of the function S_2 solution of (4.13), with $n>0$ a fixed integer, the solution is proposed in the following form:

$$S_2(\tau)=\sum_{0\le i\le n} f_i(\tau)\delta^i+O(\delta^{n+1}),\quad \tau\in[\tau_R,\tau_L],$$

where the functions $f_i\in \mathrm{C}^1([\tau_R,\tau_L],\mathbb{R})$ only depend on τ and $(\sigma,\mathbf{v}_L)$ and satisfy $f_0(\tau_L)=(s_2)_L$ and $f_i(\tau_L)=0$ for $i>0$. Now, we need give an expansion of $S_2(\tau)^{1-\theta}$. Let us set $\omega_i(\{k_j\})=\left\{(k_j)_{1\le j\le i-1}:\ k_j\in\mathbb{N},\ \sum_{j=1}^{i-1} j\ k_j=i\right\}$ for $i>0$ and let us introduce

$$\varphi_0(\tau)=(s_2)_L^{1-\theta},$$
$$\varphi_1(\tau)=(1-\theta)\frac{f_1(\tau)}{(s_2)_L^{\theta}},$$
$$\varphi_i(\tau)=(1-\theta)\frac{f_i(\tau)}{(s_2)_L^{\theta}}+(s_2)_L^{1-\theta}\sum_{\omega(\{k_j\})}\alpha_{\{k_j\}}\prod_{j=1}^{i-1}\left(\frac{f_j(\tau)}{(s_2)_L}\right)^{k_j},\quad i>1$$
$$\text{where}\ \ \alpha_{\{k_j\}}=\left(\prod_{j=0}^{\sum_{j=1}^{i}k_j-1}(1-\theta-j)\right)\Big/\left(\prod_{j=1}^{i}k_j!\right),$$

to obtain

$$S_2(\tau)^{1-\theta}=\sum_{i=0}^{n}\varphi_i(\tau)\delta^i+O(\delta^{n+1}). \quad (4.14)$$

Now, we turn proving that the functions f_i and φ_i write in the generic form Θ given by (4.11). Clearly, there exists a generic function as proposed in (4.11) so that:

$$\tau^{\gamma_1-2}(\gamma_2-1)\mathcal{F}(\tau)=\Theta(\tau).$$

Then the smooth function f_1 satisfies:

$$\frac{1}{1-\theta}\ \frac{\tau^{(\gamma_\tau-1)(\theta-1)+\gamma}}{(s_2)_L^{\theta}}\ \frac{df_1(\tau)}{d\tau}+(s_2)_L\ \frac{d\left(\frac{1}{\tau^{\gamma_\tau-\gamma}}\right)}{d\tau}=\Theta(\tau).$$

Hence, there exists a generic function Θ so that $d_\tau f_1(\tau)=\Theta(\tau)$ which implies the existence of Θ_1 so that $f_1(\tau)=\Theta_1(\tau)$. Let us assume that there exists Θ_i, with

$0 \leq i < n$, so that $f_i(\tau) = \Theta_i(\tau)$. Concerning the last step of this recurrence, the function f_n satisfies:

$$\frac{\tau^{(\gamma_\tau-1)(\theta-1)+\gamma}}{(1-\theta)(s_2)_L^\theta}\left(\frac{df_n(\tau)}{d\tau} + (s_2)_L \sum_{\omega(\{k_j\})} \alpha_{\{k_j\}} \frac{d}{d\tau}\prod_{j=1}^{n-1}\left(\frac{f_j(\tau)}{(s_2)_L}\right)^{k_j}\right) + \frac{d\frac{f_{n-1}(\tau)}{\tau^{\gamma_\tau-\gamma}}}{d\tau} = 0.$$

We immediately deduce the existence of a generic smooth function Θ_n so that $f_n(\tau) = \Theta_n(\tau)$. This above recurrence ensures the following asymptotic expansion:

$$S_2(\tau) = (s_2)_L + \sum_{1\leq i\leq n} \Theta_i(\tau)\delta^i + O(\delta^{n+1}), \quad \tau \in [\tau_R, \tau_L].$$

In addition, the asymptotic expansion (4.12) of $(s_2)_R$ is thus obtained since we have $S_2(\tau_R) = (s_2)_R$. The proof is thus concluded. □

As soon as we have $\gamma_1 = \gamma_2$, the proof of Theorem 4.2 is completed. Indeed, $\tau_R = \tau^\star$ where $\tau^\star$ only depends on $(\sigma, \mathbf{v}_L)$. Now, let us assume $\gamma_1 \neq \gamma_2$ to conclude the proof with the following result:

Lemma 4.3. *Assume $\gamma_1 \neq \gamma_2$. There exists two sequences $(\alpha_i)_{1\leq i\leq n}$ and $(\beta_i)_{1\leq i\leq n}$ of real number only depending on $(\sigma, \mathbf{v}_L)$, so that*

$$\rho_R = \rho_R^0 + \sum_{1\leq i\leq n} \alpha_i\delta^i + O(\delta^{n+1}), \tag{4.15}$$

$$(s_2)_R = (s_2)_L + \sum_{1\leq i\leq n} \beta_i\delta^i + O(\delta^{n+1}). \tag{4.16}$$

Proof. The following form is assumed for the end state $\tau_R = 1/\rho_R$:

$$\tau_R = \tau_R^0 + \sum_{i=1}^n \alpha_i\delta^i + O(\delta^{n+1}), \tag{4.17}$$

where $\tau_R^0 = 1/\rho_R^0$ and $(\alpha_i)_{1\leq i\leq n}$ denotes a real sequence which only depends on $(\sigma, \mathbf{v}_L)$. First, let us develop (4.12) when involving the expansion (4.17) to write:

$$(s_2)_R = (s_2)_L + \sum_{i=1}^n \beta_i\delta^i + O(\delta^{n+1}),$$

where the coefficients β_i depend on $(\sigma, \mathbf{v}_L)$ and $(\alpha_1, ..., \alpha_i)$. As a consequence, the coefficients of the $(s_2)_R$'s expansion only depends on $(\sigma, \mathbf{v}_L)$. The proof will be concluded when establishing the existence and uniqueness of the sequence $(\alpha_i)_{1\leq i\leq n}$. To access such an issue, the identity (3.22) is considered to link τ_R and $(s_2)_R$. It is thus necessary to be more precise about the dependence on $(\alpha_1, ..., \alpha_i)$ concerning the coefficients β_i. For the sake of simplicity in the notations, we set c_i a constant which only depends on $(\sigma, \mathbf{v}_L)$ while $\tilde{c}_i$ denotes a constant with a dependence on $(\sigma, \mathbf{v}_L)$ but also on $(\alpha_1, ..., \alpha_{i-1})$. With some abuse in the notations, we set $\tilde{c}_1 = 0$.

Let us consider a generic function Θ in the form (4.11) to write the following expansion:

$$\Theta(\tau_R) = \Theta(\tau_R^0) + \sum_{j=1}^{n} (c_j \alpha_j + \tilde{c_j}) \delta^j + O(\delta^{n+1}). \tag{4.18}$$

Indeed, we have

$$\Theta(\tau_R) = \sum_{l=0}^{L} a_l \tau_R^{b_l}, \tag{4.19}$$

where the following expansion is easily obtained:

$$\tau_R^{b_l} = (\tau_R^0)^{b_l} + \sum_{i=1}^{n} (c_i \alpha_i + \tilde{c}_i) \delta^i + O(\delta^{n+1}). \tag{4.20}$$

The required relation (4.18) is deduced from (4.19) and (4.20).

Now, let us consider the expansion (4.12) where the coefficients, given by $\Theta(\tau_R)$, are substituted by there own expansion (4.18) to obtain:

$$(s_2)_R = (s_2)_L + \Theta_1(\tau_R^0)\delta + \sum_{i=1}^{n-1} \left(\Theta_{i+1}(\tau_R^0) + c_i\alpha_i + \tilde{c}_i \right) \delta^{i+1} + O(\delta^{n+1}).$$

Since the pair $(\tau_R, (s_2)_R)$ belongs to $G^{-1}(0)$, defined by (3.22), we obtain:

$$\begin{aligned} (s_2)_L + \Theta_1(\tau_R^0)\delta + \sum_{i=1}^{n-1} \left(\Theta_{i+1}(\tau_R^0) + c_i\alpha_i + \tilde{c}_i \right) \delta^{i+1} + O(\delta^{n+1}) = \\ -\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} (\tau_R)^{\gamma_2 - 1} \mathcal{F}(\tau_R). \end{aligned} \tag{4.21}$$

Let us underline that the smooth function $\tau \to \tau^{\gamma_2 - 1}\mathcal{F}(\tau)$ reads under the generic form (4.11). Hence, the expansion (4.18) can be once again applied:

$$-\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} (\tau_R)^{\gamma_2 - 1} \mathcal{F}(\tau_R) = -\frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} (\tau_R^0)^{\gamma_2 - 1} \mathcal{F}(\tau_R^0) + \sum_{i=1}^{n} (c_i^{\mathcal{F}} \alpha_i + \tilde{c}_i^{\mathcal{F}}) \delta^i + O(\delta^{n+1}),$$

where $c_i^{\mathcal{F}} := c_i^{\mathcal{F}}(\sigma, \mathbf{v}_L)$, $\tilde{c}_1^{\mathcal{F}} := 0$ and $\tilde{c}_i^{\mathcal{F}} := \tilde{c}_i^{\mathcal{F}}(\sigma, \mathbf{v}_L, \alpha_1, ..., \alpha_{i-1})$. The equation (4.21) thus reads:

$$\begin{aligned} \left\{ (s_2)_L + \frac{\gamma_2 - 1}{\gamma_2 - \gamma_1} (\tau_R^0)^{\gamma_2 - 1} \mathcal{F}(\tau_R^0) \right\} + \left\{ \Theta_1(\tau_R^0) - c_1^{\mathcal{F}} \alpha_1 \right\} \delta \\ + \sum_{i=2}^{n} \left\{ \Theta_i(\tau_R^0) + c_{i-1}\alpha_{i-1} + \tilde{c}_{i-1} - c_i^{\mathcal{F}}\alpha_i - \tilde{c}_i^{\mathcal{F}} \right\} \delta^i = O(\delta^{n+1}). \end{aligned} \tag{4.22}$$

Now, let us apply Lemma 3.1 with $\mu_2 \equiv 0$ to establish that the pair $(\tau_R^0, (s_2^0)_R)$ only depends on $(\sigma, \mathbf{v}_L)$. Indeed, $(\tau_R^0, (s_2^0)_R)$ is solution of the following system:

$$\begin{cases} (s_2^0)_R = (s_2)_L, \\ \mathcal{F}(\tau_R^0) + \dfrac{\gamma_2 - \gamma_1}{\gamma_2 - 1} (s_2^0)_R (\tau_R^0)^{1-\gamma_2} = 0. \end{cases}$$

We deduce from (4.22) that the n-uplet $(\alpha_1, ..., \alpha_n)$ is the unique solution of the following $n \times n$ system of algebraic relations:

$$\begin{cases} c_1^{\mathcal{F}} \alpha_1 = \Theta_1(\tau_R^0), \\ c_i^{\mathcal{F}} \alpha_i = \Theta_i(\tau_R^0) + c_{i-1}\alpha_{i-1} + \tilde{c}_{i-1}(\alpha_1, ..., \alpha_{i-2}) - \tilde{c}_i^{\mathcal{F}}(\alpha_1, ..., \alpha_{i-1}), \quad 2 \leq i \leq n. \end{cases}$$

As expected, the solution only depends on $(\sigma, \mathbf{v}_L)$. The proof is thus completed. □

5. Proof of Theorem 4.1

This section is devoted to the proof of our main result, namely Theorem 4.1. For the sake of clarity in the notations, we omit with no confusion the notation $\widehat{\ }$ associated with the definition of the traveling wave. Let the pair $(\sigma, \mathbf{v}_L)$ be fixed in $(\mathbb{R} \times \Omega)$. The smooth function $\mathbf{v}^\eta \in \mathrm{C}^1(\mathbb{R}, \Omega)$ denotes the traveling wave solution of (2.4) with $\lim_{\xi \to -\infty} \mathbf{v}^\eta(\xi) = \mathbf{v}_L$, while $\mathbf{v}^0 \in \mathrm{C}^1(\mathbb{R}, \Omega)$ is the traveling wave solution of (4.3) which satisfies $\lim_{\xi \to -\infty} \mathbf{v}^0(\xi) = \mathbf{v}_L$. The parameterization of $\mathbf{v}^0$ is assumed to be fixed and will be specified below.

In the sequel, we set $\tau^\eta = 1/\rho^\eta$ and $\tau^0 = 1/\rho^0$. Theorem 4.1 will be established from the following three propositions:

Proposition 5.1. *Invoking a suitable parameterization of the traveling wave $\mathbf{v}^\eta$, let us assume that the $\mathrm{C}^1(\mathbb{R})$ function sequences $(\tau^\eta)_{\eta>0}$ and $(s_2^\eta)_{\eta>0}$ tend respectively to the smooth functions $\xi \to \tau^0(\xi)$ and $\xi \to s_2^0(\xi) = (s_2)_L$ as η goes to zero where the convergence is uniform on $\mathbb{R}$. Then we have*

$$\lim_{\eta \to 0} \mathbf{v}^\eta = \mathbf{v}^0,$$

the convergence being uniform on $\mathbb{R}$.

Put in other words, the proof of Theorem 4.1 follows from the $\mathbb{R}$-uniform convergence of $(\tau^\eta)_{\eta>0}$ and $(s_2^\eta)_{\eta>0}$ to the smooth functions $\xi \to \tau^0(\xi)$ and $\xi \to s_2^0(\xi)$. These convergences are stated in the two following results. Just, let us recall that $s_2^0(\xi) = (s_2)_L$ for all $\xi \in \mathbb{R}$ since we have $d_\xi s_2^0(\xi) = 0$ with $\mu_2 = 0$.

Proposition 5.2. *The function sequence $(s_2^\eta)_{\eta>0}$ tends to the real constant $(s_2)_L$ as η goes to zero. The convergence is uniform on $\mathbb{R}$ and does not depend on the choice of the parameterization.*

This result may be completed when proving the convergence of the orbits, defined by the traveling wave solutions of (2.4), to orbit defined by the traveling wave solution of (4.3). To be more precise, let us denote $\mathcal{L}^\eta$ (respectively $\mathcal{L}^0$) the integral curve defined by (τ^η, s_2^η) (resp. $(\tau^0, (s_2)_L)$) solution of (3.13) for all $\eta > 0$ (resp. $\eta = 0$). Both orbits $\mathcal{L}^\eta$ and $\mathcal{L}^0$ are closed arc which join $(\tau_L, (s_2)_L)$ to $(\tau_R^\eta, (s_2^\eta)_R)$ or $(\tau_R^0, (s_2)_L)$ respectively. The following result ensures the convergence of $\mathcal{L}^\eta$ to $\mathcal{L}^0$ as η goes to zero.

Lemma 5.1. *For all parameterization of the traveling waves $(\mathbf{v}^\eta)_{\eta>0}$, the following limit holds:*

$$\lim_{\eta\to 0} \tau_R^\eta = \tau_R^0. \tag{5.1}$$

Then for small enough values of η, the integral curve $\mathcal{L}^\eta$ is entirely included a neighborhood of $\mathcal{L}^0$.

Proof. The limit (5.1) follows from the definition of the end states of the traveling waves. Indeed and after (3.22), the pair $(\tau_R^\eta, (s_2^\eta)_R)$ satisfies

$$\mathcal{F}(\tau_R^\eta) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(s_2^\eta)_R(\tau_R^\eta)^{1-\gamma_2} = 0,$$

where $\mathcal{F}$ is defined by (3.14), while the pair $(\tau_R^0, (s_2)_L)$ is defined by

$$\mathcal{F}(\tau_R^0) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(s_2)_L(\tau_R^0)^{1-\gamma_2} = 0.$$

First, let us assume $\gamma_1 = \gamma_2$. We have $\tau_R^\eta = \tau^\star = \tau_R^0$ for all $\eta > 0$ and the conclusion is obvious. Next, with $\gamma_1 \neq \gamma_2$, let us denote $\overline{\tau}$ the limit of τ_R^η as η goes to zero. After Proposition 5.2 we have $(s_2^\eta)_R \to (s_2)_L$ as η tends to zero then, in the limit of η to zero, we obtain:

$$\mathcal{F}(\overline{\tau}) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(s_2)_L(\overline{\tau})^{1-\gamma_2} = 0. \tag{5.2}$$

Assume $\gamma_1 < \gamma_2$ to exhibit two distinguished positive roots of (5.2), namely τ_L and τ_R^0. After Lemma 3.3, $\overline{\tau} < \tau_L$ then we have $\overline{\tau} = \tau_R^0$.

Now, assume $\gamma_1 > \gamma_2$. This specific case turns out to be more complex since a straightforward computation ensures the existence of three distinguished roots for (5.2). These roots are τ_L, τ_R^0 and an other we denote $\check{\tau}$. Hence, we have two possibilities: $\overline{\tau} = \tau_R^0$ or $\overline{\tau} = \check{\tau}$. In order to eliminate $\overline{\tau} = \check{\tau}$, it is necessary to specify τ^m, defined in Lemma 3.3, to obtain $\check{\tau} < \tau^m \leq \tau_R^\eta \leq \tau_R^0$ and thus $\overline{\tau} = \tau_R^0$.

Let us recall that the curve $G^{-1}(0)$, defined by (3.24), defines a continuum set of end points of (3.13). The orbits $\mathcal{L}^0$ and $(\mathcal{L}^\eta)_{\eta>0}$ connect two distinguished points of $G^{-1}(0)$, namely $(\tau_L, (s_2)_L)$ and $(\tau_R^0, (s_2)_L)$ respectively $(\tau_R^\eta, (s_2^\eta)_R)$. These orbits do not intersect $G^{-1}(0)$ excepted in their ends. Now, the manifold $G^{-1}(0)$ is known to coincide with the curve

$$\Gamma(\tau) = -\frac{\gamma_2-1}{\gamma_2-\gamma_1}\tau^{\gamma_2-1}\mathcal{F}(\tau), \quad 0 < \tau \leq \tau_L. \tag{5.3}$$

A straightforward study of Γ ensures the existence of a unique local minimum (in τ^m) and a unique local maximum (in τ^M) such that $0 < \tau^m < \tau_R^0 < \tau^M < \tau_L$. Let us note from now on that τ^m defines a minimum of the admissible τ and it defines a maximum of the function Γ. Since Γ, and thus $G^{-1}(0)$, does not depend on η then τ^m and τ^M once again do not depend on η. Arguing the monotony of $\mathcal{L}^0$ and $(\mathcal{L}^\eta)_{\eta>0}$, the end state $(\tau_R^\eta, (s_2^\eta)_R)$ necessarily belongs to the closed arc $\{(\tau, s_2 = \Gamma(\tau)) : \ \tau \in [\tau^m, \tau_R^0]\}$ and thus we obtain $\tau^m \leq \tau_R^\eta \leq \tau_R^0$.

To conclude the proof, we show that $\check{\tau} < \tau^m$. Studying the function $\tau \to \mathcal{F}(\tau) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(s_2)_L\tau^{1-\gamma_2}$ over $(0,\tau_L]$, an easy computation ensures that this function is positive if $\tau \in (\check{\tau}, \tau_R^0)$ and non positive otherwise. When evaluated on τ^m, this function is shown to be positive. Indeed, we have

$$\mathcal{F}(\tau^m) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(s_2)_L(\tau^m)^{1-\gamma_2} = \frac{\gamma_2-\gamma_1}{\gamma_2-1}(\tau^m)^{1-\gamma_2}\left((s_2)_L - \Gamma(\tau^m)\right) > 0,$$

where $\Gamma(\tau^m) = \max_{(0,\tau_L]}\Gamma(\tau) > \Gamma(\tau_R^0) = (s_2)_L$. The proof is thus completed. □

A specific parameterization must be fixed to ensure the uniform convergence of the sequence $(\tau^\eta)_{\eta>0}$.

Proposition 5.3. *When a suitable parameterization is assumed, the function sequence $(\tau^\eta)_{\eta>0}$ tends to the smooth function $\xi \to \tau^0(\xi)$ as η goes to zero. The convergence is uniform on $\mathbb{R}$.*

The proof of Theorem 4.1 readily follows from the above three propositions.

Proof of Proposition 5.1. Let the suitable parameterization of $\mathbf{v}^\eta$ be assumed. Invoking the convergence of $(\tau^\eta)_{\eta>0}$ and $(s_2^\eta)_{\eta>0}$, the uniform convergence of (ρ^η) and (p_2^η) is proved. Indeed, we have

$$\rho^\eta - \rho^0 = \frac{\tau^0-\tau^\eta}{\tau^0\tau^\eta} \quad \text{and} \quad p_2^\eta - p_2^0 = (s_2^\eta - s_2^0)(\tau^\eta)^{-\gamma_2} + ((\rho^\eta)^{\gamma_2} - (\rho^0)^{\gamma_2})s_2^0,$$

while the smooth functions $\xi \to \tau^\eta(\xi)$ and $\xi \to \tau^0(\xi)$ are strictly positive for all $\xi \in \mathbb{R}$ and $\eta > 0$. Since the traveling waves $\mathbf{v}^\eta$ and $\mathbf{v}^0$ satisfy Lemma 3.3, we obtain:

$$|\rho^\eta - \rho^0| \le \frac{|\tau^0-\tau^\eta|}{(\tau^m)^2} \quad \text{and} \quad |p_2^\eta - p_2^0| \le \frac{|s_2^\eta - s_2^0|}{(\tau^m)^{\gamma_2}} + |(\rho^\eta)^{\gamma_2} - (\rho^0)^{\gamma_2}|s_2^M, \quad \forall\xi \in \mathbb{R},$$

where the constants $\tau^m > 0$ and $s_2^M > 0$ do not depend on η. The $\mathbb{R}$-uniform convergence of the sequences $(\rho^\eta)_{\eta>0}$ and $(p_2^\eta)_{\eta>0}$ to the functions ρ^0 and p_2^0 is established. Concerning the $\mathbb{R}$-uniform convergence of $((\rho u)^\eta)_{\eta>0}$ and $(p_1^\eta)_{\eta>0}$ to the smooth function $(\rho u)^0$ and p_2^0, the expected result directly deduces from (3.12). Indeed, we have:

$$\begin{aligned}
&(\rho u)^\eta - (\rho u)^0 = \sigma(\rho^\eta - \rho^0),\\
&\frac{p_1^\eta - p_1^0}{\gamma_1 - 1} = \frac{p_2^\eta - p_2^0}{\gamma_2-1} + \frac{\rho^\eta-\rho^0}{\rho_L}\left(\frac{\gamma_1}{\gamma_1-1}(p_1)_L + \frac{\gamma_2}{\gamma_2-1}(p_2)_L - \frac{M^2}{2}\right),
\end{aligned}$$

and the proof is achieved. □

Proof of Proposition 5.2. Let $\mathbf{v}^\eta \in \mathrm{C}^1(\mathbb{R},\Omega)$ be a traveling wave solution of (2.4) for all fixed $\eta > 0$. The proof is based on a relevant estimation of the smooth function $\xi \to d_\xi s_2^\eta(\xi)$. After Lemma 3.1, both functions $\xi \to \tau^\eta(\xi)$ and $\xi \to s_2^\eta(\xi)$

satisfy the autonomous ODE system (3.13). After easy computations, we obtain to the following equation satisfied by s_2^η:

$$\frac{ds_2^\eta(\xi)}{d\xi} = (\gamma_2-1)\tau^\eta(\xi)^{\gamma_2-2}\frac{\mu_2(\mathbf{v}^\eta)}{\mu_1(\mathbf{v}^\eta)+\mu_2(\mathbf{v}^\eta)} \times \left(\mathcal{F}(\tau^\eta(\xi)) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}s_2^\eta(\xi)\tau^\eta(\xi)^{1-\gamma_2}\right)\frac{d\tau^\eta(\xi)}{d\xi}, \quad \forall\xi\in\mathbb{R},\ \eta>0,$$

where $\mathcal{F}$ is a quadratic function defined by (3.14). When considering a traveling wave solution, the function $\xi \to d_\xi\tau^\eta(\xi)$ is negative for all $\xi\in\mathbb{R}$ with $\eta>0$. Hence, the function $\xi \to \mathcal{F}(\tau^\eta(\xi)) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}s_2^\eta(\xi)\tau^\eta(\xi)^{1-\gamma_2}$ is once again negative for all $\xi\in\mathbb{R}$ with $\eta>0$.

Now, we deduce from Lemma 3.3 the following inequality:

$$(\gamma_2-1)\tau^\eta(\xi)^{\gamma_2-2}\frac{\mu_2(\mathbf{v}^\eta(\xi))}{\mu_1(\mathbf{v}^\eta(\xi))+\mu_2(\mathbf{v}^\eta(\xi))} \le (\gamma_2-1)\min(\tau_L^{\gamma_2-2},(\tau^m)^{\gamma_2-2})\frac{\nu_2(\mathbf{v}^\eta(\xi))}{\mu_1(\mathbf{v}^\eta(\xi)}\eta.$$

Moreover, since each component of the map $\xi\to\mathbf{v}^\eta(\xi)$ can be bounded independently on η (see Lemma 3.3), there exists an open subset of Ω, independent of η, such that $\xi\to\mathbf{v}^\eta(\xi)$ belongs to this subset for all $\eta>0$. As a consequence, arguing the smoothness of the maps μ_1 and ν_2, there exists two constants, denoted $C_{\mu_1}>0$ and $C_{\nu_2}>0$, such that $\mu_1(\mathbf{v}^\eta(\xi))>C_{\mu_1}$ and $\nu_2(\mathbf{v}^\eta(\xi))<C_{\nu_2}$ for all $\xi\in\mathbb{R}$. Then, there exists a constant $C>0$ which does not depend on η such that the following inequality is obtained:

$$\frac{ds_2^\eta(\xi)}{d\xi} \le \eta C\left(\mathcal{F}(\tau^\eta(\xi)) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}s_2^\eta(\xi)\tau^\eta(\xi)^{1-\gamma_2}\right)\frac{d\tau^\eta(\xi)}{d\xi}, \quad \forall\xi\in\mathbb{R},\ \eta>0.$$

Next, since the quadratic function $\mathcal{F}$ does not depend on η, there exists a positive constant, denoted $\tilde{C}$ independent of η, such that

$$0 \ge \mathcal{F}(\tau^\eta(\xi)) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}s_2^\eta(\xi)\tau^\eta(\xi)^{1-\gamma_2} \ge -\tilde{C}, \quad \forall\xi\in\mathbb{R},\ \eta>0.$$

We thus obtain

$$0 \le \frac{ds_2^\eta(\xi)}{d\xi} \le \eta C\tilde{C}\left|\frac{d\tau^\eta(\xi)}{d\xi}\right|, \quad \forall\xi\in\mathbb{R},\ \eta>0, \tag{5.4}$$

when integrating the relation (5.4) over $(-\infty,\xi)$ for all $\xi\in\mathbb{R}$, we obtain the following expected inequality:

$$0 \le s_2^\eta(\xi) - (s_2)_L \le \eta C\tilde{C}(\tau_L - \tau^\eta(\xi)) \quad \forall\xi\in\mathbb{R},\ \eta>0. \tag{5.5}$$

Since we have $0\le\tau_L-\tau^\eta(\xi)\le\tau_L-\tau^m$ for all $\xi\in\mathbb{R}$ with $\eta>0$, the proof is completed. □

Now, we turn proving Proposition 5.3. To access this issue, a particular form of the ODE satisfied by the function $\xi\to\tau^\eta(\xi)-\tau^0(\xi)$ is exhibited. The main interest of this ODE is to recast the system under a *linear* form (in a sense to be prescribed). The linear ODE will be shown to satisfy sufficient conditions to ensure the expected convergence result.

5.1. *Proof of Proposition 5.3: A simplified problem*

Let us illustrate our purpose when enforcing restrictive assumptions: $\gamma_1 = \gamma_2$, $\mu_1(\mathbf{v}) = \mu_1^0 > 0$ and $\mu_2(\mathbf{v}) = \eta$. The general setting will be developed below. After Lemma 3.1, the functions $\tau^0, (\tau^\eta)_{\eta>0} \in \mathrm{C}^1(\mathbb{R}, \mathbb{R}_+^*)$ are known to be unique up to a parameterization and to satisfy for all $\xi \in \mathbb{R}$ and $\eta > 0$:

$$M\tau^\eta(\mu_1^0 + \eta)\frac{d\tau^\eta(\xi)}{d\xi} = \mathcal{F}(\tau^\eta) \quad \text{and} \quad M\tau^0\mu_1^0\frac{d\tau^0(\xi)}{d\xi} = \mathcal{F}(\tau^0),$$

where $\mathcal{F}$ is defined by (3.14). We thus obtain

$$\frac{d}{d\xi}(\tau^\eta - \tau^0) = \frac{1}{M\mu_1^0}\left(\frac{\mathcal{F}(\tau^\eta)}{\tau^\eta} - \frac{\mathcal{F}(\tau^0)}{\tau^0}\right) - \frac{\eta}{\mu_1^0}\frac{d\tau^\eta}{d\xi}, \tag{5.6}$$

but for

$$\frac{\mathcal{F}(\tau^\eta)}{\tau^\eta} - \frac{\mathcal{F}(\tau^0)}{\tau^0} = M^2\left(\frac{\gamma_1+1}{2}\right)(\tau^\eta - \tau^0)\left(1 - \frac{\tau_L\tau^\star}{\tau^\eta\tau^0}\right).$$

As a consequence, the equation (5.6) reads:

$$\frac{d}{d\xi}(\tau^\eta - \tau^0) = (\tau^\eta - \tau^0)\mathcal{A}^\eta(\xi) + \mathcal{B}^\eta(\xi), \quad \forall \xi \in \mathbb{R},\ \eta > 0, \tag{5.7}$$

where the functions $\mathcal{A}^\eta, \mathcal{B}^\eta \in \mathrm{C}^1(\mathbb{R}, \mathbb{R})$ are defined as follows:

$$\mathcal{A}^\eta(\xi) = \frac{M}{\mu_1^0}\left(\frac{\gamma_1+1}{2}\right)\left(1 - \frac{\tau_L\tau^\star}{\tau^\eta\tau^0}\right), \quad \mathcal{B}^\eta(\xi) = -\frac{\eta}{\mu_1^0}\frac{d\tau^\eta}{d\xi}, \quad \forall \xi \in \mathbb{R}, \eta > 0. \tag{5.8}$$

Note that the *linear* form (5.7) satisfied by $\tau^\eta - \tau^0$ arises when assuming the existence of the functions $\xi \to \tau^\eta(\xi)$ and $\xi \to \tau^0(\xi)$. Of course, this non standard form (5.7) never can be used to prove some existence problem for ODE theory. In the present setting, it can be understood as an other representation of $\tau^\eta - \tau^0$ justified by the knowledge of the involved functions τ^η and τ^0.

Now, a Cauchy problem can be associated with (5.7) when giving an additional initial data. Let us emphasize that the solutions $(\tau^\eta)_{\eta>0}$ and τ^0 of the system (3.13) are defined up to a parameterization. This parameterization is thus fixed when enforcing an initial data for both functions τ^η and τ^0. The parameterization of τ^0 is fixed as follows:

$$\tau^0(0) = \mathcal{T}_0, \tag{5.9}$$

where $\mathcal{T}_0 \in (\tau_R^0, \tau_L)$ is arbitrarily given. For choice of interest, $\mathcal{T}_0$ will be considered in a neighborhood of τ_L. By virtue of Lemma 5.1, with η small enough, the parameterization of τ^η can be assumed as follows:

$$\tau^\eta(0) = \mathcal{T}_0 + \epsilon_\eta < \tau_L, \tag{5.10}$$

where the small parameter $\epsilon_\eta > 0$ tends to zero as η goes to zero. With some abuse in the notation, the inequality $\mathcal{T}_0 + \epsilon_\eta < \tau_L$ will be always assumed, *i.e.* η is assumed to be small enough.

When involving this suitable parameterization, the smooth function $\xi \to \tau^\eta(\xi) - \tau^0(\xi)$ satisfies the following linear Cauchy problem:

$$\begin{cases} \dfrac{d}{d\xi}(\tau^\eta - \tau^0) = (\tau^\eta - \tau^0)\mathcal{A}^\eta(\xi) + \mathcal{B}^\eta(\xi), \quad \forall \xi \in \mathbb{R}, \eta > 0, \\ (\tau^\eta - \tau^0)(0) = \epsilon_\eta, \end{cases} \tag{5.11}$$

where $\mathcal{A}^\eta$ and $\mathcal{A}^\eta$ are defined by (5.8).

The expected convergence, *i.e.* $\lim_{\eta\to 0} \sup_{\xi\in\mathbb{R}} |\tau^\eta(\xi) - \tau^0(\xi)| = 0$, is ensured by the following result. Indeed, we have

Theorem 5.1. *Consider the following linear Cauchy problem:*

$$\begin{cases} \dfrac{dX^\eta(\xi)}{d\xi} = \mathcal{A}^\eta(\xi)X^\eta(\xi) + \mathcal{B}^\eta(\xi), \forall \xi \in \mathbb{R}, \eta > 0, \\ X^\eta(0) = \epsilon_\eta, \end{cases} \tag{5.12}$$

where $\mathcal{A}^\eta, \mathcal{B}^\eta \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$ for all $\eta > 0$ and $\epsilon_\eta > 0$ tends to zero as η goes to zero. Let C denotes a generic constant independent of η. Assume that $\mathcal{A}^\eta$ satisfies $|\mathcal{A}^\eta(\xi)| < C$ for all $\xi \in \mathbb{R}$ with $\eta > 0$, and the following inequalities:

$$\mathcal{A}^\eta(\xi) > C, \ \forall \xi < -a \quad \textit{and} \quad \mathcal{A}^\eta(\xi) < -C, \ \forall \xi > a, \tag{5.13}$$

where $a > 0$ denotes a large enough constant independent of η. Assume the following inequality satisfied by $\mathcal{B}^\eta$:

$$|\mathcal{B}^\eta(\xi)| \le K^\eta(\xi)|X^\eta(\xi)| + |f^\eta(\xi)|, \quad \forall \xi \in \mathbb{R}, \eta > 0, \tag{5.14}$$

where $f^\eta \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$ tends to zero as η goes to zero but for a $\mathbb{R}$-uniform convergence. Let $K^\eta \in \mathrm{C}^0(\mathbb{R},\mathbb{R}_+) \cap \mathrm{L}^1(\mathbb{R})$ be a function such that the following inequality holds with ξ_0 fixed in $\mathbb{R}$:

$$\int_{\xi_0}^{+\infty} K^\eta(\xi)d\xi \le |X^\eta(\xi_0)| + C^0(\xi_0) + C^\eta, \tag{5.15}$$

where $C^0(\xi_0)$ (respectively C^η) depends solely of ξ_0 (resp. η) and

$$\lim_{\xi_0\to+\infty} C^0(\xi_0) = 0, \qquad \lim_{\eta\to 0} C^\eta = 0. \tag{5.16}$$

Then, the nonextensible solution of (5.12), $X^\eta \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, satisfies:

$$\lim_{\eta\to 0} \sup_{\xi\in\mathbb{R}} |X^\eta(\xi)| = 0.$$

Let us note from now on that the assumption (5.14) will give the required limit of the sequence $(X^\eta)_{\eta>0}$ but for a uniform convergence over every compact intervals. The control of the convergence near the infinities is obtained by the assumptions (5.13), (5.15) and (5.16).

The proof of Proposition 5.3 readily follows from the above statement as soon as $\mathcal{A}^\eta$ and $\mathcal{B}^\eta$, defined by (5.8), satisfy the assumptions stated Theorem 5.1.

Concerning the function $\mathcal{B}^\eta$, when assuming our restrictive assumptions, since $\tau^\eta \in [\tau^\star, \tau_L]$ we have

$$|\mathcal{B}^\eta(\xi)| = \frac{\eta}{\mu_1^0}\left|\frac{d\tau^\eta}{d\xi}\right| \leq \frac{\eta}{\mu_1^0+\eta}\frac{1}{M\tau^\star\mu_1^0}\left|\mathcal{F}\left(\frac{\tau^\star+\tau_L}{2}\right)\right|, \quad \forall\xi\in\mathbb{R}, \eta>0.$$

Let us set $K^\eta \equiv 0$ to write $|\mathcal{B}^\eta(\xi)| \leq f^\eta(\xi) = C\eta$ for all $\xi \in \mathbb{R}$ and $\eta > 0$ with C a generic positive constant. Hence, $\mathcal{B}^\eta$ satisfies the assumptions stated Theorem 5.1.

Concerning the properties satisfied by $\mathcal{A}^\eta$, we easily deduce that $\mathcal{A}^\eta$ is bounded independently on η since $\tau^m \leq (\tau^\eta(\xi), \tau^0(\xi)) \leq \tau_L$ for all $\xi \in \mathbb{R}$. The expected estimations (5.13) are established in the following result:

Lemma 5.2. *Let $\mathcal{A}^\eta$ be defined by (5.8). There exists two smooth functions, denoted $\mathcal{A}^\pm$, independent of η, such that*

$$\begin{aligned} \mathcal{A}^\eta(\xi) &\leq \mathcal{A}^+(\xi), \quad \forall\xi>0, \eta>0,\\ \mathcal{A}^\eta(\xi) &\geq \mathcal{A}^-(\xi), \quad \forall\xi<0, \eta>0, \end{aligned}$$

where the functions $\mathcal{A}^\pm$ satisfy $\lim_{\xi\to\pm\infty} \pm\mathcal{A}^\pm(\xi) < 0$.

Proof. Arguing the monotony of $\xi \to \tau^\eta(\xi)$, we have

$$\tau^\eta(\xi) > \mathcal{T}_0 + \epsilon_\eta > \mathcal{T}_0, \quad \forall\xi<0, \eta>0.$$

We thus obtain

$$\mathcal{A}^\eta(\xi) \geq \frac{M}{\mu_1^0}\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L\tau^\star}{\mathcal{T}_0\tau^0(\xi)}\right) = \mathcal{A}^-(\xi), \quad \forall\xi<0, \eta>0.$$

The following limit holds:

$$\lim_{\xi\to-\infty}\mathcal{A}^-(\xi) = \frac{M}{\mu_1^0}\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau^\star}{\mathcal{T}_0}\right) > 0.$$

Reversely, $\tau^\eta(\xi) < \mathcal{T}_0 + \epsilon_\eta$ for all $\xi > 0$ and $\eta > 0$. With ϵ_η small enough, we have $\mathcal{T}_0 + \epsilon_\eta < (\tau_L + \mathcal{T}_0)/2$ to write:

$$\mathcal{A}^\eta(\xi) \leq \frac{M}{\mu_1^0}\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L\tau^\star}{\tau^0(\xi)(\tau_L+\mathcal{T}_0)/2}\right) = \mathcal{A}^+(\xi), \quad \forall\xi>0, \eta>0.$$

Since we have $\lim_{\xi\to+\infty}\tau^0(\xi) = \tau_R^0 = \tau^\star$, the following limit is obtained:

$$\lim_{\xi\to+\infty}\mathcal{A}^+(\xi) = \frac{M}{\mu_1^0}\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L}{(\mathcal{T}_0+\tau_L)/2}\right) < 0.$$

The proof is thus completed. □

5.2. *Proof of Proposition 5.3: The general setting*

Now, we turn proving Proposition 5.3 in the general setting when extending the linear representation (5.7). First, an easy computation gives the following identity:

$$\begin{aligned}\frac{d}{d\xi}(\tau^\eta-\tau^0)=&\frac{1}{M\mu_1(\mathbf{v}^0)}\left(\frac{\mathcal{F}(\tau^\eta)}{\tau^\eta}-\frac{\mathcal{F}(\tau^0)}{\tau^0}\right)+\\&\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{1}{M\mu_1(\mathbf{v}^0)}\left(\frac{s_2^\eta}{(\tau^\eta)^{\gamma_2}}-\frac{(s_2)_L}{(\tau^0)^{\gamma_2}}\right)-\\&\frac{\mu_1(\mathbf{v}^\eta)-\mu_1(\mathbf{v}^0)+\mu_2(\mathbf{v}^\eta)}{\mu_1(\mathbf{v}^0)}\frac{d\tau^\eta}{d\xi},\quad\forall\xi\in\mathbb{R},\eta>0,\end{aligned}$$

where for all $\xi\in\mathbb{R}$ and $\eta>0$ we have:

$$\begin{aligned}&\frac{\mathcal{F}(\tau^\eta)}{\tau^\eta}-\frac{\mathcal{F}(\tau^0)}{\tau^0}=(\tau^\eta-\tau^0)\left(M^2\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L\tau^\star}{\tau^\eta\tau^0}\right)+\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{\tau_L(p_2)_L}{\tau^\eta\tau^0}\right),\\&\frac{s_2^\eta}{(\tau^\eta)^{\gamma_2}}-\frac{(s_2)_L}{(\tau^0)^{\gamma_2}}=\frac{(s_2)_L}{(\tau^\eta\tau^0)^{\gamma_2}}\left((\tau^0)^{\gamma_2}-(\tau^\eta)^{\gamma_2}\right)+\frac{1}{(\tau^\eta)^{\gamma_2}}(s_2^\eta-(s_2)_L).\end{aligned}$$

When assuming the parameterization (5.9)-(5.10), the linear linear Cauchy problem (5.11), satisfied by $\tau^\eta-\tau^0$, is thus obtain where the functions $\mathcal{A}^\eta$ and $\mathcal{B}^\eta$ are defined as follows:

$$\begin{aligned}\mathcal{A}^\eta(\xi)=&\frac{1}{M\mu_1(\mathbf{v}^0)}\left(M^2\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L\tau^\star}{\tau^\eta\tau^0}\right)+\right.\\&\left.\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{(p_2)_L\tau_L}{\tau^\eta\tau^0}+\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{(s_2)_L}{(\tau^\eta\tau^0)^{\gamma_2}}\frac{(\tau^0)^{\gamma_2}-(\tau^\eta)^{\gamma_2}}{\tau^\eta-\tau^0}\right),\end{aligned}\tag{5.17}$$

$$\begin{aligned}\mathcal{B}^\eta(\xi)=&\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{1}{M\mu_1(\mathbf{v}^0)}\frac{1}{(\tau^\eta)^{\gamma_2}}(s_2^\eta-(s_2)_L)-\\&\frac{\mu_1(\mathbf{v}^\eta)-\mu_1(\mathbf{v}^0)}{\mu_1(\mathbf{v}^0)}\frac{d\tau^\eta}{d\xi}-\frac{\mu_2(\mathbf{v}^\eta)}{\mu_1(\mathbf{v}^0)}\frac{d\tau^\eta}{d\xi}.\end{aligned}\tag{5.18}$$

Note from now on that both functions $\mathcal{A}^\eta$ and $\mathcal{B}^\eta$ belong to $\mathrm{C}^1(\mathbb{R},\mathbb{R})$. Indeed, the involved functions τ^η, τ^0, s_2^η are $\mathrm{C}^1(\mathbb{R},\mathbb{R}_+^*)$ and the function $\xi\to\frac{(\tau^0)^{\gamma_2}-(\tau^\eta)^{\gamma_2}}{\tau^\eta-\tau^0}$ is obviously shown to be $\mathrm{C}^1(\mathbb{R},\mathbb{R})$ while $\xi\to\mu_1(\mathbf{v}^0(\xi))$ is a smooth positive function for all $\xi\in\mathbb{R}$.

The proof of Proposition 5.3 is thus completed when establishing the following statement:

Proposition 5.4. *Let $\mathcal{A}^\eta$ and $\mathcal{B}^\eta$ be defined by (5.17) and (5.18) respectively. Then the functions $\mathcal{A}^\eta,\mathcal{B}^\eta\in\mathrm{C}^1(\mathbb{R},\mathbb{R})$ satisfy the assumptions stated in Theorem 5.1.*

The proof of the above result readily follows from the three following technical lemmas. The first one is devoted to establish the expected estimation concerning $\mathcal{B}^\eta$.

Lemma 5.3. *Let $\mathcal{B}^\eta$ be defined by (5.18). There exists a positive constant C, independent of η, and a positive function $K^\eta\in\mathrm{C}^1(\mathbb{R},\mathbb{R}_+)\cap\mathrm{L}^1(\mathbb{R})$ such that*

$$|\mathcal{B}^\eta(\xi)|\le K^\eta(\xi)|\tau^\eta(\xi)-\tau^0(\xi)|+C\eta,\quad\forall\xi\in\mathbb{R},\eta>0,$$

where K^η satisfies (5.15)-(5.16).

The estimations (5.13) are deduced from the two following results:

Lemma 5.4. *Let $\mathcal{A}^\eta$ be defined by (5.17). Assume $|\mathcal{T}_0 - \tau_L|$ small enough, then there exists a smooth function independent of η, denoted $\mathcal{A}^- \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, such that*

$$\mathcal{A}^\eta(\xi) \geq \mathcal{A}^-(\xi), \quad \forall \xi < 0, \eta > 0,$$

where $\lim_{\xi\to-\infty} \mathcal{A}^-(\xi) > 0$.

Lemma 5.5. *Let $\mathcal{A}^\eta$ be defined by (5.17) with the same parameterization as involved in Lemma 5.4. Assume $\eta > 0$ small enough, then there exists a constant $\xi^\star > 0$ and a function $\mathcal{A}^+ \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, independent of η, such that*

$$\mathcal{A}^\eta(\xi) \leq \mathcal{A}^+(\xi), \quad \forall \xi > \xi^\star, \eta > 0,$$

where $\lim_{\xi\to+\infty} \mathcal{A}^+(\xi) < 0$.

Since $\tau^m \leq (\tau^\eta(\xi), \tau^0(\xi)) \leq \tau_L$ for all $\xi \in \mathbb{R}$ and $\eta > 0$, the function $\mathcal{A}^\eta$ is easily shown to be bounded independently on η. Hence, Proposition 5.4 readily follows from Lemmas 5.3, 5.4 and 5.5.

Now, the three above results are established successively. Let us note from now on that the lost of symmetry in the limit system implies distinguished studies for each case $\gamma_1 < \gamma_2$, $\gamma_1 = \gamma_2$ and $\gamma_1 > \gamma_2$, since the role played by each pressure cannot be reversed. In fact, the specific case $\gamma_1 = \gamma_2$ turns out to be similar to the simplified model involving restrictive assumptions. As a consequence, in the sequel we consider $\gamma_1 \neq \gamma_2$.

Proof of Lemma 5.3. Since the hypothesis (3.3)-(4.1) is assumed for the viscosities and $0 < \tau^m \leq \tau^\eta(\xi)$ (see Lemma 3.3) for all $\xi \in \mathbb{R}$ and $\eta > 0$, we easily obtain:

$$|\mathcal{B}^\eta(\xi)| \leq \frac{1}{M\nu}\left|\frac{\gamma_2-\gamma_1}{\gamma_2-1}\right| |s_2^\eta - (s_2)_L| + \frac{|\mu_1(\mathbf{v}^\eta) - \mu_1(\mathbf{v}^0)|}{\nu}\left|\frac{d\tau^\eta}{d\xi}\right| + \frac{\eta}{\nu}|\nu_2(\mathbf{v}^\eta)|\left|\frac{d\tau^\eta}{d\xi}\right|,$$

where $\nu = \inf_{\xi\in\mathbb{R}} \mu_1(\mathbf{v}^0(\xi)) > 0$. After (5.5), let us recall the existence of a constant $C > 0$, independent of η, such that $|s_2^\eta - (s_2)_L| \leq C\eta$. In addition, we have $|d_\xi \tau^\eta| \leq C$ for all $\xi \in \mathbb{R}$ and $\eta > 0$. From (3.13) we write

$$\begin{aligned}\left|\frac{d\tau^\eta}{d\xi}\right| &\leq \frac{1}{\mu_1(\mathbf{v}^\eta)+\mu_2(\mathbf{v}^\eta)}\frac{1}{M\tau^\eta}\left(|\mathcal{F}(\tau^\eta)| + \left|\frac{\gamma_2-\gamma_1}{\gamma_2-1}\right| s_2^\eta (\tau^\eta)^{1-\gamma_2}\right),\\ &\leq \frac{1}{\nu M\tau^m}\left(\max_{\tau\in[\tau^m,\tau_L]} |\mathcal{F}(\tau)| + \left|\frac{\gamma_2-\gamma_1}{\gamma_2-1}\right| s_2^M (\tau^m)^{1-\gamma_2}\right). \end{aligned} \tag{5.19}$$

Moreover, we have $|\nu_2(\mathbf{v}^\eta)| \leq C$ since ν_2 is a positive smooth function and each component of $\mathbf{v}^\eta$ is bounded independently of η. We thus obtain:

$$|\mathcal{B}^\eta(\xi)| \leq |\mu_1(\mathbf{v}^\eta) - \mu_1(\mathbf{v}^0)|\frac{1}{\nu}\left|\frac{d\tau^\eta}{d\xi}\right| + C\eta, \quad \forall \xi \in \mathbb{R}, \eta > 0.$$

Since the functions $\mathbf{v}^0$ and $\mathbf{v}^\eta$ are valued in a compact set of Ω, the smoothness property of the viscosity function μ_1 ensures the following inequality:

$$|\mu_1(\mathbf{v}^\eta) - \mu_1(\mathbf{v}^0)| \leq C|\tau^\eta - \tau^0| + C\eta, \quad \forall \xi \in \mathbb{R}, \eta > 0,$$

where $C > 0$ denotes a constant. The required inequality (5.14) is thus obtained when the function K^η is defined by

$$K^\eta(\xi) = \frac{C}{\nu}\left|\frac{d\tau^\eta}{d\xi}\right|, \quad \text{for all } \xi \in \mathbb{R}, \eta > 0.$$

To achieve the proof, let us establish that (5.15)-(5.16) hold. Indeed, with $\xi_0 \in \mathbb{R}$ fixed, we have:

$$\begin{aligned}\int_{\xi_0}^{+\infty} K^\eta(\xi)d\xi &= \frac{C}{\nu}|\tau^\eta(\xi_0) - \tau_R^\eta|, \quad \forall \eta > 0,\\ &\leq \frac{C}{\nu}\left(|\tau^\eta(\xi_0) - \tau^0(\xi_0)| + |\tau^0(\xi_0) - \tau_R^0| + |\tau_R^0 - \tau_R^\eta|\right), \ \forall \eta > 0,\end{aligned}$$

and the proof is completed. □

Proof of Lemma 5.4. For the sake of simplicity in the notations, we introduce two functions, $\mathcal{A}_1^\eta$ and $\mathcal{A}_2^\eta$ in $\mathrm{C}^1(\mathbb{R},\mathbb{R})$, defined as follows:

$$\mathcal{A}_1^\eta(\xi) = M^2\left(\frac{\gamma_1+1}{2}\right)\left(1 - \frac{\tau_L\tau^\star}{\tau^\eta\tau^0}\right) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\frac{\tau_L}{\tau^\eta\tau^0}, \tag{5.20}$$

$$\mathcal{A}_2^\eta(\xi) = \frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{(s_2)_L}{(\tau^\eta\tau^0)^{\gamma_2}}\frac{(\tau^0)^{\gamma_2} - (\tau^\eta)^{\gamma_2}}{\tau^\eta - \tau^0}, \tag{5.21}$$

to write

$$\mathcal{A}^\eta(\xi) = \frac{1}{M\mu_1(\mathbf{v}^0)}\left(\mathcal{A}_1^\eta(\xi) + \mathcal{A}_2^\eta(\xi)\right), \quad \forall \xi \in \mathbb{R}, \eta > 0. \tag{5.22}$$

First, the monotony of $\mathcal{A}_1^\eta$ and $\mathcal{A}_2^\eta$ is studied to show that $\mathcal{A}_1^\eta$ is a decreasing function and $\mathcal{A}_2^\eta$ decreases (respectively increases) as $\gamma_1 < \gamma_2$ (resp. $\gamma_1 > \gamma_2$). Indeed, the derivative of $\mathcal{A}_1^\eta$ reads:

$$\frac{d}{d\xi}\mathcal{A}_1^\eta(\xi) = \left(M^2\left(\frac{\gamma_1+1}{2}\right)\tau_L\tau^\star - \frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L\right)\frac{\tau^\eta d_\xi\tau^0 + \tau^0 d_\xi\tau^\eta}{(\tau^\eta\tau^0)^2}.$$

Since $d_\xi\tau^\eta < 0$, $d_\xi\tau^0 < 0$ and $\tau^\star$ is defined by (3.15) to write:

$$\begin{aligned}&M^2\left(\frac{\gamma_1+1}{2}\right)\tau_L\tau^\star - \frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L =\\ &\qquad(\gamma_1-1)\tau_L\left(\frac{M^2}{2}\tau_L + \frac{\gamma_1}{\gamma_1-1}(p_1)_L + \frac{\gamma_2}{\gamma_2-1}(p_2)_L\right) > 0.\end{aligned} \tag{5.23}$$

thus we have $d_\xi\mathcal{A}_1^\eta < 0$.

Concerning the monotony of $\mathcal{A}_2^\eta$ we write:

$$\frac{d}{d\xi}\mathcal{A}_2^\eta(\xi) = \frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{(s_2)_L}{(\tau^\eta\tau^0)^{\gamma_2+1}}\frac{1}{(\tau^\eta-\tau^0)^2}\times \left(\tau^0(\tau^\eta)^{\gamma_2+1}g\left(\frac{\tau^0}{\tau^\eta}\right)d_\xi\tau^\eta+\tau^\eta(\tau^0)^{\gamma_2+1}g\left(\frac{\tau^\eta}{\tau^0}\right)d_\xi\tau^0\right),$$

where we have introduced the function $g\in \mathrm{C}^1(\mathbb{R}_+^*,\mathbb{R}_+)$ defined as follows:

$$g(\zeta)=\gamma_2(\zeta^{\gamma_2+1}-1)+(\gamma_2+1)(1-\zeta^{\gamma_2}),\quad \forall\zeta\geq 0.$$

A straightforward study of g gives $g(\zeta)\geq 0$ for all $\zeta>0$ with $g(1)=0$, and the monotony of $\mathcal{A}_2^\eta$ is as expected.

Now, let us assume $\gamma_1<\gamma_2$ to obtain the following inequality:

$$\mathcal{A}^\eta(\xi)\geq\frac{1}{M\mu_1(\mathbf{v}^0)}\left(\mathcal{A}_1^\eta(0)+\mathcal{A}_2^\eta(0)\right),\quad \forall\xi<0,\eta>0.$$

In fact, the values $\mathcal{A}_1^\eta(0)$ and $\mathcal{A}_2^\eta(0)$ can be estimated independently of η. Indeed, we have:

$$\mathcal{A}_1^\eta(0)=M^2\left(\frac{\gamma_1+1}{2}\right)\left(1-\frac{\tau_L\tau^\star}{(\mathcal{T}_0+\epsilon_\eta)\mathcal{T}_0}\right)+\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\frac{\tau_L}{(\mathcal{T}_0+\epsilon_\eta)\mathcal{T}_0},\tag{5.24}$$

$$\mathcal{A}_2^\eta(0)=-\frac{\gamma_2-\gamma_1}{\gamma_2-1}\frac{(s_2)_L}{(\mathcal{T}_0+\epsilon_\eta)^{\gamma_2}\mathcal{T}_0^{\gamma_2}}\frac{\mathcal{T}_0^{\gamma_2}-(\mathcal{T}_0+\epsilon_\eta)^{\gamma_2}}{\mathcal{T}_0-(\mathcal{T}_0+\epsilon_\eta)},\tag{5.25}$$

but for $\epsilon_\eta>0$. In one hand, invoking (5.23) we obtain:

$$\begin{aligned}\mathcal{A}_1^\eta(0)&=M^2\left(\frac{\gamma_1+1}{2}\right)-\frac{1}{(\mathcal{T}_0+\epsilon_\eta)\mathcal{T}_0}\left(M^2\left(\frac{\gamma_1+1}{2}\right)\tau_L\tau^\star-\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L\right),\\ &\geq M^2\left(\frac{\gamma_1+1}{2}\right)-\frac{1}{\mathcal{T}_0^2}\left(M^2\left(\frac{\gamma_1+1}{2}\right)\tau_L\tau^\star-\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L\right),\end{aligned}\tag{5.26}$$

In the second hand, we have:

$$\begin{aligned}\mathcal{A}_2^\eta(0)&=-\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L\left(\frac{\tau_L}{\mathcal{T}_0}\right)^{\gamma_2-1}\frac{1}{(\mathcal{T}_0+\epsilon_\eta)\mathcal{T}_0}\frac{\left(\frac{\mathcal{T}_0}{\mathcal{T}_0+\epsilon_\eta}\right)^{\gamma_2}-1}{\left(\frac{\mathcal{T}_0}{\mathcal{T}_0+\epsilon_\eta}\right)-1},\\ &\geq-\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau_L\left(\frac{\tau_L}{\mathcal{T}_0}\right)^{\gamma_2-1}\frac{\gamma_2}{\mathcal{T}_0^2}.\end{aligned}\tag{5.27}$$

Now, for the sake of simplicity in the notations, let us introduce

$$\overline{\mathcal{A}}(\tau)=M^2\left(\frac{\gamma_1+1}{2}\right)+\frac{\tau_L}{\tau^2}\left(-M^2\left(\frac{\gamma_1+1}{2}\right)\tau^\star+\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\left(1-\gamma_2\left(\frac{\tau_L}{\tau}\right)^{\gamma_2-1}\right)\right).$$

The function $\tau\to\overline{\mathcal{A}}(\tau)$ is defined in a neighborhood of τ_L, and we have $\mathcal{A}_1^\eta(0)+\mathcal{A}_2^\eta(0)\geq\overline{\mathcal{A}}(\mathcal{T}_0)$ for all $\eta>0$. In addition, an easy computation gives:

$$\overline{\mathcal{A}}(\tau_L)=\frac{1}{\tau_L}\left(M^2\tau_L-\gamma_1(p_1)_L-\gamma_2(p_2)_L\right),$$

where we have $\overline{\mathcal{A}}(\tau_L) > 0$ as soon as (3.4) is assumed. As a consequence, the continuity of $\overline{\mathcal{A}}$ in a neighborhood of $\tau_L > 0$ ensures $\overline{\mathcal{A}}(\mathcal{T}_0) > 0$ for $|\mathcal{T}_0 - \tau_L|$ small enough. Then we deduce

$$\mathcal{A}^\eta(\xi) \geq \frac{1}{M\mu_1(\mathbf{v}^0)}\overline{\mathcal{A}}(\mathcal{T}_0) > 0, \quad \forall \xi < 0, \eta > 0,$$

and the proof is completed as soon as $\gamma_1 < \gamma_2$.

Now, we turn proving a similar inequality with $\gamma_1 > \gamma_2$. Arguing the monotony properties of $\mathcal{A}_1^\eta$ and $\mathcal{A}_2^\eta$, we have

$$\mathcal{A}^\eta(\xi) \geq \frac{1}{M\mu_1(\mathbf{v}^0)}\left(\mathcal{A}_1^\eta(0) + \lim_{\xi\to-\infty}\mathcal{A}_2^\eta(\xi)\right) \quad \forall \xi < 0, \eta > 0,$$

where

$$\lim_{\xi\to-\infty}\mathcal{A}_2^\eta(\xi) = -\frac{\gamma_2-\gamma_1}{\gamma_2-1}\gamma_2\frac{(p_2)_L}{\tau_L}.$$

Since (5.26) holds, we obtain for all $\xi < 0$ and $\eta > 0$:

$$\mathcal{A}^\eta(\xi) \geq \frac{1}{M\mu_1(\mathbf{v}^0)}\left(M^2\left(\frac{\gamma_1+1}{2}\right)\left(1 - \frac{\tau_L\tau^\star}{\mathcal{T}_0^2}\right) + \frac{\gamma_2-\gamma_1}{\gamma_1-1}(p_2)_L\tau_L\left(\frac{1}{\mathcal{T}_0^2} - \frac{\gamma_2}{\tau_L^2}\right)\right).$$

Indeed, in the limit of $\mathcal{T}_0$ to τ_L, the right hand side is easily shown to be strictly positive. As a consequence, the right hand side of the above inequality stays positive as soon as $|\mathcal{T}_0 - \tau_L|$ is small enough. The proof is thus completed. □

Proof of Lemma 5.5. Following the ideas introduced in the proof of Lemma 5.4, the expected result is obtained when considering a suitable form for $\mathcal{A}^\eta$. First, the function $\mathcal{A}^\eta$ is rewritten as follows:

$$\mathcal{A}^\eta(\xi) = \frac{1}{M\mu_1(\mathbf{v}^0)}\left(\mathcal{A}^1(\xi) + \mathcal{A}_\eta^2(\xi)\right), \quad \forall \xi \in \mathbb{R}, \eta > 0, \tag{5.28}$$

where $\mathcal{A}^1$ does not depend on η. The proof will be achieved as soon as th two following properties will be established:

(i) $\lim_{\xi\to+\infty}\mathcal{A}^1(\xi) = 0$.

(ii) For all $\delta > 0$ small enough, there exists $\eta(\delta) > 0$ and $\xi^\star > 0$ such that

$$\mathcal{A}_\eta^2(\xi) \leq \widehat{\mathcal{A}}(\xi) + C\delta, \quad \forall \xi > \xi^\star, \eta < \eta(\delta),$$

where $C \geq 0$ denotes a constant which does not depend on neither η nor δ. The function $\widehat{\mathcal{A}}$, once again independent of neither η nor δ, denotes a smooth function which obeys $\lim_{\xi\to+\infty}\widehat{\mathcal{A}}(\xi) < 0$.

First, let us detail the precise form of the functions $\mathcal{A}^1$ and $\mathcal{A}_\eta^2$. To access such an issue, let us consider the form (5.22) of $\mathcal{A}^\eta$ to recast $\mathcal{A}_1^\eta$ as follows:

$$\mathcal{A}_1^\eta(\xi) = \frac{1}{\tau^0(\tau^0-\tau_L)}\left(\mathcal{F}(\tau^0) + \frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L\tau^0\right) + \\ \frac{1}{\tau^0}\left(\frac{\tau_L}{\tau^\eta} - 1\right)\left(\frac{\gamma_2-\gamma_1}{\gamma_2-1}(p_2)_L - M^2\left(\frac{\gamma_1+1}{2}\right)\tau^\star\right),$$

where $\mathcal{F}$ is defined by (3.14) and $\tau^\star$ is given by (3.15). Now, let us introduce the positive smooth function $g \in \mathrm{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ defined as follows:

$$g(\zeta) = \zeta \frac{\zeta^{\gamma_2} - 1}{\zeta - 1},$$

in order to write:

$$\mathcal{A}_2^\eta(\xi) = -\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \frac{(p_2)_L}{\tau^0} \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2} g\left(\frac{\tau^0}{\tau^\eta}\right), \quad \forall \xi \in \mathbb{R}, \eta > 0.$$

We note from now on that the function g is positive, increasing and convex. We recast $\mathcal{A}_2^\eta$ as follows:

$$\mathcal{A}_2^\eta(\xi) = -\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \frac{(p_2)_L}{\tau^0} \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2} g(\frac{\tau^0}{\tau_L}) + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \frac{(p_2)_L}{\tau^0} \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2} \left(g(\frac{\tau^0}{\tau_L}) - g(\frac{\tau^0}{\tau^\eta})\right).$$

Since we have $\mathcal{A}_1^\eta + \mathcal{A}_2^\eta = \mathcal{A}^1 + \mathcal{A}_\eta^2$, we easily obtain:

$$\begin{aligned}
\mathcal{A}^1(\xi) &= \frac{1}{\tau^0(\tau^0 - \tau_L)} \left(\mathcal{F}(\tau^0) + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \tau^0 (p_2)_L\right) - \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \frac{(p_2)_L}{\tau^0} \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2} g(\frac{\tau^0}{\tau_L}), \\
\mathcal{A}_\eta^2(\xi) &= \frac{1}{\tau^0} \left(\frac{\tau_L}{\tau^\eta} - 1\right) \left(\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} (p_2)_L - M^2 \left(\frac{\gamma_1 + 1}{2}\right) \tau^\star\right) + \\
&\quad \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} \frac{(p_2)_L}{\tau^0} \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2} \left(g(\frac{\tau^0}{\tau_L}) - g(\frac{\tau^0}{\tau^\eta})\right).
\end{aligned}$$

The expected property (i) readily follows from the following limit:

$$\lim_{\xi \to +\infty} \mathcal{F}(\tau^0(\xi)) = -\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} (p_2)_L \tau_R^0 \left(\frac{\tau_L}{\tau_R^0}\right)^{\gamma_2}.$$

Now, we turn establishing the property (ii). First, let us assume $\gamma_1 < \gamma_2$. Since g is an increasing function, with $\tau^\eta(\xi) < \tau_L$ for all $\xi \in \mathbb{R}$ and $\eta > 0$ we have

$$\mathcal{A}_\eta^2(\xi) \leq \frac{1}{\tau^0} \left(\frac{\tau_L}{\tau^\eta} - 1\right) \left(\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} (p_2)_L - M^2 \left(\frac{\gamma_1 + 1}{2}\right) \tau^\star\right), \quad \forall \xi \in \mathbb{R}, \eta > 0.$$

Now, for all $\xi > 0$ and $\eta > 0$, we have $\tau^\eta(\xi) < \mathcal{T}_0 + \epsilon_\eta < (\mathcal{T}_0 + \tau_L)/2$ to deduce the following inequality from (5.23):

$$\mathcal{A}_\eta^2(\xi) \leq \widehat{\mathcal{A}}(\xi) \qquad \forall \xi > 0, \eta > 0,$$

with

$$\widehat{\mathcal{A}}(\xi) = \frac{1}{\tau^0} \left(\frac{\tau_L}{(\mathcal{T}_0 + \tau_L)/2} - 1\right) \left(\frac{\gamma_2 - \gamma_1}{\gamma_2 - 1} (p_2)_L - M^2 \left(\frac{\gamma_1 + 1}{2}\right) \tau^\star\right).$$

The expected property (ii) is thus obtained when considering $\gamma_1 < \gamma_2$.

Now, we turn assuming $\gamma_1 > \gamma_2$. First, let us note that the function $\mathcal{A}_\eta^2$ can be recast as follows:

$$\mathcal{A}_\eta^2(\xi) = \frac{1}{\tau^0} \left(\frac{\tau_L}{\tau^\eta} - 1\right) \mathcal{G}(\tau^0, \tau^\eta), \quad \forall \xi \in \mathbb{R}, \eta > 0, \tag{5.29}$$

where $\mathcal{G} \in \mathrm{C}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ is defined as follows:

$$\mathcal{G}(\tau^0, \tau^\eta) = -M^2 \left(\frac{\gamma_1 + 1}{2}\right) \tau^\star + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1}(p_2)_L + \frac{\gamma_1 - \gamma_2}{\gamma_2 - 1}(p_2)_L \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2 - 1} \frac{g(\tau^0/\tau_L) - g(\tau^0/\tau^\eta)}{\tau^0/\tau_L - \tau^0/\tau^\eta}.$$

As a first step, we exhibit a bound of $\mathcal{G}(\tau^0, \tau^\eta)$ independent on η. Since g is an increasing convex function, the function $\tau^\eta \to \mathcal{G}(\tau^0, \tau^\eta)$ is seen to be a decreasing function. After Lemma 3.3, we have $\tau^\eta(\xi) \geq \tau_R^\eta$ for all $\xi \in \mathbb{R}$ and $\eta > 0$ and thus $\mathcal{G}(\tau^0, \tau^\eta) \leq \mathcal{G}(\tau^0, \tau_R^\eta)$ for all $\xi \in \mathbb{R}$ and $\eta > 0$. Invoking Lemma 5.1, for all $\delta > 0$ there exists $\eta(\delta) > 0$ such that $\tau_R^\eta \geq \tau_R^0 - \delta$ for all $\eta < \eta(\delta)$. Moreover, $\delta > 0$ can be assumed small enough to ensure $\tau_R^\eta \geq \tau_R^0 - \delta > \tau^m$ where τ^m defined Lemma 3.3. As a consequence, we deduce

$$\mathcal{G}(\tau^0, \tau^\eta) \leq \mathcal{G}(\tau^0, \tau_R^0 - \delta), \quad \forall \xi \in \mathbb{R}, \eta < \eta(\delta).$$

In fact, this above estimation can be specified as follows:

$$\mathcal{G}(\tau^0, \tau^\eta) \leq \mathcal{G}(\tau^0, \tau_R^0) + C\delta, \quad \forall \xi \in \mathbb{R}, \eta < \eta(\delta), \tag{5.30}$$

where $C \geq 0$ denotes a relevant constant. Indeed, as usual we have

$$\int_{\tau_R^0 - \delta}^{\tau_R^0} \frac{\partial}{\partial \zeta} \mathcal{G}(\tau^0, \zeta) d\zeta = \mathcal{G}(\tau^0, \tau_R^0) - \mathcal{G}(\tau^0, \tau_R^0 - \delta),$$

where an easy computation gives:

$$\frac{\partial}{\partial \zeta}\mathcal{G}(\tau^0, \zeta) = \frac{\gamma_1 - \gamma_2}{\gamma_2 - 1}(p_2)_L \left(\frac{\tau_L}{\tau^0}\right)^{\gamma_2 - 1} \frac{\tau^0}{\zeta^2} \times \frac{1}{\tau^0/\tau_L - \tau^0/\zeta}\left(g'(\frac{\tau^0}{\zeta}) - \frac{g(\tau^0/\tau_L) - g(\tau^0/\zeta)}{\tau^0/\tau_L - \tau^0/\zeta}\right).$$

Arguing the monotony properties of the function g, we obtain for all ζ in the subset $[\tau_R^0 - \delta, \tau_R^0]$:

$$\frac{\partial}{\partial \zeta}\mathcal{G}(\tau^0, \zeta) \geq -\bar{C}\frac{g'(\tau^0/\tau_L) - g'(\tau^0/\zeta)}{\tau^0/\tau_L - \tau^0/\zeta} \quad \text{where} \quad \bar{C} = \frac{\gamma_1 - \gamma_2}{\gamma_2 - 1}\frac{(p_2)_L}{\tau_L}\left(\frac{\tau_L}{\tau^m}\right)^{\gamma_2 + 1} > 0.$$

Now, the study of the function of g ensures that g' is convex if $\gamma_2 > 2$, concave if $\gamma_2 < 2$ and affine if $\gamma_2 = 2$. Hence, for all $\zeta \in [\tau_R^0 - \delta, \tau_R^0]$ we have:

$$\frac{\partial}{\partial \zeta}\mathcal{G}(\tau^0, \zeta) \geq -\bar{C} \times \begin{cases} g''(\frac{\tau_L}{\tau^m}) & \text{if } \gamma_2 \geq 2, \\ g''(\frac{\tau^m}{\tau_L}) & \text{if } \gamma_2 < 2, \end{cases}$$

and the inequality (5.30) is obtained. Since we have $\tau^\eta(\xi) \leq \tau_L$ for all $\xi \in \mathbb{R}$ and $\eta > 0$, from the above estimations we deduce the following inequality satisfied by $\mathcal{A}_\eta^2$:

$$\mathcal{A}_\eta^2(\xi) \leq \frac{1}{\tau^0}\left(\frac{\tau_L}{\tau^\eta} - 1\right)\mathcal{G}(\tau^0, \tau_R^0) + \frac{1}{\tau^0}\left(\frac{\tau_L}{\tau^\eta} - 1\right)C\delta, \quad \forall \xi \in \mathbb{R}, \eta < \eta(\delta).$$

With $\tau^m < \tau^\eta(\xi) < (\mathcal{T}_0 + \tau_L)/2$ for all $\xi > 0$, let us note that we have

$$C^m \leq \frac{1}{\tau^0}\left(\frac{\tau_L}{\tau^m}\right) \leq C^0,$$

where we have set

$$C^m = \frac{1}{\tau_L}\left(\frac{\tau_L}{\tau^m} - 1\right) > 0 \quad \text{and} \quad C^0 = \frac{1}{\tau^m}\left(\frac{\tau_L}{(\mathcal{T}_0 + \tau_L)/2} - 1\right) > 0.$$

In addition, let us assume that there exists $\xi^\star \in \mathbb{R}$, independent of neither η nor δ, such that the function $\xi \to \mathcal{G}(\tau^0(\xi), \tau_R^0)$ is negative for all $\xi > \xi^\star$. Then we obtain:

$$\mathcal{A}_\eta^2(\xi) \leq C^m \mathcal{G}(\tau^0, \tau_R^0) + C^0 C\delta, \quad \xi > \max(0, \xi^\star), \eta < \eta(\delta),$$

and the property (ii) is established.

To conclude the proof, we give the existence of $\xi^\star$ which will be ensured when proving $\lim_{\xi\to+\infty} \mathcal{G}(\tau^0(\xi), \tau_R^0) < 0$, *i.e.* $\mathcal{G}(\tau_R^0, \tau_R^0) < 0$.

By definition of $\mathcal{A}^\eta$, given by (5.17), we have:

$$\mathcal{A}^\eta(\xi) = \frac{1}{M\mu_1(\mathbf{v}^0)} \frac{1}{\tau^\eta - \tau^0}\left(\frac{\mathcal{F}(\tau^\eta)}{\tau^\eta} - \frac{\mathcal{F}(\tau^0)}{\tau^0} + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1}(s_2)_L\left(\frac{1}{(\tau^\eta)^{\gamma_2}} - \frac{1}{(\tau^0)^{\gamma_2}}\right)\right).$$

Since $\lim_{\xi\to+\infty} \mathcal{A}^1(\xi) = 0$, in the limit of ξ to $+\infty$ the identity (5.28) reads:

$$\begin{aligned}
&\frac{1}{\tau_R^0}\left(\frac{\tau_L}{\tau_R^\eta} - 1\right)\mathcal{G}(\tau_R^0, \tau_R^\eta) = \\
&\frac{1}{\tau_R^\eta - \tau_R^0}\left(\frac{\mathcal{F}(\tau_R^\eta)}{\tau_R^\eta} - \frac{\mathcal{F}(\tau_R^0)}{\tau_R^0} + \frac{\gamma_2 - \gamma_1}{\gamma_2 - 1}(s_2)_L\left(\frac{1}{(\tau_R^\eta)^{\gamma_2}} - \frac{1}{(\tau_R^0)^{\gamma_2}}\right)\right).
\end{aligned}$$

With $\gamma_1 \neq \gamma_2$, let us recall that the pairs $(\tau_R^0, (s_2)_L)$ and $(\tau_R^\eta, (s_2^\eta)_R)$ belong to $G^{-1}(0)$ defined by (3.24). Involving the developed form of $\mathcal{F}$, the above relation writes:

$$\frac{1}{\tau_R^0}\left(\frac{\tau_L}{\tau_R^\eta} - 1\right)\mathcal{G}(\tau_R^0, \tau_R^\eta) = \frac{\gamma_1 - \gamma_2}{\gamma_2 - 1}\frac{1}{(\tau_R^\eta)^{\gamma_2}}\frac{(s_2^\eta)_R - (s_2)_L}{\tau_R^\eta - \tau_R^0}, \quad \forall \eta > 0, \tag{5.31}$$

where $(s_2^\eta)_R = \Gamma(\tau_R^\eta)$ and $(s_2)_L = \Gamma(\tau_R^0)$, Γ being defined by (5.3). In the limit of η to zero, the equation (5.31) reads as follows (see Lemma 5.1):

$$\frac{1}{\tau_R^0}\left(\frac{\tau_L}{\tau_R^0} - 1\right)\mathcal{G}(\tau_R^0, \tau_R^0) = \frac{\gamma_1 - \gamma_2}{\gamma_2 - 1}\frac{1}{(\tau_R^0)^{\gamma_2}}\Gamma'(\tau_R^0).$$

The derivative of Γ is negative when evaluated on τ_R^0 and thus we have $\mathcal{G}(\tau_R^0, \tau_R^0) < 0$. The proof is completed. $\square$

To conclude the present section, Theorem 5.1 is established. In fact, the proof is straightforward but we do not find such a convergence theorem in the proposed version (for instance, see Hartman[10] or Reinhard[18] for some details about the general ODE's theory).

Proof of Theorem 5.1. Let us recall that the solution of (5.12) is given by

$$X^\eta(\xi) = X^\eta(\xi_0)\mathrm{e}^{\int_{\xi_0}^{\xi} \mathcal{A}^\eta(\zeta)d\zeta} + \int_{\xi_0}^{\xi} \mathcal{B}^\eta(\zeta)\mathrm{e}^{\int_{\zeta}^{\xi} \mathcal{A}^\eta(\varsigma)d\varsigma}d\zeta, \quad \forall \xi \in \mathbb{R}, \eta > 0,$$

for all $\xi_0 \in \mathbb{R}$ fixed. Arguing the smoothness of the function $\mathcal{A}^\eta$, the following inequality holds from the assumption (5.14):

$$|X^\eta(\xi)| \leq C\left(|X^\eta(\xi_0)| + \sup_{\xi\in\mathbb{R}} |f^\eta(\xi)|\right) + \int_{\xi_0}^{\xi} K^\eta(\zeta)|X^\eta(\zeta)|d\zeta, \ \forall \xi \in \mathbb{R}, \eta > 0, \quad (5.32)$$

where $C > 0$ denotes a constant.

Let $\mathcal{K}$ be any compact interval in $\mathbb{R}$. The well-known Gronwall Lemma, applied to (5.32), ensures the $\mathcal{K}$-uniform convergence of X^η to zero as η goes to zero. Now, let us assume $\xi \geq a$ and let us set $\xi_0 = a$ to write:

$$|X^\eta(\xi)| \leq C\left(|X^\eta(a)| + \sup_{\xi\in\mathbb{R}} |f^\eta(\xi)| + \sup_{\xi\geq a} |X^\eta(\xi)| \int_a^{+\infty} K^\eta(\zeta)d\zeta\right), \ \forall \xi \geq a, \eta > 0,$$

which gives for all $\eta > 0$:

$$\left(1 - C\int_a^{+\infty} K^\eta(\zeta)d\zeta\right) \sup_{\xi\geq a} |X^\eta(\xi)| \leq C\left(|X^\eta(a)| + \sup_{\xi\in\mathbb{R}} |f^\eta(\xi)|\right).$$

Since the function K^η satisfies (5.15)-(5.16), for a large enough constant a, there exists $\eta_0 > 0$ such that

$$1 - C\int_a^{+\infty} K^\eta(\zeta)d\zeta > 0, \quad \forall 0 < \eta < \eta_0.$$

Arguing the $\mathbb{R}$-uniform convergence of $(f^\eta)_{\eta>0}$ and the $\mathcal{K}$-uniform convergence of $(X^\eta)_{\eta>0}$, we thus obtain

$$\lim_{\eta\to 0} \sup_{\xi\geq a} |X^\eta(\xi)| = 0.$$

A similar arguments used for $\xi \leq -a$ ensures the required result. □

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