How to make fully discrete entropy stability of three-point well-balanced finite-volume schemes

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Abstract

This paper presents a simple and robust method to enforce discrete entropy stability in first-order well-balanced finite volume schemes for systems of balance laws, including non-conservative terms. By leveraging an artificial viscosity technique originally introduced by Tadmor and extended in recent works, the authors propose an entropy-preserving modification applicable to a wide class of three-point schemes. The core contribution lies in the construction of a viscosity coefficient that maintains the well-balanced property while ensuring entropy dissipation at the discrete level. A theoretical framework is developed to guarantee robustness, well-balancing, and entropy stability under a suitable CFL condition. The approach is validated through numerical experiments, including applications to shallow water systems and two-layer flows, demonstrating accuracy and stability.

1 Introduction

This work is devoted to the numerical approximation of systems of balanced laws endowed with non-conservative products under the form:

$$\partial_t w + \partial_x f(w) + A(w)\partial_x w = S(w,x), \quad x \in \mathbb{R}, \quad t > 0. \tag{1}$$

The unknown state vector $w \in \mathbb{R}^d$, for $d \geq 1$, is assumed to belong to an open convex set $\Omega \subset \mathbb{R}^d$, while $f : \Omega \to \mathbb{R}^d$ is a given flux function, $A : \Omega \to \mathcal{M}_d(\mathbb{R})$ denotes a given smooth matrix and $S : \Omega \times \mathbb{R} \to \mathbb{R}^d$ represents a given source term function. For sake of simplicity, System (1) is assumed to be hyperbolic so that $(\nabla f + A)(w)$ is diagonalizable in $\mathbb{R}$ for all $w \in \Omega$. Such an assumption is usual [42, 43] but it is not necessary in forthcoming numerical developments.

System (1) is endowed with an entropy inequality that means the existence of a convex entropy-entropy flux pair (η, G) such that $G : \Omega \to \mathbb{R}$ satisfies $\nabla G = \nabla\eta \cdot (\nabla f + A)$ with $\eta(w)$ a convex function called entropy. Considering such a pair (η, G), the admissible solutions of (1) satisfy the following inequality in a weak sense:

$$\partial_t \eta(w) + \partial_x G(w) \leq \nabla\eta(w) \cdot S(w,x). \tag{2}$$

In many particular cases, the above inequality admits a conservative reformulation that reads

$$\partial_t \hat{\eta}(w,x) + \partial_x \hat{G}(w,x) \leq 0, \tag{3}$$

where $(w,x) \mapsto \hat{\eta}(w,x)$ is a convex function and $\hat{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a generalized entropy flux according to the source term. This work assumes the existence of such generalized convex entropy-entropy flux pair $(\hat{\eta}, \hat{G})$.

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The entropy inequalities for conservative systems, *i.e.* $A(w) = 0$ for all $w \in \Omega$, impose the hyperbolicity and rule out non-physical shock discontinuous solutions [25, 41, 58]. However, such properties do not occur if $\nabla f + A$ can not be written as a Jacobian matrix. In that case, (3) may be satisfied when $(\nabla f + A)(w)$ admit complex eigenvalues while the non-conservative product $A(w)\partial_x w$ has no distributional sense. Hence, weak solutions are defined according to the viscosity approach [7] or with the path-theory [26, 42, 43]. Nevertheless, entropy inequalities always yield a weighted-L^2_{loc} estimation of $w(x,t)$ because $(w,x) \mapsto \hat{\eta}(w,x)$ is a convex function and (3) obviously involves

$$\int_\alpha^\beta \hat{\eta}(w(x,t),x)\,\mathrm{d}x \leq \int_\alpha^\beta \hat{\eta}(w(x,0),x)\,\mathrm{d}x - \int_0^t \left(\hat{G}(w(\beta,s),\beta) - \hat{G}(w(\alpha,s),\alpha)\right)\mathrm{d}s, \quad \forall \alpha < \beta,\ t > 0.$$

As a consequence, according to the forthcoming numerical derivations, the interests of the entropy inequalities is twofold. On one hand, they ensure the physical relevance of weak solutions for conservative systems (see for instance [11, 32, 37]). On the other hand, they produce suitable estimations for the solutions of (1).

In addition, the presence of the source term $S(w,x)$ involves the existence of non-trivial stationary solutions for System (1). This work concerns the smooth stationary solutions which, according to (1) and (3), satisfy

$$\partial_x f(w) + A(w)\partial_x w = S(w,x), \tag{4}$$

$$\partial_x \hat{G}(w,x) = 0. \tag{5}$$

Shallow-water or Euler systems with gravity as well as the two-layer shallow-water model [16, 18, 19], adopted here, enter into such a framework.

The derivation of numerical schemes to approximate the weak solutions of (1) has been widely discussed in the literature during the last five decades. While numerous schemes have been designed for conservative hyperbolic systems (for instance, see [1, 11, 31, 32, 37, 61] and references therein for an non-exhaustive bibliography), the derivation of numerical schemes for non-conservative systems turns out to be more delicate. In this regard, a relevant framework is defined by pioneer works of Hou and LeFloch [39] along with it essential extension given by the path-consistent schemes derived by Parés in [51]. The non-conservative schemes derivation may be also considered with the single shock capturing techniques [23, 55, 56] or with a viscous approach [4, 24].

However and after [3, 34, 36], due to the presence of a source term $S(w,x)$, a fully relevant numerical scheme for (1) needs to enforce well-balanced properties to accurately approximate the steady solutions of (1). If well-balanced schemes are now well-established (for instance, see [2, 6, 14, 17, 21, 29, 40, 50, 62] and the references therein), the derivation of well-balanced entropy-satisfying schemes for non-conservative systems endowed with source term seems still rare or dedicated to a specific system [9, 28, 47].

In this regard, considering artificial viscosity technique introduced by Tadmor [59, 60], this work presents a straightforward improvement to make entropy-preserving any first-order well-balanced scheme for generic System (1). Such an additional correction can be naturally associated with [22] where the authors design a simple procedure to convert any standard consistent first-order scheme into a well-balanced scheme. As a consequence, [22] and the present work enable to transfer standard schemes into well-balanced entropy-satisfying schemes.

The paper is organized as follows. Section 2 extends [5] and presents an artificial viscosity technique to recover discrete entropy inequalities while preserving the well-balanced property. Section 3 deals with a relevant approximate Riemann solver to reformulate any 3-point finite volume scheme endowed with artificial viscosity as a Godunov-type scheme. Such a reformulation is then used in Sections 4-5 to design a sufficient artificial viscosity for discrete entropy inequalities and that preserves the stationary solutions. Finally, Section 6 assesses the proposed numerical procedure with numerical experiments.

2 Well-balanced artificial viscosity and main result

For the sake of simplicity in the notation, we shall consider constant $\Delta x > 0$ (*resp.* $\Delta t > 0$) spatial step (*resp.* time). The spatial mesh $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ is defined by $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$ and w_i^n denotes an approximation of the average $\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t^n)\,\mathrm{d}x$. The sequence of cell centers $(x_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ is defined by

$$x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}), \quad \forall i \in \mathbb{Z}. \tag{6}$$

We consider a standard 3-point finite volume numerical scheme written in the general form

$$\begin{aligned} w_i^{n+1} = w_i^n &- \frac{\Delta t}{\Delta x}\left(f_\Delta(w_i^n, w_{i+1}^n) - f_\Delta(w_{i-1}^n, w_i^n)\right) \\ &- \frac{\Delta t}{2\Delta x}\Big(A_\Delta^L(w_i^n, w_{i+1}^n)(w_{i+1}^n - w_i^n) + A_\Delta^R(w_{i-1}^n, w_i^n)(w_i^n - w_{i-1}^n)\Big) \\ &+ \frac{\Delta t}{2}\Big(S_\Delta^L(w_i^n, x_i, w_{i+1}^n, x_{i+1}) + S_\Delta^R(w_{i-1}^n, x_{i-1}, w_i^n, x_i)\Big), \end{aligned} \tag{7}$$

where $f_\Delta : (\mathbb{R}^d)^2 \to \mathbb{R}^d$, $A_\Delta^{L,R} : (\mathbb{R}^d)^2 \to \mathbb{R}^d$, $S_\Delta : (\mathbb{R}^{d+1})^2 \to \mathbb{R}^d$, respectively stands for the consistent numerical flux, the consistent discrete version of $A(w)$, $S(w,x)$ satisfying $f_\Delta(w,w) = f(w)$, $A_\Delta^{L,R}(w,w) = A(w)$ and $S_\Delta(w,x,w,x) = S(w,x)$. The numerical schemes (7) correspond to a generic form for (1) (see for instance [11,20]) and the definition of a well-balanced, robust and entropy-satisfying numerical scheme is now given for the sake of completeness.

Definition 2.1 (well-balanced, robust, entropy-satisfying numerical scheme)**.** *Let us consider System (1) endowed by a pair of convex entropy-entropy flux $(\hat\eta, \hat G)$ and the corresponding entropy inequality (3). Let us define*

$$\hat w = (w,x)^{\mathrm{T}} \in \Omega \times \mathbb{R}, \tag{8}$$

and consider $(x_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ defined by (6). Considering a relation $\mathcal{R} : (\Omega\times\mathbb{R})^2 \to \mathbb{R}^d$ that defines stationary solutions (4) at a discrete level, a three-point finite volume scheme is

(*i*) *well-balanced for the equilibria $\mathcal{R}$ if*

$$\mathcal{R}(\hat w_i^n, \hat w_{i+1}^n) \quad \text{holds} \quad \forall i \in \mathbb{Z}, \quad \text{then} \quad w_i^{n+1} = w_i^n, \quad \forall i \in \mathbb{Z}. \tag{9}$$

(*ii*) *robust if $(w_i^n)_{i\in\mathbb{Z}} \subset \Omega$ then $(w_i^{n+1})_{i\in\mathbb{Z}} \subset \Omega$.*

(*iii*) *entropy-satisfying for the convex entropy-entropy flux $(\hat\eta, \hat G)$ if there exists $\hat{\mathcal{G}} : (\Omega\times\mathbb{R})^2 \to \mathbb{R}$ such that $\hat{\mathcal{G}}(\hat w, \hat w) = \hat G(\hat w)$ and such that $\hat w_i^{n+1} = (w_i^{n+1}, x_i)_{i\in\mathbb{Z}}$ verifies*

$$\frac{\hat\eta(\hat w_i^{n+1}) - \hat\eta(\hat w_i^n)}{\Delta t} + \frac{\hat{\mathcal{G}}(\hat w_i^n, \hat w_{i+1}^n) - \hat{\mathcal{G}}(\hat w_{i-1}^n, \hat w_i^n)}{\Delta x} \le 0, \quad \forall i \in \mathbb{Z}. \tag{10}$$

The relation $\mathcal{R}$ is a local definition of a discrete stationary solution. It may be explicit or implicit algebraic equations and it may be correspond to an exact equilibrium (4) or an approximation (see [33] for the detail). In this work, we do not assume specific form for the local discrete equilibrium $\mathcal{R}$, but we assume a non-entropy production if $\mathcal{R}$ holds. This assumption writes

$$\text{if} \quad \mathcal{R}(\hat w_i^n, \hat w_{i+1}^n) \quad \text{holds} \quad \text{then} \quad \hat G(\hat w_{i+1}^n) - \hat G(\hat w_i^n) = 0. \tag{11}$$

The above assumption is a direct discrete counterpart of (5). Several first-order robust well-balanced schemes (7) satisfy Definition 2.1-(*i*)-(*ii*) [2,6,14,17,21,29,40,50,62] and the main objective is now to define an improved version of (7) in order to reach the entropy stability properties of Definition 2.1-(*iii*). In this regard, we add an artificial viscosity to the scheme (7) to get the following viscous scheme:

$$\begin{aligned} w_i^{n+1} = w_i^n &- \frac{\Delta t}{\Delta x}\left(f_\Delta(w_i^n, w_{i+1}^n) - f_\Delta(w_{i-1}^n, w_i^n)\right) \\ &- \frac{\Delta t}{2\Delta x}\Big(A_\Delta^L(w_i^n, w_{i+1}^n)(w_{i+1}^n - w_i^n) + A_\Delta^R(w_{i-1}^n, w_i^n)(w_i^n - w_{i-1}^n)\Big) \end{aligned}$$

$$+\frac{\Delta t}{2}\left(S_\Delta^L(\hat{w}_i^n,\hat{w}_{i+1}^n)+S_\Delta^R(\hat{w}_{i-1}^n,\hat{w}_i^n)\right),$$
$$+\frac{\gamma}{2}\frac{\Delta t}{\Delta x}\left(w_{i+1}^n-2w_i^n+w_{i-1}^n\right), \tag{12}$$

where $\gamma \geq 0$ stands for the artificial viscosity coefficient. Our main result given below establishes a suitable choice for γ in order to preserve the well-balanced property and to get both robustness and discrete entropy inequality. In this regard, we assume specific properties for the exact solution of the Riemann problem associated to (1). Indeed, for any two constant states $(w_L, w_R) \in \Omega^2$ and considering the initial data

$$(w(x,t=0), S(w(x,t=0),x))^\intercal = \begin{cases} (w_L, S_L)^\intercal & \text{if} \quad x<0, \\ (w_R, S_R)^\intercal & \text{if} \quad x \geq 0, \end{cases} \tag{13}$$

we assume the existence of $\lambda^\star > 0$ such that the solution $\mathcal{W}_{ex}(x/t, w_L, w_R)$ of the Riemann problem made of (1), (13) is such that

$$\mathcal{W}_{ex}(x/t, w_L, w_R) = \begin{cases} w_L, & \forall x < \lambda^\star t, \\ w_R, & \forall x > \lambda^\star t. \end{cases} \tag{14}$$

Such a condition may appear restrictive but it is satisfied by both one- and two-layer shallow-water models and also by the Euler equations with gravity. In fact, according to [6, 9, 44, 53], the above property is satisfied as soon as $S(w,x)$ can be incorporated within the hyperbolic operator as follows:

$$\partial_t \begin{pmatrix} w \\ a \end{pmatrix} + \begin{pmatrix} (\nabla f + A)(w) & -S(w,a) \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} w \\ a \end{pmatrix} = 0.$$

Our main result is now given.

Theorem 2.1 (Robustness and entropy stability for well-balanced artificial viscosity schemes)**.** *Let us consider* (1) *endowed by a pair of convex entropy-flux entropy* $(\hat{\eta}, \hat{G})$ *such that the Hessian matrix* $\nabla^2\hat{\eta}(w,x)$ *is positive definite. Let* $(w_i^n)_{i\in\mathbb{Z}}$ *be given in* Ω *and consider the following quantities*

$$(\tilde{w}_{i+\frac{1}{2}}^n, \tilde{x}_{i+\frac{1}{2}}) := \frac{1}{2}(w_i^n + w_{i+1}^n, x_i + x_{i+1}) \in \Omega \times \mathbb{R}, \tag{15}$$

$$\begin{aligned} w_{i+\frac{1}{2}}^{L,\star} &:= w_i^n - \frac{f_\Delta(w_i^n, w_{i+1}^n) - f(w_i^n)}{\lambda_{i+\frac{1}{2}}} - \frac{1}{2\lambda_{i+\frac{1}{2}}} A_\Delta^L(w_i^n, w_{i+1}^n)(w_{i+1}^n - w_i^n) \\ &\quad + \frac{\Delta x}{2\lambda_{i+\frac{1}{2}}} S_\Delta^L(\hat{w}_i^n, \hat{w}_{i+1}^n) \in \mathbb{R}^d, \end{aligned} \tag{16}$$

$$\begin{aligned} w_{i+\frac{1}{2}}^{R,\star} &:= w_{i+1}^n + \frac{f_\Delta(w_i^n, w_{i+1}^n) - f(w_{i+1}^n)}{\lambda_{i+\frac{1}{2}}} - \frac{1}{2\lambda_{i+\frac{1}{2}}} A_\Delta^R(w_i^n, w_{i+1}^n)(w_{i+1}^n - w_i^n) \\ &\quad + \frac{\Delta x}{2\lambda_{i+\frac{1}{2}}} S_\Delta^R(\hat{w}_i^n, \hat{w}_{i+1}^n) \in \mathbb{R}^d, \end{aligned} \tag{17}$$

$$\mathcal{E}_{i+\frac{1}{2}}^0 := \lambda_{i+\frac{1}{2}} \left(\hat{\eta}(w_{i+\frac{1}{2}}^{L,\star}, x_i) + \hat{\eta}(w_{i+\frac{1}{2}}^{R,\star}, x_{i+1}) - \hat{\eta}(\hat{w}_i^n) - \hat{\eta}(\hat{w}_{i+1}^n) \right) \tag{18}$$

$$+ \left(\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n) \right) \in \mathbb{R}, \tag{19}$$

$$\mathcal{D}_{i+\frac{1}{2}} := 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}^n, \tilde{x}_{i+\frac{1}{2}}) - \hat{\eta}(\hat{w}_i^n) - \hat{\eta}(\hat{w}_{i+1}^n) \in \mathbb{R}, \tag{20}$$

where $\lambda_{i+\frac{1}{2}} > \lambda^\star$*. Let* (7) *be a well-balanced scheme defined by* f_Δ, $A_\Delta^{L,R}$, S_Δ *and assume*

(h_1)*: the matrices* $A_\Delta^{L,R}$ *are such that*

$$\frac{1}{2}\left(A_\Delta^L(w_i^n, w_{i+1}^n) + A_\Delta^R(w_i^n, w_{i+1}^n)\right)(w_{i+1}^n - w_i^n) = \\ \frac{1}{\Delta t}\int_0^{\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} A\left(\mathcal{W}_{ex}\left(x/t; \hat{w}_i^n, \hat{w}_{i+1}^n\right)\right)\partial_x \mathcal{W}_{ex}\left(x/t; \hat{w}_i^n, \hat{w}_{i+1}^n\right)\, \mathrm{d}x\, \mathrm{d}t, \tag{21}$$

(h_2): the discrete source term $S_\Delta^{L,R}$ are such that

$$\frac{\Delta x}{2}\left(S_\Delta^L(\hat w_i^n,\hat w_{i+1}^n)+S_\Delta^R(\hat w_i^n,\hat w_{i+1}^n)\right)=$$
$$\frac{1}{\Delta t}\int_0^{\Delta t}\int_{-\lambda_{i+\frac12}\Delta t}^{\lambda_{i+\frac12}\Delta t} S\left(\mathcal{W}_{ex}(x/t,\hat w_i^n,\hat w_{i+1}^n),x+x_{i+\frac12}\right)\,\mathrm{d}x\,\mathrm{d}t, \tag{22}$$

(h_3): the states $w_{i+\frac12}^{L,\star}$, $w_{i+\frac12}^{R,\star}$ are such that $w_{i+\frac12}^{L,\star}=w_i^n$ and $w_{i+\frac12}^{R,\star}=w_{i+1}^n$ as soon as the local equilibrium relation $\mathcal{R}(\hat w_i^n,\hat w_{i+1}^n)$ holds.

If γ in the viscous scheme (12) *is defined as follows:*

$$\gamma=\max_{i\in\mathcal{Z}}\left(0,-\frac{\mathcal{E}_{i+\frac12}^0}{\mathcal{D}_{i+\frac12}}\right), \tag{23}$$

then $\gamma\geq 0$ is bounded and if the time step $\Delta t>0$ satisfies

$$(\lambda_{i+\frac12}+\gamma)\frac{\Delta t}{\Delta x}\leq\frac12,\quad\forall i\in\mathbb{Z}, \tag{24}$$

then the viscous scheme (12) *is*

(i) still well-balanced,

(ii) robust under a suitable CFL condition: i.e. $(w_i^{n+1})_{i\in\mathbb{Z}}\subset\Omega$,

(iii) entropy-preserving: $(w_i^{n+1})_{i\in\mathbb{Z}}$ satisfies a discrete entropy inequality (10)*.*

The proof of Theorem 2.1-(*ii*) follows the same lines as in [5], the main ideas being explained in Lemma 3.1. As a consequence, the establishment of the above main result mainly consists to prove the entropy-preservation given by Theorem 2.1-(*iii*). The role of the assumptions (h_1)-(h_3) in this proof will be highlighted in the sequel. The following section gives an equivalent reformulation for the viscous scheme (12).

3 Godunov-type scheme reformulation

In this section, the numerical scheme with the artificial viscosity (12) is reformulated as a Godunov-type scheme. This reformulation is detailed in the following lemma:

Lemma 3.1 (Reformulation as Simple Riemann Solver)**.** *Let us consider $(w_{i+\frac12}^{L,R,\star},\tilde w_{i+\frac12}^n,\lambda_{i+\frac12})\in\Omega\times(\mathbb{R}^d)^2\times\Omega\times\mathbb{R}_*^+$ respectively defined in* (16), (17) *and let us define*

$$\tilde w_{i+\frac12}^{\alpha,\star}:=\Big(1-\frac{\lambda_{i+\frac12}}{\lambda_{i+\frac12}+\gamma}\Big)\tilde w_{i+\frac12}^n+\frac{\lambda_{i+\frac12}}{\lambda_{i+\frac12}+\gamma}w_{i+\frac12}^{\alpha,\star}\in\mathbb{R}^d,\quad\forall\alpha\in\{L,R\}\,. \tag{25}$$

If the CFL restriction (24) *holds, then the viscous scheme* (12) *is equivalent to the following Godunov-type scheme:*

$$w_i^{n+1}=\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}}\tilde w(x/\Delta t,\hat w_{i-1}^n,\hat w_i^n)\,\mathrm{d}x+\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0\tilde w(x/\Delta t,\hat w_i^n,\hat w_{i+1}^n)\,\mathrm{d}x, \tag{26}$$

where $\hat w=(w,x)^\mathsf{T}\in\Omega\times\mathbb{R}$ and the approximate Riemann solver $\tilde w(\cdot,\hat w_i^n,\hat w_{i+1}^n):\mathbb{R}\times(\Omega\times\mathbb{R})^2\to\mathbb{R}^d$ satisfies

$$\tilde w(x/t,\hat w_i^n,\hat w_{i+1}^n)=\begin{cases} w_i^n & \text{if } \ x\leq-(\lambda_{i+\frac12}+\gamma)t,\\ \tilde w_{i+\frac12}^{L,\star} & \text{if } \ -(\lambda_{i+\frac12}+\gamma)t<x\leq 0,\\ \tilde w_{i+\frac12}^{R,\star} & \text{if } \ 0<x\leq(\lambda_{i+\frac12}+\gamma)t,\\ w_{i+1} & \text{if } \ (\lambda_{i+\frac12}+\gamma)t<x.\end{cases} \tag{27}$$

Moreover, under a suitable CFL condition, one can prove that $\tilde{w}^{\alpha,\star}_{i+\frac{1}{2}} \in \Omega$.

Proof. First, let us remark that the last part of the lemma is easily proved following [5]. Indeed, Ω being convex, there exists $r > 0$ such that for any $w \in \mathbb{R}^d$ satisfying $||w - \tilde{w}^n_{i+1/2}|| < r$, we have $w \in \Omega$. Now, in view of (25), it follows that $\tilde{w}^{\alpha,\star}_{i+\frac{1}{2}} \in \Omega$ for γ large enough.

Let us now then prove the main result of the lemma. Using the above definition of the approximate Riemann solver $\tilde{w}(\cdot, \hat{w}^n_i, \hat{w}^n_{i+1}) : \mathbb{R} \times (\Omega \times \mathbb{R})^2 \to \mathbb{R}^d$ we have

$$\begin{aligned}
\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}^n_{i-1}, \hat{w}^n_i)\,\mathrm{d}x &= (\lambda_{i-\frac{1}{2}} + \gamma)\frac{\Delta t}{\Delta x}\tilde{w}^{R,\star}_{i-\frac{1}{2}} + \frac{1}{\Delta x}\left(\frac{\Delta x}{2} - (\lambda_{i-\frac{1}{2}} + \gamma)\right) w^n_i, \\
&= \frac{w^n_i}{2} + (\lambda_{i-\frac{1}{2}} + \gamma)\frac{\Delta t}{\Delta x}\left(\tilde{w}^{R,\star}_{i\frac{1}{2}} - w^n_i\right), \\
&= \frac{w^n_i}{2} + (\lambda_{i-\frac{1}{2}} + \gamma)\frac{\Delta t}{\Delta x}\left(\frac{\gamma}{\lambda_{i-\frac{1}{2}}}(\tilde{w}_{i-\frac{1}{2}} - w^n_i) + \frac{\lambda_{i-\frac{1}{2}}}{\lambda_{i-\frac{1}{2}} + \gamma}(w^{R,\star}_{i-\frac{1}{2}} - w^n_i)\right), \\
&= \frac{w^n_i}{2} + \frac{\gamma}{2}\frac{\Delta t}{\Delta x}(-w^n_i + w^n_{i-1}) + \frac{\lambda_{i-\frac{1}{2}}\Delta t}{\Delta x}(w^{R,\star}_{i-\frac{1}{2}} - w^n_i).
\end{aligned} \tag{28}$$

In the same way, we also have

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, \hat{w}^n_i, \hat{w}^n_{i+1})\,\mathrm{d}x = \frac{w^n_i}{2} + \frac{\gamma}{2}\frac{\Delta t}{\Delta x}(w_{i+1} - w^n_i) + \frac{\lambda_{i+\frac{1}{2}}\Delta t}{\Delta x}(w^{L,\star}_{i+\frac{1}{2}} - w^n_i), \tag{29}$$

But, according to (16) and (17), $w^{L,\star}_{i+\frac{1}{2}} - w^n_{i+1}$ and $w^{R,\star}_{i-\frac{1}{2}} - w^n_i$ respectively read

$$\begin{aligned}
&\lambda_{i+\frac{1}{2}}(w^{L,\star}_{i+\frac{1}{2}} - w^n_i) = \\
&\qquad - f_\Delta(w^n_i, w^n_{i+1}) + f(w^n_i) - \frac{1}{2}A^L_\Delta(w^n_i, w^n_{i+1})(w^n_{i+1} - w^n_i) + \frac{\Delta x}{2}S^L_\Delta(\hat{w}^n_i, \hat{w}^n_{i+1}), \\
&\lambda_{i-\frac{1}{2}}(w^{R,\star}_{i-\frac{1}{2}} - w^n_i) = \\
&\qquad f_\Delta(w^n_{i-1}, w^n_i) - f(w^n_i) - \frac{1}{2}A^R_\Delta(w^n_{i-1}, w^n_i)(w^n_i - w^n_{i-1}) + \frac{\Delta x}{2}S^R_\Delta(\hat{w}^n_{i-1}, \hat{w}^n_i).
\end{aligned}$$

As a consequence, using the above formulations in the sum of (28) and (29), we deduce

$$\begin{aligned}
&\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}^n_{i-1}, \hat{w}^n_i)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, \hat{w}^n_i, \hat{w}^n_{i+1})\,\mathrm{d}x = \\
&\quad \frac{w^n_i}{2} + \frac{\Delta t}{\Delta x}\left(f_\Delta(w^n_{i-1}, w^n_i) - f(w^n_i) - \frac{1}{2}A^R_\Delta(w^n_{i-1}, w^n_i)(w^n_i - w^n_{i-1})\right) \\
&\quad + \frac{\Delta t}{2}S^R_\Delta(\hat{w}^n_{i-1}, \hat{w}^n_i) + \frac{\gamma}{2}\frac{\Delta t}{\Delta x}(-w^n_i + w^n_{i-1}) \\
&\quad + \frac{w^n_i}{2} + \frac{\Delta t}{\Delta x}\left(-f_\Delta(w^n_i, w^n_{i+1}) + f(w^n_i) - \frac{1}{2}A^L_\Delta(w^n_i, w^n_{i+1})(w^n_{i+1} - w^n_i)\right) \\
&\quad + \frac{\Delta t}{2}S^L_\Delta(\hat{w}^n_i, \hat{w}^n_{i+1}) + \frac{\gamma}{2}\frac{\Delta t}{\Delta x}(w^n_{i+1} - w^n_i).
\end{aligned}$$

Since a simple reorganization of the right-hand side of the above equality leads to the expected reformulation that concludes the proof. $\square$

The reformulation (26) is not unique because by analogy with the above computations, we can also define the following approximate Riemann solver

$$
\tilde{w}(x/t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} w_i^n & \text{if } x \le -(\lambda_{i+\frac{1}{2}} + \gamma)t, \\ \tilde{w}_{i+\frac{1}{2}} & \text{if } -(\lambda_{i+\frac{1}{2}} + \gamma)t \le x < -\lambda_{i+\frac{1}{2}} t, \\ w_{i+\frac{1}{2}}^{L,\star} & \text{if } -\lambda_{i+\frac{1}{2}} t < x \le 0, \\ w_{i+\frac{1}{2}}^{R,\star} & \text{if } 0 < x \le \lambda_{i+\frac{1}{2}} t, \\ \tilde{w}_{i+\frac{1}{2}} & \text{if } \lambda_{i+\frac{1}{2}} t \le x < (\lambda_{i+\frac{1}{2}} + \gamma)t, \\ w_{i+1}^n & \text{if } (\lambda_{i+\frac{1}{2}} + \gamma)t < x. \end{cases} \tag{30}
$$

The Godunov-type scheme reformulation (26) is useful to establish a discrete entropy stability. In this regard we give in the following lemma a sufficient condition that ensures a discrete entropy inequality (10). This condition is an extension of the work of Harten *et al.* [37]. For the sake of completeness, we also recall a sufficient condition for the well-balanced property.

Lemma 3.2 (Entropy-preserving well-balanced Godunov-type scheme)**.** *Lest us assume System (1) endowed by a pair of convex entropy-entropy flux $(\hat{\eta}, \hat{G})$. Let us consider a Godunov-type scheme (26) that approximates the solutions of (1) and defined by an approximate Riemann solver $\tilde{w}(\cdot, \hat{w}_i^n, \hat{w}_{i+1}^n) : \mathbb{R} \times (\Omega \times \mathbb{R})^2 \to \mathbb{R}^d$. Let us define $\tilde{x} : \mathbb{R}^3 \to \mathbb{R}$ such that*

$$
x_i = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{x}(x/\Delta t, x_{i-1}, x_i)\, \mathrm{d}x + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{x}(x/\Delta t, x_i, x_{i+1})\, \mathrm{d}x, \tag{31}
$$

where $(x_i)_{i \in \mathbb{Z}} \subset \mathbb{R}$ is defined in (6). Assume the CFL condition (24) holds.

(i) If the couple $(\tilde{w}, \tilde{x})$ satisfies the inequality

$$
\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \hat{\eta}((\tilde{w}, \tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n))\, \mathrm{d}x \le \frac{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n)}{2} - \frac{\Delta t}{\Delta x}(\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n)), \tag{32}
$$

where $\hat{w} = (w, x)^\mathsf{T} \in \Omega \times \mathbb{R}$, then the sequence $(w_i^{n+1}, x_i)_{i \in \mathbb{Z}}$ satisfies a discrete entropy inequality (10) in which the entropy numerical flux $\hat{\mathcal{G}} : (\mathbb{R}^d \times \mathbb{R})^2 \to \mathbb{R}$ is given by

$$
\begin{aligned}
\hat{\mathcal{G}}(\hat{w}_i^n, \hat{w}_{i+1}^n) = {} & \frac{\hat{G}(\hat{w}_{i+1}^n) + \hat{G}(\hat{w}_i^n)}{2} - \frac{\Delta x}{4\Delta t}(\hat{\eta}(\hat{w}_{i+1}^n) - \hat{\eta}(\hat{w}_i^n)) \\
& + \frac{1}{2\Delta t} \int_0^{\frac{\Delta x}{2}} \hat{\eta}((\tilde{w}, \tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n))\, \mathrm{d}x \\
& - \frac{1}{2\Delta t} \int_{-\frac{\Delta x}{2}}^0 \hat{\eta}((\tilde{w}, \tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n))\, \mathrm{d}x.
\end{aligned} \tag{33}
$$

(ii) If the approximate Riemann solver satisfies

$$
\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} w_i^n & \text{if } x < 0, \\ w_{i+1}^n & \text{otherwise}, \end{cases} \qquad \forall i \in \mathbb{Z}, \tag{34}
$$

as soon as the sequence $(\hat{w}_i^n)_{i \in \mathbb{Z}}$ verifies, at each interface of the mesh, the local relation $\mathcal{R}(\hat{w}_i^n, \hat{w}_{i+1}^n)$ then the Godunov-type scheme is well-balanced for the equilibria $\mathcal{R}$.

Proof. Concerning (i), considering (26) and (31) we have

$$
(w_i^{n+1}, x_i) = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} (\tilde{w}, \tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\, \mathrm{d}x + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 (\tilde{w}, \tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\, \mathrm{d}x.
$$

Since the entropy $(w,x) \mapsto \hat{\eta}(w,x)$ is a convex function, the Jensen inequality [54] applied to the above formulation leads to

$$\begin{aligned}
&\hat{\eta}(w_i^{n+1}, x_i) \leq \\
&\frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\big)\,\mathrm{d}x \\
&+ \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\big)\,\mathrm{d}x \\
&- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\big)\,\mathrm{d}x.
\end{aligned}$$

Using the entropy condition (32) in the above inequality, we obtain

$$\begin{aligned}
&\eta(w_i^{n+1}, x_i) \leq \\
&\frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\Big(\frac{\hat{\eta}(\hat{w}_{i-1}^n) + \hat{\eta}(\hat{w}_i^n)}{2} - \frac{\Delta t}{\Delta x}\Big(\hat{G}(\hat{w}_i^n) - \hat{G}(\hat{w}_{i-1}^n)\Big)\Big) \\
&+ \frac{1}{2}\Big(\frac{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n)}{2} - \frac{\Delta t}{\Delta x}\Big(\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n)\Big)\Big) \\
&- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \hat{\eta}\big((\tilde{w},\tilde{x})(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\big)\,\mathrm{d}x.
\end{aligned}$$

Reorganized the above terms with the definition of the function $\hat{\mathcal{G}}$ given by (33), we deduce the discrete entropy inequality (10).

For the statement (ii), let us assume that the sequence $(\hat{w}_i^n)_{i\in\mathbb{Z}}$ is such that the local equilibrium $\mathcal{R}(\hat{w}_i^n, \hat{w}_{i+1}^n)$ holds at each interface of the mesh. In that case, using the condition (34) in a Godunov-type scheme (26), we have

$$\begin{aligned}
w_i^{n+1} &= \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, \hat{w}_{i-1}^n, \hat{w}_i^n)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n)\,\mathrm{d}x, \\
&= \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} w_i^n\,\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{0} w_i^n\,\mathrm{d}x, \\
&= w_i^n.
\end{aligned}$$

According to Definition 2.1-(i) this last equality gives the required well-balanced property that concludes the proof. □

On the one hand, remark that the relation (34) formulates a sufficient condition for the well-balanced property. This condition is well-known (see for instance [6,8,48,49]) and direct computations could show the non-corrected scheme (7) can also be written as in (27) in the case $\gamma = 0$, so that the assumption (h_3) is equivalent to (34).

On the other hand, the entropy condition (32) needs the definition of $\tilde{x} : \mathbb{R}^3 \to \mathbb{R}$ and, in this work, we consider the following choice:

Lemma 3.3. *Let us consider* $(x_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$, $\tilde{x}_{i+\frac{1}{2}} \in \mathbb{R}$ *defined in* (6), (15) *and* $\tilde{x} : \mathbb{R}^3 \to \mathbb{R}$ *such that*

$$\tilde{x}(x/t, x_i, x_{i+1}) = \begin{cases} x_i & \text{if} \quad x \leq -(\lambda_{i+\frac{1}{2}} + \gamma)t, \\ \tilde{x}_{i+\frac{1}{2}} & \text{if} \quad -(\lambda_{i+\frac{1}{2}} + \gamma)t \leq x < -\lambda_{i+\frac{1}{2}} t, \\ x_i & \text{if} \quad -\lambda_{i+\frac{1}{2}} t < x \leq 0, \\ x_{i+1} & \text{if} \quad 0 < x \leq \lambda_{i+\frac{1}{2}} t, \\ \tilde{x}_{i+\frac{1}{2}} & \text{if} \quad \lambda_{i+\frac{1}{2}} t \leq x < (\lambda_{i+\frac{1}{2}} + \gamma)t, \\ x_{i+1} & \text{if} \quad (\lambda_{i+\frac{1}{2}} + \gamma)t < x. \end{cases} \tag{35}$$

The above definition for $\tilde{x}(x/t, x_i, x_{i+1})$ *satisfies* (31).

Proof. According to (35), straightforward computations yield to

$$\frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{x}(x/\Delta t, x_{i-1}, x_i)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 \tilde{x}(x/\Delta t, x_i, x_{i+1})\,\mathrm{d}x = x_i + \frac{\gamma \Delta t}{2\Delta x}(x_{i+1} - 2x_i + x_{i-1}).$$

But, arguing the definition of x_i given by (6), we have

$$x_{i+1} - 2x_i + x_{i-1} = (\Delta x + x_i - 2x_i + x_i - \Delta x) = 0,$$

that concludes the proof. $\square$

Equipped with Lemma 3.2, 3.3, we are now able to show Theorem 2.1. The following section concerns the proof of the well-balanced property preservation satisfied by the viscous scheme (12).

4 Well-balanced artificial viscosity

In this section, we establish Theorem 2.1-(i) which states that the viscous correction (12) does not modify the well-balanced property satisfied by the scheme (7). According to Lemma 3.2-(ii), the proof will be completed as soon as (34) is shown for the approximate Riemann solver (27). The proof is now detailed.

Proof of Theorem 2.1-(i). Assume $(\hat{w}_i^n)_{i\in\mathbb{Z}}$ is such that the local relation $\mathcal{R}(\hat{w}_i^n, \hat{w}_{i+1}^n)$ holds at each interface of the spatial mesh. As a consequence, according to the assumption (h_3), the equalities $w_{i+\frac{1}{2}}^{L,\star} = w_i^n$, $w_{i+\frac{1}{2}}^{R,\star} = w_{i+1}^n$ hold and $\mathcal{E}_{i+\frac{1}{2}}^0$ defined in (19) reads

$$\begin{aligned} \mathcal{E}_{i+\frac{1}{2}}^0 &= \lambda_{i+\frac{1}{2}}\left(\hat{\eta}(w_{i+\frac{1}{2}}^{L,\star}, x_i) + \hat{\eta}(w_{i+\frac{1}{2}}^{R,\star}, x_{i+1}) - \hat{\eta}(\hat{w}_i^n) - \hat{\eta}(\hat{w}_{i+1}^n)\right) + \left(\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n)\right), \\ &= \hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n). \end{aligned}$$

Arguing the non-entropy production (11), the above formulation simply reduces to $\mathcal{E}_{i+\frac{1}{2}}^0 = 0$. Hence, the artificial viscosity γ defined in (23) reduces to $\gamma = 0$ when the equilibrium relations $\mathcal{R}$ holds. In that case, the approximate Riemann solver (27) simply reads

$$\tilde{w}(x/t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} w_i^n & \text{if} \quad x \leq -\lambda_{i+\frac{1}{2}} t, \\ w_{i+\frac{1}{2}}^{L,\star} & \text{if} \quad -\lambda_{i+\frac{1}{2}} t < x \leq 0, \\ w_{i+\frac{1}{2}}^{R,\star} & \text{if} \quad 0 < x \leq \lambda_{i+\frac{1}{2}} t, \\ w_{i+1} & \text{if} \quad \lambda_{i+\frac{1}{2}} t < x, \end{cases}$$

where the states $w^{L,R,\star}_{i+\frac{1}{2}}$ are defined in (16), (17). Using once again the assumption (h_3), we deduce that the above approximate Riemann solver verifies

$$\tilde{w}(x/\Delta t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} w_i^n & \text{if } x<0, \\ w_{i+1}^n & \text{otherwise}, \end{cases} \qquad \forall i \in \mathbb{Z}.$$

According to Lemma 3.2 this last equality achieves the proof. □

5 Discrete entropy stability

The aim of this section is to show that γ given by (23) in the viscous scheme (12) (or in its equivalent reformulation (26), (30)) ensures (32) and thus the discrete entropy inequality (10) holds.

According (30) and (35), the viscous scheme (12) can be equivalently defined by the following couple $(\tilde{w}, \tilde{x}) : \mathbb{R} \times (\Omega \times \mathbb{R})^2 \to \mathbb{R}^{d+1}$:

$$(\tilde{w}, \tilde{x})(x/t, \hat{w}_i^n, \hat{w}_{i+1}^n) = \begin{cases} (w_i^n, x_i) & \text{if} \quad x \leq -(\lambda_{i+\frac{1}{2}} + \gamma)t, \\ (\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) & \text{if} \quad -(\lambda_{i+\frac{1}{2}} + \gamma)t \leq x < -\lambda_{i+\frac{1}{2}} t, \\ (w^{L,\star}_{i+\frac{1}{2}}, x_i) & \text{if} \quad -\lambda_{i+\frac{1}{2}} t < x \leq 0, \\ (w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}) & \text{if} \quad 0 < x \leq \lambda_{i+\frac{1}{2}} t, \\ (\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) & \text{if} \quad \lambda_{i+\frac{1}{2}} t \leq x < (\lambda_{i+\frac{1}{2}} + \gamma)t, \\ (w_{i+1}^n, x_{i+1}) & \text{if} \quad (\lambda_{i+\frac{1}{2}} + \gamma)t < x, \end{cases}$$

where $(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}, w^{L,R,\star}_{i+\frac{1}{2}}) \in \Omega \times \mathbb{R} \times (\mathbb{R}^d)^2$ are given by (15)-(17). As a consequence, a straightforward computation shows that (32) for the viscous scheme (12) reads

$$\mathcal{E}^0_{i+\frac{1}{2}} + \gamma \mathcal{D}_{i+\frac{1}{2}} \leq 0,$$

where $\mathcal{E}^0_{i+\frac{1}{2}}$ and $\mathcal{D}_{i+\frac{1}{2}}$ are respectively given by (19), (20). According to the above inequality, the proof of the entropy-preservation for the viscous scheme (12) stated in Theorem 2.1-(*iii*) will be proved as soon as an upper-bound for the ratio $-\mathcal{E}^0_{i+\frac{1}{2}}/\mathcal{D}_{i+\frac{1}{2}}$ will is established when $\mathcal{E}^0_{i+\frac{1}{2}} \geq 0$. In this regard, we first state the following lemma that concerns an additional useful inequality.

Lemma 5.1. *Let $(w_i^n)_{i\in\mathbb{Z}}$ be given in Ω and $\lambda_{i+\frac{1}{2}} > \lambda^\star$ where $\lambda^\star$ is such that (14) holds. Assume the matrices $A^{L,R}_\Delta$ that approximate the non-conservative product $A(w)\partial_x w$ satisfy (21) and the discrete source terms $S^{L,R}_\Delta$ are such that (22) holds. If $\Delta t > 0$ satisfies $\frac{\lambda_{i+\frac{1}{2}}\Delta t}{\Delta x} \leq \frac{1}{2}$ then the following inequality holds*

$$\hat{\eta}(\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) \leq \hat{\eta}^{\text{HLL}}_{i+\frac{1}{2}}, \tag{36}$$

where $\tilde{x}_{i+\frac{1}{2}} \in \mathbb{R}$ is given by (15) and where we have set

$$\begin{aligned} \hat{w}^{\text{HLL}}_{i+\frac{1}{2}} := & \frac{1}{2}(w_i^n + w_{i+1}^n) - \frac{1}{2\lambda_{i+\frac{1}{2}}}\left(f(w_{i+1}^n) - f(w_i^n)\right) \\ & - \frac{1}{4\lambda_{i+\frac{1}{2}}}\Big(A^L_\Delta(w_i^n, w_{i+1}^n) + A^R_\Delta(w_i^n, w_{i+1}^n)\Big)(w_{i+1}^n - w_i^n) \\ & + \frac{\Delta x}{4\lambda_{i+\frac{1}{2}}}\Big(S^L_\Delta(\hat{w}_i^n, \hat{w}_{i+1}^n) + S^R_\Delta(\hat{w}_i^n, \hat{w}_{i+1}^n)\Big) \in \mathbb{R}^d, \end{aligned} \tag{37}$$

$$\hat{\eta}^{\text{HLL}}_{i+\frac{1}{2}} := \frac{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n)}{2} - \frac{\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n)}{2\lambda_{i+\frac{1}{2}}} \in \mathbb{R}. \tag{38}$$

Proof. Since the CFL condition $\lambda_{i+\frac{1}{2}}{\Delta t}/{\Delta x} \leq 1/2$ holds, the exact solution of juxtaposed Riemann's problems $\left(\mathcal{W}_{ex}(({x-x_{i+\frac{1}{2}}})/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n)\right)_{i\in\mathbb{Z}}$ satisfies

$$\partial_t \mathcal{W}_{ex} + \partial_x f(\mathcal{W}_{ex}) + A(\mathcal{W}_{ex})\partial_x \mathcal{W}_{ex} = S(\mathcal{W}_{ex}, x), \tag{39a}$$
$$\partial_t \hat{\eta}(\mathcal{W}_{ex}, x) + \partial_x \hat{G}(\mathcal{W}_{ex}, x) \leq 0. \tag{39b}$$

As a consequence, applying the operator $\displaystyle\int_0^{\Delta t}\left(\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{x_{i+\frac{1}{2}}-\lambda_{i+\frac{1}{2}}\Delta t}^{x_{i+\frac{1}{2}}+\lambda_{i+\frac{1}{2}}\Delta t}\cdot\,\mathrm{d}x\right)\mathrm{d}t$ to (39) with $\lambda_{i+\frac{1}{2}}$ sufficiently large, as specified in (14), we deduce

$$\begin{aligned}
&\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right)\,\mathrm{d}x = \\
&\frac{1}{2}(w_i^n + w_{i+1}^n) - \frac{1}{2\lambda_{i+\frac{1}{2}}}\left(f(w_{i+1}^n) - f(w_i^n)\right) \\
&- \frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_0^{\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} A\left(\mathcal{W}_{ex}\left({x}/{t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right)\right)\partial_x \mathcal{W}_{ex}\left({x}/{t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_0^{\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} S\left(\mathcal{W}_{ex}({x}/{t}, \hat{w}_i^n, \hat{w}_{i+1}^n), x + x_{i+\frac{1}{2}}\right)\,\mathrm{d}x\,\mathrm{d}t, \\
&\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \hat{\eta}\left(\mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right), x + x_{i+\frac{1}{2}}\right)\,\mathrm{d}x \leq \\
&\qquad \frac{1}{2}\left(\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n)\right) - \frac{1}{2\lambda_{i+\frac{1}{2}}}\left(\hat{G}(\hat{w}_{i+1}^n) - \hat{G}(\hat{w}_i^n)\right).
\end{aligned}$$

Using the assumptions (21), (22), and the definition of $\hat{\eta}_{i+\frac{1}{2}}^{\mathrm{HLL}}$ given by (38), the above formulations write

$$\begin{aligned}
&\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right)\,\mathrm{d}x = \\
&\frac{1}{2}(w_i^n + w_{i+1}^n) - \frac{1}{2\lambda_{i+\frac{1}{2}}}\left(f(w_{i+1}^n) - f(w_i^n)\right) - \frac{1}{4\lambda_{i+\frac{1}{2}}}\left(A_\Delta^L(w_i^n, w_{i+1}^n) + A_\Delta^R(w_i^n, w_{i+1}^n)\right)(w_{i+1}^n - w_i^n) \\
&+ \frac{\Delta x}{4\lambda_{i+\frac{1}{2}}}\left(S_\Delta^L(\hat{w}_i^n, \hat{w}_{i+1}^n) + S_\Delta^R(\hat{w}_i^n, \hat{w}_{i+1}^n)\right), \\
&\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \hat{\eta}\left(\mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right), x + x_{i+\frac{1}{2}}\right)\,\mathrm{d}x \leq \hat{\eta}_{i+\frac{1}{2}}^{\mathrm{HLL}}.
\end{aligned}$$

Thanks to the definition of $\hat{w}_{i+\frac{1}{2}}^{\mathrm{HLL}}$ detailed in (37) the above relations simply reduce to

$$\hat{w}_{i+\frac{1}{2}}^{\mathrm{HLL}} = \frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right)\,\mathrm{d}x,$$
$$\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \hat{\eta}\left(\mathcal{W}_{ex}\left({x}/{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right), x + x_{i+\frac{1}{2}}\right)\,\mathrm{d}x \leq \hat{\eta}_{i+\frac{1}{2}}^{\mathrm{HLL}}.$$

But, according to (6), (15), we have

$$\tilde{x}_{i+\frac{1}{2}} = \frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t}(x + x_{i+\frac{1}{2}})\,\mathrm{d}x.$$

As a consequence, using the Jensen inequality [54], we deduce

$$\begin{aligned}
\hat{\eta}(\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) &= \hat{\eta}\Big(\frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \left(\mathcal{W}_{ex}\left({}^{x}/_{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right), x + x_{i+\frac{1}{2}}\right) \mathrm{d}x\Big),\\
&\leq \frac{1}{2\lambda_{i+\frac{1}{2}}\Delta t}\int_{-\lambda_{i+\frac{1}{2}}\Delta t}^{\lambda_{i+\frac{1}{2}}\Delta t} \hat{\eta}\left(\mathcal{W}_{ex}\left({}^{x}/_{\Delta t}, \hat{w}_i^n, \hat{w}_{i+1}^n\right), x + x_{i+\frac{1}{2}}\right) \mathrm{d}x,\\
&\leq \hat{\eta}^{\text{HLL}}_{i+\frac{1}{2}},
\end{aligned}$$

which achieves the proof. □

The state $\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}$ defined in (37) is an extension of the state derived in [37]. This state and the inequality (36) play an essential role in the proof of the entropy-preserving property verified by the viscous scheme (12) (or equivalently (26)). The proof of Theorem 2.1 is now concluded as follows.

Proof of Theorem 2.1-(iii). According to Lemma 3.2-(i), it is sufficient to establish the inequality

$$\mathcal{E}^0_{i+\frac{1}{2}} + \gamma \mathcal{D}_{i+\frac{1}{2}} \leq 0, \quad \forall i \in \mathbb{Z}, \tag{40}$$

with $\mathcal{E}^0_{i+\frac{1}{2}}$ and $\mathcal{D}_{i+\frac{1}{2}}$ defined in (19), (20). Since $\mathcal{D}_{i+\frac{1}{2}} \leq 0$, it is enough to show the ratio $-\mathcal{E}^0_{i+\frac{1}{2}}/\mathcal{D}_{i+\frac{1}{2}}$ is bounded when $\mathcal{E}^0_{i+\frac{1}{2}} \geq 0$. As a consequence, the proof will be completed as soon as an upper-bound for this ratio will be given.

From Lemma 5.1 and according to the definition of $\mathcal{E}^0_{i+\frac{1}{2}}$, $\mathcal{D}_{i+\frac{1}{2}}$, $\hat{\eta}^{\text{HLL}}_{i+\frac{1}{2}}$ given by (19), (20), (38) we have

$$\begin{aligned}
-\frac{\mathcal{E}^0_{i+\frac{1}{2}}}{\mathcal{D}_{i+\frac{1}{2}}} &= \lambda_{i+\frac{1}{2}} \frac{\hat{\eta}(w^{L,\star}_{i+\frac{1}{2}}, x_i) + \hat{\eta}(w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}) - 2\hat{\eta}^{\text{HLL}}_{i+\frac{1}{2}}}{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n) - 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}})},\\
&\leq \lambda_{i+\frac{1}{2}} \frac{\hat{\eta}(w^{L,\star}_{i+\frac{1}{2}}, x_i) + \hat{\eta}(w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}) - 2\hat{\eta}(\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}})}{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n) - 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}})},
\end{aligned}$$

where $\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}$ is given by (37). Noticing $(\hat{w}^{\text{HLL}}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) = (\frac{1}{2}(w^{L,\star}_{i+\frac{1}{2}} + w^{R,\star}_{i+\frac{1}{2}}), \frac{1}{2}(x_i + x_{i+1}))$, the above inequality reads

$$-\frac{\mathcal{E}^0_{i+\frac{1}{2}}}{\mathcal{D}_{i+\frac{1}{2}}} \leq \lambda_{i+\frac{1}{2}} \frac{\hat{\eta}(w^{L,\star}_{i+\frac{1}{2}}, x_i) + \hat{\eta}(w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}) - 2\hat{\eta}\big(\frac{1}{2}((w^{L,\star}_{i+\frac{1}{2}}, x_i) + (w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}))\big)}{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n) - 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}})}. \tag{41}$$

However, since the hessian matrix $\nabla^2\hat{\eta}$ is a definite positive matrix, $\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n) - 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}}) = 0$ if and only if $\hat{w}_{i+1}^n = \hat{w}_i^n$. Hence, arguing standard, but laborious, Taylor expansions in the neighborhood of $\hat{w}_i^n = \hat{w}_{i+1}^n$ in the right-hand side of (41) ensures the expected boundedness as follows:

$$\frac{\hat{\eta}(w^{L,\star}_{i+\frac{1}{2}}, x_i) + \hat{\eta}(w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}) - 2\hat{\eta}\big(\frac{1}{2}((w^{L,\star}_{i+\frac{1}{2}}, x_i) + (w^{R,\star}_{i+\frac{1}{2}}, x_{i+1}))\big)}{\hat{\eta}(\hat{w}_i^n) + \hat{\eta}(\hat{w}_{i+1}^n) - 2\hat{\eta}(\tilde{w}_{i+\frac{1}{2}}, \tilde{x}_{i+\frac{1}{2}})} \sim \mathcal{O}(1),$$

that concludes the proof. □

6 Numerical results

In this section we apply the technique described previously for some particular cases. Specifically, we consider first the shallow water equations as an example of a system of conservation laws with source term. Second, we address the case of the two-layer shallow water system as an example that includes non-conservative products.

6.1 Single layer Shallow Water equations

In this section, we consider the single layer shallow water equations given by

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh \end{pmatrix} \partial_x z, \quad \forall (x,t) \in I \times (0,T], \tag{42}$$

where $I \subset \mathbb{R}$ is a given interval and $T > 0$ is the final time. This model governs the water height $h \geq 0$ and the horizontal velocity $u \in \mathbb{R}$ of a fluid over a given time independent topography function $z : \mathbb{R} \to \mathbb{R}$. The gravitational constant is denoted $g > 0$. The unknown state vector $w = (h, hu)^{\mathrm{T}} \in \mathbb{R}^2$ is assumed to to belong to in the convex set $\Omega = \{(h, hu) \in \mathbb{R}^2 \mid h \geq 0, hu \in \mathbb{R}\}$. System (42) is endowed with a given initial data $w_0 : \mathbb{R} \to \Omega$ that prescribes $w(x, t = 0)$.

It is well-known [11,29] that the solutions of (42) satisfy the following conservative entropy inequality

$$\partial_t \left(\eta(w) + ghz\right) + \partial_x \left(G(w) + ghzu\right) \leq 0, \tag{43}$$

where $\eta(w) = hu^2/2 + gh^2/2$, $G(w) = hu^3/2 + gh^2 u$. However, for a given bottom topography $x \mapsto z(x)$, the entropy $(w, x) \mapsto \eta(w) + ghz(x)$ is not necessarily convex. In order to recover this convexity property, we assume the solutions of (42) are bounded *i.e.* $w \in L^\infty$ and also $(\partial_x z, \partial_x^2 z) \in (L^{+\infty})^2$ then for $C > 0$, we add $\partial_t(gCx^2) = 0$ to $\eta(w) + ghz$. As a consequence, the inequality (43) is also equivalent to

$$\partial_t \hat{\eta}_C(w, x) + \partial_x \hat{G}(w, x) \leq 0, \tag{44}$$

with the following entropy-entropy flux pair functions:

$$\hat{\eta}_C(w, x) = hu^2/2 + gh^2/2 + ghz(x) + gCx^2, \tag{45a}$$

$$\hat{G}(w, x) = hu^3/2 + gh^2 u + ghuz(x). \tag{45b}$$

Since $(w, \partial_x z, \partial_x^2 z) \in (L^\infty)^3$, a straightforward computation shows the existence of $C > 0$ large enough such that $(w, x) \mapsto \hat{\eta}_C(w, x)$ in (45a) is strictly convex.

On the other hand, the steady states of (42) are given by

$$hu = \text{cste}, \quad \text{and} \quad u^2/2 + g(h + z) = \text{cste}. \tag{46}$$

Using the above equilibrium formulation, direct computations show that $\hat{G}$ defined in (45b) satisfies the non-entropy production assumption (11). In addition, among equilibria (46), a special attention is often devoted (see for instance [10,30,38,45,46,57,63]) to the lake-at-rest

$$u = 0, \quad \text{and} \quad h + z = \text{cste}. \tag{47}$$

It is usual to say that a numerical scheme is fully well-balanced if it preserves all the steady states, while it is well-balanced if it preserves lake-at-rest solutions. Let us remark that if the base scheme is either well-balanced or fully well-balanced, the artificial viscosity added as described previously will not change this property.

We will consider different scenarios comparing several well-known numerical schemes. In particular, we consider the schemes and corresponding notations given in Table 1. In the sequel, the corresponding viscous versions (12) for the schemes in Table 1 will be denoted with the additional suffix +AV. In practice, the artificial viscosity γ defined in (23) is computed according to Algorithm 1. The regularization parameter $\varepsilon > 0$ in Algorithm 1 is fixed to $\varepsilon = 10^{-14}$ and we denote $\lambda = \max_{i \in \mathbb{Z}} \lambda_{i+\frac{1}{2}}$. An admissible choice for $C > 0$ in (45a) is computed from the exact solution (or a reference solution in a very fine mesh) for each test case.

FullWB:	Entropic well-balanced scheme defined in [47, Section 6]. This scheme satisfies a fully discrete version of (43) and it is well-balanced for the moving equilibria (46). The well-balanced property may be locally lost depending on the practical implementation of the scheme (see [47, Section 7.3] for the details).
HR:	Hydrostatic-reconstruction scheme introduced in [2]. The HR allows, from a given base numerical flux for the case of flat bottom, to define a well-balanced numerical scheme for the shallow water equations with topography. If the base scheme is positive and/or entropy-satisfying, then the resulting scheme is postive preserving and/or it verifies a semi-discrete version of (43). As base scheme we select here the VF-ROE numerical flux [13].
GenHR:	Generalized hydrostatic reconstruction introduced in [21]. This technique allows to build a fully well-balanced scheme from a given base scheme for the conservative case. Here VF-ROE numerical flux [13] is selected.
SubBM:	Subsonic well-balanced scheme introduced in [12]. This reconstruction technique allows, from a given base scheme for the conservative case, to define a numerical scheme that preserves all subsonic equilibria. If the base scheme is positive and/or entropy-satisfying, then the resulting scheme is postive preserving and/or it satisfies a semi-discrete version of (43). Here we use as base scheme VF-ROE numerical flux [13].

Table 1: Numerical schemes used for the numerical experiments devoted to the single layer shallow water model (42).

Algorithm 1 Well-balanced artificial viscosity for the single layer shallow water model

procedure
Data:
  $\varepsilon > 0$; ▷ Regularization parameter
  $(w_i^n)_{i\in\mathbb{Z}}$, $(\lambda_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$, $(\hat{\eta}_C, \hat{G})$, non-corrected scheme (7);
Initialization:
  $\gamma \leftarrow 0$;
Main Loop:
  for $i+\frac{1}{2}$ **do** ▷ Loop on interface $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$
    $\mathcal{E}^0_{i+\frac{1}{2}} \leftarrow \lambda_{i+\frac{1}{2}}\left(\frac{\hat{\eta}_C(w^{L,\star}_{i+\frac{1}{2}},x_i)-\hat{\eta}_C(\hat{w}_i^n)}{\Delta x} + \frac{\hat{\eta}(w^{R,\star}_{i+\frac{1}{2}},x_{i+1})-\hat{\eta}_C(\hat{w}_{i+1}^n)}{\Delta x}\right) + \frac{\hat{G}(\hat{w}_{i+1}^n)-\hat{G}(\hat{w}_i^n)}{\Delta x}$;
    if $\mathcal{E}^0_{i+\frac{1}{2}} > \varepsilon$ **then**
      $\mathcal{D}_{i+\frac{1}{2}} \leftarrow \frac{\hat{\eta}_C(\tilde{w}_{i+\frac{1}{2}},\tilde{x}_{i+\frac{1}{2}})-\hat{\eta}_C(\hat{w}_i^n)}{\Delta x} + \frac{\hat{\eta}_C(\tilde{w}_{i+\frac{1}{2}},\tilde{x}_{i+\frac{1}{2}})-\hat{\eta}_C(\hat{w}_{i+1}^n)}{\Delta x}$;

$$\gamma_{i+\frac{1}{2}} \leftarrow -\sqrt{2}\frac{\mathcal{E}^0_{i+\frac{1}{2}}\mathcal{D}_{i+\frac{1}{2}}}{\sqrt{\mathcal{D}^4_{i+\frac{1}{2}} + \max(\varepsilon, \mathcal{D}^4_{i+\frac{1}{2}})}}; \tag{48}$$

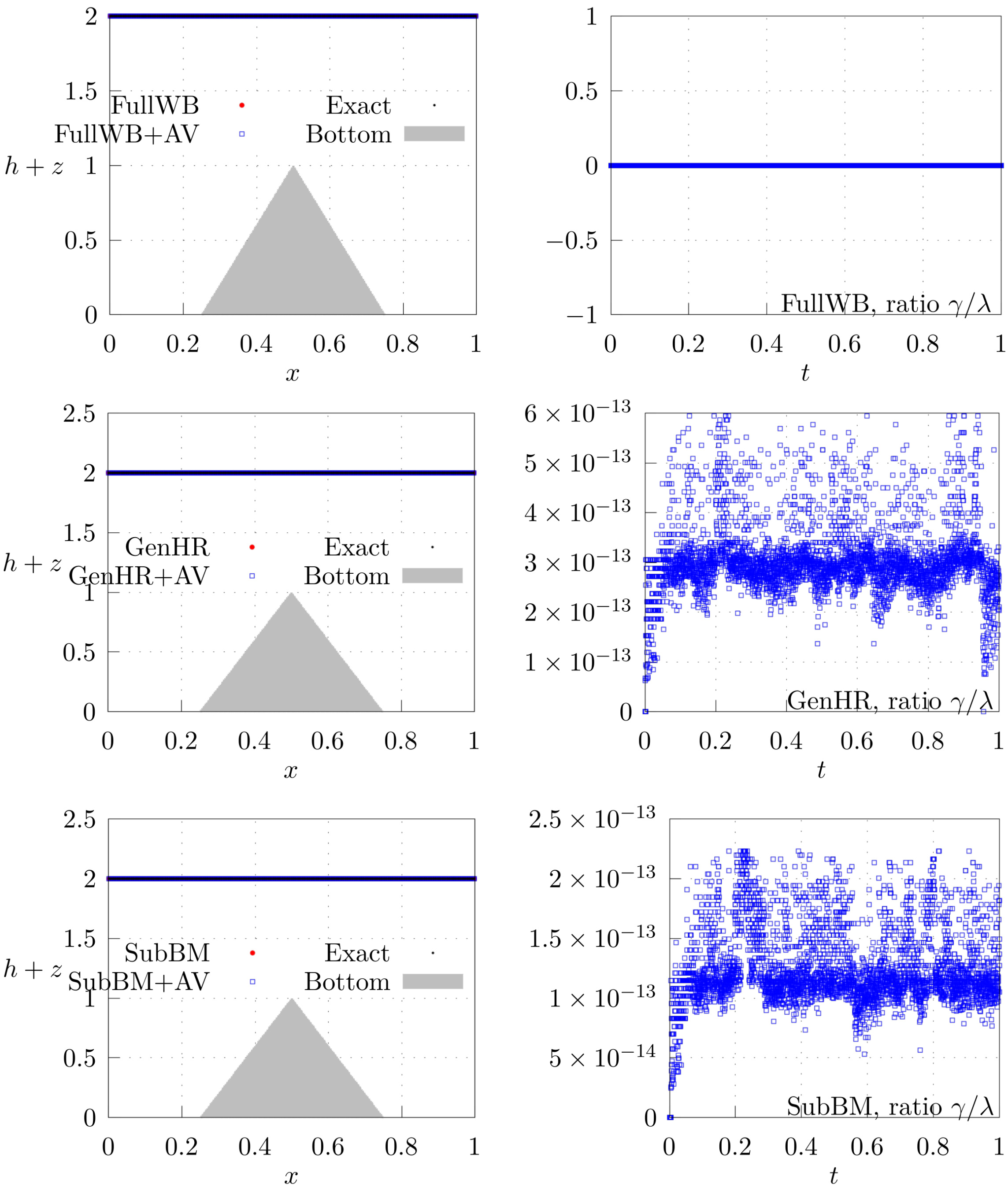
      $\gamma \leftarrow \max(\gamma_{i+\frac{1}{2}}, \gamma)$;
    end if
  end for
Result:
  return γ;
end procedure

6.1.1 Lake-at-rest steady state

Our first numerical experiment is devoted to check the well-balanced property for the lake-at-rest (47). The initial condition and the topography bottom are the following:

$$h_0(x) = \max(2 - z(x), 0), \quad u_0(x) = 0, \quad z(x) = \max(0, 1 - |4x - 2|). \tag{49}$$

The domain $[0, 1]$ is discretized with 400 cells. The unknown h and u follow periodic boundary conditions on both spatial boundaries. The constant $C > 0$ in (45a) verifies $C = 8$ and the final time is 1.0. Figure 1 and Table 2 display the compared results.



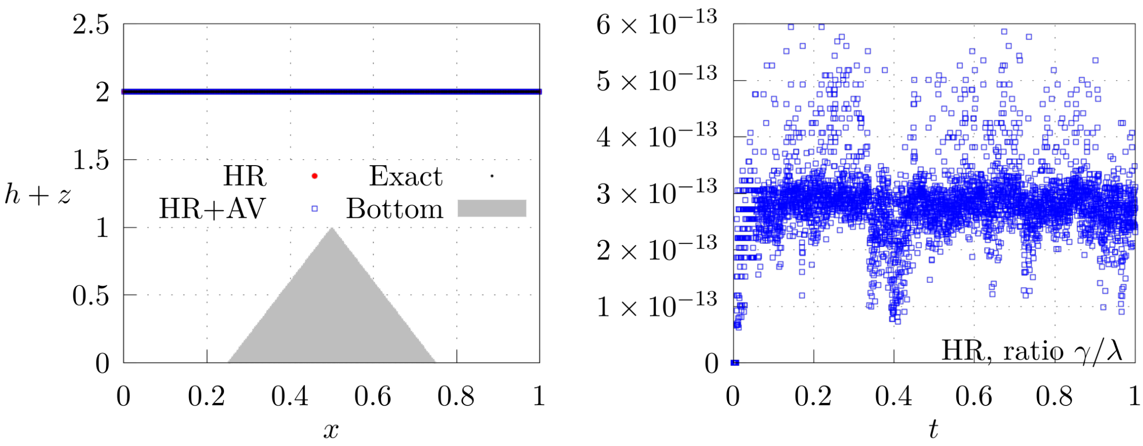


Figure 1: Numerical results and time evolution of the viscosities for the equilibrium at rest (49) in a mesh composed of 400 cells. The legend of the schemes is given in Table 1, +AV denotes the viscous version (12).

Numerical errors in (h, hu) for the lake at rest (49).

	h L^1	h L^2	h L^∞
FullWB+AV	0.0E-00	0.0E-00	0.0E-00
FullWB	0.0E-00	0.0E-00	0.0E-00
	hu L^1	hu L^2	hu L^∞
FullWB+AV	1.16E-19	1.86E-19	2.56E-21
FullWB	1.16E-19	1.86E-19	2.56E-21

Numerical errors in (h, hu) for the lake at rest (49).

	h L^1	h L^2	h L^∞
GenHR+AV	7.85E-16	8.73E-16	1.08E-15
GenHR	1.27E-18	1.38E-18	1.30E-18
	hu L^1	hu L^2	hu L^∞
GenHR+AV	9.38E-16	1.20E-15	1.32E-17
GenHR	7.82E-18	8.19E-18	6.69E-18

Numerical errors in (h, hu) for the lake at rest (49).

	h L^1	h L^2	h L^∞
SubBM+AV	3.01E-16	3.34E-16	2.85E-16
SubBM	5.41E-18	6.49E-18	3.03E-18
	hu L^1	hu L^2	hu L^∞
SubBM+AV	1.83E-15	2.16E-15	1.07E-17
SubBM	1.24E-17	1.48E-17	7.75E-18

Numerical errors in (h, hu) for the lake at rest (49).

	h L^1	h L^2	h L^∞
HR+AV	3.16E-16	3.59E-16	6.50E-16
HR	7.20E-19	7.31E-19	6.50E-19
	hu L^1	hu L^2	hu L^∞
HR+AV	1.10E-15	1.33E-15	3.14E-18
HR	8.77E-19	1.06E-18	1.40E-18

Table 2: Numerical errors in (h, hu) committed by the numerical schemes given in Table 1 and their viscous version (12) denoted +AV both for the at rest problem (49).

6.1.2 Moving equilibria

Our second numerical test concerns the preservation of the well-balanced property for moving equilibria defined by (46). Namely, we consider the following initial condition and topography:

$$(hu)_0(x) = q_0, \quad u_0^2(x)/2 + g(h_0 + z)(x) = B_0, \quad z(x) = (5/2)\cos^2(4\pi x), \tag{50}$$

where $q_0 = {}^5\!/_2$, $B_0 = {}^{25}\!/_{98} + 4g$. Such data involve a subsonic equilibrium that satisfies $u^2 \leq gh$.

The spatial domain $[0, 1]$ is dicretized with 400 cells and periodic conditions are imposed on both sides. The constant $C > 0$ in (45a) satisfies $C = 2080$ and the final time is 0.5. Figure 2 shows the compared results and Table 3 reports the numerical errors.

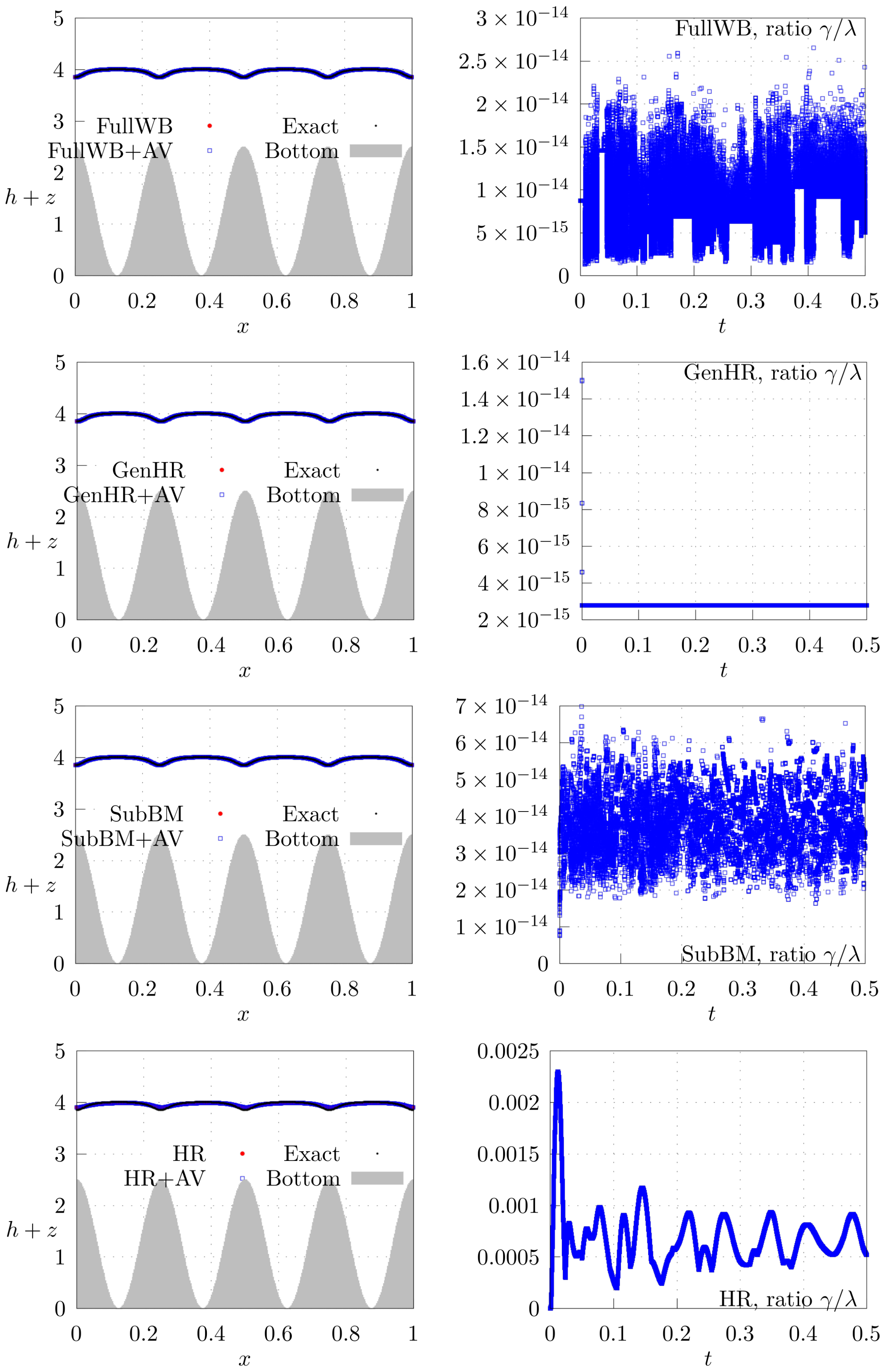

FullWB
Exact
FullWB+AV
Bottom
h + z
x
FullWB, ratio γ/λ
t
GenHR
Exact
GenHR+AV
Bottom
h + z
x
GenHR, ratio γ/λ
t
SubBM
Exact
SubBM+AV
Bottom
h + z
x
SubBM, ratio γ/λ
t
HR
Exact
HR+AV
Bottom
h + z
x
HR, ratio γ/λ
t

Figure 2: Numerical results and time evolution of the viscosities for the moving equilibrium (50) in a mesh composed of 400 cells. The legend of the numerical schemes is given in Table 1, +AV denotes the viscous version (12).

Numerical errors in (h, hu) for the moving steady states (50).

	h L^1	h L^2	h L^∞
FullWB+AV	0.0E-00	0.0E-00	0.0E-00
FullWB	0.0E-00	0.0E-00	0.0E-00
	hu L^1	hu L^2	hu L^∞
FullWB+AV	0.0E-00	0.0E-00	0.0E-00
FullWB	0.0E-00	0.0E-00	0.0E-00

Numerical errors in (h, hu) for the moving steady states (50).

	h L^1	h L^2	h L^∞
GenHR+AV	3.27E-19	4.66E-19	9.75E-19
GenHR	0.0E-00	0.0E-00	0.0E-00
	hu L^1	hu L^2	hu L^∞
GenHR+AV	0.00E-00	0.00E-00	0.00E-00
GenHR	0.00E-00	0.00E-00	0.00E-00

Numerical errors in (h, hu) for the moving steady states (50).

	h L^1	h L^2	h L^∞
SubBM+AV	3.36E-15	4.09E-15	9.72E-15
SubBM	2.11E-20	1.27E-19	1.62E-38
	hu L^1	hu L^2	hu L^∞
SubBM+AV	8.29E-14	8.33E-14	8.43E-14
SubBM	5.84E-19	1.40E-18	1.97E-36

Numerical errors in (h, hu) for the moving steady states (50).

	h L^1	h L^2	h L^∞
HR+AV	1.93E-02	2.30E-02	5.24E-02
HR	1.93E-02	2.31E-02	5.24E-02
	hu L^1	hu L^2	hu L^∞
HR+AV	0.56E-00	0.56E-00	0.58E-00
HR	0.56E-00	0.56E-00	0.58E-00

Table 3: Numerical errors in (h, hu) committed by the schemes detailed in Table 1 and by their viscous version (12) denoted +AV both for the problem (50).

Remark that this equilibrium is preserved by all of the numerical schemes used here with the exception of HR. Since the HR scheme is not fully well-balanced, it does not preserve moving steady states (50). This means that the solution computed presents an error and the equilibrium is perturbed. This results in an artificial viscosity that make in order to satisfy an entropy inequality, as seen in the last picture on the right in Figure 2.

6.1.3 Evolution to a steady state

The third case test deals with the Goutal and Maurel problem [35]. The spatial domain $[0, 25]$ is discretized with 400 cells, the initial condition and the bottom topography are

$$h_0(x) = 2, \quad (hu)_0(x) = 4.42, \quad z(x) = \max(0, 0.2 - 0.05(x - 10)^2). \tag{51}$$

On the left boundary, homogeneous Neumann condition is prescribed for water height with the discharge hu is set to 4.42. On the right boundary, the water height is set to 2 and a homogeneous Neumann condition is considered for the discharge. The initial condition will evolve until a final steady state is reached. The final time is fixed to 500 so that the steady state is attained. The exact solution is computed with the software SWASHES [27]. Figure 3 displays the compared results with $C = \frac{1}{4}$ in (45a). For the sake of clarity, the time evolutions of the ratio γ/λ is only reported for $t \in [0, 100]$. Table 4 shows the numerical errors.

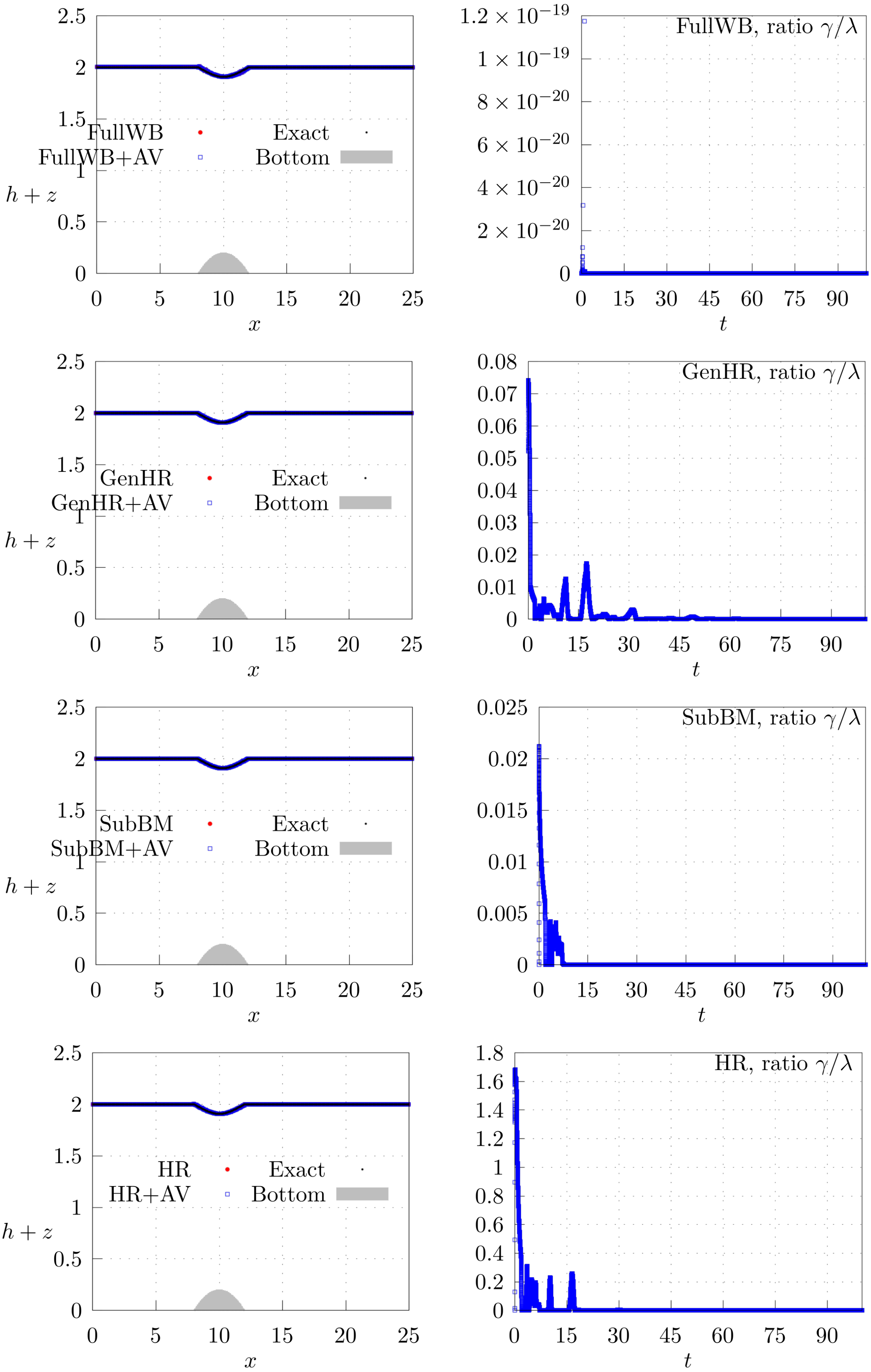

FullWB
FullWB+AV
Exact
Bottom
$h+z$
x
FullWB, ratio γ/λ
t
GenHR
GenHR+AV
Exact
Bottom
$h+z$
x
GenHR, ratio γ/λ
t
SubBM
SubBM+AV
Exact
Bottom
$h+z$
x
SubBM, ratio γ/λ
t
HR
HR+AV
Exact
Bottom
$h+z$
x
HR, ratio γ/λ
t

Figure 3: Numerical results and time evolution of the viscosities for the Goutal and Maurel's problem (51) in a mesh composed of 400 cells. The legend of the numerical schemes is given in Table 1, +AV denotes the viscous version (12).

Numerical errors in (h, hu) for Goutal and Maurel's problem (51).

	h		
	L^1	L^2	L^∞
FullWB+AV	3.86E-02	1.41E-02	2.00E-04
FullWB	3.86E-02	1.41E-02	2.00E-04
	hu		
	L^1	L^2	L^∞
FullWB+AV	3.77E-02	2.10E-02	4.43E-04
FullWB	3.77E-02	2.10E-02	4.43E-04

Numerical errors in (h, hu) for Goutal and Maurel's problem (51).

	h		
	L^1	L^2	L^∞
GenHR+AV	3.32E-14	1.01E-14	2.47E-17
GenHR	1.54E-15	3.93E-16	6.50E-19
	hu		
	L^1	L^2	L^∞
GenHR+AV	9.66E-14	2.46E-14	2.22E-15
GenHR	2.35E-15	5.28E-16	7.67E-17

Numerical errors in (h, hu) for Goutal and Maurel's problem (51).

	h		
	L^1	L^2	L^∞
SubBM+AV	3.89E-14	1.19E-14	4.07E-17
SubBM	3.18E-15	7.09E-16	6.50E-19
	hu		
	L^1	L^2	L^∞
SubBM+AV	1.02E-13	2.75E-14	4.93E-15
SubBM	2.22E-15	5.96E-16	2.85E-16

Numerical errors in (h, hu) for Goutal and Maurel's problem (51).

	h		
	L^1	L^2	L^∞
HR+AV	1.81E-02	6.89E-03	4.75E-05
HR	1.81E-02	6.89E-03	4.75E-05
	hu		
	L^1	L^2	L^∞
HR+AV	2.97E-02	1.67E-02	2.79E-04
HR	2.97E-02	1.67E-02	2.79E-04

Table 4: Numerical errors in (h, hu) committed by the schemes detailed in Table 1 and their viscous version (12) denoted +AV both for the problem (51)

The numerical results are very satisfying for the GenHR and the SubBM schemes. Once again, a large error is observed for the Hydrostatic Reconstruction (HR) scheme because this scheme is not well-balanced for the moving equilibrium induced by the Goutal and Maurel problem (51). The large numerical error observed for the FullWB scheme is an intrinsic defect of the scheme (see [47, Section 7.3] for the details). The artificial viscosity method does not correct this defect.

6.1.4 Dam-break problem

Our final test case for the shallow water system concerns the dam break problem as detailed in [49]. The spatial domain $[-1, 1]$ is discretized with 400 cells and the initial condition and topography are

$$h_0(x) + z(x) = \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{otherwise,} \end{cases} \qquad u_0(x) = 0, \quad \text{with,} \quad z(x) = \frac{1}{2}\cos^2(\pi x). \tag{52}$$

Homogeneous Neumann boundary conditions are prescribed on both boundaries. The final time is 0.1. A reference solution is computed on a fine grid made of 50 000 cells with the standard HLL scheme [37] coupled to the discrete source term $(-gh\partial_x z)(x_i, t^n) \approx -g\frac{h_i^n + h_{i+1}^n}{2}\frac{z_{i+1} - z_i}{\Delta x}$. Figure 4 displays the results with $C = 17$ in (45a).

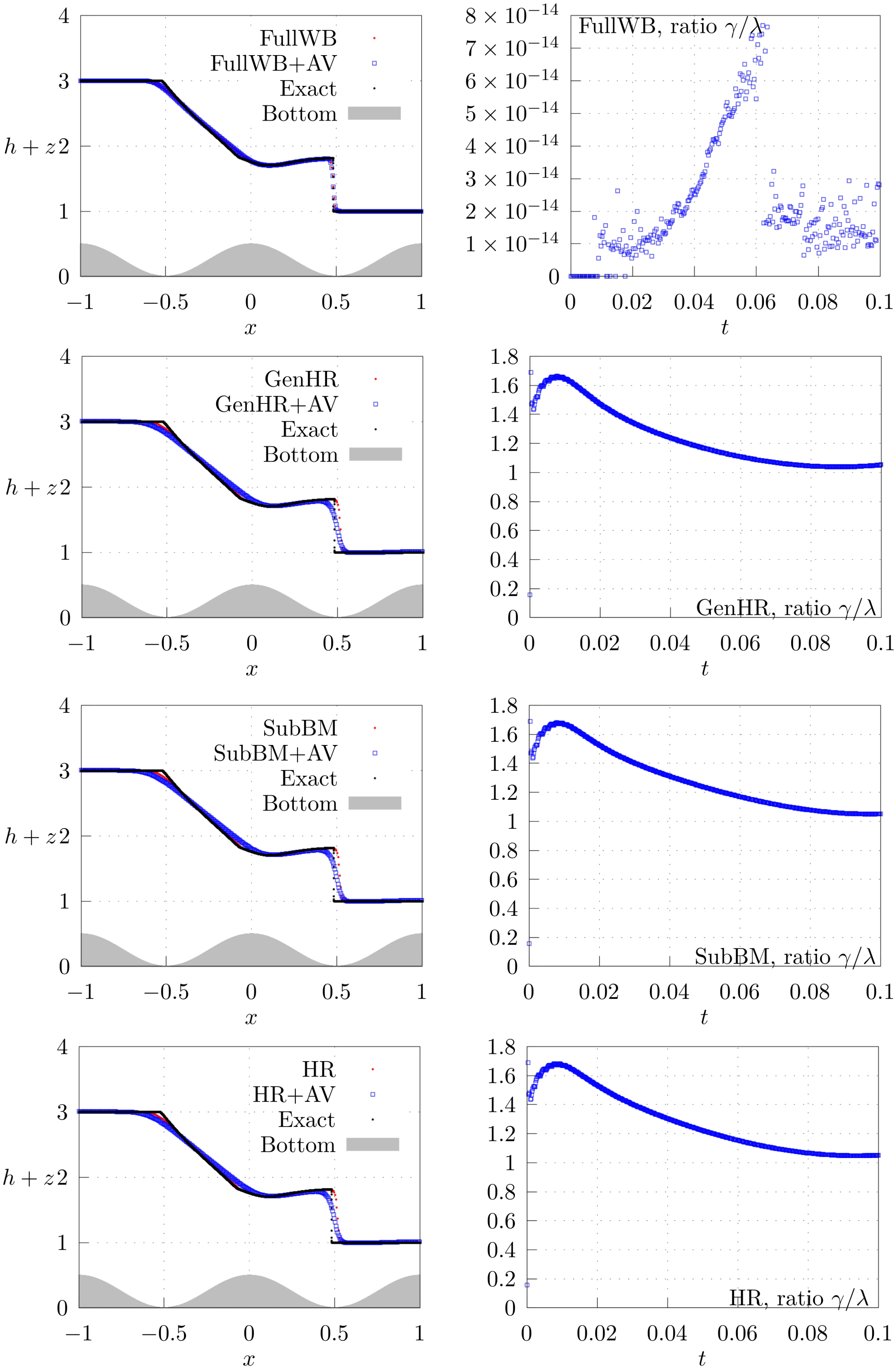

FullWB
FullWB+AV
Exact
Bottom
$h + z$
x
FullWB, ratio γ/λ
t
GenHR
GenHR+AV
Exact
Bottom
$h + z$
x
GenHR, ratio γ/λ
t
SubBM
SubBM+AV
Exact
Bottom
$h + z$
x
SubBM, ratio γ/λ
t
HR
HR+AV
Exact
Bottom
$h + z$
x
HR, ratio γ/λ
t

Figure 4: Numerical results and time evolution of the viscosities for the dam break problem (52) in a mesh composed of 400 cells. The legend of the numerical schemes is given in Table 1, +AV denotes the viscous version (12).

Since the non-corrected FullWB scheme satisfies a discrete entropy inequality, the viscous version coincides with the initial scheme.

6.2 Two-layer shallow water equations

Let us now consider the two-layer shallow water system that models two inmiscible fluids in a shallow regime of constant density ρ_j and thickness h_j $(j = 1, 2)$. We assume $\rho_1 < \rho_2$ so that index $j = 1$ refers to the upper layer, while $j = 2$ corresponds to the lower layer. We shall denote by u_j the horizontal velocity within each layer and by $q_j = h_j u_j$ the discharge. The fixed bottom topography will be denoted as $z(x)$.

The equations of the system may be written as

$$\begin{cases} \partial_t h_1 + \partial_x q_1 = 0, \\ \partial_t q_1 + \partial_x \left(\dfrac{q_1^2}{h_1} + \dfrac{1}{2} g h_1^2 \right) = -g h_1 \partial_x h_2 - g h_1 \partial_x z, \\ \partial_t h_2 + \partial_x q_2 = 0, \\ \partial_t q_2 + \partial_x \left(\dfrac{q_2^2}{h_2} + \dfrac{1}{2} g h_2^2 \right) = -\dfrac{\rho_1}{\rho_2} g h_2 \partial_x h_1 - g h_2 \partial_x z. \end{cases} \tag{53}$$

This system can also be written in a compact form

$$\partial_t w + \partial_x f(w) + A(w) \partial_x w = S(w) \partial_x z, \tag{54}$$

where

$$w = \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix}, \quad f(w) = \begin{pmatrix} q_1 \\ \dfrac{q_1^2}{h_1} + \dfrac{1}{2} g h_1^2 \\ q_2 \\ \dfrac{q_2^2}{h_2} + \dfrac{1}{2} g h_2^2 \end{pmatrix},$$

$$A(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g h_1 & 0 \\ 0 & 0 & 0 & 0 \\ g r h_2 & 0 & 0 & 0 \end{pmatrix}, \quad S(w) = \begin{pmatrix} 0 \\ -g h_1 \\ 0 \\ -g h_2 \end{pmatrix},$$

with $r = \rho_1 / \rho_2$.

The usual entropy-pair associated with the above two-layer shallow water system is given by

$$\hat{\eta}(w, z) = \sum_{j=1}^{2} \rho_j h_j \left(\frac{u_j^2}{2} + g \frac{h_j}{2} + z \right) + g \rho_1 h_1 h_2 \tag{55a}$$

$$\hat{G}(w, z) = \sum_{j=1}^{2} \rho_j h_j u_j \left(\frac{u_j^2}{2} + g h_j + z \right) + \rho_1 g h_1 h_2 (u_1 + u_2). \tag{55b}$$

We remark that the function $(w, x) \mapsto \eta(w, z(x))$ does not satisfy the required convexity and we may introduce some additional convex correction with respect to x as proposed in (45). However, we take the opportunity to test our numerical method out of the validity domain imposed by Theorem 2.1. As a consequence, we here suggest to adopt the non-convex with respect to z but convex with respect to w entropy function given by (55).

Concerning the steady solutions, we shall focus on lake-at-rest steady states given by

$$u_1 = u_2 = 0, \quad h_2 + z = \text{cste}, \quad h_1 = \text{cste}.$$

We consider a path-conservative numerical scheme [52], with segments as family of paths combined with a hydrostatic reconstruction [2, 21]. More explicitly, we define

$$w_{i+1/2}^{\pm} = ((h_1)_{i+1/2}^{\pm}, (q_1)_{i+1/2}^{\pm}, (h_2)_{i+1/2}^{\pm}, (q_2)_{i+1/2}^{\pm})^t,$$
$$(h_2)_{i+1/2}^{-} = \max(0, (h_2)_i + z_i - z_{i+1/2}), \quad (h_2)_{i+1/2}^{+} = \max(0, (h_2)_{i+1} + z_{i+1} - z_{i+1/2}),$$
$$(h_1)_{i+1/2}^{-} = (h_1)_i, \quad (h_1)_{i+1/2}^{+} = (h_1)_{i+1}, \quad z_{i+1/2} = \max(z_i, z_{i+1}),$$

and we consider a Roe path-conservative scheme written in a PVM formalism [15]:

$$\begin{aligned} w_i^{n+1} = w_i^n &- \frac{\Delta t}{\Delta x}\left(f_\Delta(w_{i+1/2}^-, w_{i+1/2}^+) - f_\Delta(w_{i-1/2}^-, w_{i-1/2}^+) + \Pi_{i+1/2}^- + \Pi_{i-1/2}^+\right) \\ &- \frac{\Delta t}{2\Delta x}\left(A_\Delta^L(w_{i+1/2}^-, w_{i+1/2}^+)\cdot(w_{i+1/2}^+ - w_{i+1/2}^-) + A_\Delta^R(w_{i-1/2}^-, w_{i-1/2}^+)\cdot(w_{i-1/2}^+ - w_{i-1/2}^-)\right), \end{aligned} \tag{56}$$

where the numerical flux is given by

$$f_\Delta(w_{i+1/2}^-, w_{i+1/2}^+) = \frac{1}{2}\left(f(w_{i+1/2}^-) + f(w_{i+1/2}^+)\right) - \frac{1}{2}|\mathcal{A}_\Delta^n|\cdot(w_{i+1/2}^+ - w_{i+1/2}^-), \tag{57}$$

$$\Pi_{i+1/2}^- = \left(0, 0, 0, \frac{g}{2}((h_2^2)_{i+1/2}^- - (h_2^2)_i)\right)^t, \quad \Pi_{i-1/2}^+ = \left(0, 0, 0, \frac{g}{2}((h_2^2)_i - (h_2^2)_{i-1/2}^+)\right)^t,$$

with $|\mathcal{A}_\Delta^n|$ the absolute value of the Roe matrix

$$\mathcal{A}_\Delta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh_{\Delta,1} - u_{\Delta,1}^2 & 2u_{\Delta,1} & gh_{\Delta,2} & 0 \\ 0 & 0 & 0 & 1 \\ rgh_{\Delta,1} & 0 & gh_{\Delta,2} - u_{\Delta,2}^2 & 2u_{\Delta,2} \end{pmatrix}, \tag{58}$$

$h_{\Delta,j}$ and $u_{\Delta,j}$, $j = 1,\ 2$ are the Roe averages and are defined as

$$h_{\Delta,j} = \frac{h_{i,j} + h_{i+1,j}}{2}, \quad u_{\Delta,j} = \frac{\sqrt{h_{i,j}}u_{i,j} + \sqrt{h_{i+1,j}}u_{i+1,j}}{\sqrt{h_{i,j}} + \sqrt{h_{i+1,j}}}, \quad j = 1,\ 2.$$

Finally,

$$A_\Delta^L(w_i^n, w_{i+1}^n) = A_\Delta^R(w_i^n, w_{i+1}^n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & gh_{\Delta,1} & 0 \\ 0 & 0 & 0 & 0 \\ grh_{\Delta,2} & 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

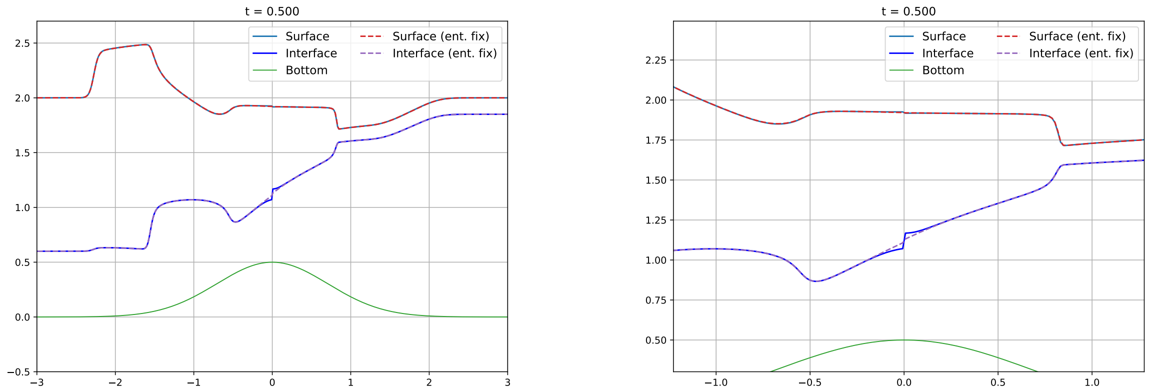
6.2.1 Internal dam break with small density ratio

Let us first consider a test case with $r = 0.2$. The spatial domain is set to $[-3, 3]$ which is discretized with 500 volumes. The initial condition and bottom are set as

$$z(x) = 0.5\exp(-x^2), \quad h_1(x,0) + h_2(x,0) + z(x) = 2, \quad h_2(x,0) + z(x) = \begin{cases} 0.6, & \text{if } x < 0, \\ 1.85, & \text{otherwise,} \end{cases}$$

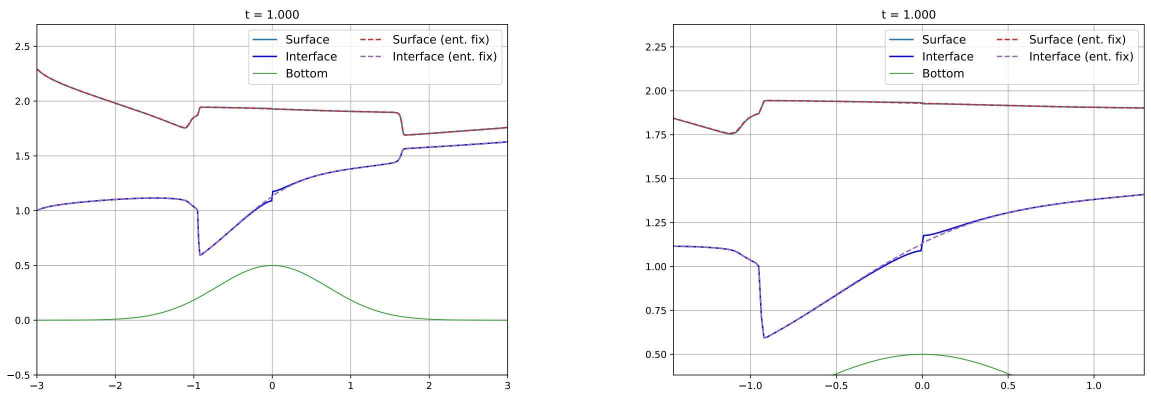
with zero discharge $q_1(x,0) = q_2(x,0) = 0$. Open boundary conditions are considered and the final time is set to $t = 1$.

Figures 5 and 6 show the surface and interface evolution for $t = 0.5$ and $t = 1$ respectively. As we see, the Roe scheme does not satisfy an entropy inequality and it results in a non-physical shock in the interface corresponding to the maximum of the topography. Nevertheless, the AV correction allows to overcome this problem. The corresponding AV correction is shown in Figure 7.



(a) (b) Zoom

Figure 5: Surface $h_1 + h_2 + z$ and interface $h_2 + z$ evolution of an internal dam break problem for $r = 0.2$. Comparison for Roe scheme with and without the artificial viscosity method.



(a) (b) Zoom

Figure 6: Surface $h_1 + h_2 + z$ and interface $h_2 + z$ evolution of an internal dam break problem for $r = 0.2$. Comparison for Roe scheme with and without the artificial viscosity method.

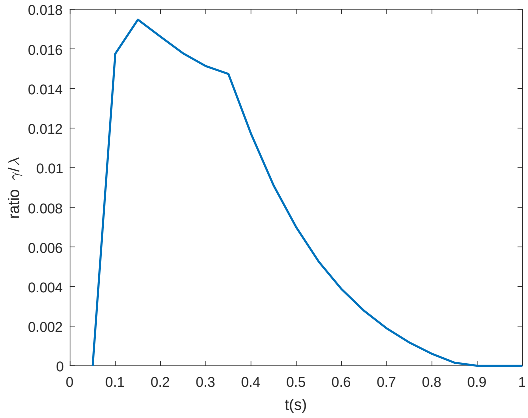
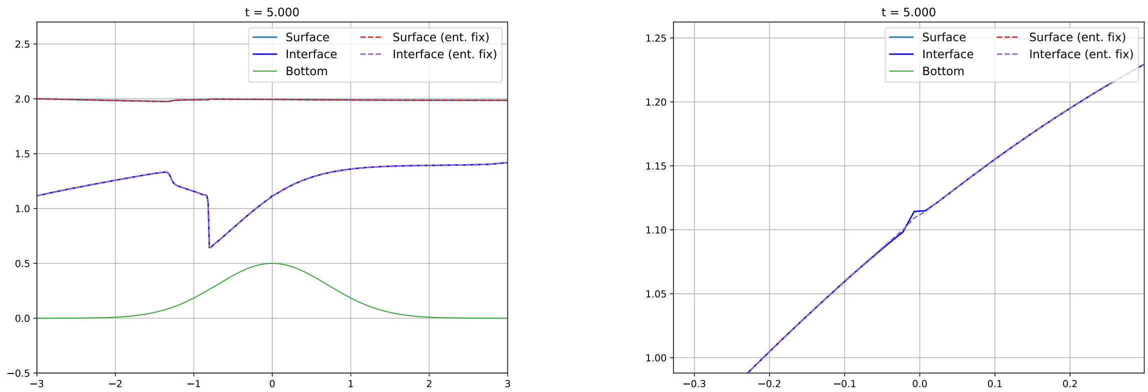


Figure 7: Time evolution of the maximum ratio γ/λ for $r = 0.2$.



(a)

(b) Zoom

Figure 8: Surface $h_1 + h_2 + z$ and interface $h_2 + z$ evolution of an internal dam break problem for $r = 0.9$. Comparison for Roe scheme with and without the artificial viscosity method.

6.2.2 Internal dam break with big density ratio

Let us consider now the same initial condition used in previous example, but now we set the ratio of densities to $r = 0.9$. When the ratio of densities is near 1, the perturbations on the free surface due to the internal dam break are less noticeable. The internal waves propagate with smaller speed, while the overall dynamics is similar, observing an internal shock that is formed in the interface. This is shown in Figure 8.

In this case, the artifact that can be observed in previous case is now less noticeable and we have to zoom into the interface to observe it. In any case, the AV correction allows to overcome this problem again. The corresponding AV correction is shown in Figure 9.

7 Conclusions

In this work, we have presented a general and efficient framework to ensure discrete entropy stability for first-order well-balanced finite volume schemes applied to systems of balance laws, including those with non-conservative products. Building upon the artificial viscosity approach introduced by Tadmor and further developed in recent literature, we have proposed a minimal correction that can be systematically added to a wide range of existing well-balanced schemes.

The proposed method maintains the well-balanced property and ensures robustness and guarantees entropy stability through a carefully constructed artificial viscosity coefficient. The theoretical analysis

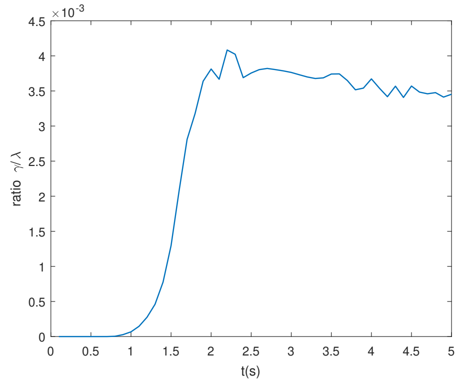


Figure 9: Time evolution of the maximum ratio γ/λ for $r = 0.9$.

confirms that the corrected scheme satisfies discrete entropy inequalities, preserves stationary solutions, and remains robust under a suitable CFL condition.

A reformulation of the modified scheme as a Godunov-type method allowed us to derive sufficient conditions for entropy preservation, well-balancing and robustness. Numerical experiments, particularly on the shallow water equations and the two-layer shallow water model, have demonstrated the effectiveness of the proposed method. The results confirm that entropy stability can be achieved without compromising the accuracy or well-balanced properties of the base schemes.

An important contribution of this work is its flexibility and generality: the proposed correction can be combined, for example, with the strategy described in [22] to extend any consistent first-order solver for the homogeneous problem—*i.e.*, without source terms—into a well-balanced, robust, and entropy-satisfying solver for general one-dimensional balance laws. To the best of our knowledge, this is the first time such a general and constructive procedure is presented, offering a unified framework to develop stable and physically relevant numerical methods for a wide range of balance law systems.

Overall, this approach provides a practical and theoretically sound procedure to enhance existing well-balanced schemes with entropy-preserving capabilities, offering potential for future extensions to higher-order methods.

Acknowledgments

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