WELL-BALANCED SCHEMES TO CAPTURE NON-EXPLICIT STEADY STATES IN THE EULER EQUATIONS WITH GRAVITY

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Abstract. We consider the numerical approximation of the solution of the Euler equations endowed with a gravitational potential. The gravitational source term is known to introduce severe difficulties as soon as the numerical simulations are concerned by steady solutions. Indeed, steady states are here governed by strongly nonlinear PDE relations, and their approximations turn out to be very challenging. By adopting relaxation techniques, we propose to design a finite volume method, which satisfies robustness, accuracy and stability properties. Actually, the obtained numerical method is proved to preserve the positiveness of both density and internal energy. Moreover, we establish a full set of discrete entropy inequalities. Finally, concerning the approximation of the steady states, the proposed scheme exactly preserves a suitable approximation of the partial differential equation governing the steady states under consideration. In fact, an additional property is given since the scheme exactly captures the hydrostatic (or isothermal) steady states independently from the gravitational potential definition. Several simulations illustrate the relevance of the designed numerical procedure.

1. Introduction.

The present work is devoted to the numerical approximation of the solutions of the Euler equations endowed with a gravity source term. In one space dimension, the system under consideration writes as follows:

$$\partial_t \rho + \partial_x \rho u = 0, \tag{1.1a}$$

$$\partial_t \rho u + \partial_x (\rho u^2 + p) = -\rho \partial_x \Phi, \tag{1.1b}$$

$$\partial_t E + \partial_x (E + p) u = -\rho u \partial_x \Phi, \tag{1.1c}$$

where $\rho(x,t) > 0$ denotes the density, $u(x,t) \in \mathbb{R}$ the velocity and $E(x,t) > 0$ the total energy. Here, the function $\Phi : \mathbb{R} \to \mathbb{R}$ stands for a given smooth gravitational potential. Concerning the pressure, it is given by a general pressure law $p(\tau, e)$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ where we have introduce $\tau = 1/\rho$ the specific volume and $e > 0$ the internal energy. Adopting such notations, the total energy of the system rewrites

$$E = \rho e + \frac{1}{2}\rho u^2.$$

The pressure function obeys the second law of thermodynamics. As a consequence, there exists a specific entropy $s(\tau, e) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies, for some temperature $T(\tau, e) > 0$, the following relation:

$$-T ds = de + p d\tau. \tag{1.2}$$

It results the following equalities:

$$\partial_\tau s(\tau, e) = -\frac{p(\tau, e)}{T(\tau, e)} < 0 \quad \text{and} \quad \partial_e s(\tau, e) = -\frac{1}{T(\tau, e)} < 0. \tag{1.3}$$

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In this work, we impose the application $(\tau, e) \mapsto s(\tau, e)$ to be strictly convex.

To shorten the notations, the system (1.1) can be rewritten in the following condensed form:

$$\partial_t w + \partial_x f(w) = s(w), \tag{1.4}$$

where we have set

$$w = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}, \quad s(w) = \begin{pmatrix} 0 \\ -\rho \partial_x \Phi \\ -\rho u \partial_x \Phi \end{pmatrix}. \tag{1.5}$$

The system (1.4) is associated with the following phase space:

$$\Omega = \left\{ w \in \mathbb{R}^3; \quad \rho > 0, \quad e > 0 \right\}. \tag{1.6}$$

In addition, we impose a positive acoustic impedance as follows:

$$\tau(p\partial_e p - \partial_\tau p) > 0,$$

in order to enforce the system (1.1) to be hyperbolic.

Now, since the system (1.1) is hyperbolic, discontinuous solutions may occur in finite time. In order to rule out the unphysical solutions, the system must be endowed with entropy inequalities that are stated in the following result.

Lemma 1.1. *The smooth solutions of (1.1) satisfy the additional conservation laws*

$$\partial_t \rho \mathcal{F}(s) + \partial_x \rho \mathcal{F}(s) u = 0, \tag{1.7}$$

for all smooth function $\mathcal{F}$.
Moreover, assume

$$\mathcal{F}'(s) > 0 \quad \text{and} \quad \frac{1}{c_p}\mathcal{F}'(s) + \mathcal{F}''(s) > 0, \tag{1.8}$$

where c_p is the specific heat at constant pressure, defined by

$$c_p = -T \left(\frac{\partial s}{\partial T} \right)_p,$$

then the application $w \mapsto \rho\mathcal{F}(s)$ is strictly convex. As a consequence, the pair $(\rho\mathcal{F}(s), \rho\mathcal{F}(s)u)$ defines a Lax entropy pair for system (1.1). Hence, the weak solutions of (1.1) satisfy in addition:

$$\partial_t \rho \mathcal{F}(s) + \partial_x \rho \mathcal{F}(s) u \leq 0. \tag{1.9}$$

Proof. First, let us consider smooth solutions of (1.1). From the mass equation (1.1a), we get

$$\partial_t \tau + u \partial_x u - u \partial_x \tau = 0, \tag{1.10}$$

and from both momentum equation (1.1b) and energy equation (1.1c), we get

$$\partial_t e + u \partial_x e + p\tau \partial_x u = 0. \tag{1.11}$$

Next, multiplying (1.10) by $-\frac{p}{T}$ and (1.11) by $-\frac{1}{T}$ and using the relations (1.3), we obtain

$$\partial_\tau s \partial_t \tau + u\partial_\tau s \partial_x \tau - \tau \partial_\tau s \partial_x u = 0, \tag{1.12}$$

$$\partial_e s \partial_t e + u\partial_e s \partial_x e + \tau \partial_\tau s \partial_x u = 0. \tag{1.13}$$

The sum of (1.12) and (1.13) easily gives

$$\partial_t s + u\partial_x s = 0.$$

The result is then achieved by multiplying this relation by $\rho\mathcal{F}'(s)$ and combining with the mass equation.

The establishment of the Lax entropy pair comes from a straightforward study of the Hessian matrix associated to the function $w \mapsto \rho\mathcal{F}(s)$ (for instance, see [23, 31, 21, 26] and references therein). □

Let us emphasize that the additional conservation laws (1.7) are satisfied by the smooth solutions of the system (1.1) with source term. However, the gravity source term does not participate to the entropy laws associated with (1.1).

Nevertheless, the source term drastically modifies the steady states. In the present work, we focus our attention on the steady states at rest, *i.e.* with $u \equiv 0$. These steady solutions of particular interest are characterized as follows:

$$\begin{cases} u = 0, \\ \partial_x p = -\rho \partial_x \Phi. \end{cases} \tag{1.14}$$

This PDE system can be easily solved as soon as the pressure law only depends on the density. For instance the reader is referred to works devoted to the well-known shallow-water system [3, 9, 24, 35] and some related model [14, 17, 18, 37] (see also [28] for isentropic steady states associated with (1.1)). Unfortunately, since the pressure function p depends on both τ and e, we cannot exhibit algebraic relations satisfied by the solutions of (1.14), except under restrictive assumptions.

However, among the whole set of solutions of (1.14), a family is of prime importance, especially for astrophysics applications. It is known as the polytropic equilibrium, defined by

$$u(x) = 0, \qquad p(x) = K\rho(x)^\Gamma, \tag{1.15}$$

for $K > 0$ and $\Gamma \in (0, +\infty]$. Let us underline that here, Γ represents the polytropic coefficient, which is in general different from the adiabatic coefficient γ coming from an ideal gas law. Equation (1.14), together with the additional condition (1.15), can be solved explicitly. However, the formulation is different according to the value of Γ:

- For $\Gamma = 1$, we obtain the isothermal equilibrium, which is defined by

$$\begin{cases} u(x) = 0, \\ \rho(x) = \exp\left(\dfrac{C - \Phi(x)}{K}\right), \\ p(x) = K \exp\left(\dfrac{C - \Phi(x)}{K}\right), \end{cases} \tag{1.16}$$

  for a given constant $C \in \mathbb{R}$.

- The case $\Gamma = +\infty$ represents the incompressible equilibrium:

$$\begin{cases} u(x) = 0, \\ \rho(x) = \text{constant}, \\ p(x) + \rho(x)\Phi(x) = \text{constant}. \end{cases} \tag{1.17}$$

- Finally, for $\Gamma \in (0,1) \cup (1,+\infty)$, we get the following expression:

$$\begin{cases} u(x) = 0, \\ \rho(x) = \left(\dfrac{\Gamma - 1}{\Gamma K}(C - \Phi(x))\right)^{\frac{1}{\Gamma-1}}, \\ p(x) = K^{\frac{1}{1-\Gamma}} \left(\dfrac{\Gamma - 1}{\Gamma}(C - \Phi(x))\right)^{\frac{\Gamma}{\Gamma-1}}, \end{cases} \tag{1.18}$$

  for a given constant $C \in \mathbb{R}$.

Some authors also consider the specific case of the isentropic equilibrium. We underline that in the case of an ideal case law, with adiabatic coefficient γ, the isentropic equilibrium coincides with the polytropic equilibrium (1.15), with a polytropic coefficient $\Gamma = \gamma$.

The objective of this paper is to derive a finite volume method to approximate the weak solutions of (1.1), which is able to capture exactly the polytropic equilibria (1.16), (1.17) and (1.18). In addition, we also require that the numerical scheme approximate accurately all the solutions of (1.14). Moreover, the derived scheme must be entropy preserving.

During the last two decades, numerous techniques were proposed in the literature to derive well-balanced schemes. Most of them concerned the shallow-water equations or related models. For a non-exhaustive list, the reader is referred, for instance, to [3, 9, 24, 35, 14, 17, 18, 37, 28, 36, 41, 15, 8, 7, 38] and references therein.

However, adopting the Euler equations with gravity, the derivation of well-balanced schemes turns out to be more delicate since the steady states are not explicitly known and just given by the PDE system (1.14). Nevertheless, we can mention the pioneering work by Cargo and LeRoux [13]. By introducing cleverly an evolution equation for the hydrostatic potential, they are able to reformulate the system in a much simpler way. From this equivalent reformulation, they derive a simple and relevant well-balanced scheme. Unfortunately, this reformulation technique only works for a constant gravity field, namely $\Phi(x) = gx$ with $g > 0$ a given constant. Moreover, it is quite difficult to extend this scheme to two or three dimensions.

The Cargo-LeRoux's technique was recently revisited in the work by Chalons *et al* [15]. These authors proposed a suitable well-balanced relaxation scheme and they establish that the resulting numerical method is entropy preserving.

Another technique, based on a local hydrostatic reconstruction, was also developed by Käppeli and Mishra [28]. Although this method extends very easily to second-order and to two space dimensions, it only preserves the *isentropic* steady states. In our work, we would like to preserve exactly a much wider family of steady states.

Following the ideas introduced in the companion paper [22] devoted to the Ripa model, we propose to design a Godunov-type scheme based on a relevant approximate Riemann solver. The key point in the derivation of the approximate Riemann solver stays in a correct insertion of the source term in order to preserve the steady states defined by (1.14) (see [28, 36, 42]).

To obtain an approximate Riemann solver consistent with the gravity source term, we consider a Suliciu-type relaxation strategy. The resulting numerical scheme turns out to be positive preserving, entropy satisfying and well-balanced with the exact capture of the polytropic, isothermal and incompressible equilibria.

The paper is organized as follows. The next section is devoted to the derivation of relaxation models (see [27, 20, 15, 2, 4, 5, 9, 12, 11, 10, 34, 19]) in order to get the required approximate Riemann solver. Here we derive a Suliciu type model by enforcing a *transport property* to be satisfied by the source term. Such a model is governed by a system, which is *more linear* than the usual Suliciu model [20, 4, 9]. The main advantage of this new relaxation model stays for an easy algebra. However, the Riemann problem for this relaxation system is under-determined and a closure relation is missing. Actually, this failure turns out to be beneficial, since the missing relation can be relevantly defined to enforce the well-balanced property in a sense to be prescribed. As a consequence, we obtain an approximate Riemann solver which contains the source term. Next, we show that the closure equation used to ensure that the well-balanced property is satisfied, can be reformulated as relaxation equations. In other terms, we are able to exhibit a full relaxation model which admits a unique Riemann solution and which leads to the same approximate Riemann solver. We conclude section 2 by proposing an extension of the work of Cargo and LeRoux [13]. We show that the relaxation model introduced by Chalons *et al.* [15] can be extended to take into account general gravitational potentials. This extended model, once again, gives the same approximate Riemann solver.

Section 3 is devoted to prove technical results to establish that the derived approximate Riemann solver is consistent with the entropy inequalities (1.7) in the sense of Harten, Lax and van Leer [25].

In Section 4, we present the Godunov-type scheme associated with the derived approximate solver. In addition, we prove the main properties satisfied by the obtained scheme. As expected, we establish that the scheme preserves the set of admissible states. Moreover, we show that the scheme is entropy preserving. Concerning the well-balanced property, we prove that the scheme is able to capture *exactly* the polytropic (1.18), isothermal (1.16) and incompressible (1.17) equilibria, as well as a suitable approximation of the solutions of (1.14)

Finally, Section 5 is devoted to illustrate the good behavior of the scheme and several numerical experiments are performed.

2. Relaxation models. In order to derive a Godunov-type scheme, the present section is devoted to derive an approximate Riemann solver. To address such an issue, we here adopt a Suliciu-type relaxation strategy [4, 9]. More precisely, we approximate the weak solutions of the initial system (1.1) by the weak solutions of a first-order system endowed with a relaxation source term, namely the relaxation model. In a limit of a relaxation parameter, the relaxation model must restore, in a sense to be prescribed, the initial system. In the companion paper 22, that studies the shallow water equatipons with horizontal temperature gradients, also called RIPA model, we have developed a well-balanced relaxation solver. The analysis in the remainderof this chapter has some similarity in 22. Therefore we will give a short version of the computations. For more details check analogues in 22. We now design three distinct relaxation strategies to derive the required approximate Riemann solver.

2.1. An incomplete Suliciu model. According to the Suliciu relaxation approach (see [4, 9, 1, 16]), we suggest to approximate the pressure p by a new variable

π governed by the following evolution law:

$$\partial_t \pi + u\partial_x \pi + \frac{a^2}{\rho}\partial_x u = \frac{1}{\varepsilon}\left(p(\tau,e) - \pi\right).$$

The relaxation parameter $a > 0$ will be fixed later in order to satisfy some robustness and stability conditions.

Now we decide to approximate the gravity in order to plug the source term within the approximate Riemann solver. Since we do not consider self gravitational effects, the potential satisfies the following relation:

$$\partial_t \Phi = 0.$$

After [24, 9, 30], by introducing this stationary relation into the relaxation model directly leads to the approximate Riemann solver. Unfortunately, this stationary equation involves some strong nonlinearities and thus the resolution of the associated Riemann problem turns out to be very difficult. Following the ideas introduced in [22], we avoid such a failure, through approximating the gravity by a new variable Z governed by a transport relaxation equation as follows:

$$\partial_t Z + u\partial_x Z = \frac{1}{\varepsilon}\left(\Phi - Z\right).$$

This choice makes the algebra associated with the relaxation model much easier, but as we shall see, it will result in a missing Riemann invariant to solve the associated Riemann problem. Actually, it adds a degree of freedom that we will take advantage from in order to preserve the steady states of interest.

We are able to give the full relaxation model we now adopt to approximate the weak solutions of (1.1):

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + \pi) = -\rho\partial_x Z, \\ \partial_t E + \partial_x (E+\pi)u = -\rho u \partial_x Z, \\ \partial_t \rho\pi + \partial_x(\rho\pi + a^2)u = \dfrac{\rho}{\varepsilon}(p(\tau,e) - \pi), \\ \partial_t \rho Z + \partial_x \rho Z u = \dfrac{\rho}{\varepsilon}(\Phi - Z). \end{cases} \tag{2.1}$$

Let us notice that, as soon as the parameter ε goes to zero, we formally recover the system (1.1) from the first three equations of the above system (2.1).

In order to simplify the notations, we introduce

$$W = (\rho, \rho u, E, \rho\pi, \rho Z)^T$$

to designate the state vectors in the phase space

$$\mathcal{O} = \left\{W \in \mathbb{R}^5, \quad \rho > 0, \quad e > 0\right\}.$$

From each state $w \in \Omega$, defined by (1.5) and a given gravity function Φ, we define an equilibrium state for the relaxation model as follows:

$$W^{eq}(w) = (\rho, \rho u, E, \rho p(\tau,e), \rho\Phi)^T. \tag{2.2}$$

We no give the algebra of the homogeneous system extracted from (2.1), which we denote with unambiguous notations by $(2.1)_{\varepsilon=\infty}$. The following statement is given without proof (see for instance [23, 31, 32]). The system exhibits the same properties as the relaxation system tackeled in [22].

LEMMA 2.1. *Let $a>0$ be given. The homogeneous system extracted from (2.1) is hyperbolic for all $W\in\mathcal{O}$. The eigenvalues of the system are $\lambda^{\pm}=u\pm\frac{a}{\rho}$ and $\lambda^u=u$. The eigenvalue $\lambda^u=u$ is of multiplicity three. All the fields are linearly degenerate.*

We no want to compute the solution to the Riemann problem associated to $(2.1)_{\varepsilon=\infty}$. Thus we consider an initial data made of two constant states separated by a discontinuity located at $x=0$:

$$W_0(x)=\begin{cases} W_L & \text{if } x<0,\\ W_R & \text{if } x>0.\end{cases} \tag{2.3}$$

According to the algebra described bellow, if the solution of the Riemann problem (2.3) exists, then it is made of four constant states separated by three contact discontinuities. Therefore, we seek solutions in the form

$$W_{\mathcal{R}}\left(\frac{x}{t};W_L,W_R\right)=\begin{cases} W_L & \text{if } x/t<\lambda^-,\\ W_L^\star & \text{if } \lambda^-<x/t<\lambda^u,\\ W_R^\star & \text{if } \lambda^u<x/t<\lambda^+,\\ W_R & \text{if } \lambda^+<x/t.\end{cases} \tag{2.4}$$

Since all the eigenvalues are linear degenerate, one can solve the Riemann problem by considering the Riemann invariants across each wave. Similar to [22] the contact discontinuity assiciatecd with the eigenvlue $\lambda=u$ admits only one invariant instead of two.

For the characteristic fields defined by λ^-,λ^u and λ^+, we easily obtain the following Riemann invariants:

$$I_1^{\pm}=u\pm\frac{a}{\rho},\qquad I_2^{\pm}=\pi\mp au,\qquad I_3^{\pm}=e-\frac{\pi^2}{2a^2},\qquad I_4^{\pm}=Z,\qquad I_1^u=u. \tag{2.5}$$

The resulting Riemann problem (2.3) is therefore under-determined, since the 10 unknowns $W_L^\star,W_R^\star$ are connected only by 9 equations defined by the Riemann Invariants (2.5). To cure this defect, we suggest, similar to the work in [22], to impose an additional relation:

$$\pi_R^\star-\pi_L^\star=-\overline{\rho}(W_L,W_R)(Z_R^\star-Z_L^\star), \tag{2.6}$$

where the function $\overline{\rho}:\mathcal{O}\times\mathcal{O}\to\mathbb{R}^+$ denotes a ρ-average function. The supplementary relation (2.6) has been fixed according to the definition of the steady states at rest given by (1.14). Indeed, since both relaxation unknowns π and Z will have the behavior of $p(\tau,e)$ and Φ, we notice that the identity (2.6) can be understood, in a sense to be prescribed, as a discretization of (1.14).

For now, we assume that this function satisfies the following consistency property:

$$\rho_L=\rho_R=\rho\quad\Rightarrow\quad\overline{\rho}(W_L,W_R)=\rho \tag{2.7}$$

and the symmetry property

$$\overline{\rho}(W_R,W_L)=\overline{\rho}(W_L,W_R).$$

A precise definition of $\overline{\rho}$ will be given later.

With the closure equation (2.6), we are now able to give the Riemann solution of $(2.1)_{\varepsilon=\infty}$ –(2.3).

LEMMA 2.2. *The Riemann solution of the system $(2.1)_{\varepsilon=\infty}$ –(2.3) completed by the relation (2.6) is given by (2.4) where the intermediate states $W_L^\star$ and $W_R^\star$ are defined by*

$$Z_L^\star = Z_L, \qquad Z_R^\star = Z_R, \tag{2.8a}$$

$$\begin{aligned} u^\star &= u_L^\star = u_R^\star \\ &= \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(\pi_R - \pi_L) - \frac{1}{2a}\overline{\rho}(W_L, W_R)(Z_R - Z_L), \end{aligned} \tag{2.8b}$$

$$\pi_L^\star = \pi_L + a(u_L - u^\star), \qquad \pi_R^\star = \pi_R + a(u^\star - u_R), \tag{2.8c}$$

$$\frac{1}{\rho_L^\star} = \frac{1}{\rho_L} + \frac{1}{a}(u^\star - u_L), \qquad \frac{1}{\rho_R^\star} = \frac{1}{\rho_R} + \frac{1}{a}(u_R - u^\star), \tag{2.8d}$$

$$e_L^\star = e_L + \frac{1}{2a^2}\left({\pi_L^\star}^2 - {\pi_L}^2\right), \qquad e_R^\star = e_R + \frac{1}{2a^2}\left({\pi_R^\star}^2 - {\pi_R}^2\right). \tag{2.8e}$$

Proof. The calculations are straightforward and left to the reader. □

The main properties of the numerical scheme will be implied by the Riemann solution (2.4)-(2.8) with an initial data given by (2.3) satisfying the equilibrium condition (2.2). We define

$$w^{eq}\left(\frac{x}{t}; w_L, w_R\right) = W_{\mathcal{R}}^{(\rho,\rho u,E)}\left(\frac{x}{t}; W^{eq}(w_L), W^{eq}(w_R)\right) \tag{2.9}$$

as such a specific approximate solution. Therefore w^{eq} is nothing but an approximate Riemann solver for (1.1).

Now we want to analyze the approximate Riemann solution w^{eq} for its properties. Lemma (2.3) concerns the robustness of w^{eq}.

LEMMA 2.3. *Given w_L and w_R in Ω, given by (1.6). If the relaxation parameter a is large enough to ensure the following inequalities:*

$$u_L - \frac{a}{\rho_L} < u^\star < u_R + \frac{a}{\rho_R}, \tag{2.10}$$

$$e_L + \frac{{\pi_L^\star}^2 - {\pi_L}^2}{2a^2} > 0, \qquad e_R + \frac{{\pi_R^\star}^2 - {\pi_R}^2}{2a^2} > 0, \tag{2.11}$$

where $u^\star$ and $\pi_{L,R}^\star$ are defined by (2.8b) and (2.8c), respectively.
Then $w^{eq}(x/t; w_L, w_R)$ belongs to Ω.

Proof. A straightforward application of the results stated in [22] gives the desired results. □

The next Lemma concerns the well-balanced property of the approximate Riemann solver w^{eq}.

LEMMA 2.4. *Let w_L and w_R be given in Ω such that*

$$u_L = u_R = 0, \tag{2.12}$$

$$p_R - p_L + \overline{\rho}(W_L, W_R)(\Phi_R - \Phi_L) = 0. \tag{2.13}$$

Then we get w^{eq} at rest:

$$w^{eq}(x/t; w_L, w_R) = \begin{cases} w_L & \text{if } x/t < 0, \\ w_R & \text{if } x/t > 0. \end{cases} \tag{2.14}$$

Proof. By assuming the left and right states satisfy (2.12) and (2.13) and the solutions to the intermediate states, it suffices to proof that $u^\star = 0$. $u^\star$ reads now

$$u^\star = -\frac{p_R - p_L}{2a} - \frac{1}{2a}\overline{\rho}(\rho_L, \rho_R)(\Phi_R - \Phi_L).$$

The result immediately comes from (2.13). ◻

Lemma 2.4 is free from the definition of the ρ–average $\overline{\rho}$ and therefore quite general. To the authors knwoledge, our's is a scheme that maintains the hydrostatic equilibrium at least to second order for all hydrostatic equilibria. In addition we will maintain the hydrostatic equilibrium to machine accuracy for certain families of equilibria. Hence, in the following Lemma we will specify quadratures for the ρ–average $\overline{\rho}$ to be consistent with the isothermal, incompressible and polytropic hydrostatic equilibria.

Lemma 2.5.

(i) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_{L,R} = \exp\left(\dfrac{C - \Phi_{L,R}}{K}\right), \\ p_{L,R} = K \exp\left(\dfrac{C - \Phi_{L,R}}{K}\right), \end{cases} \tag{2.15}$$

with $K > 0$ and $C \in \mathbb{R}$. Assume the ρ–average $\overline{\rho}$ is defined by

$$\overline{\rho}(W_L, W_R) = \begin{cases} \dfrac{\rho_R - \rho_L}{\ln(\rho_R) - \ln(\rho_L)} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases} \tag{2.16}$$

Then we get w^{eq} at rest given by (2.14).

(ii) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_L = \rho_R, \\ p_L + \rho_L \Phi_L = p_R + \rho_R \Phi_R. \end{cases} \tag{2.17}$$

Then we get w^{eq} at rest given by (2.14).

(iii) Let w_L and w_R be two states in Ω which satisfy

$$\begin{cases} u_L = u_R = 0, \\ \rho_{L,R} = \left(\dfrac{\Gamma - 1}{\Gamma K}(C - \Phi_{L,R})\right)^{\frac{1}{\Gamma - 1}}, \\ p_{L,R} = K^{\frac{1}{1-\Gamma}} \left(\dfrac{\Gamma - 1}{\Gamma}(C - \Phi_{L,R})\right)^{\frac{\Gamma}{\Gamma - 1}}, \end{cases} \tag{2.18}$$

with $\Gamma \in (0,1) \cup (1,+\infty)$, $K > 0$ *and* $C \in \mathbb{R}$. *Assume the* ρ*–average* $\overline{\rho}$ *is defined by*

$$\overline{\rho}(W_L, W_R) = \begin{cases} \dfrac{\Gamma - 1}{\Gamma} \dfrac{\rho_R^\Gamma - \rho_L^\Gamma}{\rho_R^{\Gamma-1} - \rho_L^{\Gamma-1}} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases} \tag{2.19}$$

Then we get w^{eq} *at rest given by (2.14).*

Proof. As in Lemma 2.4, we just need to prove that $u^\star = 0$. Since in all cases we have $u_L = u_R = 0$, the intermediate speed $u^\star$ writes

$$u^\star = -\frac{1}{2a}\left(p_R - p_L + \overline{\rho}(W_L, W_R)(\Phi_R - \Phi_L)\right).$$

In case (i), the hypothesis (2.15) implies

$$\Phi_R - \Phi_L = K(\ln(\rho_R) - \ln(\rho_L)).$$

In addition, we have

$$p_R - p_L = K(\rho_R - \rho_L).$$

Combining the last two relations together with the definition (2.16), we get $u^\star = 0$.

Case (ii) is immediate thanks to the consistency property (2.7) satisfied by $\overline{\rho}$.

Finally, in case (iii), the hypothesis (2.18) leads to

$$\Phi_R - \Phi_L = K\frac{\Gamma}{\Gamma - 1}\left(\rho_R^{\Gamma-1} - \rho_L^{\Gamma-1}\right)$$

and

$$p_R - p_L = K\left(\rho_R^\Gamma - \rho_L^\Gamma\right).$$

We conclude from the two last equations and from the definition (2.19) that $u^\star = 0$, which concludes the proof. □

After showing some basic properties of the approximate Riemann solver w^{eq}, we now want to justify the closure of the system $(2.1)_{\varepsilon=\infty}$ –(2.3).

2.2. A complete Suliciu relaxation model. Following the paper [22], we modify the relaxation model (2.1) by introducing new relaxation variables X^-, X^+ in order to obtain a well-posed relaxation system. Equivallently, we consider the following modification for the relaxation momentum equation:

$$\partial_t \rho u + \partial_x(\rho u^2 + \pi) + \overline{\rho}(X^-, X^+)\partial_x Z = 0,$$

where the dynamics for X^-, X^+ are given by

$$\begin{aligned} \partial_t X^+ + (u + \delta)\partial_x X^+ &= \frac{1}{\varepsilon}(\rho - X^+), \\ \partial_t X^- + (u - \delta)\partial_x X^- &= \frac{1}{\varepsilon}(\rho - X^-), \end{aligned}$$

and $\delta > 0$ is a small parameter.

The new relaxation system then reads:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \pi) + \overline{\rho}(X^-, X^+)\partial_x Z = 0, \\ \partial_t E + \partial_x (E + \pi) u + \overline{\rho}(X^-, X^+) u \partial_x Z = 0, \\ \partial_t \rho\pi + \partial_x (\rho\pi + a^2 u) = \dfrac{\rho}{\varepsilon}(p(\rho, e) - \pi), \\ \partial_t \rho Z + \partial_x \rho Z u = \dfrac{\rho}{\varepsilon}(\Phi - Z), \\ \partial_t \rho X^+ + \partial_x \rho X^+ u + \delta\rho\partial_x X^+ = \dfrac{\rho}{\varepsilon}(\rho - X^+), \\ \partial_t \rho X^- + \partial_x \rho X^- u - \delta\rho\partial_x X^- = \dfrac{\rho}{\varepsilon}(\rho - X^-). \end{cases} \tag{2.20}$$

Under a stability condition for the parameter δ and following the analysis in [22] we find, that the relaxation system (2.20) will give the same results for the solution of the Riemann problem as given in (2.8). But note that here we do not have to impose a closure equation to solve for the intermediate states. In this section we have justified the inclusion of the equation (2.6) in the analysis of the incomplete relaxation model of the previous section.

2.3. A Cargo-LeRoux's formulation [13]. Finally, we propose a reformulation of the approximate Riemann solver (2.9) based on the work by Cargo and LeRoux [13], which has recently been revisited by Chalons *et al.* [15]. By assuming a linear potential Φ,i.e. $\Phi(x) = gx$, their approach consists in introducing a potential function $q(x, t)$ as follows:

$$\partial_x q = g\rho.$$

Therefore it is possible to establish the following law satisfied by q:

$$\partial_t \rho q + \partial_x \rho q u = 0. \tag{2.21}$$

Then, by extension of the work by Cargo and LeRoux [13], the weak solutions of the Euler model with gravity (1.1) are solutions of the following model:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \phi) = 0, \\ \partial_t H + \partial_x (H + \phi) u + = 0, \\ \partial_t \rho q + \partial_x \rho q u = 0. \end{cases}$$

, where $\phi = p + q$ and $H = E - q$ are defined to play the role of a new pressure and Energy respectively and therefore deriving a equaivalent conservative system.

Chalons *et al.* [15] derive a Suliciu relaxation model on a similar reformulation:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) + \partial_x q = 0, \\ \partial_t E + \partial_x (E + p) u + u\partial_x q = 0, \\ \partial_t \rho q + \partial_x \rho q u = 0, \\ \partial_t \rho\pi + \partial_x (\rho\pi + a^2) u = \dfrac{\rho}{\varepsilon}(p(\rho, e) - \pi). \end{cases} \tag{2.22}$$

, where they do not keep the time derivative q_t but deal with the nonconservative term $u\partial_x q$ instead. The interested reader is reffered to Chalons *et al.* [15] for details.

In this work, we consider a general potential Φ and therefore have to consider a relation for the potential q as

$$\partial_x q = \rho \partial_x \Phi.$$

Since $\partial_x \Phi \neq const$, we can not hope for a conservative transport for the potential q as given in (2.21). Thus, we ca not directly apply the same derivationn of the Suliciu-type model like (2.22). We therefore consider a modification of the model (2.22) but for a suitable relaxation of q.

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \phi) = 0, \\ \partial_t H + \partial_x (H + \phi) u + = 0, \\ \partial_t \rho \pi + \partial_x (\rho \pi + a^2) u = \dfrac{\rho}{\varepsilon}(p(\rho, e) - \pi), \\ \partial_t \rho q + \partial_x \rho q u = \dfrac{\rho}{\varepsilon}\left(\displaystyle\int^x \rho \partial_x \Phi dx - q\right). \end{cases} \tag{2.23}$$

If we solve in the spirit of the previous sections for the Riemann solution we get

$$u^\star = \frac{1}{2}(u_L + u_R) - \frac{1}{2a}(\pi_R + q_R - \pi_L - q_L). \tag{2.24}$$

Therefore only the quantity $q_R - q_L$ is used in the Riemann solution. If one approximates the relaxation unknown q with the following formula:

$$q_R - q_L = \overline{\rho}(\rho_L, \rho_R)(\Phi_R - \Phi_L), \tag{2.25}$$

where $\overline{\rho}$ is a ρ–average function once again defined according to (2.6), then the previous introduced approximate Riemann solver is recovered , where q_L and q_R satisfy (2.25).

3. Consistency with the entropy inequalities. This section is dedicated to prove that the approximate Riemann solver w^{eq}, given by (2.9), is consistent with the entropy inequalities (1.7) in the sense of Harten, Lax and van Leer [25]. Indeed, such entropy consistency turns out to be the main ingredient establishing the required discrete entropy inequalities satisfied by the numerical scheme associated to the approximate Riemann solver w^{eq}.

We first state the main result of this section, where w^{eq}, given by (2.9), is shown to be entropy consistent in the sense of Harten, Lax and van Leer.

THEOREM 3.1. *Let w_L and w_R be two states of Ω. We consider a smooth function $\mathcal{F}$ such that the hypotheses (1.8) are satisfied. Let $a > 0$ be a parameter such that the following sub-characteristic Whitham conditions hold:*

$$a^2 > p(\tau_L, e_L)\, \partial_e p(\tau_L, e_L) - \partial_\tau p(\tau_L, e_L), \tag{3.1a}$$
$$a^2 > p(\tau_L^*, e_L^*)\, \partial_e p(\tau_L^*, e_L^*) - \partial_\tau p(\tau_L*, e_L^*), \tag{3.1b}$$
$$a^2 > p(\tau_R^*, e_R^*)\, \partial_e p(\tau_R^*, e_R^*) - \partial_\tau p(\tau_R^*, e_R^*), \tag{3.1c}$$
$$a^2 > p(\tau_R, e_R)\, \partial_e p(\tau_R, e_R) - \partial_\tau p(\tau_R, e_R). \tag{3.1d}$$

Fix $\Delta t > 0$ and $\Delta x > 0$ two constants such that the following (CFL) restriction is satisfied:

$$\frac{\Delta t}{\Delta x} \max\left\{ \left| u_L - \frac{a}{\rho_L} \right|, \left| u_R + \frac{a}{\rho_R} \right| \right\} \leq \frac{1}{2}. \tag{3.2}$$

Moreover, we assume that the pressure law satisfies Assumption 3.3 given later on.

Then the approximate Riemann solver w^{eq}, defined by (2.9), satisfies the inequalities

$$\frac{1}{\Delta x} \int_0^{\Delta x/2} (\rho \mathcal{F}(s)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{\rho_R \mathcal{F}(s_R)}{2} - \frac{\Delta t}{\Delta x} \left(\rho_R \mathcal{F}(s_R) u_R - \{\rho \mathcal{F}(s) u\}_{L,R} \right), \tag{3.3}$$

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{0} (\rho \mathcal{F}(s)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{\rho_L \mathcal{F}(s_L)}{2} - \frac{\Delta t}{\Delta x} \left(\{\rho \mathcal{F}(s) u\}_{L,R} - \rho_L \mathcal{F}(s_L) u_L \right), \tag{3.4}$$

where we have set

$$\{\rho \mathcal{F}(s) u\}_{L,R} = \begin{cases} \rho_L \mathcal{F}(s_L) u_L & \text{if } 0 < u_L - \frac{a}{\rho_L}, \\ \rho_L^\star \mathcal{F}(s_L) u^\star & \text{if } u_L - \frac{a}{\rho_L} < 0 < u^\star, \\ \rho_R^\star \mathcal{F}(s_R) u^\star & \text{if } u^\star < 0 < u_R + \frac{a}{\rho_R}, \\ \rho_R \mathcal{F}(s_R) u_R & \text{if } u_R + \frac{a}{\rho_R} < 0. \end{cases} \tag{3.5}$$

Let us notice that the usual Harten, Lax and van Leer entropy consistency [25] reads:

$$\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} (\rho \mathcal{F}(s)) \left(w^{eq} \left(\frac{x}{\Delta t}; w_L, w_R \right) \right) dx \leq \frac{1}{2} \left(\rho_L \mathcal{F}(s_L) + \rho_R \mathcal{F}(s_R) \right) - \frac{\Delta t}{\Delta x} \left(\rho_R \mathcal{F}(s_R) u_R - \rho_L \mathcal{F}(s_L) u_L \right).$$

Of course this last inequality directly comes from the sum of (3.3) and (3.4). In fact, the formulation (3.3) and (3.4) will be more convenient to derive the expected discrete entropy inequalities.

In order to establish Theorem 3.1, we now give four successive technical lemmas devoted to establish and analyze a suitable relaxation entropy function (see [4, 6] to similar arguments).

Here, we adopt the relaxation model (2.1) supplemented by the relation (2.6). Clearly, all the forthcoming developments can be derived by considering one of the models (2.20) or (2.23).

First, we set

$$I(W) := I(\pi, \tau) = \pi + a^2 \tau, \tag{3.6}$$

$$J(W) := J(\pi, e) = e - \frac{\pi^2}{2a^2}. \tag{3.7}$$

In fact, these quantities are strong Riemann invariants of the first-order homogeneous system $(2.1)_{\varepsilon=\infty}$ in the following sense:

LEMMA 3.2. *The weak solutions of the relaxation model* $(2.1)_{\varepsilon=\infty}$ *satisfy*

$$\partial_t \rho \Psi(I,J) + \partial_x \rho \Psi(I,J) u = 0, \tag{3.8}$$

for all smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$.

Proof. First, let us consider a smooth solution W of the system $(2.1)_{\varepsilon=\infty}$. An easy computation gives the following evolution law satisfied by the internal energy $e = E/\rho - u^2/2$:

$$\partial_t e + \pi\tau\partial_x u + u\partial_x e = 0.$$

From the density equation, we deduce the evolution equation for the specific volume given by

$$\partial_t \tau + u\partial_x \tau - \tau\partial_x u = 0.$$

Finally, from the evolution equation of π, we get

$$\partial_t \frac{\pi^2}{2} + a^2\pi\tau\partial_x u + u\partial_x \frac{\pi^2}{2} = 0.$$

From the three above identities, we immediately obtain

$$\partial_t I + u\partial_x I = 0 \qquad \text{and} \qquad \partial_t J + u\partial_x J = 0.$$

As a consequence, for all smooth function $\Psi : \mathbb{R}^2 \to \mathbb{R}$, the following transport equation is then satisfied:

$$\partial_t \Psi(I,J) + u\partial_x \Psi(I,J) = 0,$$

to directly deduce the conservation equation (3.8).

To conclude the proof, we just have to mention that the system $(2.1)_{\varepsilon=\infty}$ is only made of linearly degenerate fields. Then the relation (3.8) is also satisfied by the weak solutions of $(2.1)_{\varepsilon=\infty}$. □

Next, we introduce the set of all the admissible pairs (I,J) given by

$$\begin{aligned}
\mathcal{A} = \Big\{ & (I,J) \in \mathbb{R}^2, \quad \exists \tau > 0, \quad \exists e > 0 \text{ such that:} \\
& I = p(\tau,e) + a^2\tau, && (3.9)\\
& J = e - \frac{p(\tau,e)^2}{2a^2}, && (3.10)\\
& a^2 > p(\tau,e)\,\partial_e p(\tau,e) - \partial_\tau p(\tau,e) \Big\}. && (3.11)
\end{aligned}$$

We underline that the inequality (3.11), known as the sub-characteristic Whitham condition [40], imposes that the sound speed $a\tau$ of the system $(2.1)_{\varepsilon=\infty}$ has to be greater than the sound speed $c = \tau\sqrt{p\partial_e p - \partial_\tau p}$ of the original model (1.1).

Moreover, let us notice that, for all $\tau > 0$ and $e > 0$ satisfying (3.11), the definitions (3.6) and (3.7) of I and J imply that the pair $(I(p(\tau,e),\tau), J(p(\tau,e),e))$ belongs to $\mathcal{A}$.

Next, for all pair (I,J) in $\mathcal{A}$, we introduce the function $f_{I,J}:\mathbb{R}^+\to\mathbb{R}$ defined as follows:

$$f_{I,J}(\tau)=\tau p\left(\tau, J+\frac{(I-a^2\tau)^2}{2a^2}\right)+a^2\tau^2-I\tau. \tag{3.12}$$

At this level, an additional assumption must be satisfied by the pressure law.

ASSUMPTION 3.3. *We assume the pressure law is such that the function $\tau\mapsto f_{I,J}(\tau)$, defined by (3.12), is strictly convex for all pair (I,J) fixed in $\mathcal{A}$.*

Nevertheless, such a pressure restriction is satisfied for most usual pressure laws. For instance, in the case of a perfect gas law, given by

$$p(\tau,e)=(\gamma-1)\frac{e}{\tau},$$

the function $f_{I,J}$ writes

$$f_{I,J}(\tau)=\frac{\gamma+1}{2}a^2\tau^2-\gamma I\tau+(\gamma-1)J+\frac{\gamma-1}{2a^2}I^2.$$

We immediately see that $f_{I,J}$ is a second-order polynomial with a positive highest degree coefficient, and thus it is strictly convex.

Next, when the function $f_{I,J}$ admits roots within $\mathbb{R}^+$, we denote by $\overline{\tau}(I,J)$ the larger one and we define

$$\overline{e}(I,J)=J+\frac{(I-a^2\overline{\tau}(I,J))^2}{2a^2}. \tag{3.13}$$

The following result establishes some important properties satisfied by $\overline{\tau}(I,J)$ and $\overline{e}(I,J)$.

LEMMA 3.4. *Let $a>0$ be given. When they are defined, the functions $\overline{\tau}$ and $\overline{e}$ satisfy for all (I,J) in $\mathcal{A}$:*

$$I=p(\overline{\tau}(I,J),\overline{e}(I,J))+a^2\overline{\tau}(I,J), \tag{3.14}$$

$$J=\overline{e}(I,J)-\frac{p(\overline{\tau}(I,J),\overline{e}(I,J))^2}{2a^2}, \tag{3.15}$$

$$a^2>p\left(\overline{\tau}(I,J),\overline{e}(I,J)\right)\partial_e p\left(\overline{\tau}(I,J),\overline{e}(I,J)\right)-\partial_\tau p\left(\overline{\tau}(I,J),\overline{e}(I,J)\right). \tag{3.16}$$

Moreover, for all $\tau>0$ and $e>0$ which satisfy the Whitham condition (3.11), the reals $\overline{\tau}\left(I(p(\tau,e),\tau)),J(p(\tau,e),e)\right)$ and $\overline{e}\left(I(p(\tau,e),\tau)),J(p(\tau,e),e)\right)$ are well-defined and satisfy

$$\overline{\tau}\left(I(p(\tau,e),\tau),J(p(\tau,e),e)\right)=\tau, \tag{3.17}$$

$$\overline{e}\left(I(p(\tau,e),\tau),J(p(\tau,e),e)\right)=e. \tag{3.18}$$

Proof. By definition of $\overline{\tau}$ as root of $f_{I,J}$, we immediately get the relation (3.14). We then deduce relation (3.15) from (3.14) and the definition of $\overline{e}$ given by (3.13).

To establish inequality (3.16), let us compute the derivative of $f_{I,J}$:

$$\frac{d}{d\tau}f_{I,J}(\tau) = \left(p + \tau\partial_\tau p - (I - a^2\tau)\tau\partial_e p + 2a^2\tau - I\right)\left(\tau, J + \frac{(I-a^2\tau)^2}{2a^2}\right).$$

Using relations (3.14) and (3.15), we get

$$\frac{d}{d\tau}f_{I,J}(\overline{\tau}) = \overline{\tau}\left(a^2 + \partial_\tau p(\overline{\tau},\overline{e}) - p(\overline{\tau},\overline{e})\partial_e p(\overline{\tau},\overline{e})\right).$$

Since $\overline{\tau}$ is assumed to be the largest root of the strictly convex function $f_{I,J}$, then we have $\frac{d}{d\tau}f_{I,J}(\overline{\tau}) > 0$ and we deduce immediately (3.16).

Finally, given $\tau > 0$ and $e > 0$, we easily see that

$$(f_{I,J})\Big|_{\substack{I=I(p(\tau,e),\tau)\\ J=J(p(\tau,e),e)}}(\tau) = 0$$

and

$$\left(\frac{d}{d\tau}f_{I,J}\right)\Big|_{\substack{I=I(p(\tau,e),\tau)\\ J=J(p(\tau,e),e)}}(\tau) = \tau(a^2 + \partial_\tau p(\tau,e) - p(\tau,e)\partial_e p(\tau,e) > 0,$$

since the Whitham condition (3.11) is satisfied. The function $f_{I,J}$ being strictly convex, it cannot have more than one root where the derivative is positive. As a consequence, we have (3.17). The relation (3.18) comes directly from the definition (3.13) of $\overline{e}$. □

Now, we introduce the set

$$\mathcal{E} = \left\{W \in \mathcal{O};\quad (I(W), J(W)) \in \mathcal{A},\quad a^2 > p\,(\tau,e)\,\partial_e p\,(\tau,e) - \partial_\tau p\,(\tau,e)\right\}. \tag{3.19}$$

Let us notice that, according to Lemma 3.4, both quantities $\overline{\tau}(I(W),J(W))$ and $\overline{e}(I(W),J(W))$ are well-defined as soon as W belongs to the set $\mathcal{E}$. As a consequence, for all state $W \in \mathcal{E}$, we set

$$\overline{s}(W) = s(\overline{\tau}(I(W),J(W)), \overline{e}(I(W),J(W))), \tag{3.20}$$

where s is the specific entropy defined according to (1.2). From Lemma 3.4, we can immediately remark that the functions s and $\overline{s}$ coincide as soon as relaxation equilibrium states are assumed; namely $I := I(p(\tau,e),\tau)$ and $J := J(p(\tau,e),e)$. Indeed, under the sub-characteristic Whitham condition (3.11) imposed by definition of $\mathcal{E}$, from (3.17) and (3.18) we get

$$\overline{\tau}\left(W_{|\pi=p(\tau,e)}\right) = \tau \qquad \text{and} \qquad \overline{e}\left(W_{|\pi=p(\tau,e)}\right) = e.$$

As a consequence, we have

$$\overline{s}\left(W_{|\pi=p(\tau,e)}\right) = s\,(\tau,e)\,.$$

For the sake of simplicity in the forthcoming developments, we short the notations as follows:

$$\overline{\tau} = \overline{\tau}(I(W),J(W)), \quad \overline{e} = \overline{e}(I(W),J(W)), \quad \overline{p} = p(\overline{\tau},\overline{e}).$$

We now prove that the function $\overline{s}$ reaches its minimum when the relaxation equilibrium holds.

Lemma 3.5. *For all $W \in \mathcal{E}$, we have*

$$\overline{s}(W) \geq s\left(\tau, e\right). \tag{3.21}$$

Proof. To establish this result, we first evaluate the successive derivative of the function $\pi \mapsto \overline{s}(W)$. To access such an issue, we detail the following sequence of derivation.

First, we derive relation (3.14) with respect to π to get

$$\partial_\pi I = \partial_\pi \overline{\tau} \partial_\tau \overline{p} + \partial_\pi \overline{e} \partial_e \overline{p} + a^2 \partial_\pi \overline{\tau}.$$

But from (3.6), we have $\partial_\pi I = 1$ and thus we deduce

$$\partial_\pi \overline{e} = \frac{1}{\partial_e \overline{p}} \left(1 - (\partial_\tau \overline{p} + a^2) \partial_\pi \overline{\tau}\right). \tag{3.22}$$

Next, we derive relation (3.15) with respect to π to get

$$-\frac{\pi}{a^2} = \left(1 - \frac{\overline{p} \partial_e \overline{p}}{a^2}\right) \partial_\pi \overline{e} - \frac{\overline{p}}{a^2} \partial_\tau \overline{p} \partial_\pi \overline{\tau}.$$

Plugging (3.22) into this relation gives

$$\partial_\pi \overline{\tau} = \frac{(\overline{p} - \pi) \partial_e \overline{p} - a^2}{a^2 \left(\overline{p} \partial_e \overline{p} - \partial_\tau \overline{p} - a^2\right)}. \tag{3.23}$$

We now consider the function $\overline{s}$ defined by (3.20). Deriving $\overline{s}$ with respect to π, we obtain

$$\partial_\pi \overline{s} = \partial_\tau \overline{s} \partial_\pi \overline{\tau} + \partial_e \overline{s} \partial_\pi \overline{e},$$

where we substitute $\partial_\pi \overline{e}$ by the expression (3.22) to get

$$\partial_\pi \overline{s} = \left(\partial_\tau \overline{s} - \frac{\partial_e \overline{s}}{\partial_e \overline{p}} (\partial_\tau \overline{p} + a^2)\right) \partial_\pi \overline{\tau} + \frac{\partial_e \overline{s}}{\partial_e \overline{p}}.$$

Next, we plug the definition of $\partial_\pi \overline{\tau}$, given by (3.23), into this relation to obtain, after a straightforward calculation:

$$\partial_\pi \overline{s} = \frac{(\overline{p} - \pi)(\partial_\tau \overline{s} \partial_e \overline{p} - \partial_\tau \overline{p} \partial_e \overline{s} - a^2 \partial_e \overline{s}) + a^2 (\overline{p} \partial_e \overline{s} - \partial_\tau \overline{s})}{a^2 (\overline{p} \partial_e \overline{p} - \partial_\tau \overline{p} - a^2)}.$$

Since by definition of the specific entropy, given by (1.3), we have

$$\partial_\tau s = p \partial_e s, \tag{3.24}$$

we immediately deduce the expected first-order derivative of $\overline{s}$ with respect to π as follows:

$$\partial_\pi \overline{s} = \frac{\overline{p} - \pi}{a^2} \partial_e \overline{s}. \tag{3.25}$$

Finally, deriving again this expression with respect to π, we obtain

$$\partial_{\pi\pi}\overline{s} = \frac{\partial_e\overline{s}}{a^2}\left(\partial_e\overline{s}\partial_\tau\overline{p} + (\overline{p}-\pi)\partial_{\tau e}\overline{s}\right)\partial_\pi\overline{\tau} + \frac{1}{a^2}\left(\partial_e\overline{s}\partial_e\overline{p} + (\overline{p}-\pi)\partial_{ee}\overline{s}\right)\partial_\pi\overline{e} - \frac{\partial_e\overline{s}}{a^2}.$$

Once again we substitute the expressions (3.22) of $\partial_\pi\overline{e}$ and (3.23) of $\partial_\pi\overline{\tau}$ to get

$$\partial_{\pi\pi}\overline{s} = \frac{1}{a^4(\overline{p}\partial_e\overline{p} - \partial_\tau\overline{p} - a^2)}\left((\overline{p}-\pi)^2\left(\partial_e\overline{p}\partial_{\tau e}\overline{s} - \partial_{ee}\overline{s}\partial_\tau\overline{p} - a^2\partial_{ee}\overline{s}\right)\right.$$
$$\left. + a^2(\overline{p}-\pi)\left(\overline{p}\partial_{ee}\overline{s} - \partial_{\tau e}\overline{s} - \partial_e\overline{p}\partial_e\overline{s}\right) + a^4\partial_e\overline{s}\right). \quad (3.26)$$

Let us notice that deriving (3.24) with respect to e gives

$$\partial_{\tau e}s = p\partial_{ee}s + \partial_e p\partial_e s,$$

to write (3.26) as follows:

$$\partial_{\pi\pi}\overline{s} = \frac{(\overline{p}-\pi)^2}{a^4}\partial_{ee}\overline{s} + \frac{\partial_e\overline{s}}{a^4(\overline{p}\partial_e\overline{p} - \partial_\tau\overline{p} - a^2)}\left((\overline{p}-\pi)\partial_e\overline{p} - a^2\right)^2. \quad (3.27)$$

Since $(\tau, e) \mapsto s(\tau, e)$ is a strictly convex function, we have $\partial_{ee}\overline{s} > 0$. On the other hand, from the inequalities (1.3), we get $\partial_e\overline{s} < 0$. Finally, under the sub-characteristic Whitham condition (3.16), we deduce that $\partial_{\pi\pi}\overline{s} \geq 0$. As a consequence, the function $\pi \mapsto \overline{s}$ is convex. Therefore, the proof will be concluded as soon as we establish $\partial_\pi\overline{s}|_{\pi=p(\tau,e)} = 0$.

Let us notice that we have

$$\begin{aligned}\overline{p}_{|\pi=p(\tau,e)} &= p\left(\overline{\tau}\left(p(\tau,e) + a^2\tau, e - \frac{p(\tau,e)^2}{2a^2}\right), \overline{e}\left(p(\tau,e) + a^2\tau, e - \frac{p(\tau,e)^2}{2a^2}\right)\right),\\ &= p(\tau, e).\end{aligned}$$

After (3.25), we see that $\partial_\pi\overline{s}$ vanishes at the point $\pi = p(\tau, e)$. So we have

$$\overline{s}(W) \geq \overline{s}\left(W_{\pi=p(\tau,e)}\right).$$

Since the state $W_{|\pi=p(\tau,e)}$ coincides to the relaxation equilibrium, necessarily we have $\overline{s}\left(W_{|\pi=p(\tau,e)}\right) = s(\tau, e)$, which concludes the proof. □

Next, we prove that the solution of the Riemann problem (2.3) satisfies a property of decreasing entropy.

Lemma 3.6. *Let W_L and W_R be two states in $\mathcal{O}$ at the relaxation equilibrium: $\pi_L = p(\tau_L, e_L)$ and $\pi_R = p(\tau_R, e_R)$. Assume that the sub-characteristic Whitham conditions (3.1) are satisfied. Let $W_{\mathcal{R}}$, defined by (2.4) and (2.8), be the solution of the Riemann problem (2.3)–(2.6). Then $\overline{s}\left(W_{\mathcal{R}}\left(x/t; W_L, W_R\right)\right)$ is defined for all (x,t) in $\mathbb{R}\times\mathbb{R}^+$ and satisfies*

$$\overline{s}\left(W_{\mathcal{R}}\left(\frac{x}{t}; W_L, W_R\right)\right) \geq s\left(\left(\tau_{\mathcal{R}}, e_{\mathcal{R}}\right)\left(\frac{x}{t}; W_L, W_R\right)_{|\pi=p(\tau,e)}\right). \quad (3.28)$$

Proof. Since the function $W_{\mathcal{R}}$ is made of four constant states W_L, $W_L^\star$, $W_R^\star$ and W_R, according to (2.4), the proof will be achieved as soon as the following inequalities will be established:

$$\overline{s}(W_{L,R}) \geq s\left(\tau_{L,R}, e_{L,R}\right) \qquad \text{and} \qquad \overline{s}(W_{L,R}^\star) \geq s\left(\tau^\star_{L,R|\pi=p^\star_{L,R}}, e^\star_{L,R|\pi=p^\star_{L,R}}\right).$$

After Lemma 3.5, such inequalities are satisfied whenever the involved states $W_{L,R}$ and $W^\star_{L,R}$ stay in $\mathcal{E}$.

With W_L and W_R given at the relaxation equilibrium and the Whitham conditions (3.1a) and (3.1d), the states W_L and W_R immediately belong to the set $\mathcal{E}$. Next, we turn establishing that $W_L^\star$ belongs to $\mathcal{E}$. According to (3.19), we have to prove that $(I(W_L^\star), J(W_L^\star))$ is in $\mathcal{A}$.

According to Lemma 3.2, I and J are strong Riemann invariants, so we have

$$I(W_L^\star) = I(W_L) \qquad \text{and} \qquad J(W_L^\star) = J(W_L).$$

Next, since $\pi_L = p(\tau_L, e_L)$, we get the two following relations:

$$I(W_L^\star) = p(\tau_L, e_L) + a^2\tau_L \qquad \text{and} \qquad J(W_L^\star) = e_L - \frac{p(\tau_L, e_L)^2}{2a^2}.$$

Finally, the Whitham condition (3.1a) enforces $(I(W_L^\star), J(W_L^\star))$ to belong to $\mathcal{A}$. To conclude, the condition (3.1b) then ensures that $W_L^\star$ belongs to $\mathcal{E}$. A similar reasoning leads to $W_R^\star \in \mathcal{E}$. Arguing Lemma 3.5, the proof is completed. □

Equipped with the above entropy minimum principle, given Lemma 3.6, we are able to establish Theorem 3.1

Proof of Theorem 3.1. First, let us consider the weak solutions of the relaxation model (2.1), with an initial condition given by

$$W(x,0) = \begin{cases} W^{eq}(w_L) & \text{if } x < 0, \\ W^{eq}(w_R) & \text{if } x > 0. \end{cases}$$

The function $W \mapsto \overline{s}(W)$, defined by (3.20), only depends of I and J, so Lemma 3.2 ensures that the weak solutions of (2.1) satisfy the additional following conservation law:

$$\partial_t \rho\mathcal{F}(\overline{s}) + \partial_x \rho\mathcal{F}(\overline{s})u = 0.$$

We integrate this equation over $[0, \Delta x/2) \times [0, \Delta t)$ to get

$$\begin{aligned} \int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{s})) \left(W_{\mathcal{R}} \left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R) \right) \right) dx = \int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{s}))(W(x,0)) dx \\ - \Delta t (\rho\mathcal{F}(\overline{s})u) \left(W_{\mathcal{R}} \left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R) \right) \right) \\ + \Delta t (\rho\mathcal{F}(\overline{s})u) \left(W_{\mathcal{R}} \left(0; W^{eq}(w_L), W^{eq}(w_R) \right) \right). \end{aligned} \tag{3.29}$$

Since the state $W^{eq}(w_R)$ is at the relaxation equilibrium, the following sequence of equalities holds for $x \in [0, \Delta x/2)$:

$$(\rho\mathcal{F}(\overline{s}))(W(x,0)) = (\rho\mathcal{F}(\overline{s}))(W^{eq}(w_R)) = (\rho\mathcal{F}(s))(W^{eq}(w_R)) = \rho_R \mathcal{F}(s_R). \tag{3.30}$$

In the other hand, the CFL restriction (3.2) implies for all $x \in [0, \Delta x/2)$:

$$W_{\mathcal{R}} \left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R) \right) = W^{eq}(w_R).$$

As a consequence, the first flux term in (3.29) writes

$$(\rho\mathcal{F}(\overline{s})u) \left(W_{\mathcal{R}} \left(\frac{\Delta x}{2\Delta t}; W^{eq}(w_L), W^{eq}(w_R) \right) \right) = \rho_R \mathcal{F}(s_R) u_R. \tag{3.31}$$

Now, since $\overline{s}$ only depends of I and J and these two variables are continuous through the waves of speed $u_L - a/\rho_L$ and $u_R + a/\rho_R$, we have

$$\overline{s}\left(W_{\mathcal{R}}\left(0; W^{eq}(w_L), W^{eq}(w_R)\right)\right) = \begin{cases} \overline{s}(W^{eq}(w_L)) = s(w_L) & \text{if } u^\star > 0, \\ \overline{s}(W^{eq}(w_R)) = s(w_R) & \text{if } u^\star < 0. \end{cases}$$

We immediately deduce that the second flux term in (3.29) writes

$$(\rho\mathcal{F}(\overline{s})u)\left(W_{\mathcal{R}}\left(0; W^{eq}(w_L), W^{eq}(w_R)\right)\right) = \{\rho\mathcal{F}(s)u\}_{L,R}, \tag{3.32}$$

where $\{\rho\mathcal{F}(s)u\}_{L,R}$ is defined by (3.5).

We plug (3.30), (3.31) and (3.32) into (3.29), to obtain

$$\begin{aligned}\frac{1}{\Delta x}\int_0^{\Delta x/2} (\rho\mathcal{F}(\overline{s}))\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) dx = \frac{\rho_R\mathcal{F}(s_R)}{2} \\ - \frac{\Delta t}{\Delta x}\left(\rho_R\mathcal{F}(s_R)u_R - \{\rho\mathcal{F}u\}_{L,R}\right).\end{aligned} \tag{3.33}$$

Finally, Lemma 3.5 ensures that

$$\overline{s}\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) \geq s\left((\tau^{eq}, e^{eq})\left(\frac{x}{\Delta t}; w_L, w_R\right)\right).$$

Moreover, from (1.8), the function $\mathcal{F}$ is increasing, and thus we get

$$\mathcal{F}(\overline{s})\left(W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W^{eq}(w_L), W^{eq}(w_R)\right)\right) \geq \mathcal{F}(s)\left(w^{eq}\left(\frac{x}{\Delta t}; w_L, w_R\right)\right),$$

to obtain the expected inequality (3.3).

The inequality (3.4) is proven in a very similar way by adopting an integration over $(-\Delta x/2, 0] \times [0, \Delta t)$. □

4. The relaxation scheme. Equipped with the approximate Riemann solver $w^{eq}(x/t; w_L, w_R)$, defined by (2.9), we now derive a finite volume scheme to discretize the Euler equation with gravity (1.1). First, we introduce some usual mesh notations. Concerning the space discretization, we adopt a uniform mesh with cells $(x_i - \Delta x/2, x_i + \Delta x/2)$ where $\Delta x >$ denotes the constant size of the mesh. The time discretization is given by $t^{n+1} = t^n + \Delta t$ where $\Delta t > 0$ is the time step restricted according the following CFL like condition:

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(\left|u_i^n - \frac{a_{i+1/2}}{\rho_i^n}\right|, \left|u_{i+1}^n + \frac{a_{i+1/2}}{\rho_{i+1}^n}\right|\right) \leq \frac{1}{2}, \tag{4.1}$$

where $a_{i+1/2}$ will be detailed later on.

From an approximation of the solution at time t^n, given by

$$w^n(x, t^n) = w_i^n, \qquad x \in (x_{i-1/2}, x_{i+1/2}),$$

we define the updated state at time t^{n+1} as follows:

$$w_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w^n(x, t^n + \Delta t) dx. \tag{4.2}$$

Here, the function $w^n(x,t^n+t)$ is nothing but the sequence of the approximate Riemann solver (2.9) stated at each interface $x_{i+1/2}$:

$$w^n(x,t^n+t) = w^{eq}\left(\frac{x-x_{i+1/2}}{t}; w_i^n, w_{i+1}^n\right), \qquad x \in (x_i, x_{i+1}) \quad t \in (0,\Delta t). \quad (4.3)$$

In fact, for all t in $(0,\Delta t)$, the successive approximate Riemann solvers, involved to define $w^n(x,t^n+t)$, do not interact as long as the parameter $a_{i+1/2}$, introduced in (4.1), coincides with a local (defined interface per interface) relaxation parameter. As a consequence, $a_{i+1/2}$ is asked to satisfy at each interface both Ω-preserving conditions (2.10)-(2.11) and sub-characteristic Whitham conditions (3.1).

After a straightforward computation (for instance, see [32, 25, 39, 29]), the updated state w_i^{n+1} reads in the following more convenient form:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right) + \frac{\Delta t}{2}\left(S_{i-1/2} + S_{i+1/2}\right), \quad (4.4)$$

where the numerical source term reads:

$$S_{i+1/2} = \left(0, -\overline{\rho}(\rho_i^n,\rho_{i+1}^n)\frac{\Phi_{i+1}-\Phi_i}{\Delta x}, -\overline{\rho}(\rho_i^n,\rho_{i+1}^n)u^\star_{i+1/2}\frac{\Phi_{i+1}-\Phi_i}{\Delta x}\right)^T. \quad (4.5)$$

The detailed form of the numerical flux function $f_{i+1/2} := f_\Delta(w_i^n,\Phi_i,w_{i+1}^n,\Phi_{i+1})$ is given by

$$f_\Delta(w_L,\Phi_L,w_R,\Phi_R) =$$
$$\begin{cases} \left(\rho_L u_L, \rho_L u_L^2 + p_L + s_{LR}, (E_L+p_L)u_L + u^\star s_{LR}\right)^T & \text{if } u_L - \dfrac{a}{\rho_L} > 0, \\ \left(\rho_L^\star u^\star, \rho_L^\star (u^\star)^2 + \pi_L^\star + s_{LR}, (E_L^\star+\pi_L^\star)u^\star + u^\star s_{LR}\right)^T & \text{if } u_L - \dfrac{a}{\rho_L} < 0 < u^\star, \\ \left(\rho_R^\star u^\star, \rho_R^\star (u^\star)^2 + \pi_R^\star - s_{LR}, (E_R^\star+\pi_R^\star)u^\star - u^\star s_{LR}\right)^T & \text{if } u^\star < 0 < u_R + \dfrac{a}{\rho_R}, \\ \left(\rho_R u_R, \rho_R u_R^2 + p_R - s_{LR}, (E_R+p_R)u_R - u^\star s_{LR}\right)^T & \text{if } u_R + \dfrac{a}{\rho_R} < 0, \end{cases} \quad (4.6)$$

where we have introduced

$$s_{LR} = -\overline{\rho}(\rho_L,\rho_R)(\Phi_R - \Phi_L) \quad (4.7)$$

and the intermediate states are given by (2.8).

To conclude this presentation of the relaxation scheme (4.4)-(4.5)-(4.6)-(4.7), we now give the main its main properties. Indeed, in the following result we summarize the robustness, the stability and the well-balancedness of the derived numerical method.

Theorem 4.1. *For all i in $\mathbb{Z}$, assume that the local relaxation parameter $a_{i+1/2}$ satisfies, at each interface $x_{i+1/2}$, the order conditions (2.10)-(2.11) and the sub-characteristic Whitham condition (3.1). Assume the pressure law satisfies Assumption 3.3. Assume w_i^n belongs to Ω for all $i \in \mathbb{Z}$. Then, under the CFL condition (4.1), the updated state w_i^{n+1}, defined by the relaxation scheme (4.4), satisfies the following properties:*

1. *Robustness: For all i in $\mathbb{Z}$, w_i^{n+1} belongs to Ω.*

2. *Entropy preserving: For all smooth function $\mathcal{F}$ such that (1.8) is verified, w_i^{n+1} satisfies the following discrete entropy inequality:*

$$\rho_i^{n+1}\mathcal{F}(s_i^{n+1}) \le \rho_i^n\mathcal{F}(s_i^n) - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(s)u\}_{i+1/2}^n - \{\rho\mathcal{F}(s)u\}_{i+1/2}^n\right), \quad (4.8)$$

where the entropy numerical flux are defined by

$$\{\rho\mathcal{F}(s)u\}_{i+1/2}^n = f_{i+1/2}^\rho \times \begin{cases}\mathcal{F}(s_i^n) & \text{if } F_{i+1/2}^\rho > 0,\\ \mathcal{F}(s_{i+1}^n) & \text{if } F_{i+1/2}^\rho < 0.\end{cases} \quad (4.9)$$

3. *General steady state preserving: Let us consider an initial data w_i^0 given by*

$$\frac{1}{\Delta x}(p_{i+1}^0 - p_i^0) + \overline{\rho}(\rho_i^0, \rho_{i+1}^0)\frac{\Phi_{i+1}-\Phi_i}{\Delta x} = 0. \quad (4.10)$$

Then the updated state w_i^{n+1} stays at rest, and thus satisfies $w_i^{n+1} = w_i^n$ for all $i \in \mathbb{Z}$

Proof.

First, let us establish the robustness of the scheme. Since we have imposed (2.10) and (2.11), Lemma 2.3 can be applied locally to each function $w^{eq}(x/t; w_i^n, w_{i+1}^n)$, which thus stays in Ω. The CFL restriction (4.1) ensures that the Riemann problems do not interact, so the function $x \mapsto w^n(x, t^n + \Delta t)$ is valued in Ω. Then the update formula (4.2) and the convexity of Ω imply that w_i^{n+1} belongs to Ω.

Next, we prove the stability property satisfied by the scheme. Since the CFL restriction (4.1) and the Whitham conditions (3.1) are satisfied, we can apply Theorem 3.1. Inequality (3.3), where we have set $w_L = w_{i-1}^n$ and $w_R = w_i^n$, reads

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_i}(\rho\mathcal{F}(s))\left(w^{eq}\left(\frac{x - x_{i-1/2}}{\Delta t}; w_{i-1}^n, w_i^n\right)\right)dx \\ \le \frac{\rho_i^n\mathcal{F}(s_i^n)}{2} - \frac{\Delta t}{\Delta x}\left(\rho_i^n\mathcal{F}(s_i^n)u_i^n - \{\rho\mathcal{F}(s)u\}_{i-1/2}^n\right)dx, \quad (4.11)$$

whereas inequality (3.4), where we have set $w_L = w_i^n$ and $w_R = w_{i+1}^n$, reads

$$\frac{1}{\Delta x}\int_{x_i}^{x_{i+1/2}}(\rho\mathcal{F}(s))\left(w^{eq}\left(\frac{x - x_{i+1/2}}{\Delta t}; w_i^n, w_{i+1}^n\right)\right) \\ \le \frac{\rho_i^n\mathcal{F}(s_i^n)}{2} - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(s)u\}_{i+1/2}^n - \rho_i^n\mathcal{F}(s_i^n)u_i^n\right). \quad (4.12)$$

Summing (4.11) and (4.12), we obtain

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}(\rho\mathcal{F}(s))\left(w^n(x, t^n + \Delta t)\right)dx \le \rho_i^n\mathcal{F}(s_i^n) \\ - \frac{\Delta t}{\Delta x}\left(\{\rho\mathcal{F}(s)u\}_{i+1/2}^n - \{\rho\mathcal{F}(s)u\}_{i-1/2}^n\right). \quad (4.13)$$

Now, according to assumption (1.8), the function $w \mapsto \rho\mathcal{F}(s)$ is strictly convex, so we can apply the Jensen inequality to get

$$\rho\mathcal{F}(s)\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w^n(x, t^n + \Delta t)dx\right) \le \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}}(\rho\mathcal{F}(s))\left(w^n(x, t^n + \Delta t)\right)dx. \quad (4.14)$$

Using the definition (4.2) of the numerical scheme together with inequalities (4.13) and (4.14), we obtain the required discrete entropy inequality (4.8).

About the establishment of the well-balanced property, the proof directly comes from Lemma 2.4. Indeed, at each interface, the initial data satisfies (2.12)-(2.13) and Lemma 2.4 can be applied. As a consequence, at each interface the approximate Riemann solver stays at rest. Since the updated state w_i^1, at time $t = \Delta t$, is defined by (4.2), we immediately deduce $w_i^1 = w_i^0$ for all i in $\mathbb{Z}$. Arguing an induction procedure, the proof of the steady state preserving property is then completed. ▯

Concerning the steady state preserving property, we notice that no additional definition is imposed to the ρ-average. We just impose an initial data given by a specific approximation of the partial differential equation (1.14), given by (4.10). Then, this initial data is exactly preserved. Concerning the capture of the polytropic steady states, the ρ-average must be specified as presented next.

THEOREM 4.2. *Assume one of the following conditions occurs:*

1. *Isothermal equilibrium: The average function $\overline{\rho}$ is defined by (2.16) and the initial data w_i^0 satisfies*

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = \exp\left(\dfrac{C - \Phi_i}{K}\right), \\ p_i^0 = K \exp\left(\dfrac{C - \Phi_i}{K}\right), \end{cases} \qquad \text{with } K > 0 \quad \text{and} \quad C \in \mathbb{R};$$

2. *Incompressible equilibrium: The initial data w_i^0 satisfies*

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = M, \\ p_i^0 + \rho_i^0 \Phi_i = C, \end{cases} \qquad \text{with } M > 0 \quad \text{and} \quad C \in \mathbb{R};$$

3. *General polytropic equilibrium: The average function $\overline{\rho}$ is defined by (2.19) and the initial data w_i^0 satisfies*

$$\begin{cases} u_i^0 = 0, \\ \rho_i^0 = \left(\dfrac{\Gamma - 1}{\Gamma K}(C - \Phi_i)\right)^{\frac{1}{\Gamma - 1}}, \\ p_i^0 = K^{\frac{1}{1-\Gamma}} \left(\dfrac{\Gamma - 1}{\Gamma}(C - \Phi_i)\right)^{\frac{\Gamma}{\Gamma - 1}}, \end{cases} \qquad \begin{array}{l} \text{with } \Gamma \in (0,1) \cup (1, +\infty), \\ K > 0 \quad \text{and} \quad C \in \mathbb{R}. \end{array}$$

Then the updated state w_i^{n+1} stays at rest, and thus satisfies $w_i^{n+1} = w_i^n$ for all $i \in \mathbb{Z}$.

Proof. Since the initial data w_i^0 satisfies one of the conditions 1, 2 or 3, we can apply Lemma 2.5, so each approximate Riemann solver stays at rest. As a consequence, each approximate Riemann solver is given by (2.14). With the updated state w_i^1 defined by (4.2), we get immediately $w_i^1 = w_i^0$.

The proof is then easily achieved by an induction procedure. ▯

5. Numerical results. This section is devoted to illustrate the good behavior of the numerical scheme (4.4)–(4.5)–(4.6). In all the applications, the pressure will be given by an ideal gas law

$$p = (\gamma - 1)\rho e,$$

N	Density	Velocity
100	1.01E-16	1.41E-16
200	2.24E-16	1.42E-16
400	3.61E-16	1.38E-16
800	5.39E-16	1.71E-16
1600	9.28E-16	2.49E-16
3200	1.60E-15	2.50E-16

TABLE 5.1
L^1 error in density and velocity for the isothermal atmosphere

where the adiabatic coefficient is set to $\gamma = 1.4$.

5.1. Isothermal atmosphere. The aim of this experiment is to illustrate the exact preservation of the isothermal equilibrium. To underline that this property does not depend on the gravitational potential, we consider a gravity source term given by

$$\Phi(x) = x^2$$

on the computational domain $[0, 1]$. The initial condition is fixed to the isothermal equilibrium

$$(\rho_0, u_0, p_0)(x) = \left(e^{-x^2}, \quad 0, \quad e^{-x^2}\right).$$

As an isothermal equilibrium is considered, we choose the definition (2.16) for $\overline{\rho}$. At final time $T = 0.25$, we compute the L^1 error between the approximated solution and the exact solution. The results are given on Table 5.1 in density and velocity for an increasing number of cells N.

The results show that the isothermal atmosphere is preserved up to machine precision. This is coherent with Theorem 4.2.

5.2. General steady state. Next, we investigate the behavior of the scheme on a general steady state, i.e. a steady state which does not belong to the polytropic family described by (1.15). We consider the computational domain $[0, 1]$ with periodic boundary conditions. The gravitational potential is here defined by

$$\Phi(x) = -\sin(2\pi x).$$

One can easily check that the initial condition

$$(\rho_0, u_0, p_0)(x) = (3 + 2\sin(2\pi x), \quad 0, \quad 3 + 3\sin(2\pi x) - 0.5\cos(4\pi x)) \tag{5.1}$$

defined a steady state solution.

Here, we choose the definition (2.16) for the ρ–average $\overline{\rho}$. We compute the L^1 error at time $T = 1$ for an increasing number of cells N. The results, as well as the convergence orders, are displayed in Table 5.2 for the density and the velocity. We observe that a second-order is achieved, although the scheme is only first-order. This can be formally explained by the fact that relation (4.10) is satisfied up to second-order. Indeed, a straightforward computation leads to

$$p_0(x + \Delta x) - p_0(x) + \overline{\rho}(\rho_0(x), \rho_0(x + \Delta x))(\Phi(x + \Delta x) - \Phi(x)) = O(\Delta x^2).$$

N	Density		Velocity	
100	4.46E-05		2.03E-05	
200	7.11E-06	2.65	5.29E-06	1.94
400	1.23E-06	2.53	1.34E-06	1.98
800	2.35E-07	2.39	3.37E-07	1.99
1600	5.02E-08	2.23	8.44E-08	2.00
3200	1.15E-08	2.13	2.11E-08	2.00

TABLE 5.2
L^1 error in density and velocity for general steady state

FIG. 5.1. *Perturbation in pressure (left) and in velocity (right)*

However, we notice that for this specific case, defining $\overline{\rho}$ by

$$\overline{\rho}(W_L, W_R) = \frac{\rho_L + \rho_R}{2}$$

leads to the exact satisfaction of relation (4.10). Thus with this choice, the steady state (5.1) would be exactly preserved.

5.3. Perturbation of an isothermal atmosphere. The following test case has been introduced in [33, 42]. Here we consider a classical constant gravitational field described by the potential $\Phi(x) = x$.The computational domain $[0, 1]$ is initially filled with a gas staying in an isothermal equilibrium

$$\rho_0(x) = p_0(x) = e^{-x}, \quad u_0(x) = 0.$$

At time $t = 0$, the pressure is perturbed by

$$p(x, t = 0) = p_0(x) + 0.01e^{-100(x-0.5)^2}.$$

For the average $\overline{\rho}$, we choose here the definition (2.16).

We compute the solution at time $T = 0.25$ using 100 cells. The results are compared with a reference solution, which is obtained using 30.000 cells. On Figure 5.1, we display the results in pressure perturbation (i.e. $p(x) - p_0(x)$) and in velocity perturbation.

The initial perturbation splits into two waves moving in opposite directions. We can see that the scheme does not create spurious oscillations near the bumps. In additions, the size of the bumps decrease with time. This shows the stability of the numerical scheme with respect to this steady state. Finally, comparison with the reference solution shows that the general shape is well captured by the numerical scheme, although it is only a first order scheme

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