Hydrostatic Upwind Schemes for Shallow–Water Equations

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Abstract We consider the numerical approximation of the shallow–water equations with non–flat topography. We introduce a new topography discretization that makes all schemes to be well–balanced and robust. At the discrepancy with the well–known hydrostatic reconstruction, the proposed numerical procedure does not involve any cut–off. Moreover, the obtained scheme is able to deal with dry areas. Several numerical benchmarks are performed to assert the interest of the method.

Key words: shallow–water equations, finite volume schemes, source term approximations, well–balanced schemes, positive preserving schemes
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1 Introduction

The present work concerns the derivation of finite volume methods to approximate the solutions of the shallow–water equations when involving non–flat topography. The model under interest is given as follows:

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) = -gh\partial_x z, \end{cases} \tag{1}$$

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where $h > 0$ is the local water–depth and $u \in \mathbb{R}$ is the depth–average velocity. Here, $z : \mathbb{R} \to \mathbb{R}^+$ denotes the topography while $g > 0$ is the gravitational constant.

To shorten the notations, let us set

$$w = \begin{pmatrix} h \\ hu \end{pmatrix}, \qquad f(w) = \begin{pmatrix} hu \\ hu^2 + g\dfrac{h^2}{2} \end{pmatrix},$$

respectively the state vector $\mathbb{R} \times \mathbb{R}^+ \to \Omega$ and the flux function $\Omega \to \mathbb{R}^2$ where Ω stands for the convex space of admissible states:

$$\Omega = \{w \in \mathbb{R}^2;\ h > 0,\ hu \in \mathbb{R}\}.$$

Let us recall that the steady state of the lake at rest, defined by $u = 0$ and $h + z = \text{cste}$, is of primary importance and it must be preserved by the numerical schemes.

The objective is now to derive schemes which preserve positive the water depth and exactly capture the lake at rest. Several approaches have been recently introduced in the literature to approximate the solutions of (1). Most of them propose to consider the discretization of the associated homogeneous system:

$$\partial_t w + \partial_x f(w) = 0, \tag{2}$$

to suggest a relevant approximation of the topography source term able to restore the expected lake at rest. For instance, we refer to [10] where a suitable correction of the VFRoe scheme is proposed (see also [11, 12, 4]). Other approaches consider systematic corrections to enforce the required *well–balanced* property; i.e. to enforce the exact capture of the lake at rest. The hydrostatic reconstruction [1] (for instance, see also [18, 5]) is certainly one of the most celebrate *well–balanced* technique. However, to be water–depth positive preserving, this approach introduces a cut–off procedure which may involve some failures. Indeed, after the work by Delestre [9], let us consider a small water–depth on a topography with constant slope. Next, let us increase the slope to reduce the water–depth. Considering coarse grids, the hydrostatic reconstruction introduces a wrong behavior since the water–depth increases (see figure 4).

In this paper, we derive a new strategy to systematically enforce the well–balanced property. In fact, the proposed scheme is able to deal with vanishing water height without any additional correction and it is proved to be robust. Arguing [2, 15, 16, 17], we suggest a relevant upwind form of the topography source term but by involving the free surface.

The paper is organized as follows. In the next section, we introduce a new formulation of (1) and we present the suggested scheme. Section 3 is devoted to some essential properties as consistency and robustness. The last section presents numerical experiments to illustrate the interest of the method. Specifically, the adopted numerical scheme is shown to capture the correct behavior for large topography slopes.

2 Upwind source term discretization

In order to derive our scheme, we need considering a reformulation of system (1) by introducing the free surface $H = h + z$ and a fraction of water $X = \frac{h}{H}$. As usual, one can chose arbitrarily the topography origin (see [6, 13] and references therein), and thus we impose a bottom reference so that $H > 0$. As a consequence, X is well–defined. Involving such notations, the weak solutions of (1) satisfy the following system:

$$\begin{cases} \partial_t h + \partial_x XHu = 0, \\ \partial_t hu + \partial_x \left(X \left(Hu^2 + g\dfrac{H^2}{2} \right) - \dfrac{g}{2} hz \right) + gh\partial_x z = 0. \end{cases} \tag{3}$$

Let us remark the following identity:

$$\frac{1}{2}\partial_x(hz) - h\partial_x z = \frac{H^2}{2}\partial_x X,$$

to simplify the discharge equation. Then the smooth solutions of (1) also satisfy:

$$\begin{cases} \partial_t h + \partial_x XHu = 0, \\ \partial_t hu + \partial_x \left(X \left(Hu^2 + g\dfrac{H^2}{2} \right) \right) - g\dfrac{H^2}{2}\partial_x X = 0. \end{cases} \tag{4}$$

For the sake of simplicity in the notations, let us set

$$W = (H, Hu)^T,$$

to rewrite (4) as follows:

$$\partial_t w + \partial_x X f(W) - \begin{pmatrix} 0 \\ g\frac{H^2}{2}\partial_x X \end{pmatrix} = 0.$$

Now, we propose a discretization of (4), but let us note from now on that the discrete form we will obtain for (4) will be consistent with (3). As a consequence, the suggested discretization will be relevant to approximate the weak solutions of (3) or equivalently (1). We suggest to modify any scheme associated with the homogeneous system (2). Hence, let us consider $f^{\Delta x} : \Omega \times \Omega \to \mathbb{R}^2$ a consistent numerical flux function, i.e. $f^{\Delta x}(w, w) = f(w)$. We restrict ourselves to the regular meshes of size Δx such that $\Delta x = x_{i+1/2} - x_{i-1/2}$ for all $i \in \mathbb{Z}$, and we denote the time step by Δt, with $t^{n+1} = t^n + \Delta t$ for all $n \in \mathbb{N}$. The proposed scheme thus reads:

$$\begin{aligned} w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(X^n_{i+1/2} f^{\Delta x}(W_i^n, W_{i+1}^n) - X^n_{i-1/2} f^{\Delta x}(W_{i-1}^n, W_i^n) \right) \\ + \begin{pmatrix} 0 \\ \frac{\Delta t}{\Delta x}\frac{g}{2} H^n_{i+1/2} H^n_{i-1/2} (X^n_{i+1/2} - X^n_{i-1/2}) \end{pmatrix}, \end{aligned} \tag{5}$$

where we have set

$$H^n_{i+1/2} = \begin{cases} H^n_i, & if\ f_h^{\Delta x}(W_i^n, W_{i+1}^n) > 0, \\ H^n_{i+1}, & otherwise, \end{cases} \tag{6}$$

$$X^n_{i+1/2} = \begin{cases} X^n_i, & if\ f_h^{\Delta x}(W_i^n, W_{i+1}^n) > 0, \\ X^n_{i+1}, & otherwise, \end{cases} \tag{7}$$

where $f_h^{\Delta x}$ denotes the first component of the numerical flux function. Here, we have set $W_i^n = X_i^n h_i^n$ where $X_i^n = h_i^n/(h_i^n + Z_i)$.

Before we establish the main properties of the scheme, let us emphasize that the suggested numerical procedure is an extremely easy way to consider the topography from a relevant discretization of the homogeneous shallow–water equation (2). The reader is referred to [15] where similar ideas were introduced.

3 Main properties

First of all, let us remark that the scheme (5) is obviously consistent with (4). Since (4) turns out to be a non conservative formulation of (3), or equivalently (1), after the work by Dal Maso, LeFLoch and Murat [8] (see also [14, 3, 7]), this may suggest that our approach cannot deal with weak solutions of (1). As a consequence, we first prove the consistency of (5) with (1).

Lemma 1. *Let $(w_i^{n+1})_{i\in\mathbb{Z}}$ be given by (5)-(6)-(7). Then $(w_i^{n+1})_{i\in\mathbb{Z}}$ satisfies in addition:*

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x}\left(X^n_{i+1/2} f_h^{\Delta x}(W_i^n, W_{i+1}^n) - X^n_{i-1/2} f_h^{\Delta x}(W_{i-1}^n, W_i^n)\right), \tag{8}$$

$$\begin{aligned}(hu)_i^{n+1} = {} & (hu)_i^n - \frac{\Delta t}{\Delta x}\left(X^n_{i+1/2} f_{hu}^{\Delta x}(W_i^n, W_{i+1}^n) - X^n_{i-1/2} f_{hu}^{\Delta x}(W_{i-1}^n, W_i^n)\right) \\ & + \frac{\Delta t}{\Delta x}\frac{g}{2}\left(h^n_{i+1/2} z^n_{i+1/2} - h^n_{i-1/2} z^n_{i-1/2}\right) \\ & - \frac{\Delta t}{\Delta x}\frac{g}{2}(h^n_{i+1/2} + h^n_{i-1/2})(z^n_{i+1/2} + z^n_{i-1/2}),\end{aligned} \tag{9}$$

where $h^n_{i+1/2} = H^n_{i-1/2} X^n_{i-1/2}$ and $z^n_{i+1/2} = H^n_{i-1/2}(1 - X^n_{i-1/2})$. As a consequence, the scheme (5) is consistent with (3) and thus with (1).

We skip the proof of this result since it is just a reformulation of (5).

At this level, the adopted scheme turns out to be relevant when approximating the weak solutions of (1). Moreover, let us note that the above scheme derivation does not need some smoothness argument concerning the topography function z. Now, let us state our main result.

Theorem 1. *Let w_i^n belongs to Ω for all i in $\mathbb{Z}$. Under a suitable CFL like restriction, let us assume that the numerical flux function $f^{\Delta x}$ is Ω–preserving as follows:*

$$w_i^n - \frac{\Delta t}{\Delta x}\left(f^{\Delta x}(w_i^n, w_{i+1}^n) - f^{\Delta x}(w_{i-1}^n, w_i^n)\right) \in \Omega.$$

1. *Assume an additional CFL restriction given by*

$$\frac{\Delta t}{\Delta x}\left(\max(0, f_h^{\Delta x}(W_i^n, W_{i+1}^n) - \min(0, f_h^{\Delta x}(W_{i-1}^n, W_i^n)\right) < H_i^n,$$

   then w_i^{n+1} given by (5) stays in Ω for all i in $\mathbb{Z}$.
2. *The scheme (5) is well–balanced. Assume $u_i^n = 0$ and $h_i^n + z_i = H$ a positive constant, then $u_i^{n+1} = 0$ and $h_i^{n+1} + z_i = H$ for all i in $\mathbb{Z}$.*

We here do not prove the above result but we just establish the preservation of the lake at rest. Let us assume $u_i^n = 0$ and $h_i^n + z_i = H > 0$ for all $i \in \mathbb{Z}$. As a consequence, we have $W_i^n = (H,0)^T$ for all $i \in \mathbb{Z}$. Arguing the consistency of the numerical flux function $f^{\Delta x}$, we have

$$f^{\Delta x}(W_i^n, W_{i+1}^n) = f(W) = \begin{pmatrix} 0 \\ g\dfrac{H^2}{2} \end{pmatrix}, \qquad \forall i \in \mathbb{Z}.$$

Concerning the water heigh, we immediately deduce $h_i^{n+1} = h_i^n$ for all $i \in \mathbb{Z}$ to obtain $h_i^{n+1} + z_i = h_i^n + z_i = H$. Now, let us rewrite the discretization of the discharge:

$$\begin{aligned}(hu)_i^{n+1} =& (hu)_i^n - \frac{\Delta t}{\Delta x}\left(X_{i+1/2}^n g\frac{H^2}{2} - X_{i-1/2}^n g\frac{H^2}{2}\right) \\ &+ \frac{\Delta t}{\Delta x}\frac{g}{2} H_{i+1/2}^n H_{i-1/2}^n (X_{i+1/2}^n - X_{i-1/2}).\end{aligned}$$

Since $H_{i+1/2}^n = H$ for all $i \in \mathbb{Z}$, we get $(hu)_i^{n+1} = 0$. The scheme is thus well–balanced.

4 Numerical results

The numerical experiments are performed on a grid made of 500 cells. Here, the CFL number is systematically fixed to 0.5. To validate the derived numerical scheme, we first propose to consider transcritical flow with shock over a bump. Figure 1 shows a comparison between the classic hydrostatic reconstruction and our scheme. From now on, let us underline that the scheme formulation (5) depends on the choice of the bottom origin. Hence, the comparison is performed by involving three distinct origins. Since the consistency property is not modified, the approximated solutions stay similar. In Figure 2, we give the approximation obtained when considering a numerical flux function of HLLC type, VFRoe type and Lax–Friedrichs type. In Figure 3 we present a comparison between first and second order schemes. Here, a MUSCL second order extension has been performed. The second

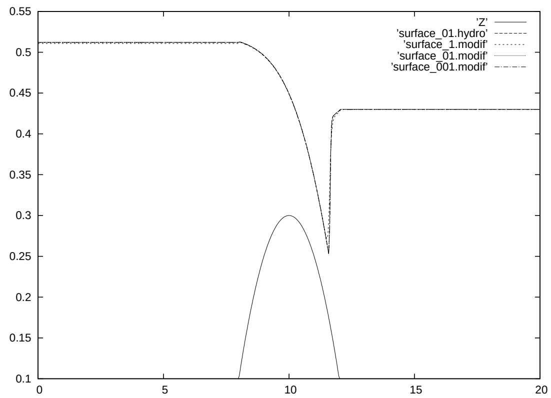

Fig. 1 Comparison hydrostatic reconsturction / hydrostatic upwind involving an arbitrary topography origin so that $\min(z) = 1$, 0.1 and 0.001.

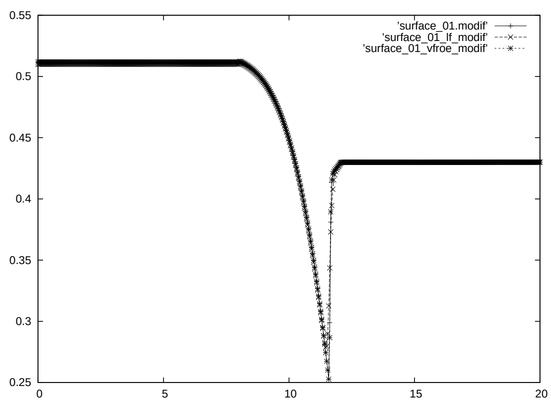

Fig. 2 Comparison between HLLC, VFRoe and Lax–Friedrichs flux functions.

test concerns the known hydrostatic reconstruction failure. Here, the topography is made of a constant slope. As the slope increases, the expected water height must decrease. Because of the cut–off involved into the hydrostatic reconstruction, a wrong water behavior is noted. The same simulation obtained with the hydrostatic upwind scheme gives the required water behavior.

The last experiment, presented in Figure 5, concerns a drain on a non flat bottom in order to simulate dry areas. Once again we obtain an excellent behavior of the derived numerical scheme. As a perspective of the derived 1D technique, we must now propose an extension for 2D unstructured meshes. To address such an issue, arguing the rotational invariance of the model, we should write the 2D formulation assoicated with (4) in the x–direction to obtain the required flux approximation per edge. The interest of such an extension should be an easy topography discretization to obtain 2D well–balnced schemes.

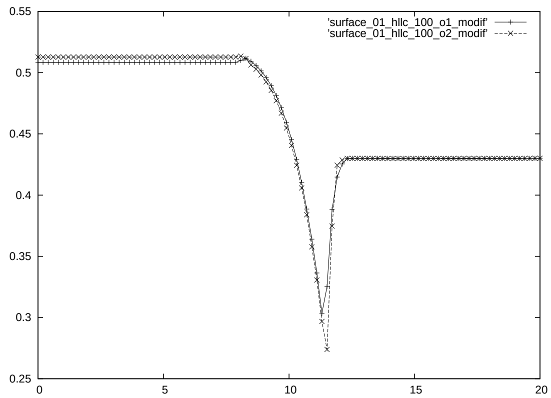

Fig. 3 Comparison first and second order.

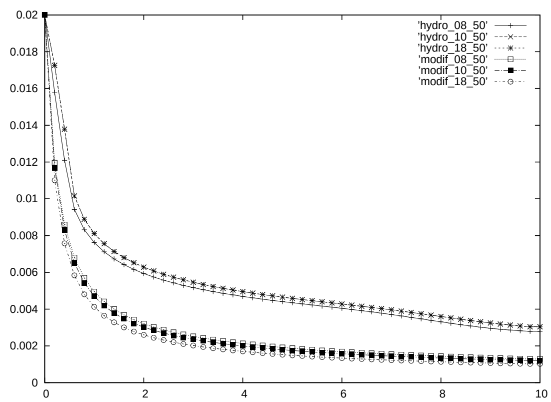

Fig. 4 Failure of the hydrostatic reconstruction. Wrong behavior of h for 8%, 10% and 18% topography slopes. Because of the cut–off the water heigt increase. Morover, with 10% and 18% slope, h exactly coincides and thus the obtained approximation is non–physical.

References

1. Audusse E., Bouchut F., Bristeau M.O., Klein R., Perthame B.: A fast and stable well–balanced scheme with hydrostatic reconstruction for shallow water flows, SIAM J.Sci.Comp., **25**, 2050–2065 (2004).
2. Bermudez A., Vazquez–Cendon M.E., Upwind Methods for Hyperbolic Conservation Laws with Source Terms, Computers and Fluids. **23** 1049–1071 (1994).
3. Berthon C., Coquel F.: Nonlinear projection methods for multi–entropies Navier–Stokes systems, Math. Comput., **76**, 1163–1194 (2007).
4. Berthon C., Dubois J., Dubroca B., Nguyen–Bui T.H., Turpault R.: A Free Streaming Contact Preserving Scheme for the M1 Model, Adv. Appl. Math. Mech., **3**, 259–285 (2010).
5. Berthon C., Marche F.: A positive well–balanced VFRoe–ncv scheme for non–homogeneous shallow–water equations, ISCM–EPMESC proceedings, AIP Conference Proceedings, 1495–1500 (2010).

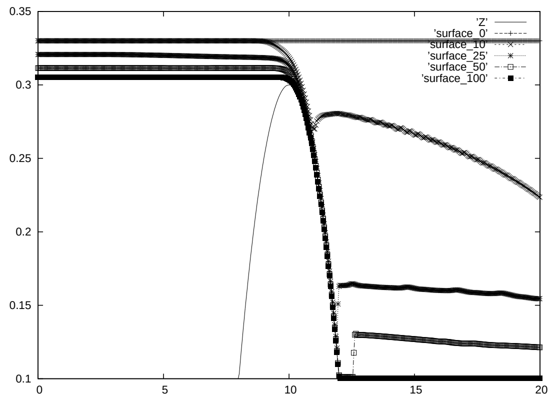

Fig. 5 Drain on a nonflat bottom: Water height at time $t = 0$, 1, 10, 25, 50 and 100.

6. Bouchut F.: Non–linear stability of finite volume methods for hyperbolic conservation laws and well–balanced schemes for sources, Frontiers in Mathematics, Birkhauser, 2004.
7. Castro M., LeFloch P., Munoz–Ruiz M.L., Parés C.: Why many theories of shock waves are necessary: convergence error in formally path–consistent schemes, J. Comput. Phys. **227**, 8107-8129 (2008).
8. Dal Maso G., LeFloch P., Murat F., Definition and weak stability of a non conservative product, J. Math. Pures Appl., **74**, 483–548 (1995).
9. Delestre O., Simulation du ruissellement d'eau de pluie sur des surfaces agricoles, PhD Thesis, University of Orléans, http://tel.archives-ouvertes.fr/tel-00531377/en (2010).
10. Gallouët T., Hérard J.M., Seguin N.: Some recent Finite Volume schemes to compute Euler equations using real gas EOS, Int J. Num. Meth. Fluids, **39**, 1073–1138 (2002).
11. Gallouet T., Hérard J.M., Seguin N.: Some approximate Godunov schemes to compute shallow–water equations with topography, Computers and Fluids, **32**, 479–513 (2003).
12. Gallouet T., Hérard J.M., Seguin N.: On the use of some symetrizing variables to deal with vacuum, Calcolo, **40**, 163–194 (2003).
13. Greenberg J.M., Leroux A.Y.: A well–balanced scheme for the numerical processing of source terms in hyperbolic equations, SIAM J. Numer. Anal., **33**, 1–16 (1996).
14. Hou T.Y., LeFloch P.: Why non–conservative schemes converge to wrong solutions: error analysis, Math. Comput., **206**, pp. 497–530 (1994).
15. Jin S.: A steady–state capturing method for hyperbolic systems with geometrical source terms, M2AN Math. Model. Numer. Anal., **35**, 631–645 (2001).
16. Jin S., Wen X.: Two interface–type numerical methods for computing hyperbolic systems with geometrical source terms having concentrations, SIAM J. Sci. Comput., **26**, 2079–2101 (2005).
17. Jin S., Wen X.: An efficient method for computing hyperbolic systems with geometrical source terms having concentrations, Special issue dedicated to the 70th birthday of Professor Zhong–Ci Shi. J. Comput. Math., **22**, 230–249 (2004).
18. Marche F., Bonneton P., Fabrie P., Seguin N.: Evaluation of well–balanced bore–capturing schemes for 2D wetting and drying processes, Int. J. Numer. Meth. Fluids, **53**, 867–894 (2007).

The paper is in final form and no similar paper has been or is being submitted elsewhere.