WHY MANY THEORIES OF SHOCK WAVES ARE NECESSARY. KINETIC RELATIONS FOR NONCONSERVATIVE SYSTEMS

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Abstract. We consider a class of nonconservative hyperbolic systems of partial differential equations endowed with a strictly convex mathematical entropy. We formulate a well-posed Riemann problem by supplementing it with a kinetic relation, that is, by prescribing the rate of entropy dissipation across any shock wave. Our condition can be regarded as a generalization to nonconservative systems of a similar concept introduced by Truskinovsky and Abeyaratne-Knowles for subsonic phase transitions and generalized by LeFloch for undercompressive waves to general hyperbolic systems. The proposed kinetic relation for nonconservative systems turns out to be equivalent, for the class of systems under consideration, to Dal Maso, LeFloch, and Murat's definition based on a prescribed family of Lipschitz continuous paths.

In agreement with previous theories, the kinetic relation should be derived from a phase plane analysis of traveling solutions associated with an augmented version of the nonconservative system. We illustrate with several examples that nonconservative systems arising in the applications fit in our framework. For a typical model of turbulent fluid dynamics we provide a detailled analysis of the existence and properties of traveling waves and we derive the corresponding kinetic function. Numerical experiments illustrate the properties of the kinetic relations, which can serve to assess the efficiently of nonconservative schemes.

Contents

1. Introduction

Certain nonlinear hyperbolic models arising in continuum physics and describing complex fluid flows or solid materials do not take the standard form of conservation laws but, instead, the form of nonconservative hyperbolic systems

$$\partial_t u + A(u)\,\partial_x u = 0, \qquad x \in \mathbb{R},\ t \geq 0. \tag{1.1}$$

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Here, $u = u(x,t) \in \Omega$ is an unknown field and takes values in a convex and open domain $\Omega \subset \mathbb{R}^N$, while the matrix-valued field $A = A(u)$ is given and has, for each value u, N real and distinct eigenvalues and a basis of N eigenvectors. It is well-known that nonlinear hyperbolic systems do not admit smooth solutions since propagating discontinuities arise in finite time even from smooth initial data. In recent years, nonconservative systems have been the subject of active research. A rather complete theory based on the pioneering definition by Dal Maso, LeFloch, and Murat [19] is now available which covers the definition of weak solutions [52, 34, 35, 19, 40], the existence of solutions to the Riemann problem [34, 19] the initial value problem [34, 42, 18], the uniqueness of bounded variation solutions [5, 39], and their approximation via finite difference schemes [28, 13, 43]. In addition, many nonconservative models arising in continuum mechanics have been systematically investigated, as such model play an important role in the modeling of multi-phase flows and turbulent fluid dynamics [2, 6, 7, 8, 14, 15].

In the present paper, we consider a class of nonconservative systems of the form (1.1), characterized by the property that a large family of additional entropy functions (conservation laws) exist. In other words, the systems to be considered below *formally* have a conservative form if nonlinear combinations of the given equations are allowed. However, the physical modeling dictate that nonconservative equations should be kept and it is precisely under these conditions that a “kinetic relation”, as we propose in the present paper, should enter into play. The kinetic relation has so far been introduced for systems of conservation laws and to handle nonclassical shocks; see LeFloch [39] for a presentation of this theory.

The concept of a kinetic relation for nonconservative systems discussed now was actually introduced first in 1999 by the authors of the present article in an unpublished note. Later on, this concept was investigated numerically in several papers by Aubert, Berthon, Chalons, and Coquel; see [2, 8, 15], and the control of the numerical dissipation of finite difference schemes was extensively addressed. Our purpose in the present paper is to provide the required theoretical set-up, and demonstrate that the kinetic relation is an efficiently tool to assess the properties of schemes for nonconservative systems.

Recall that the design and the properties of difference schemes suitable for the numerical approximation of nonconservative systems (1.1) is very challenging. The main source of difficulty lies in the fact that shock waves to nonconservative systems are small-scale dependent and the dissipation terms induced by the numerical discretization tend to drive the propagation of the shocks. This phenomena was rigorously analyzed for scalar equations by Hou and LeFloch [28]. On the other hand we emphasize that Glimm scheme and front-tracking algorithms do not contain any numerical dissipation and, actually, have been proven to converge to the correct solutions [34, 42, 39].

We begin with a general discussion of nonconservative hyperbolic systems arising in continuum physics in order to motivate our general approach proposed in the forthcoming section and developed on selected examples in the rest of this paper. The models of interest here naturally stand in a nonconservative form, and this is a direct consequence of simplifying assumptions which are made in the derivation of these models; these assumptions are also necessary if a tractable model is to be found. Such assumptions typically originate in averaging procedures that intend to bypass the description of intricate mechanisms taking place at microscopic scales. The small scale fluctuations that are thought to be of lesser interest induce dissipative and/or relaxation phenomena at the macroscopic level, and can also be accounted for as source-terms.

Most (if not all) nonconservative hyperbolic models arising in the applications admit (several distinct) entropy balance laws which are consistent with the underlying dissipative and relaxation mechanisms. These additional balance equations, as we will show, provide a natural approach to formulating additional generalized jump conditions built from entropy rate productions. Moreover, these entropy functions are sufficient in number to allow for a complete set of jump relations.

The objectives and results in this paper are as follows:

- First of all, as mentioned above, we restrict attention to a class of nonconservative systems (defined in Section 3 below) which encompasses, however, most of the models encountered in the applications. To motivate the proposed class of systems, we observe that, in the applications we have in mind (e.g. multi-phase and multi-fluid models): (1) all but one of the equations (1.1) can be rewritten in a conservative form and, moreover, (2) the system (1.1) is endowed with a mathematical entropy, i.e. a (strictly convex) nonlinear function $U = U(u)$ corresponding to an additional conservation law satisfied by all smooth solutions.
- For such systems, the concept of weak solutions introduced by Dal Maso, LeFloch, and Murat [19] can be simplified. Therein, a family of Lipschitz continuous paths was necessary to uniquely define the nonconservative product $A(u)\,\partial_x u$ associated with the vector-valued field u. In contrast, for our particular class of nonconservative systems, one nonconservative product between scalar-valued functions, only, must be defined. This structure allows us to simply supplement the model (1.1) with an additional algebraic scalar equation which, for each shock wave, determines the *entropy dissipation rate* associated with the entropy U. We call this additional jump condition a *kinetic relation* and the entropy dissipation function a *kinetic function*. In Section 3 below, a precise definition is given. The main result of this section is a proof of the existence of a solution to the Riemann problem for (1.1) which satisfies the prescribed kinetic relation. Our proof is a generalization of an argument given in Dal Maso, LeFloch and Murat [19] in the setting of general families of paths.
- It is remarquable that many models of interest arising in the applications take the form considered in Section 3 below, and this will illustrated in the following Section 2. In Section 4, we focus on a model of particular importance, which arises in turbulent fluid dynamics. Taking into account the dissipation terms induced by the physical modeling, the existence and properties of associated traveling waves are established. In turn, this provides us with the kinetic function needed to apply the general theory in Section 3.

2. Nonconservative systems in fluid dynamics

To show the structure of the nonconservative systems of interest, it is worth to begin with the *shallow water equations with topography*

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho v) = 0, \\ &\partial_t(\rho v) + \partial_x\big(\rho v^2 + g\,\frac{\rho^2}{2}\big) - g\rho\,\partial_x a = 0, \end{aligned} \tag{2.1}$$

where ρ and v are the mass density and the velocity of the fluid, respectively, and the (prescribed) topography function $a : \mathbb{R} \to \mathbb{R}_+$ depends on the spatial variable x and is assumed to be solely piecewise Lipschitz continuous. Here, g is the gravity constant. The product $g\rho\,\partial_x a$ is nothing but a nonconservative product which is not defined in a classical sense at points of discontinuity.

By setting $\mathbf{u} := (\rho, \rho v)$, weak solutions should obey the following entropy inequality

$$\begin{aligned} &\partial_t \mathcal{U}(\mathbf{u}, a) + \partial_x \mathcal{F}(\mathbf{u}, a) \leq 0, \\ &\mathcal{U}(\mathbf{u}, a) := \rho E(v) + \rho a, \qquad e'(\rho) = \frac{p(\rho)}{\rho^2}, \\ &\mathcal{F}(\mathbf{u}, a) := \rho \frac{v^3}{2} + \rho e(\rho)\, v + p(\rho) v + \rho v\, a. \end{aligned} \tag{2.2}$$

Another model with a closely related structure is

$$\begin{aligned} \partial_t(a\rho) + \partial_x(a\rho v) = 0, \\ \partial_t(a\rho v) + \partial_x(a\rho v^2 + a\, p(\rho)) - p(\rho)\, \partial_x a = 0, \end{aligned} \tag{2.3}$$

which describes *one-dimensional nozzle flows* as well as *compressible flows in porous media*. Again, the function $a : \mathbb{R} \to \mathbb{R}$ is solely piecewise Lipschitz continuous, and denotes here the nozzle cross-section or the porosity function, respectively.

By setting $\mathbf{u} := (a\rho, a\rho v)$, weak solutions to (2.3) should obey the entropy inequality

$$\begin{aligned} &\partial_t \mathcal{U}(\mathbf{u}, a) + \partial_x \mathcal{F}(\mathbf{u}, a) \leq 0, \\ &\mathcal{U}(\mathbf{u}, a) = a^2 \rho \frac{v^2}{2} + a\rho\, e(\rho), \\ &\mathcal{F}(\mathbf{u}, a) = \big(\mathcal{U}(\mathbf{u}, a) + p(\rho)\big)\, v. \end{aligned} \tag{2.4}$$

The systems (2.1) and (2.3) and closely related models with source-terms have received considerable attention over the past decade, from, both, analytical and numerical standpoints. We refer the reader to [35] (connection with the theory of nonconservative systems), [21, 22, 3, 13, 10, 20, 30] (approximation by finite difference or finite volume schemes), and [46, 47, 29, 23, 44, 45] (construction of a Riemann solver). In particular, we refer the reader to Bouchut [10] for a comprehensive review and to the references therein.

We observe here that both models (2.1) and (2.3) fall within the following class of nonconservative hyperbolic models with singular source-term

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, a) - \mathbf{g}(\mathbf{u}, a) \partial_x a = 0, \tag{2.5}$$

where a is a given (piecewise Lipschitz continuous) function of the spatial variable x and the unknown map $\mathbf{u}$ takes values in a convex and open domain $\Omega_{\mathbf{u}} \subset \mathbb{R}^N$, while $\mathbf{f} : \Omega_{\mathbf{u}} \times \mathbb{R} \to \mathbb{R}^N$ and $\mathbf{g} : \Omega_{\mathbf{u}} \times \mathbb{R} \to \mathbb{R}^N$ are given smooth mappings.

Motivated by the structure of the above two examples, especially the entropy inequalities (2.2) and (2.4) and in order to develop a general theory we assume that the hyperbolic system (2.5) is endowed with a (sufficiently smooth) entropy function $\mathcal{U} : \Omega_{\mathbf{u}} \times \mathbb{R} \to \mathbb{R}$ and a corresponding entropy flux $\mathcal{F} : \Omega_{\mathbf{u}} \times \mathbb{R} \to \mathbb{R}$, so that solutions to (2.5) satisfy the entropy inequality

$$\partial_t \mathcal{U}(\mathbf{u}, a) + \partial_x \mathcal{F}(\mathbf{u}, a) \leq 0. \tag{2.6}$$

The principal examples of interest arising in the form (2.5) in the applications do admit such an entropy.

The above class is known to include, after the seminal work by Greenberg and Leroux [21], the class of hyperbolic systems with source terms:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{g}(\mathbf{u}), \tag{2.7}$$

which, by introducing the (rather trivial function) $a(x) = x$, indeed take the form (cf. (2.5))

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) - \mathbf{g}(\mathbf{u})\, \partial_x a = 0. \tag{2.8}$$

This nonconservative reformulation is useful for designing "well-balanced schemes", which properly account for the competition between the source term and the differential hyperbolic operator in the large time asymptotic $t \to +\infty$. (See [21, 22, 10].) The (somewhat artificial but useful) system (2.8) admits a conserved entropy in the scalar case $n = 1$, *provided* the source g does not vanish, namely it suffices to define $\mathcal{U}'(u) := 1/g(u)$ and $\mathcal{F}'(u) := f'(u)/g(u)$.

As advocated by LeFloch [35] for the equations for nozzle flows (2.3), the prescribed function a, being independent of the time variable, can be regarded as an independent unknown of the following extended version of (2.5) in the extended variable $u := (\mathbf{u}, a)$

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}, a) - \mathbf{g}(\mathbf{u}, a)\, \partial_x a &= 0, \\ \partial_t a &= 0. \end{aligned} \tag{2.9}$$

Assuming from now on that the matrix $D_{\mathbf{u}}\mathbf{f}(\mathbf{u}, a)$ is diagonalizable for all $\mathbf{u} \in \Omega_{\mathbf{u}}$ and $a \in \mathbb{R}$ with real eigenvalues $\lambda_1(\mathbf{u}, a), \dots, \lambda_n(\mathbf{u}, a)$ and a full set of eigenvectors $r_1(\mathbf{u}, a), \dots, r_n(\mathbf{u}, a)$, it is obvious that (2.9) admits the same eigenvalues plus 0 (with multiplicity 1). Moreover, it admits a full set of eigenvectors *if and only if* $\lambda_j(\mathbf{u}, a) \neq 0$ for all $j = 1, \dots, n$. In general, (2.9) is solely weakly hyperbolic and, due to possible resonance effects, difficulties arise even in tackling the simplest initial value problem, i.e. the Riemann problem; see the pioneering work of Isaacson and Temple [29], as well as Goatin and LeFloch [23] for a general Riemann solver. In the rest of this paper, we tacitly assume that no resonance takes place in solutions under consideration.

While a rigorous definition of nonconservative products will wait until the following section, we can here already provide some preliminary discussion, based on an observation by LeFloch [35] for the nozzle flow equations and on the presentation in Bouchut [10] for more general systems.

With this non-resonance assumption enforced, we then observe from (2.9) that the variable a is a Riemann invariant associated with the eigenvalue $\lambda_{n+1}(\mathbf{u}, a) := 0$. In other words, a stays constant across waves associated with any other (non-vanishing) eigenvalue and, consequently, the nonconservative product $\mathbf{g}(\mathbf{u}, a)\partial_x a$ only needs to be defined for $(n + 1)$-contact discontinuities.

The entropy inequality (2.6) should be satisfied as an equality in the sense of distributions across standing waves, hence

$$\mathcal{F}(\mathbf{u}_+, a_+) - \mathcal{F}(\mathbf{u}_-, a_-) = 0. \tag{2.10}$$

From the physical viewpoint, we can further investigate the validity of (2.10), obtained as a direct consequence of the augmented form (2.9). To that purpose, we specialize (2.10) first to the case of the shallow water equations (2.1)

$$\frac{m^2}{2\rho_+^2} + e(\rho_+) + \frac{p(\rho_+)}{\rho_+} + a_+ = \frac{m^2}{2\rho_-^2} + e(\rho_-) + \frac{p(\rho_-)}{\rho_-} + a_-, \tag{2.11}$$

where $m =: \rho_- v_- = \rho_+ v_+$ denote the mass, and second in the case of the nozzle flow equations (2.3)

$$\frac{m^2}{2a_+^2\rho_+^2} + e(\rho_+) + \frac{p(\rho_+)}{\rho_+} = \frac{m^2}{2a_+^2\rho_-^2} + e(\rho_-) + \frac{p(\rho_-)}{\rho_-}, \tag{2.12}$$

in which $m := a_- \rho_- v_- = a_+ \rho_+ v_+$. The above equations can be implicitly solved in ρ_+ away from resonance (see [10, 23, 44, 45] for details) and stand at the very basis of the design of well-balanced schemes.

Furthermore, in concrete experiments with fluid flows, for instance in nozzles, it is observed that an abrupt change (modeled therefore by a discontinuity) in either the topography function, the duct cross-section, or the porosity function, generally produces fine-scale features in solutions which may enter in competition with complex dissipation phenomena such as friction. To account for such dissipation mechanisms, the entropy law (2.2) or (2.4) cannot be any longer expressed as a conservation law across the standing wave. The associated entropy dissipation rates are the so-called "singular loss of momentum" used by engineers and well-documented in the applied literature. It is necessary, on the ground of physical experiments, to replace (2.12) by the more general condition

$$\begin{aligned}&(a\rho v)_+ - (a\rho v)_- = 0\\ &\mathcal{J} = -(a\rho v)_- \,\kappa(\mathbf{u}_-, a_-),\end{aligned} \tag{2.13}$$

with

$$\mathcal{J} := (a\rho v)_+ \Big(\frac{v_+^2}{2} + e(\rho_+) + \frac{p(\rho_+)}{\rho_+}\Big) - (a\rho v)_- \Big(\frac{v_-^2}{2} + e(\rho_-) + \frac{p(\rho_-)}{\rho_-}\Big),$$

where the prescribed function $\kappa : \Omega_{\mathbf{u}} \times \mathbb{R} \to \mathbb{R}_+$ defines mathematically the singular loss of momentum. The extension relative to (2.11) is completely similar.

We continue this section with a more sophisticated model of compressible flows, describing *multi-fluid mixtures*, where the variable a introduced previously now stands for a fluid mass fraction. Following the celebrated review by Steward and Wendroff [51], (stratified) multi-fluid models may be regarded as two distinct fluids evolving with distinct velocities and distinct thermodynamic properties, each propagating within "nozzles" whose cross sections denoted by $a \in (0,1)$ and $1 - a(x,t)$, respectively depend on the spatial as well as the time variables (see [1] for instance). For notational convenience, a is traditionally denoted by α_1 (void fraction of the fluid 1) and $(1-a)$ by α_2 (void fraction of the fluid 2), with

$$\alpha_1(x,t) + \alpha_2(x,t) = 1.$$

Furthermore, the evolution of a is now described either via an algebraic closure equation (based on the isobaric assumption; see [51] for details) or by considering it as an independent variable governed by a supplementary evolution equation. From the point of view of the present paper and in order to avoid instability issues due to lack of hyperbolicity of the model, we adopt the second strategy, following here Ransom and Hicks [48] and Baer and Nunziato [4]. This approach was extensively investigated in recent years; see Gallouet, Hérard, and Seguin [20], Berthon and Nkonga [9], and the references therein.

In turn, the multi-fluid model under consideration takes the form

$$\begin{aligned}&\partial_t \alpha_1 + V_I(\mathbf{u}) \partial_x \alpha_1 = \lambda(p_2 - p_1),\\ &\partial_t(\alpha_1\rho_1) + \partial_x(\alpha_1\rho_1\, u_1) = 0,\\ &\partial_t(\alpha_1\rho_1\, u_1) + \partial_x(\alpha_1\rho_1\, u_1^2 + \alpha_1\, p_1) - P_I(\mathbf{u})\,\partial_x\alpha_1 = \lambda\,(u_2 - u_1) + \epsilon\,\partial_x(\mu_1\,\partial_x u_1),\\ &\partial_t(\alpha_2\rho_2) + \partial_x(\alpha_2\rho_2\, u_2) = 0,\\ &\partial_t(\alpha_2\rho_2 u_2) + \partial_x(\alpha_2\rho_2 u_2^2 + \alpha_2\, p_2) - P_I(\mathbf{u})\,\partial_x\alpha_2 = -\lambda\,(u_2 - u_1) + \epsilon\,\partial_x(\mu_2\,\partial_x u_2),\end{aligned} \tag{2.14}$$

where $\mathbf{u} := (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \alpha_2\rho 2, \alpha_2\rho_2 u_2)$ is the vector-valued unknown. Here, the barotropic pressure laws $p_i = p_i(\rho_i)$ are assumed to satisfy the monotonicity condition $p_i'(\rho_i) > 0$. The relaxation parameter $\lambda > 0$ may take arbitrarily large values, depending of the multi-fluid flow regime under consideration, while $\epsilon > 0$ (the inverse of a Reynolds number) is usually small. Moreover, the (smooth) functions $V_I : \Omega_{\mathbf{u}} \to \mathbb{R}$ and $P_I : \Omega_{\mathbf{u}} \to \mathbb{R}$ represent the interfacial velocity and interfacial pressure, respectively. Following the original work by Ransom and Hicks

[48], one can set *for instance*

$$V_I(\mathbf{u}) := \frac{1}{2}(u_1 + u_2), \qquad P_I(\mathbf{u}) := \frac{1}{2}(p_1 + p_2). \tag{2.15}$$

It turns out that, independently of the precise form of the constitutive equations, the system (2.14) admits five real eigenvalues, i.e.

$$V_I(\mathbf{u}), \quad u_i \pm c_i(\rho_i),$$

where $c_i^2(\rho_i) := p'(\rho_i) > 0$ (as well as a basis of right eigenvectors) *if and only if*

$$|V_I(\mathbf{u}) - u_i| \neq c_i(\rho_i), \quad i = 1, 2. \tag{2.16}$$

In other words, like the (much simpler) model (2.9), the principal (first-order) part of (2.14) is only weakly hyperbolic if (2.16) is violated. Here again, we tacitly assume that solutions under consideration do not develop resonance phenomena.

One key constraint that arises in choosing the required closure laws for $V_I(\mathbf{u})$ and $P_I(\mathbf{u})$ is the existence of a mathematical entropy pair associated with (2.14). Interestingly, the total energy

$$\mathcal{U} := \alpha_1\rho_1 E_1(\mathbf{u}) + \alpha_2\rho_2 E_2(\mathbf{u})$$

with $E_i(\mathbf{u}) := \frac{u_i^2}{2} + e_i(\rho_i)$ is an entropy for (2.14) if and only if the interfacial closure laws $V_I(\mathbf{u})$ and $P_I(\mathbf{u})$ satisfy the *interfacial compatibility condition*

$$V_I(\mathbf{u})(p_2 - p_1) + P_I(\mathbf{u})(u_2 - u_1) = p_2 u_1 - p_1 u_2 \tag{2.17}$$

for all states under consideration (see [20] and [16] for instance). Indeed, under the assumption (2.17), smooth solutions of (2.14) satisfy the entropy balance law

$$\begin{aligned}
&\partial_t\mathcal{U}(\mathbf{u}) + \partial_x\mathcal{F}(\mathbf{u}) = -\lambda(u_2 - u_1)^2 - \lambda(p_2 - p_1)^2 - \mathcal{D},\\
&\mathcal{U}(\mathbf{u}) := (\alpha_1\rho_1 E_1(\mathbf{u}) + \alpha_2\rho_2 E_2(\mathbf{u}))\\
&\mathcal{F}(\mathbf{u}) := \Big((\alpha_1\rho_1 E_1(\mathbf{u}) + \alpha_1 p_1)u_1 + (\alpha_2\rho_2 E_2(\mathbf{u}) + \alpha_2 p_2)u_2\Big),\\
&\mathcal{D}(\mathbf{u}) := \epsilon\mu_1(\partial_x u_1)^2 + \epsilon\mu_2(\partial_x u_2)^2 - \epsilon\,\partial_x(\mu_1\alpha_1\partial_x u_1 + \mu_2\alpha_2\partial_x u_2).
\end{aligned} \tag{2.18}$$

The dissipation $\mathcal{D}$ formally converges to a non-positive measure when $\epsilon \to 0$ and/or $\lambda \to +\infty$, so that in this limit we do have the entropy inequality

$$\partial_t\mathcal{U}(\mathbf{u}) + \partial_x\mathcal{F}(\mathbf{u}) \leq 0. \tag{2.19}$$

We conclude this section with another setting for complex compressible materials which naturally gives rise to hyperbolic equations with viscous perturbations in non conservation form. The models under consideration may be regarded as natural extensions of the usual Navier-Stokes equations. Such extensions make use of N independent internal energies $(e_i)_{1\leq i\leq N}$ for governing N independent pressure laws $(p_i(\tau, e_i))_{1\leq i\leq N}$. These PDE models take the generic form:

$$\begin{aligned}
&\partial_t\rho + \partial_x(\rho u) = 0,\\
&\partial_t(\rho u) + \partial_x\Big(\rho u^2 + \sum_{i=1}^{N} p_i(\tau, e_i)\Big) = \epsilon\,\partial_x\Big(\sum_{i=1}^{n} \mu_i(\tau, e_i)\partial_x u\Big),\\
&\partial_t(\rho e_i) + \partial_x(\rho u e_i) + p_i(\tau, e_i)\partial_x u = \epsilon\,\mu_i(\tau, e_i)(\partial_x u)^2,
\end{aligned} \tag{2.20}$$

where $\rho > 0$ denotes the density, $u \in \mathbb{R}$ is the velocity, and $\tau = 1/\rho > 0$ is the specific volume. Here, $\epsilon > 0$ stands the inverse of the Reynolds number.

Several models from the Physics actually enter the proposed framework and can be distinguished according to the precise definition of the constitutive closure laws for the pressures and the viscosities. Precise assumptions on the required state laws will be addressed in Section 4 devoted to the analysis of the traveling wave solutions of (2.20).

Models from the plasma physics where the temperature of the electron gases must be distinguished from the temperature of the other heavy species, typically take the form (2.20) with $N = 2$ (see [17] for a presentation). Models from the physics of compressible turbulent flows can also be seen to fall within the frame of PDEs (2.20). We refer the reader to [6, 7, 8, 14, 15] for the mathematical and the numerical analysis of several models ranging from two distinct internal energies, the so-called laminar and turbulent ones, to $N > 2$ different energies to account for a refined description of the turbulent energy cascade. We also emphasize that multi-fluid models, as those studied in [9], enter the proposed framework.

In most if not all the applications to complex compressible materials, the inverse of the Reynolds number ϵ modulating the strenght of the viscous perturbation is a small parameter. Solutions of interest therefore exhibit stiff zones of transitions, namely viscous shock layers and boundary layers. Viscous shocks cannot be properly resolved for mesh refinements of practical interest and we are thus led to study the limit $\epsilon \to 0^+$ in the system (2.20).

There exists several ways to tackle the system (2.20) in the limit $\epsilon \to 0^+$ depending on suitable change of variables. It can be seen that (4.1) can recast equivalently as:

$$\mathcal{A}_0(\mathbf{w}^\epsilon)\partial_t \mathbf{w}^\epsilon + \mathcal{A}_1(\mathbf{w}^\epsilon)\partial_x \mathbf{w}^\epsilon = \epsilon \partial_x(\mathcal{D}(\mathbf{w}^\epsilon)\partial_x \mathbf{w}^\epsilon), \tag{2.21}$$

with $\mathcal{A}_0$ regular, or

$$\partial_t \mathbf{u}^\epsilon + \partial_x \mathcal{F}(\mathbf{u}^\epsilon) = \epsilon\, \mathcal{R}(\mathbf{u}^\epsilon, \partial_x \mathbf{u}^\epsilon, \partial_{xx}\mathbf{u}^\epsilon). \tag{2.22}$$

Namely in (4.16), the diffusive operator writes in conservation form while $\mathcal{A}_0(\mathbf{w})$ and $\mathcal{A}_1(\mathbf{w})$ are not Jacobian matrices of some flux function. By contrast in (2.22), the first order operator stands in conservation form but not the regularization terms.

The precise definitions of the change of unknown $\mathbf{w}$ and $\mathbf{u}$ is addressed in Section 4. We just highlight at this stage that concerning the equivalent form (2.21) and provided that suitable estimates on the sequence of solutions $\mathbf{w}_\epsilon$ hold true, the right-hand side is expected to vanish in the limit $\epsilon \to 0^+$ in the usual sense of the distributions. By contrast the left-hand side in non conservation form may be handled thanks to the theory of family of paths introduced by LeFloch [35], Dal Maso, LeFloch and Murat [19]. As far as the next equivalent form (2.22 is considered, the left-hand side now stands in conservation form and can be treated in the usual sense of the distributions. In opposition, the right-hand side cannot any longer be expected to converge to 0, generally speaking, but merely to a bounded Borel measure concentrated on the shocks of the limit solutions. The next section provides a convenient framework for handling the required passage to the limit in the PDEs (2.22).

3. Kinetic relations for nonconservative systems

Having in mind the examples described in the previous section, we present one of the main contributions of the present paper, i.e. the concept of kinetic relation, which that allows us to rigorously define certain nonconservative products arising in the applications.

Recall that weak solutions to nonconservative systems are defined in the class of functions with bounded variation (BV). By standard regularity theorems, such functions can be handled essentially as if they were piecewise Lipschitz continuous. Throughout the present paper and for simplicity in the presentation, we restrict attention to piecewise Lipschitz continuous functions and refer to [19] for details of the the DLM theory.

For simplicity in the presentation, we restrict attention to solutions defined in a neighborhood of a constant state in $\mathbb{R}^N$ which can be normalized to be the origin. We denote by $\mathcal{B}_{\delta_0}$ the ball centered at the origin and of small radius $\delta_0 > 0$. Dal Maso, LeFloch and Murat's definition is based on prescribing a *family of Lipschitz continuous paths* $\phi = \phi(s; u_0, u_1) \in \mathcal{B}_{\delta_0}$ ($s \in [0,1]$), which allows one to connect any two points u_0, u_1 in $\mathcal{B}_{\delta_1}$ for some $\delta_1 \le \delta_0$. In particular, it is assumed that

$$\phi(0; u_0, u_1) = u_0, \qquad \phi(1; u_0, u_1) = u_1. \tag{3.1}$$

(See [19, 40] for the precise conditions, omitted in this short review.) As proposed in LeFloch [35], this family of paths should be determined from traveling wave solutions of an augmented model. Indeed, it has been recognized that weak solutions of (1.1) depend on the effect of small scales that have been neglected at the hyperbolic level of modeling, but are taken into account in the augmented version

$$\partial_t u + A(u)\, \partial_x u = R(u, \epsilon\, u_x, \epsilon^2 u_{xx}, \cdots), \tag{3.2}$$

where $R^\epsilon = 0$ if $\epsilon = 0$. The family of paths determined by traveling wave trajectories precisely yields the "missing information" required to set-up the hyperbolic theory.

A (piecewise Lipschitz continuous) function $u = u(x,t)$ is called a *weak solution* of the nonconservative system (1.1) if u satisfies the equations (1.1) in a classical sense in the regions where it is Lipschitz continuous and, additionally, the following generalization of the Rankine-Hugoniot jump relation holds along every curve of discontinuity of u. Precisely, for any shock wave connecting two states u_0, u_1 at the speed $\overline{\Lambda} = \overline{\Lambda}(u_0, u_1)$,

$$u(x,t) = \begin{cases} u_0, & x < \overline{\Lambda}\, t, \\ u_1, & x > \overline{\Lambda}\, t, \end{cases} \tag{3.3}$$

we impose the *generalized jump relation*

$$-\overline{\Lambda}\,(u_1 - u_0) + \int_0^1 A(\phi(s; u_0, u_1))\, \partial_s \phi(s; u_0, u_1)\, ds = 0. \tag{3.4}$$

Note that, in the conservative case when $A(u) = Df(u)$ for some flux-function f, this relation reduces to

$$-\overline{\Lambda}\,(u_1 - u_0) + f(u_1) - f(u_0) = 0,$$

which is *independent* of the paths ϕ and is nothing but the standard jump relation.

Based on the above definition, one can solve [19] the Riemann problem for (1.1), corresponding to the piecewise constant initial data

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{3.5}$$

where u_l, u_r are constants in $\mathcal{B}_{\delta_2}$ with $\delta_2 \le \delta_1$. This construction generalizes Lax's standard construction for conservative systems [32, 33]. Recall that (admissible) shock waves must be constraint by Lax shock inequalities (for some $j = 1, \dots, N$)

$$\lambda_j(u_0) > \overline{\Lambda} > \lambda_j(u_1). \tag{3.6}$$

The Riemann solver can then be used to design numerical schemes for the approximation of the general initial value problem, e.g. Glimm or front tracking schemes.

In certain applications, it has been found convenient to avoid introducing the whole family of paths ϕ. It is precisely our purpose in the present paper to introduce, for a particular class of nonconservative systems, a new definition of weak solutions, which imposes Rankine-Hugoniot jump relations in the form of "kinetic relations" and does not require the knowledge of any "internal structure" for shock waves.

We will assume that the nonconservative system under consideration *formally* admits N conservation laws, so we consider the system

$$\partial_t u + \partial_x f(u) = 0, \tag{3.7}$$

which consists of conservation laws valid for *smooth* solutions, only. Our goal is to describe singular limits (3.2), where $R = R^\epsilon$ is a *nonconservative regularization*. Precisely, we are going to supplement (3.7) with N jump relations, referred to as "kinetic relation", which determines the dynamics of shocks in weak solutions to (3.7).

We suppose that in an open and convex domain $\mathcal{U} \subset \mathbb{R}^N$ of the phase space, the system (1.1) is strictly hyperbolic, with eigenvalues $\lambda_1(u) < \ldots < \lambda_N(u)$ and basis of eigenvectors $l_i(u), r_i(u)$. Let $\mathcal{L} \subset \mathbb{R}$ be a compact set containing all speeds under consideration in the problem.

Definition 3.1. *A kinetic function is a Lipschitz continuous map $\Phi : \mathcal{U} \times \mathcal{L} \to \mathbb{R}^N$ satisfying (for $j = 1, \ldots, N$)*

$$\begin{aligned} &\Phi(u, \lambda_j(u)) = 0, && u \in \mathcal{U}, \\ &|l_j(u) \cdot \partial_\Lambda \Phi(u, \Lambda)| \le c_1\, |\Lambda - \lambda_j(u)|, && (u, \Lambda) \in \mathcal{U} \times \mathcal{L}, \end{aligned} \tag{3.8}$$

for some constant $c_1 > 0$.

Given a a kinetic function Φ, a piecewise Lipschitz solution $u = u(x,t) \in \mathcal{U}$ is called a Φ-admissible weak solution if the differential equations (3.7) are satisfied in each region of continuity of u and moreover, along any curve of discontinuity of u, connecting some values u_-, u_+ at the speed Λ, the following kinetic relation holds

$$-\Lambda\,(u_+ - u_-) + f(u_+) - f(u_-) = \Phi(u_-, \Lambda). \tag{3.9}$$

In certain applications, it may be more convenient to express the kinetic functions in terms of the left- and right-hand states, that is, $\Phi = \Phi(u_-, u_+)$. In the applications, the kinetic function Φ should be determined from traveling wave solutions of a specific system (3.2) and should be thought of as a "correction" to the standard Rankine-Hugoniot relation.

By introducing the Borel measure denoted by μ_u^Φ that vanishes in the regions of continuity of u and has the mass $\Phi(u_-, \Lambda)$ along its curves of discontinuity, we easily see that Definition 3.1 is equivalent to the requirement (see [37])

$$\partial_t u + \partial_x f(u) = \mu_u^\Phi, \tag{3.10}$$

which is regarded as an equality between bounded measures. Note that we recover the usual conservative case by simply choosing both Φ and μ_u^Φ to vanish identically.

In the rest of this section we study the case of genuinely nonlinear systems. This assumption allows us to use the shock speed as a regular parameter along the (generalized) Hugoniot curve.

Theorem 3.2 (Riemann problem for nonconservative systems with kinetic relations)**.** *Suppose that (3.7) is a strictly hyperbolic system in a neighborhood $\mathcal{B}_{\delta_0}$ of the origin 0 and admits genuinely nonlinear characteristic fields only, i.e.*

$$(\nabla \lambda_j \cdot r_j)(0) > 0, \qquad j = 1, \cdots, N.$$

Let $\Phi = \Phi(u, \Lambda)$ be a (Lipschitz continuous) kinetic function defined in the neighborhood $\mathcal{B}_{\delta_0} \times \mathcal{L}$, with defined for some sufficiently small $\delta > 0$ by

$$\mathcal{L} := \bigcup_j \mathcal{L}_j, \qquad \mathcal{L}_j := (\lambda_j(0) - \delta, \lambda_j(0) + \delta).$$

Then, there exists $\delta_1 \le \delta_0$ such that the Riemann problem (3.5), (3.7) with data $u_l, u_r \in \mathcal{B}_{\delta_1}$ admits a unique Φ-admissible weak solution in the class of piecewise *smooth solutions consisting of a combination of rarefaction waves and shock waves satisfying the kinetic relation. Moreover, the corresponding wave curves are solely Lipschitz continuous.*

Clearly, under the assumptions of the above theorem, (3.8) implies that, for $j = 1, \dots, N$,

$$|l_j(u) \cdot \Phi(u, \Lambda)| \le c_2 \, |\Lambda - \lambda_j(u)|^2, \qquad (\lambda, u) \in \mathcal{U} \times \mathcal{L}_j \tag{3.11}$$

for some $c_2 > 0$.

Proof. We want to generalize the proof given in [19] for general families of paths; see also the related proof in [26] for nonclassical shocks. We are going to show that the given set of jump conditions (3.9) suffices to determine a (generalized) Hugoniot curve uniquely, and we will investigate whether its tangency and regularity properties. The rest of the proof (selection of the admissible part of the Hugoniot curve, actual construction of the wave curves, Riemann solution) then follows as in [19] and will be omitted.

We denote by $\overline{\lambda}_j(u_0, u_1)$ and $\overline{l}_j(u_0, u_1)$ the eigenvalues and left-eigenvectors of the averaged matrix

$$A(u_0, u_1) := \int_0^1 Df(u_0 + m\,(u_1 - u_0))\, dm.$$

In a neighborhood of the point $(u_0, \lambda_j(u_0))$, we consider the kinetic relation

$$G(\Lambda, u_1) := -\Lambda\,(u_1 - u_0) + f(u_1) - f(u_0) - \Phi(u_0, \Lambda) = 0. \tag{3.12}$$

Fix some index i and let us restrict attention to the (nonlinear) cone-like region K determined by the two conditions on $u_1 \in \mathcal{B}_{\delta_1}$

$$\begin{aligned}
&\big|(u_1 - u_0) \cdot l_i(u_0, u_1)\big| \ge C_* \, |\Lambda - \lambda_i(u_0)|,\\
&|u_1 - u_0| + |\Lambda - \lambda_i(u_0)| < \delta_2,
\end{aligned}$$

where a condition on $C_* > 0$ will be imposed below. Observe that $G(u_0, \lambda_i(u_0)) = 0$.

Multiplying the generalized jump relation (3.12) by $\overline{l}_i(u_0, u_1)$ we find

$$\begin{aligned}
0 &= \overline{l}_i(u_0, u_1) \cdot \big(A(u_0, u_1) - \Lambda\big)\,(u_1 - u_0) - \overline{l}_i(u_0, u_1) \cdot \Phi(u_0, \Lambda)\\
&= \big(\overline{\lambda}_i(u_0, u_1) - \Lambda\big)\, \overline{l}_i(u_0, u_1) \cdot (u_1 - u_0) - \overline{l}_i(u_0, u_1) \cdot \Phi(u_0, \Lambda).
\end{aligned}$$

Therefore, we can express the shock speed $\Lambda = \overline{\Lambda}(u_0, u_1)$ in the form

$$0 = \overline{\Lambda} - \overline{\lambda}_i(u_0, u_1) + \frac{\overline{l}_i(u_0, u_1) \cdot \Phi(u_0, \overline{\Lambda})}{\overline{l}_i(u_0, u_1) \cdot (u_1 - u_0)} =: \Omega(u_1, \overline{\Lambda}). \tag{3.13}$$

Now, observe that the function Ω satisfies

$$\begin{aligned}
\Big|\frac{\partial \Omega}{\partial \lambda}(u_1, \overline{\Lambda}) - 1\Big| &= \frac{\overline{l}_i(u_0, u_1) \cdot \partial_\lambda \Phi(u_0, \overline{\Lambda})}{\overline{l}_i(u_0, u_1) \cdot (u_1 - u_0)}\\
&\ge -c_2 \, O(1) \, \frac{|u_1 - u_0| + |\overline{\Lambda} - \overline{\lambda}_i(u_0)|}{|\overline{l}_i(u_0, u_1) \cdot (u_1 - u_0)|},
\end{aligned}$$

where the constant $O(1)$ depends only on the flux. Hence, we have

$$\frac{\partial \Omega}{\partial \Lambda}(u_1, \overline{\Lambda}) = 1 + \frac{c_2}{C_*}\, O(1),$$

which is positive provided c_1 is sufficiently small. As a consequence, the implicit function for Lipschitz continuous mappings applies and shows that the implicit equation (3.13) determine the shock speed $\overline{\Lambda} = \overline{\Lambda}(u_0, u_1)$ uniquely.

Next, we consider the remaining components, corresponding to $j \neq i$:

$$H(u_0, u_1) := l_j(u_0, u_1) \cdot (u_1 - u_0) - \frac{l_j(u_0, u_1) \cdot \Phi(u_0, \overline{\Lambda})}{\overline{\Lambda}(u_0, u_1) - \lambda_j(u_0, u_1)}.$$

Denoting by $L(u_0)$ the $N \times (N-1)$ matrix of vectors $l_j(u_0)$ for $j \neq i$, we can compute the differential of H, as follows:

$$\begin{aligned}\frac{DH}{Du_1}(u_0, u_1) = L(u_0) + O(1)\,|u_1 - u_0| &+ O(1) C_1 \frac{|\overline{\Lambda}(u_0, u_1) - \lambda_i(u_0)|^2}{|\overline{\Lambda}(u_0, u_1) - \lambda_j(u_0)|} \\ &+ O(1)\, C_1 |\overline{\Lambda}(u_0, u_1) - \lambda_i(u_0)| \left| \frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1) \right| \\ &+ O(1)\, C_1 |\overline{\Lambda}(u_0, u_1) - \lambda_i(u_0)|^2 \left| \frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1) - \frac{1}{2}\nabla \lambda_i(u_0) \right|,\end{aligned}$$

where we have used that $\overline{\Lambda}(u_0, u_1) - \lambda_j(u_0, u_1)$ is bounded away from 0. Hence, we find

$$\frac{\partial H}{\partial u_1}(u_0, u_1) = L(u_0) + o(1) + o(1) \left| \frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1) \right|.$$

On the other hand, the u_1-derivative of the shock speed satisfies

$$\begin{aligned}&\frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1) \\ &= \frac{1}{2}\nabla \lambda_i(u_0) + O(1) \frac{|\lambda - \lambda_i(u_0)|^{b+1}}{|l_i(u_0, u_1) \cdot (u_1 - u_0)|} \\ &\quad + O(1) \frac{|\overline{\Lambda} - \lambda_i(u_0)|^2}{|l_i(u_0, u_1) \cdot (u_1 - u_0)|^2} + O(1) \frac{|\overline{\Lambda} - \lambda_i(u_0)|}{|l_i(u_0, u_1) \cdot (u_1 - u_0)|} \frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1),\end{aligned}$$

which shows that

$$\frac{\partial \overline{\Lambda}}{\partial u_1}(u_0, u_1) = \frac{1}{2}\nabla \lambda_i(u_0) + o(1).$$

In conclusion, $\frac{\partial H}{\partial u_1}(u_0, u_1) = L(u_0) + o(1)$, and the implicit function theorem applies to the set of equations $H(u_0, u_1) = 0$, which therefore determines a unique shock curve $s \mapsto u_1 = u_1(s; u_0)$, defined locally near u_0. Near the base point $u(0) = u_0$, the tangent of this curve is defined almost everywhere and, due to the smallness of the constant c_1 in (3.8), takes its values in a small neighborhood of the eigenvector $r_i(u_0)$. □

We now introduce a class of nonconservative system to which the framework in the previous subsection can be applied.

We assume that the first $N - p$ equations in (1.1) have a conservative form while the remaining p equations are nonconservative. In other word, we set $u = (v, w)$ and we consider the nonconservative systems

$$\begin{aligned}&\partial_t v + \partial_x g(v, w) = 0, \\ &\partial_t w + B(v, w)\, \partial_x v + C(v, w)\, \partial_x w = 0.\end{aligned} \tag{3.14}$$

Here $g = g(v, w) \in \mathbb{R}^{N-1}$ while $B = B(v, w), C = C(v, w)$ are $p \times (N - p)$ and $p \times p$ matrix-valued mappings, respectively.

It must be stressed that the assumption made here refers directly to the set of equations listed in (1.1) or to *linear* combinations of them. Of course, nonlinear functions of the original variable u cannot be considered at this level of the analysis, in general, since discontinuous solutions are sought.

Our second assumption is the existence of p mathematical entropy pairs. That is, we assume that there exist k strictly convex functions $U_k = U_k(v,w)$ together with their associated flux $F_k = F_k(v,w)$ such that

$$\partial_t U_k(v,w) + \partial_x F_k(v,w) = 0, \qquad k = 1, \cdots, p, \tag{3.15}$$

holds for all *smooth* solutions to (3.8). We search for solutions satisfying the *entropy inequality*

$$\partial_t U_k(v,w) + \partial_x F_k(v,w) \le 0, \qquad k = 1, \cdots, p. \tag{3.16}$$

Many of the models of interest take the form (3.8)–(3.16).

Definition 3.3. *Nonlinear hyperbolic systems in nonconservative form that have the structure* (3.14) *and admits at least* p *mathematical entropies and satisfying the non-degenerascy condition*

$$det\left(\nabla_w U_1(v,w), \cdots, \nabla_w U_p(v,w)\right) \ne 0. \tag{3.17}$$

are called nonconservative systems endowed with a full set of entropies.

We now focus on the entropy dissipation associated with the entropies U_k. The basic idea is to replace the nonconservative equations in the system (3.8) with conservative equations for the entropy dissipation but the latter involving a measure source-term. Of course it is necessary for $(v,w) \mapsto (v, U(v,w))$ to define a change of variable, say $U_w \ne 0$.

Observe first that the inequality (3.16) implies a constrain on shock waves, i.e., with the notation introduced earlier in (3.3),

$$\begin{aligned} E_k(\Lambda; u_0, u_1) &:= -\Lambda\left(U_k(u_1) - U_k(u_0)\right) + F_k(u_1) - F_k(u_0) \\ &\le 0, \end{aligned} \tag{3.18}$$

for all $k = 1, \ldots p$. On the other hand, the first $N-p$ equations in (2.6) yield $N-p$ jump relations in the fully explicit form

$$-\Lambda\,(v_1 - v_0) + g(v_1, w_1) - g(v_0, w_0) = 0. \tag{3.19}$$

Since p jump relations are "missing", we supply it in the form

$$E_i(\Lambda; u_0, u_1) = \Phi_i(\Lambda; u_0) \le 0, \qquad i = 1, \cdots p, \tag{3.20}$$

which we refer to as a kinetic relation and where Φ is a given "constitutive" function, called a "kinetic function", to be determined case by case in the examples.

Definition 3.4. *Let* $\Phi = (0, \ldots, 0, \Phi_1, \ldots \Phi_p)$ *be a kinetic function. A piecewise Lipschitz continuous function* $u = (v,w)$ *is called a* Φ-admissible solution of the nonconservative system (3.8) *if it satisfies the equations in a classical sense in the regions of continuity and if each propagating discontinuity satisfies the* $N-p$ *jump relations* (3.19) *together with the kinetic relations* (3.20)*.*

We reformulate the main result, in a slightly weaker form which is adapted to the present context, since it is natural to assume that the entropy dissipation is of cubic order near the base point.

Corollary 3.5 (Riemann problem for nonconservative systems endowed with a full set of entropies)**.** *Consider a nonconservative system endowed with a full set of entropies. Suppose that the system is strictly hyperbolic and genuinely nonlinear in the neighborhood of some state* $u_* = (v_*, w_*)$. *Let* $\Phi_i = \Phi_i(u_0, u_1)$ *be a regular function defined in the neighborhood of each speed* $\lambda_j(u_*)$ *for* $j = 1, \cdots, N$ *and satisfying for all* u_0, u_1

$$\begin{aligned} &\Phi_i(u_0, \lambda_i(u_0)) = 0, \\ &\partial_\Lambda \Phi_i(u_0, \Lambda) = O(1)\,(\Lambda - \lambda_j(u_0))^2, \end{aligned} \tag{3.21}$$

where $O(1)$ denotes a positive and bounded function. Then, the corresponding Riemann problem admits a unique admissible solution in the class of piecewise smooth solutions consisting of a combination of rarefaction waves and admissible shock waves.

4. Multi-pressure Navier-Stokes system

In this section, we establish the existence and uniqueness of the traveling wave solutions of the multi-pressure Navier-Stokes equations introduced in Section **??**, under fairly general assumptions on the pressure and viscosity closure laws. The equations under consideration read

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x\Big(\rho u^2 + \sum_{i=1}^N p_i(\tau, e_i)\Big) = \epsilon\,\partial_x\Big(\sum_{i=1}^n \mu_i(\tau, e_i)\partial_x u\Big), \\ &\partial_t(\rho e_i) + \partial_x(\rho u e_i) + p_i(\tau, e_i)\partial_x u = \epsilon\,\mu_i(\tau, e_i)(\partial_x u)^2. \end{aligned} \tag{4.1}$$

Here, $\epsilon > 0$ denotes the inverse of the Reynolds number. Each smooth pressure law $p_i(\tau, e_i)$, $1 \le i \le N$, is assumed to obey the second principle of the thermodynamics, namely

$$T_i(\tau, e_i)\,ds_i = de_i + p_i(\tau, e_i)\,d\tau, \tag{4.2}$$

where $T_i(\tau, e_i) > 0$ is the corresponding temperature variable and $s_i > 0$ denotes the specific entropy.

The map $(\tau, s_i) \mapsto e_i(\tau, s_i)$ is thus well-defined and is assumed to be strictly convex. In addition, the following asymptotic conditions are assumed

$$\lim_{\tau\to 0^+} e_i(\tau, s_i) = +\infty, \qquad \lim_{s_i\to +\infty} e_i(\tau, s_i) = +\infty, \qquad \lim_{\tau\to +\infty} e_i(\tau, s_i) = 0. \tag{4.3}$$

It follows that

$$p_i(\tau, s_i) = -\frac{\partial e_i}{\partial \tau}(\tau, s_i) > 0, \qquad T_i(\tau, s_i) = \frac{\partial e_i}{\partial s_i}(\tau, s_i) > 0. \tag{4.4}$$

Furthermore, the following assumptions are introduced for any given $\tau > 0$:

$$\frac{\partial p_i}{\partial s_i}(\tau, s_i) > 0, \tag{4.5}$$

$$\sum_{i=1}^N \frac{\partial^2 p_i}{\partial \tau^2}(\tau, \mathbf{s}) > 0, \tag{4.6}$$

$$\lim_{\tau\to 0^+}\sum_{i=1}^N p_i(\tau, \mathbf{s}) = +\infty, \quad \lim_{\tau\to +\infty}\sum_{i=1}^N p_i(\tau, \mathbf{s}) = 0, \tag{4.7}$$

$$\lim_{\tau\to 0^+}\sum_{i=1}^N \frac{\partial p_i}{\partial \tau}(\tau, \mathbf{s}) = -\infty, \quad \lim_{\tau\to +\infty}\sum_{i=1}^N \frac{\partial p_i}{\partial \tau}(\tau, \mathbf{s}) = 0, \tag{4.8}$$

$$\lim_{s_i\to +\infty} \frac{\partial p_i}{\partial \tau}(\tau, s_i) = -\infty, \tag{4.9}$$

where $\mathbf{s} = (s_1, ..., s_N)$.

Next, the viscosity laws are given smooth functions with

$$\mu_i(\tau, s_i) \ge 0, \quad 1 \le i \le N, \qquad \text{and} \qquad \mu(\tau, s) := \sum_{i=1}^N \mu_i(\tau, s_i) > 0. \tag{4.10}$$

To shorten the notation, the PDE's system (4.1) is given in the condensed form

$$\partial_t \mathbf{v}^\epsilon + \mathcal{A}(\mathbf{v}^\epsilon)\partial_x \mathbf{v}^\epsilon = \epsilon\mathcal{B}(\mathbf{v}^\epsilon, \partial_x\mathbf{v}^\epsilon, \partial_{xx}\mathbf{v}^\epsilon),$$

with $\mathbf{v}$ in the phase space:

$$\Omega_{\mathbf{v}} = \{\mathbf{v} = (\rho, \rho u, (\rho e_i)_{1\le i\le N}) \in \mathbb{R}^{N+2}; \rho > 0, \rho u \in \mathbb{R}, \rho e_i > 0, 1 \le i \le N\}.$$

The basic properties of (4.1) are summarized in the following statement.

Lemma 4.1. *The underlying first-order part from (4.1) is hyperbolic in $\Omega_{\mathbf{v}}$ and admits three distinct eigenvalues*

$$\lambda_1(\mathbf{v}) = u - c(\mathbf{v}),\ \lambda_2(\mathbf{v}) = ... = \lambda_{N+1}(\mathbf{v}) = u,\ \lambda_{N+2}(\mathbf{v}) = u - c(\mathbf{v}), \tag{4.11}$$

(4.12)

where we have set

$$c^2(\mathbf{v}) = \sum_{i=1}^{N} -\tau^2 \frac{\partial p_i}{\partial \tau}(\tau, s_i). \tag{4.13}$$

The extreme fields are genuinely nonlinear while the intermediate ones are linearly degenerate. Then, smooth solutions of (4.1) satisfy the additional conservation law

$$\partial_t (\rho E)^\epsilon + \partial_x \left(\{\rho E\}(\mathbf{v}^\epsilon) + \sum_{i=1}^{N} p_i(\tau^\epsilon, s_i^\epsilon) u^\epsilon \right) = \epsilon \partial_x \left(\sum_{i=1}^{N} \mu_i(\tau^\epsilon, s_i^\epsilon) u^\epsilon \partial_x u^\epsilon \right),$$

where the total energy reads

$$(\rho E) = \frac{(\rho u)^2}{2\rho} + \sum_{i=1}^{N} \rho e_i. \tag{4.14}$$

At last, the smooth solutions of (4.1) obey the N balance equations

$$\partial_t (\rho s_i)^\epsilon + \partial_x \left((\rho s_i)^\epsilon u^\epsilon \right) = \epsilon \frac{\mu_i(\tau^\epsilon, s_i^\epsilon)}{T_i(\tau^\epsilon, s_i^\epsilon)} (\partial_x u^\epsilon)^2. \tag{4.15}$$

As already claimed, changes of variables with distinctive features allow to recast (4.1) either in the equivalent form:

$$\mathcal{A}_0(\mathbf{w}^\epsilon)\partial_t \mathbf{w}^\epsilon + \mathcal{A}_1(\mathbf{w}^\epsilon)\partial_x \mathbf{w}^\epsilon = \epsilon \partial_x (\mathcal{D}(\mathbf{w}^\epsilon)\partial_x \mathbf{w}^\epsilon), \tag{4.16}$$

with $\mathcal{A}_0$ regular, or in the form

$$\partial_t \mathbf{u}^\epsilon + \partial_x \mathcal{F}(\mathbf{u}^\epsilon) = \epsilon\, \mathcal{R}(\mathbf{u}^\epsilon, \partial_x \mathbf{u}^\epsilon, \partial_{xx} \mathbf{u}^\epsilon). \tag{4.17}$$

We briefly discuss the changes of variables involved in (4.16) and (4.17). Concerning (4.16), we first observe that summing the N governing equations for the internal energies yields

$$\partial_t \rho e + \partial_x \rho e u + \sum_{i=1}^{N} p_i \partial_x u = \epsilon \mu(\tau, \mathbf{s})(\partial_x u)^2,$$

so that the following identities are easily checked:

$$\begin{aligned} &\mu(\tau, \mathbf{s})\Big(\partial_t \rho e_i + \partial_x(\rho e_i u) + p_i(\tau, s_i)\partial_x u\Big) - \\ &\mu_i(\tau, s_i)\Big(\partial_t \rho e + \partial_x(\rho e u) + \sum_{i=1}^{N} p_i \partial_x u\Big) = 0, \quad 1 \le i \le N-1. \end{aligned} \tag{4.18}$$

Since $\rho e = \rho E - (\rho u)^2/(2\rho)$, the conservation laws for ρ, ρu and ρE supplemented by the $(N-1)$ balance equations in (4.18) can be seen to give the equivalent form stated in (4.16) when defining $\mathbf{w} = (\rho, \rho, \rho E, (\rho e_i)_{1\le i\le N-1})$. A direct calculation shows that $\det \mathcal{A}_0(\mathbf{w}) = \mu(\tau, \mathbf{s})^{n-1} > 0$.

Concerning system (4.17), several change of variables can be used and we advocate in the sequel the change of variables $\mathbf{v} \in \Omega_{\mathbf{v}} \mapsto \mathbf{u}(\mathbf{v}) \in \Omega_{\mathbf{u}}$, with $\mathbf{u}(\mathbf{v}) = (\rho, \rho u, (\rho s_i)_{1\le i\le N})$.

As underlined in Section 3, both approaches relies on the study of the traveling wave solutions of (4.1). Due to the frame invariance properties satisfied by the PDE model (4.1), it suffices to analyze traveling waves solutions associated with the first extreme field. With this respect, the main result of this section is as follows.

Theorem 4.2 (Traveling wave solutions to the multi-pressure Navier-Stokes system)**.** *Let $\mathbf{u}_L \in \Omega_{\mathbf{u}}$ and $\sigma \in \mathbb{R}$ be given such that*

$$\frac{u_L - \sigma}{c(\mathbf{u}_L)} > 1, \quad c^2(\mathbf{u}_L) = \sum_{i=1}^{N} -\tau_L^2 \frac{\partial p}{\partial \tau}(\tau_L, (s_i)_L). \tag{4.19}$$

Then, there exists a unique traveling wave solution to (4.1) *issuing from the left-hand state $\mathbf{u}_L$ and reaching some right-hand state $\mathbf{u}_R \in \Omega_{\mathbf{u}}$ with*

$$0 < \frac{u_R - \sigma}{c(\mathbf{u}_R)} < 1 \tag{4.20}$$

The proof of this result will follow from the characterization of a positively invariant compact set of $\Omega_{\mathbf{u}}$. Then the Lasalle invariance principle applied in connection with a suitable Lyapunov function ensures the existence of a traveling wave. Uniqueness is obtained as a simple consequence of the center manifold theorem.

We gather here some of the notation used repeatedly hereafter and give the precise form of the autonomous system which governs the viscous profiles we study for existence. Simple but useful geometrical properties induced by the corresponding vector field will be then put forward.

Due to Galilean invariance, it suffices to consider the case of a null velocity σ. The precise form of the PDE system governing the traveling wave solutions then follows when restricting attention to solutions which depend solely on x:

$$\begin{aligned} (\rho u)_x &= 0, \\ (\rho u^2 + p(\tau, \mathbf{s}))_x &= (\mu(\tau, \mathbf{s}) u_x)_x, \\ T_i(\tau, s_i)(\rho s_i u)_x &= \mu_i(\tau, \mathbf{s})(u_x)^2, \quad 1 \le i \le N. \end{aligned} \tag{4.21}$$

The first equation in (4.21) implies that the relative mass flux ρu has a constant value denoted by $m = \rho_L u_L$. As already underlined, we focus ourselves on traveling wave solutions associated with the first GNL field; namely we consider $m > 0$. Observe that the Lax condition (4.19) expressed expressed for a null velocity σ reads

$$m > \rho_L c_L.$$

Next by integrating once the second equation in (4.21), the identity $u = m\tau$ allows one to derive the following $(N+1)$-dimensional autonomous system:

$$\begin{aligned} \dot{\tau} &= \frac{1}{\mu(\tau, \mathbf{s})}\Big(p(\tau, \mathbf{s}) - p(\tau_L, \mathbf{s}_L) + m^2(\tau - \tau_L)\Big) := \frac{1}{\mu(\tau, \mathbf{s})} \mathcal{F}(\tau, \mathbf{s}), \\ \dot{s}_i &= \frac{\mu_i(\tau, s_i)}{\mu^2(\tau, \mathbf{s}) T_i(\tau, s_i)} \mathcal{F}^2(\tau, \mathbf{s}), \qquad 1 \le i \le N, \end{aligned} \tag{4.22}$$

where dots denote differentiation with respect to the rescaled variable x/m that we shall refer with little abuse as a time in the sequel.

This dynamical system is endowed with the following open subset of $\mathbb{R}^{N+1}$ which will serve as a natural phase space:

$$\Omega = \left\{ \omega := (\tau, \mathbf{s}) \in \mathbb{R}^{N+1}; \tau > 0 \right\}. \tag{4.23}$$

To shorten the notation, a given function Ψ of τ and $\mathbf{s}$ is simply denoted hereafter by $\Psi(\omega)$.

Recall that the total viscosity $\mu(\omega)$ is assumed to stay strictly positive over Ω. Then, the regularity assumptions made on all the thermodynamic and viscosity mappings ensure that the vector field in (4.22) is continuously differentiable

The unique nonextensible solution of (4.22) with initial data ω_0 in Ω is referred as to the flow $\omega_0 \cdot t$ for the times t in the maximal interval of existence $(t^-(\omega_0), t^+(\omega_0))$. The positive (respectively negative) semi-orbit $\gamma^+(\omega_0)$ (resp. $\gamma^-(\omega_0)$) classically denotes the set of states $\omega_0 \cdot [0, t^+(\omega_0)) = \{\omega_0 \cdot t : 0 \leq t < t^+(\omega_0)\}$ (resp. $\omega_0 \cdot (t^-(\omega_0), 0] = \{\omega_0 \cdot t : t^-(\omega_0) < t \leq 0\}$), the orbit being then defined as $\gamma(\omega_0) = \gamma^-(\omega_0) \cup \gamma^+(\omega_0)$. At last, for each ω_0 in Ω, the positive limit set (the so-called ϖ-limit set in what follows) of ω_0 finds the definition $\varpi(\omega_0) := \cap_{t>0} \overline{\gamma^+(\omega_0 \cdot t)}$, such a set is thus empty as soon as $t^+(\omega_0)$ is finite.

Before we enter the central part of the analysis, let us underline that the $(N+1)$ constitutive variables of (4.22) are necessarily kept in there evolution in time in a N-dimensional sub-manifold of Ω, the latter being entirely prescribed by the choice of the initial data $\omega_0 \in \Omega$. This is the matter of the following statement which essentially reflects the conservation property met by the total energy (4.14).

Proposition 4.3. *Let ω_0 be a given state in Ω. Then the flow $\omega_0 \cdot t$ satisfies for all time in its maximal interval of existence:*

$$\mathcal{H}(\omega_0 \cdot t) = \mathcal{H}(\omega_0), \tag{4.24}$$

where the regular mapping $\mathcal{H} : \Omega \to \mathbb{R}$ is defined by:

$$\mathcal{H}(\omega) = e(\omega) - e(\omega_L) - \frac{m^2}{2}(\tau^2 - \tau_L^2) + (m^2\tau_L + p(\omega_L))(\tau - \tau_L). \tag{4.25}$$

Proof. All the flows under consideration are at least continuously differentiable in their maximal interval of existence. The additional conservation law (4.14) for the total energy therefore applies and its differential form reads

$$\{(E + \tau p(\omega))\rho u\}_x = \left\{\frac{\mu}{2}(u^2)_x\right\}_x. \tag{4.26}$$

In view of the algebraic invariant $\rho u = m$, (4.26) once integrated for a prescribed ω_0 in Ω between time zero and a given time t in $(t^-(\omega_0), t^+(\omega_0))$ can be seen to read

$$\begin{aligned} \{E + \tau p(\omega))\rho u\}(t) - \{E + \tau p(\omega))\rho u\}(0) &= \{\tau(\mu\dot{\tau})\}(t) - \{\tau(\mu\dot{\tau})\}(0), \\ &= \{\tau\mathcal{F}(\omega)\}(t) - \{\tau\mathcal{F}(\omega)\}(0). \end{aligned} \tag{4.27}$$

Since E writes $m^2\tau^2/2 + \epsilon(\omega)$, the definition of $\mathcal{F}$ given in (4.22) easily yields the required identity (4.24) after some rearrangements in the terms while subtracting for convenience to both sides the constant $\epsilon_L + m^2\tau_L^2/2 + \tau_L p_L$. □

The above statement clearly implies that all the possible heteroclinic orbits of (4.22) which connect the critical point ω_L in the past are only made of states ω such that

$$\mathcal{H}(\omega) = \mathcal{H}(\omega_L) = 0. \tag{4.28}$$

To end up with these preliminary remarks, we point out an elementary but useful geometrical property of the flows associated with (4.22) which will put restriction on possible right connecting states.

Lemma 4.4. *Let ω_0 be given in Ω, then the subset of Ω defined by*

$$\Omega(\omega_0) = \{\omega \in \Omega; \mathbf{s} \geq \mathbf{s}_0, \mathcal{H}(\omega) = \mathcal{H}(\omega_0)\} \tag{4.29}$$

is positively invariant.

The invariance of this region with respect to all positive semi-flows immediately follows from the non-negativeness of the N^{th}-last components of the vector field entering the definition of (4.22). As a consequence, eventual heteroclinic orbits connecting ω_L in the past necessarily entirely lie in

$$\Omega(\omega_L) = \{\omega \in \Omega; \mathbf{s} \geq \mathbf{s}_L, \mathcal{H}(\omega) = 0\}\,. \tag{4.30}$$

The region (4.30) will have a central role in the derivation of positively invariant compact sets.

Here, we exhibit some important features of the linearization $LX(\omega_c)$ of the vector field X at equilibrium points ω_c, i.e. at states satisfying $\mathcal{F}(\omega_c) = 0$. We check in particular that such states are always non-hyperbolic points for which the space $\mathbb{R}^{N+1}$ writes as a direct sum of the eigenspaces associated with $LX(\omega_c)$ under the following non-degeneracy condition:

$$\partial_\tau \mathcal{F}(\omega_c) = m^2 + \partial_\tau p(\tau_c, \mathbf{s}_c) \neq 0. \tag{4.31}$$

In that aim, let us state some basic facts concerning the linearization $LX(\omega_c)$. The requirement $\mathcal{F}(\omega_c) = 0$ is easily seen to enforce all the partial derivatives of the N^{th}-last components of X to be identically zero (since these components are all proportional to $\mathcal{F}^2$). Under the nondegeneracy condition (4.31), there consequently exists only one non trivial eigenvalue namely $\partial_\tau \mathcal{F}(\omega_c)/\mu(\omega_c)$ while $\lambda = 0$ is a semisimple eigenvalue of $LX(\omega_c)$ of multiplicity N. Furthermore, the corresponding eigenspaces $T(\omega_c)$ and $T^c(\omega_c)$ are respectively the span of e_1 and $e_2, ..., e_N$ where $\{e_i\}_{1\leq i\leq N+1}$ stands for the canonical orthonormal basis of $\mathbb{R}^{N+1}$.

Equipped with these results, the center manifold theorem ensures the existence of two locally invariant manifolds $\mathcal{W}(\omega_c)$ and $\mathcal{W}^c(\omega_c)$ (the so-called center manifold) of class at least C^1 and C^0 respectively which go through ω_c and are respectively tangent to $T(\omega_c)$ and $T^c(\omega_c)$ at this point. The regularity properties we have mentioned, are indeed inherited from the continuous differentiability of the vector field, according to this theorem.

Assuming $\partial_\tau \mathcal{F}(\omega_c)$ to be positive (respectively negative), *i.e.* assuming the corresponding sign for the unique nontrivial eigenvalue of $LX(\omega_c)$; $\mathcal{W}(\omega_c)$ is classically referred as to the unstable (resp. stable) manifold with superscript u (resp. s). Recall that the unstable manifold of a point is the manifold composed of the totality of the orbits which tend exponentially fast to the point in negative time; the stable manifold being defined conversely. Then by well-known topological considerations, two rest points namely ω_L and ω_R are connected by a heteroclinic orbit γ precisely if $\gamma \subset \mathcal{W}^u(\omega_L) \cap \mathcal{W}^s(\omega_R)$.

An obvious requirement for the existence of a heteroclinic orbit connecting ω_L in the past is then

$$\partial_\tau \mathcal{F}(\omega_L) = m^2 + \partial_\tau p(\omega_L) < 0,$$

but the validity of such an inequality is precisely the matter of the Lax condition (4.19). Conversely, a possible connecting point ω_R in the future is necessarily subject to the condition $\partial_\tau \mathcal{F}(\omega_R) > 0$.

Now and since the unstable manifold $\mathcal{W}^u(\omega_L)$ is one dimensional, there exists locally exactly two solutions of (4.22) which approach ω_L as $t \to -\infty$. Arguing about the property of $\mathcal{W}^u(\omega_L)$ to be tangent to e_1, the associated almost horizontal orbits approach ω_L from the two opposite directions $\tau \geq \tau_L$ and $\tau \leq \tau_L$. With clear notation, $\gamma_>(\omega_L)$ (respectively $\gamma_<(\omega_L)$) will denote the first (resp. the second) orbit.

The following assertion discards the solution converging to ω_L for negative times from the region $\tau \geq \tau_L$. Note that such a result precisely precluded expansion shocks to admit viscous profiles.

Proposition 4.5. *There is no heteroclinic orbit of the dynamical system (4.22) in the domain* $\mathcal{N} := \{\omega \in \Omega : \tau \geq \tau_L, \mathbf{s} \geq \mathbf{s}_L\}$.

Consequently, only the second solution can give rise to a heteroclinic orbit. Since the vector field $X : \Omega \to \mathbb{R}^{N+1}$ is Lipschitz-continuous, the uniqueness part of the celebrated Picard-Lindelöf theorem readily gives the following.

Corollary 4.6. *There exists at most one heteroclinic orbit of the dynamical system (4.22) which connects* ω_L *in the past.*

The proof of Proposition 4.5 will follow from the following statement.

Lemma 4.7. *Any given state* ω *distinct from* ω_L *in the set* $\{\tau \geq \tau_L, \mathbf{s} \geq \mathbf{s}_L\}$, *obeys* $\mathcal{F}(\omega) > 0$.

Proof. Observe that the positiveness assumption (4.6) on the Grüneisen numbers implies that for all ω in the region under interest ($\mathbf{s} \geq \mathbf{s}_L$):

$$f(\tau) := \mathcal{F}(\tau, \mathbf{s}_L) \leq \mathcal{F}(\omega) \tag{4.32}$$

with equality iff $\mathbf{s} = \mathbf{s}_L$. We have in particular for $\tau = \tau_L$: $\mathcal{F}(\tau_L, \mathbf{s}) > 0$ as soon as $\mathbf{s} > \mathbf{s}_L$. Next, considering $\tau > \tau_L$, the following identity

$$f'(\tau) = m^2 + \partial_\tau p(\tau, \mathbf{s}_L)$$

clearly yields, under the assumption (4.5) of positive fundamental derivatives, that $\partial^2_{\tau\tau} p(\tau, \mathbf{s}) > 0$ for all $\omega \in \Omega$ andn therefore,

$$f'(\tau) \geq f'(\tau_L) = m^2 - (\rho_L c_L)^2 > 0$$

thanks to the Lax condition (4.19). It immediately follows that $f(\tau) \geq f(\tau_L) = \mathcal{F}(\omega_L) = 0$ as soon as $\tau \geq \tau_L$ with equality to zero iff $\tau = \tau_L$. The inequality (4.32) then gives the required conclusion. □

Proof of Proposition 4.5. Let $\mathbf{n}_\omega$ be the unit inward normal at the following hypersurfaces $\{\tau = \tau_L, \mathbf{s} \geq \mathbf{s}_L\}$ and $\{\tau \geq \tau_L, \mathbf{s} = \mathbf{s}_L\}$ for all states in these sets. Note that these sets are the lower part of the boundary of $\mathcal{N}$. For such states, the Lemma 4.7 implies from the definition of the vector field: $X(\omega) \cdot \mathbf{n}_\omega \geq 0$. As a well-known consequence, $\mathcal{N}$ stays invariant for all positive semi-flows. The required conclusion follows again from Lemma 4.7 which says that no critical point exists in $\mathcal{N}$. □

We now stress another important consequence of the local properties of the phase portrait at the rest point ω_L. By opposition to the states in the orbit $\gamma_>(\omega_L)$; the second orbit $\gamma_<(\omega_L)$ emanating from the region $\tau \leq \tau_L$ is by definition made of states ω that at least when close enough to but distinct from ω_L give rise to a compression: namely locally $\mathcal{F}(\omega) < 0$ in view of the governing equation for τ. This simple observation implies in turn that the viscous profile we study for existence stays necessarily uniformly compressive. This claim is a consequence of the following statement.

Lemma 4.8. *The following set*

$$\mathcal{I} = \{\omega \in \Omega : \tau < \tau_L,\ \mathbf{s} \geq \mathbf{s}_L,\ \mathcal{F}(\omega) \leq 0\} \tag{4.33}$$

is positively invariant under the action of the dynamical system (4.22).

Proof. The above assertion is trivial for states $\omega_0 \in \mathcal{I}$ which satisfy $\mathcal{F}(\omega_0) = 0$. Considering states ω_0 with the property $\mathcal{F}(\omega_0) < 0$, we observe that the positive semi-flow $\omega_0 \cdot t$ necessarily satisfies $\mathcal{F}(\omega_0 \cdot t) < 0$ for all time in $[0, t^+(\omega_0))$. Indeed assuming the existence of a finite time t_c in this interval with the property $\mathcal{F}(\omega_0 \cdot t_c) = 0$ would result in a critical point $\omega_0 \cdot t_c$ for the dynamical system (4.22). But by the Lipschitz-continuity property of the vector field in Ω, it is well-known that such a point cannot be reached in finite time. □

The orbit $\gamma_<(\omega_L)$ is therefore trapped in the region $\mathcal{I}$. We now establish that in addition this orbit necessarily stays within a compact subset K of $\mathcal{I}$. This will imply that $\gamma_<(\omega_L)$ is relatively compact. Well-known considerations imply that the associated ϖ-limit set is nonempty, compact and connected. The existence of K primary stems from the following result.

Lemma 4.9. *Let ω_0 be a given state in Ω. Then the positive semi-orbit $\gamma^+(\omega_0)$ has no limit point in the set $\{\tau = 0\}$.*

This assertion immediately gives that the orbit $\gamma_<(\omega_L)$ has the same property.

Proof. For all time t in $[0, t^+(\omega_0))$, the positive semi-flow $\omega_0 \cdot t$ is known to obey $\mathcal{H}(\omega) = \mathcal{H}(\omega_0) < \infty$ and $\mathbf{s} \geq \mathbf{s}_0$. Arguing about the positivity of all the temperatures $T_i = \partial_{s_i}\mathcal{H}$, we immediately have

$$h(\tau) := \mathcal{H}(\tau, \mathbf{s}_0) \leq \mathcal{H}(\tau, \mathbf{s}),$$

with the property that $h(\tau)$ goes to infinity as τ goes to zero (see the asymptotic condition (4.7)).

Assume that $\overline{\gamma^+(\omega_0)} \cap \{\tau = 0\}$ is nonempty. As a consequence, for all $\varepsilon > 0$ there exists $t_\varepsilon \in (0, t^+(\omega_0))$ such that $0 < \tau|_{\omega_0 \cdot t_\varepsilon} < \varepsilon$. Necessarily there exists $\varepsilon_0 > 0$ so that $h(\tau|_{\omega_0 \cdot t_{\varepsilon_0}}) > \mathcal{H}(\omega_0)$ and this rises the contradiction with the preservation of $\mathcal{H}(\omega_0)$ along the orbit. □

We have proven that any given positive semi-flow of the dynamical system (4.22) with initial data ω_0 in Ω satisfies $t^+(\omega_0) = \infty$ (since $\mathbf{s} > \mathbf{s}_0$). We now conclude with the existence (and therefore uniqueness) of the required viscous profile.

Proposition 4.10. *There exists a state ω_R in $\mathcal{H}^{-1}(0) \cap \mathcal{F}^{-1}(0)$ which is connected by $\gamma_<(\omega_L)$ in the future.*

Proof. We first establish that the specific entropy vector $\mathbf{s}$ stays upper-bounded along all positive semi-flows with initial data in $\mathcal{I}$. For fixed τ in $(0, \tau_L)$, the conditions (4.4) and (4.6) shows that $\mathcal{H}(\tau, \mathbf{s})$ rises arbitrarily with $\mathbf{s}$. The same steps as the ones involved in the previous proof, apply to give the required result. As a consequence, $\gamma_<(\omega_L)$ is necessarily included in a compact subset, namely K, of the positively invariant region $\mathcal{I}$. This orbit is therefore relatively compact and its ϖ-limit set is nonempty. This limit set is necessarily included in $\mathcal{H}^{-1}(0)$. To conclude, let us observe that τ, when understood as a mapping of the variable ω trivially yields a Lyapunov function on $\mathcal{I}$ where we have by construction $\mathcal{F}(\omega) \leq 0$. The LaSalle invariance principle applied in connection with this Lyapunov function then ensures that the nonempty ϖ-limit set is included in $\{\omega \in \mathcal{I} : \mathcal{F}(\omega) = 0\}$. This establishes the existence of ω_R. □

5. End states for viscous layers with varying viscosity

Existence and uniqueness (up to a translation) of traveling wave solutions for the multi-pressure Navier-Stokes equations have been obtained in the previous section for N general smooth viscosity laws prescribed under the non degeneracy condition (4.10). Being given a fixed state $\omega_L \in \Omega$ and a velocity σ according to the condtion (4.19), we aim here at characterizing the subset of Ω made of all the states ω_R that can be reached in the future by a traveling wave with speed σ and connecting ω_L in the past. We naturally expect the exit state ω_R to depend on the close form of expression of the N-uple of viscosity laws. The dynamical system (4.22) shows that such a dependence more precisely occurs in term of the ratios of the viscosity laws. As a consequence, possible states ω_R to be reached in the future from a fixed ω_L in the past, at speed σ, generically depend on $N-1$ degrees of freedom. The set of exit states we are seeking is thus expected to have $(N-1)$ Hausdorff dimension.

It turns convenient to study the projection of this set onto the following positive cone of $\mathbb{R}^N$, (understod as the space of the specific entropies $\mathbf{s} = (s_1, ..., s_N)$) with origin $\mathbf{s}_L$:

$$S^+(\mathbf{s}) = \left\{\mathbf{s} \in \mathbb{R}^N \ / \ \mathbf{s} = \mathbf{s}_L + \lambda\mathbf{a},\ \mathbf{a} \in \mathcal{S}_+^N,\ \lambda \geq 0\right\}. \tag{5.1}$$

Here, $\mathcal{S}_+^N$ denotes the (positive) part of the unit sphere in $\mathbb{R}^N$ defined by

$$\mathcal{S}_+^N = \left\{\mathbf{a} \in \mathbb{R}_+^N;\ \| \mathbf{a} \|= 1\right\}.$$

Investigating for existence all the possible entropies $\mathbf{s}_R$ in the proposed cone (5.1) simply comes from the property that the heterocline solutions of Theorem 4.2 necessarily obey $\mathbf{s}_R \geq \mathbf{s}_L$ with strict inequality for at least one specific entropy s_i for some index $1 \leq i \leq N$.

We show hereafter that the projection in the half cone (5.1) of the states ω_R that can be reached when varying the definition of the N viscosity laws, is a smooth manifold with co-dimension one:

$$\mathcal{C} = \left\{\mathbf{s} \in S^+(\mathbf{s}) \ / \ \mathbf{s} = \mathbf{s}_L + \Lambda_0(\mathbf{a})\mathbf{a},\ \mathbf{a} \in \mathcal{S}_+^N\right\}, \tag{5.2}$$

for some suitable mapping $\Lambda_0(\mathbf{a}) : \mathbf{a} \in \mathcal{S}_+^N \mapsto \Lambda_0(\mathbf{a}) \in \mathbb{R}$ which precise definition will be given latter on. The derivation of the proposed manifold is performed in two steps. In a first step, we analyse closely all the critical points $(\tau_c, \mathbf{s}_c)$ of the dynamical system (4.22), *i.e.* the solutions of

$$\mathcal{F}(\tau_c, \mathbf{s}_c) = 0, \tag{5.3}$$

without reference to a precise N-uple of viscosity laws.

We emphasize that eligible critical points that can be reached from the state ω_L in the past must preserve the total energy as stated in (4.28). Such states must therefore solve in addition

$$\mathcal{H}(\tau_c, \mathbf{s}_c) = 0, \quad \text{with} \quad \mathbf{s}_R \geq \mathbf{s}_L. \tag{5.4}$$

Analyzing the solution of (5.3)-(5.4) will give birth to the manifold (5.2).

In a second step, we will establish that any given value $\mathbf{s}$ in the proposed manifold can be actually achieved for at least one suitable N-uple of viscosity laws. As a consequence, the manifold (5.2) is entirely made of all the specific entropy $\mathbf{s}_R$ that can be reached in the future by a traveling wave solution with speed σ and issued from ω_L, when varying the definition of the N viscosity laws.

Let us outline the content of this section. We first analyze the mappings $\mathbf{s} \in S^+(\mathbf{s}) \mapsto \tau_{\mathcal{F}}(\mathbf{s}) \in \mathbb{R}_+$ that solve

$$\mathcal{F}(\tau_{\mathcal{F}}(\mathbf{s}), \mathbf{s}) = 0. \tag{5.5}$$

We then characterize the mapping $\mathbf{s} \in S^+(\mathbf{s}) \mapsto \tau_{\mathcal{H}}(\mathbf{s}) \in \mathbb{R}_+$ solving

$$\mathcal{H}(\tau_{\mathcal{H}}(\mathbf{s}), \mathbf{s}) = 0. \tag{5.6}$$

Equipped with these two families of functions we will study for existence values of the specific entropy $\mathbf{s}_c$ in $S^+(\mathbf{s})$ that satisfy $\tau_{\mathcal{F}}(\mathbf{s}_c) = \tau_{\mathcal{H}}(\mathbf{s}_c)$, namely values of $\mathbf{s}$ that simultaneously solve (5.5) and (5.6). We now state the main result of this section.

Theorem 5.1. *Assume that (4.3)-(4.9) on the thermodynamics are satisfied. Then there exists a unique map* $\mathcal{T} \in \mathcal{C}^0(\bar{\mathcal{K}}, \mathbb{R}_+^*) \cap \mathcal{C}^1(\mathcal{K}, \mathbb{R}_+^*)$ *where* $\mathcal{K} \subset S^+(\mathbf{s})$ *reads*

$$\mathcal{K} = \left\{\mathbf{s} \in S^+(\mathbf{s}) \ / \ \mathbf{s} = \mathbf{s}_L + \lambda\mathbf{a},\ \mathbf{a} \in \mathcal{S}_+^N,\ \lambda \in]0, \Lambda_0(\mathbf{a})[\right\}, \tag{5.7}$$

for some smooth application $\Lambda_0 \in \mathcal{C}^1(\mathcal{S}_+^N, \mathbb{R}_+^*)$ *with the following properties:*

(i) $\mathcal{H}(\mathcal{T}(\mathbf{s}), \mathbf{s}) = 0$*, for all* $\mathbf{s} \in \bar{\mathcal{K}}$*,*

(ii) $\mathcal{F}\big(\mathcal{T}(\mathbf{s}),\mathbf{s}\big)=0$, *for all* $\mathbf{s}\in\mathcal{C}$ *where*

$$\mathcal{C}=\big\{\mathbf{s}\in S^{+}(\mathbf{s})\ /\ \mathbf{s}=\mathbf{s}_L+\Lambda_0(\mathbf{a})\mathbf{a},\ \mathbf{a}\in\mathcal{S}_{+}^{N}\big\}. \tag{5.8}$$

In addition $\mathcal{T}$ *obeys*

(iii) $\mathcal{F}\big(\mathcal{T}(\mathbf{s}_L),\mathbf{s}_L\big)=0$,
(iv) $\mathcal{F}\big(\mathcal{T}(\mathbf{s}),\mathbf{s}\big)<0$, *for all* $\mathbf{s}\in\mathcal{K}$.

The mapping $\Lambda_0:\mathcal{S}_{+}^{N}\to\mathbb{R}_{+}^{*}$ will be built in the course of the proof. Rephrasing the above result, the function $\mathcal{T}(\mathbf{s})$ with $\mathbf{s}\in\mathcal{C}$, simultaneously makes vanish $\mathcal{F}$ and $\mathcal{H}$, so that all the values $\mathbf{s}$ in the smooth manifold $\mathcal{C}$ are candidate for being reached in the future from the state ω_L via a traveling wave with speed σ for suitable choice of the N viscosity laws.

We now show that all the values $\mathbf{s}$ in the manifold $\mathcal{C}$, defined by (5.8), are actually eligible candidates for entering the definition of the specific entropy in exit states ω_R.

Lemma 5.2. *A state* ω_L *being given in* Ω *and a velocity* σ *being prescribed according to (4.19). For any given* $\mathbf{s}\in\mathcal{C}$*, there exists at least one relevant definition of the* N*-uple of viscosity laws which yields a traveling wave solution with speed* σ *issued from* ω_L *and connecting a state* ω_R *in the future with* $\mathbf{s}_R=\mathbf{s}$*.*

Proof. The proof of this result makes use of particular viscosity laws under the form

$$\mu_i(\tau,s_i)=\mu_i^0T_i(\tau,\mathbf{s}_i),\quad \mu_i^0\geq 0,\quad 1\leq i\leq N. \tag{5.9}$$

The non degeneracy condition (4.10) is satisfied as soon as

$$\sum_{i=1}^{N}\mu_i^0>0, \tag{5.10}$$

since each of the temperature law $T_i(\tau,\mathbf{s}_i)$ is assumed to be positive. Without lost of generality, we assume $\mu_N^0>0$.

We stress that viscosity laws which linearly depend on the temperature naturally arise in the kinetic theory for dilute gases. We refer the reader to the book by Hirschefelder, Curtiss and Bird [27].

Observe that viscosity laws in the special form (5.9) let evolve each specific entropy according to

$$\dot{s}_i=\frac{\mu_i^0}{\mu^2(\tau,\mathbf{s})}\mathcal{F}^2(\tau,\mathbf{s}),\qquad 1\leq i\leq N.$$

We thus infer the following $(N-1)$ balance equations linking the evolution of the first $(N-1)$ specific entropies s_i to the last one:

$$\dot{s}_i=\frac{\mu_i^0}{\mu_N^0}\dot{s}_N,\qquad 1\leq i\leq N-1.$$

Since the ratios μ_i^0/μ_N^0 are constant real numbers, we deduce:

$$\left(s_i-\frac{\mu_i^0}{\mu_N^0}s_N\right)(\xi)=s_i^L-\frac{\mu_i^0}{\mu_N^0}s_N^L,\qquad \text{for all}\quad \xi\in\mathbb{R}.$$

We therefore end up with $(N-1)$ jump relations

$$s_i^R-s_i^L=\frac{\mu_i^0}{\mu_N^0}\left(s_N^R-s_N^L\right),\quad 1\leq i\leq N. \tag{5.11}$$

We emphasize at this stage that $s_N^R - s_N^L > 0$ in view of our assumption $\mu_N^0 > 0$. From the jump relation (5.11), we therefore get

$$\mathbf{s}_R - \mathbf{s}_L = \frac{s_N^R - s_N^L}{\mu_N^0} \begin{pmatrix} \mu_1^0 \\ \vdots \\ \mu_N^0 \end{pmatrix}, \tag{5.12}$$

which is obviously in the form $\Lambda_0(\mathbf{a})\mathbf{a}$, for $\mathbf{a}$ in $\mathcal{S}_+^N$, given by

$$\mathbf{a} = \frac{1}{\sqrt{\sum_{j=1}^N (\mu_j^0)^2}} \left(\mu_j^0\right)_{1\leq j\leq N}.$$

Next and up to some relabelling in the viscosity in order to allow μ_N^0 to vanish, any given $\mathbf{a} \in \mathcal{S}_+^N$ gives rise to an admissible N-uple of viscosity coefficients. This concludes the proof. $\square$

Remark 5.3. *The identity (5.12) shows in addition that the mapping $\mathbf{a} \mapsto \Lambda_0(\mathbf{a})$ can be built as soon as the jump in the last specific entropy $s_N^R - s_L^N$ is known. This evaluation can be performed numerically (see for instance* [7]*).*

We now give a proof of the main result of this section, namely Theorem 5.1. In that aim, and as already claimed, we propose to first study for existence in $\mathcal{S}_+^N$ the roots $\tau(\mathbf{s})$ of $\mathcal{F}(\tau, \mathbf{s}) = 0$. Then, we shall study their distinctive properties by investigating the values of $\mathcal{H}(\tau(\mathbf{s}), \mathbf{s})$.

Proposition 5.4. *There exists two maps, we denote by $\tau^\pm$ belonging to $\mathcal{C}^1\big(\mathcal{D} \cup \{\mathbf{s}_L\}, \mathbb{R}_+^*\big) \cap \mathcal{C}^0\big(\bar{\mathcal{D}}, \mathbb{R}_+^*\big)$, where $\mathcal{D}$ is the subset of $S^+(\mathbf{s}_L)$ defined by*

$$\mathcal{D} = \big\{\mathbf{s} \in S^+(\mathbf{s}_L) \ / \ \mathbf{s} = \mathbf{s}_L + \lambda\mathbf{a},\ \mathbf{a} \in \mathcal{S}_+^N,\ \lambda \in]0, \bar{\Lambda}(\mathbf{a})[\big\}, \tag{5.13}$$

for some $\bar{\Lambda} \in \mathcal{C}^1\big(\mathcal{S}_+^N, \mathbb{R}_+^*\big)$ with $\bar{\Lambda}(\mathbf{a}) > \Lambda_0(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{S}_+^N$, so that

$$\mathcal{F}(\tau^\pm(\mathbf{s}), \mathbf{s}) = 0, \qquad \mathbf{s} \in \bar{\mathcal{D}}.$$

In addition, these two families of roots are interlaced according to

(i) $\tau^-(\mathbf{s}) < \tau^+(\mathbf{s}) < \tau_L$, *for all* $\mathbf{s} \in \mathcal{D}\backslash\{\mathbf{s}_L\}$,
(ii) $\tau^+(\mathbf{s}) = \tau_L$ *in* $\bar{\mathcal{D}}$ *iff* $\mathbf{s} = \mathbf{s}_L$ *with* $\tau^-(\mathbf{s}_L) < \tau^+(\mathbf{s}_L) = \tau_L$,
(iii) $\tau^-(\mathbf{s}) = \tau^+(\mathbf{s})$ *in* $\bar{\mathcal{D}}$ *iff* $\mathbf{s} = \mathbf{s}_L + \bar{\Lambda}(\mathbf{a})\mathbf{a}$, $\mathbf{a} \in \mathcal{S}_+^N$.

Again, the map $\bar{\Lambda} : \mathcal{S}_+^N \to \mathbb{R}_+^*$ will be built in the course of the proof. But from now on, notice that $\bar{\mathcal{K}} \subset \mathcal{D}$. We shall show that for fixed $\mathbf{s} \in \bar{\mathcal{D}}$, $\mathcal{F}(\tau, \mathbf{s}) = 0$ only admits $\tau^\pm(\mathbf{s})$ as roots and cannot be solved in τ for values of $\mathbf{s}$ in $S^+(\mathbf{s}_L)\backslash\bar{\mathcal{D}}$. As a consequence, all the critical points $(\tau(\mathbf{s}), \mathbf{s})$ of (4.22) are necessarily achieved for $\mathbf{s} \in \bar{\mathcal{D}}$ so that $\tau(\mathbf{s})$ must coincide with either $\tau^-(\mathbf{s})$ or $\tau^+(\mathbf{s})$ for suitable values of $\mathbf{s} \in \bar{\mathcal{D}}$: *i.e.* such that $\mathcal{H}(\tau^-(\mathbf{s}), \mathbf{s}) = 0$ or $\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) = 0$. In this way, let us state some properties of $\mathcal{H}$ with respect to the above two families of roots.

Proposition 5.5. *Using the notation of Propositions 5.1 and 5.4, we have*

(i) $\mathcal{H}(\tau^+(\mathbf{s}_L), \mathbf{s}_L) = 0$,
(ii) $\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) > 0$, *for all* $\mathbf{s} \in \bar{\mathcal{D}}\backslash\{\mathbf{s}_L\}$,

while

(iii) $\mathcal{H}(\tau^-(\mathbf{s}), \mathbf{s}) < 0$, *for all* $\mathbf{s} \in \mathcal{K} \cup \{\mathbf{s}_L\}$,
(iv) $\mathcal{H}(\tau^-(\mathbf{s}), \mathbf{s}) = 0$, *for all* $\mathbf{s} \in \mathcal{C} = \big\{\mathbf{s} \in \mathcal{D} \ / \ \mathbf{s} = \mathbf{s}_L + \Lambda_0(\mathbf{a})\mathbf{a},\ \mathbf{a} \in \mathcal{S}_+^N\big\}$,
(v) $\mathcal{H}(\tau^-(\mathbf{s}), \mathbf{s}) > 0$, *for all* $\mathbf{s} \in \bar{\mathcal{D}}\backslash\bar{\mathcal{K}}$.

Put in other words, the critical points of the differential system (4.22) necessarily coincide with the set $\{\tau^+(\mathbf{s}_L), \mathbf{s}_L\}$ and $\{((\tau^-(\mathbf{s}), \mathbf{s}) \ / \ \mathbf{s} \in \mathcal{C}\}$. Keeping this mind, we next analyze the roots $\tau(\mathbf{s})$ of $\mathcal{H}(\tau(\mathbf{s}), \mathbf{s})$. The following claim states that $\mathcal{H}$ admits three distinct branches of roots in $\bar{\mathcal{K}}$. A particular attention is paid to single out a branch $\mathcal{T}$ obeying the requirements:

$$\mathcal{T}(\mathbf{s}_L) = \tau^+(\mathbf{s}_L) \qquad \text{together with} \qquad \mathcal{T}(\mathbf{s}) = \tau^-(\mathbf{s}) \qquad \text{for all} \quad \mathbf{s} \in \mathcal{C}, \tag{5.14}$$

as put forward in Proposition 5.5.

Proposition 5.6. *There exist three maps in $\mathcal{C}^0(\bar{K}, \mathbb{R}_+^*) \cap \mathcal{C}^1(\mathcal{K}, \mathbb{R}_+^*)$ respectively denoted by $\check{\mathcal{T}}$, $\mathcal{T}$, $\hat{\mathcal{T}} : \bar{\mathcal{K}} \to \mathbb{R}_+^*$, so that:*

(i) $\mathcal{H}(\mathcal{T}(\mathbf{s}), \mathbf{s}) = \mathcal{H}(\check{\mathcal{T}}(\mathbf{s}), \mathbf{s}) = \mathcal{H}(\hat{\mathcal{T}}(\mathbf{s}), \mathbf{s}) = 0$, *for all* $\mathbf{s} \in \bar{\mathcal{K}}$.

These are interlaced with the roots $\tau^\pm(\mathbf{s})$ of $\mathcal{F}$ as follows:

(ii) $\check{\mathcal{T}}(\mathbf{s}) < \tau^-(\mathbf{s}) < \mathcal{T}(\mathbf{s}) < \tau^+(\mathbf{s}) < \hat{\mathcal{T}}(\mathbf{s})$, *for all* $\mathbf{s} \in \mathcal{K}$,
(iii) $\check{\mathcal{T}}(\mathbf{s}_L) < \mathcal{T}(\mathbf{s}_L) = \tau^+(\mathbf{s}_L) = \hat{\mathcal{T}}(\mathbf{s}_L)$,
(iv) $\check{\mathcal{T}}(\mathbf{s}) = \tau^-(\mathbf{s}) = \mathcal{T}(\mathbf{s}) < \hat{\mathcal{T}}(\mathbf{s})$, *for all* $\mathbf{s} \in \mathcal{C}$.

Observe that the intermediate mapping $\mathcal{T}$ fulfills the requirements (5.14) so that Theorem 5.1 is established.

We now give the proofs of Propositions 5.4 to 5.6. Proposition 5.4 relies on the following two technical lemma.

Lemma 5.7. *For all fixed $\mathbf{s} \in S^+(\mathbf{s}_L)$, $\mathcal{F}(., \mathbf{s})$ admits a unique minimum in τ we denote $\bar{\tau}(\mathbf{s})$ where $\bar{\tau} \in \mathcal{C}^1(S^+(\mathbf{s}_L), \mathbb{R}_+^*)$ with $\bar{\tau}(\mathbf{s}) < \tau_L$ for all $\mathbf{s} \in \bar{\mathcal{D}}$. This minimum obeys:*

(i) $\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) < 0$, *for all* $\mathbf{s} \in \mathcal{D} \cup \{\mathbf{s}_L\}$,
(ii) $\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = 0$, *for all* $\mathbf{s} \in \Gamma := \{\mathbf{s} \in S^+(\mathbf{s}_L) \ / \ \mathbf{s} = \mathbf{s}_L + \bar{\Lambda}(\mathbf{a})\mathbf{a}, \ \mathbf{a} \in \mathcal{S}_+^N\}$,
(iii) $\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) > 0$, *for all* $\mathbf{s} \in S^+(\mathbf{s}_L) \backslash \bar{\mathcal{D}}$,

where the set $\mathcal{D}$ has been defined in Proposition 5.4.

Lemma 5.8. *For all fixed $\mathbf{s} \in S^+(\mathbf{s}_L)$, $\mathcal{F}(., \mathbf{s})$ is strictly decreasing (respectively strictly increasing) for all $\tau \in (0, \bar{\tau}(\mathbf{s}))$ (resp. for all $\tau > \bar{\tau}(\mathbf{s})$) and achieves the following limits*

$$\lim_{\tau \to 0^+} \mathcal{F}(\tau, \mathbf{s}) = +\infty, \qquad \lim_{\tau \to +\infty} \mathcal{F}(\tau, \mathbf{s}) = +\infty.$$

As a consequence, $\mathcal{F}(\tau, \mathbf{s}) = 0$ can be solved in τ only when $\mathbf{s} \in \bar{\mathcal{D}}$, with exactly one solution when $\mathbf{s} \in \Gamma$ and exactly two solutions $\tau^\pm(\mathbf{s})$ for $\mathbf{s} \in \bar{\mathcal{D}} \backslash \Gamma$. These solutions define two maps $\tau^\pm \in \mathcal{C}^0(\bar{\mathcal{D}}, \mathbb{R}_+^*) \cap \mathcal{C}^1(\mathcal{D} \cup \{\mathbf{s}_L\}, \mathbb{R}_+^*)$ with the following properties:

(1) $\tau^-(\mathbf{s}) < \bar{\tau}(\mathbf{s}) < \tau^+(\mathbf{s}) < \tau_L$, *for all* $\mathbf{s} \in \mathcal{D}$,
(2) $\tau^-(\mathbf{s}) = \bar{\tau}(\mathbf{s}) = \tau^+(\mathbf{s}) < \tau_L$, *for all* $\mathbf{s} = \mathbf{s}_L + \bar{\Lambda}(\mathbf{a})\mathbf{a}, \ \mathbf{a} \in \mathcal{S}_+^N$,
(3) $\tau^-(\mathbf{s}_L) < \bar{\tau}(\mathbf{s}_L) < \tau^+(\mathbf{s}_L) = \tau_L$.

We now establish Lemma 5.7 underlining that the set $\mathcal{D}$ entering the Proposition 5.5 will be explicitly derived in the course of the proof.

Proof of lemma 5.7. Let $\mathbf{s}$ be fixed in $S^+(\mathbf{s})$. Arguing about the smoothness of the internal energies, the map $\tau \mapsto \mathcal{F}(., \mathbf{s})$ is at least of class $\mathcal{C}^2(\mathbb{R}_+^*)$. Easy calculations then yield for all $\tau > 0$:

$$\frac{\partial \mathcal{F}}{\partial \tau}(\tau, \mathbf{s}) = \frac{\partial p}{\partial \tau}(\tau, \mathbf{s}) + m^2, \frac{\partial^2 \mathcal{F}}{\partial \tau^2}(\tau, \mathbf{s}) = \frac{\partial^2 p}{\partial \tau^2}(\tau, \mathbf{s}).$$

On the one hand, the map $\tau \mapsto \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \mathbf{s})$ is strictly increasing in view of the genuine nonlinearity assumption (4.5) for the total pressure. On the other hand, assumptions (4.8) on the asymptotic behaviour of $\frac{\partial p}{\partial \tau}$ imply that:

$$\lim_{\tau \to 0^+} \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \mathbf{s}) = -\infty \quad \text{and} \quad \lim_{\tau \to +\infty} \frac{\partial \mathcal{F}}{\partial \tau}(\tau, \mathbf{s}) = m^2 > 0.$$

As a consequence, for all $\mathbf{s} \in S^+(\mathbf{s}_L)$, there exists a unique $\bar{\tau}(\mathbf{s}) > 0$ so that $\frac{\partial \mathcal{F}}{\partial \tau}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = 0$. This defines a map $\bar{\tau}$ in $\mathcal{C}^1\big(S^+(\mathbf{s}_L), \mathbb{R}_+^*\big)$ thanks to the implicit function theorem. Note that the assumption (4.19) on the relative Mach number implies that $\partial_\tau \mathcal{F}(\tau_L, \mathbf{s}_L) > 0$ while $\partial_\tau \mathcal{F}(\bar{\tau}(\mathbf{s}_L), \mathbf{s}_L) = 0$, therefore hypothesis (4.6) ensures:

$$\bar{\tau}(\mathbf{s}_L) < \tau_L. \tag{5.15}$$

Next, we construct the set $\mathcal{D} \subset S^+(\mathbf{s}_L)$ introduced in Proposition 5.5 when studying for existence the zeros of $\mathbf{s} \in S^+(\mathbf{s}_L) \mapsto \mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s})$. In this way, we first notice that by definition of $\bar{\tau}(\mathbf{s})$, we have for all $\mathbf{s} \in S^+(\mathbf{s}_L)$:

$$\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = p(\bar{\tau}(\mathbf{s}), s) - p(\tau_L, \mathbf{s}_L) - \frac{\partial p}{\partial \tau}(\bar{\tau}(\mathbf{s}), \mathbf{s})\big(\bar{\tau}(\mathbf{s}) - \tau_L\big).$$

Introducing the auxiliary function $\phi : \mathbb{R}_+ \times \mathcal{S}_+^N \to S^+(\mathbf{s}_L)$ defined by

$$\phi(\lambda, \mathbf{a}) = \mathcal{F}\big(\bar{\tau}(\mathbf{s}_L + \lambda \mathbf{a}), \mathbf{s}_L + \lambda \mathbf{a}\big)$$

straightforward calculations give

$$\frac{\partial \phi}{\partial \lambda}(\lambda, \mathbf{a}) = \sum_{1 \le i \le N} \left(\frac{\partial p_i}{\partial s_i} a_i + \left(\sum_{1 \le j \le N} \frac{\partial^2 p_j}{\partial \tau^2} \frac{\partial \bar{\tau}}{\partial s_i} + \frac{\partial^2 p_i}{\partial \tau \partial s_i} \right) a_i \right). \tag{5.16}$$

But differentiating the identity $\partial_\tau p(\bar{\tau}(\mathbf{s}), \mathbf{s}) = -m^2$, valid for all $\mathbf{s} \in S^+(\mathbf{s}_L)$, easily implies that (5.16) reduces to

$$\frac{\partial \phi}{\partial \lambda}(\lambda, \mathbf{a}) = \sum_{1 \le i \le N} \frac{\partial p_i}{\partial s_i} \left(\bar{\tau}(\mathbf{s}_L + \lambda \mathbf{a}), \mathbf{s}_L + \lambda \mathbf{a} \right) a_i. \tag{5.17}$$

This derivative is therefore strictly positive for all $\mathbf{a} \in \mathcal{S}_+^N$ because of (4.6). Next, arguing about the strict convexity in τ of the total pressure and the property (5.15) expressing that $\bar{\tau}(\mathbf{s}_L) < \tau_L$, we get

$$\phi(0, \mathbf{a}) = p(\bar{\tau}(\mathbf{s}_L), \mathbf{s}_L) - p(\tau_L, \mathbf{s}_L) - \frac{\partial p}{\partial \tau}(\bar{\tau}(\mathbf{s}_L), \mathbf{s}_L)\big(\bar{\tau}(\mathbf{s}_L) - \tau_L\big) < 0.$$

To conclude, we have to check that for all $\mathbf{a} \in \mathcal{S}_+^N$, the map $\lambda \mapsto \phi(\lambda, \mathbf{a})$ achieves positive values for finite values of λ. Indeed, the implicit function theorem will thus ensure the existence of a map $\bar{\Lambda} \in \mathcal{C}^1\big(\mathcal{S}_+^N, \mathbb{R}_+^*\big)$ well defined in $\mathcal{D}$ with the following property:

$$\phi(\bar{\Lambda}(\mathbf{a}), \mathbf{a}) = 0, \qquad \text{for all} \quad \mathbf{a} \in \mathcal{S}_+^N \tag{5.18}$$

with $\phi(\lambda, \mathbf{a}) < 0$ (respectively > 0) for all $\lambda < \bar{\Lambda}(\mathbf{a})$ (resp. $\lambda > \bar{\Lambda}(\mathbf{a})$). This is nothing but the required result. To establish the validity of (5.18), we show that for all $\mathbf{a} \in \mathcal{S}_+^N$ there exists $\lambda_\star(\mathbf{a}) > 0$ so that

$$\bar{\tau}(\mathbf{s}_L + \lambda_\star(\mathbf{a})) = \tau_L. \tag{5.19}$$

Indeed for such values of λ, ϕ boils down to

$$\phi(\lambda_\star(\mathbf{a}), \mathbf{a}) = p(\tau_L, \mathbf{s}) - p(\tau_L, \mathbf{s}_L) > 0,$$

because of (4.6). To derive (5.19), we introduce the auxiliary smooth function $\theta \in \mathcal{C}^1\big(\mathbb{R}_+, \mathbb{R}\big)$ defined for all $\mathbf{a} \in \mathcal{S}_+^N$ by

$$\theta(\lambda) = \frac{\partial p}{\partial \tau}(\tau_L, \mathbf{s}_L + \lambda \mathbf{a}) + m^2.$$

For any given $\mathbf{a}$ in $\mathcal{S}_+^N$, we establish the existence of solutions to $\theta(\lambda) = 0$, $\lambda_\star(a)$ being chosen to be for instance the smallest one for definiteness. Existence of such solution(s) readily follows from the assumption (4.19) on the relative Mach number ensuring that $\theta(0) > 0$ while the asymptotic condition (4.8) ensures $\theta(\lambda) < 0$ for large enough λ. Note that the solutions under consideration are strictly positive. In addition and since $\bar\Lambda(\mathbf{a}) < \lambda_\star(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{S}_+^N$, we have obtained

$$\bar\tau(\mathbf{s}) < \tau_L, \qquad \mathbf{s} \in \bar{\mathcal{D}}. \tag{5.20}$$

This concludes the proof of Lemma 5.7. □

Proof of Lemma 5.8. Let be given $\mathbf{s}$ in $S^+(\mathbf{s}_L)$. By definition of $\bar\tau(\mathbf{s})$, $\mathcal{F}(.,\mathbf{s})$ achieves the monotonicity properties stated in the Lemma 5.8, the required limits immediately follows from the asymptotic conditions (4.7). The study of the sign of $\mathcal{F}(\bar\tau(\mathbf{s}), \mathbf{s})$ as we have proposed previously, obviously implies that for fixed $\mathbf{s} \in S^+(\mathbf{s}_L)$, the equation $\mathcal{F}(\tau, \mathbf{s}) = 0$ has exactly two solutions τ^-, τ^+ in $\bar{\mathcal{D}}\backslash\Gamma$ so that $\tau^- < \bar\tau(\mathbf{s}) < \tau^+$; this equation has exactly one solution, namely $\bar\tau(\mathbf{s})$, when $\mathbf{s} \in \Gamma$ and has no solution whenever $\mathbf{s} \in S^+(\mathbf{s}_L)\backslash\bar{\mathcal{D}}$. In addition, using the notation introduced in the proof of Lemma 5.7, it can be easily seen that the following limits hold true:

$$\lim_{\lambda \to \bar\Lambda(\mathbf{a})^-} \tau^\pm(\mathbf{s}_L + \lambda \mathbf{a}) = \bar\tau(\mathbf{s}_L + \bar\Lambda(\mathbf{a})\mathbf{a}), \qquad \text{for all} \quad \mathbf{a} \in \mathcal{S}_+^N \tag{5.21}$$

These observations allow for the definition of two maps $\tau^\pm : \bar{\mathcal{D}} \to \mathbb{R}_+^*$ satisfying:

$$\mathcal{F}(\tau^\pm(\mathbf{s}), \mathbf{s}) = 0, \quad \text{for all} \quad \mathbf{s} \in \bar{\mathcal{D}},$$

and so that

$$\tau^-(\mathbf{s}) < \bar\tau(\mathbf{s}) < \tau^+(\mathbf{s}), \quad \mathbf{s} \in \bar{\mathcal{D}}\backslash\Gamma; \quad \tau^-(\mathbf{s}) = \bar\tau(\mathbf{s}) = \tau^+(\mathbf{s}), \quad \mathbf{s} \in \Gamma. \tag{5.22}$$

Then the above inequalities yield $\partial_\tau \mathcal{F}(\tau^\pm(\mathbf{s}), \mathbf{s}) \neq 0$ for all $\mathbf{s} \in \mathcal{D} \cup \{\mathbf{s}_L\}$ in view of (5.22) so that the implicit function theorem ensures that $\tau^\pm \in \mathcal{C}^1(\mathcal{D} \cup \{\mathbf{s}_L\}, \mathbb{R}_+^*)$ while (5.21) gives that $\tau^\pm \in \mathcal{C}^0(\bar{\mathcal{D}}, \mathbb{R}_+^*)$.

Next, focusing to some given $\mathbf{s} \in \bar{\mathcal{D}}\backslash\{\mathbf{s}_L\}$, we observe that

$$\mathcal{F}(\tau_L, \mathbf{s}) = p(\tau_L, \mathbf{s}) - p(\tau_L, \mathbf{s}_L) > 0$$

so that necessarily, either $\tau_L < \tau^-(\mathbf{s})$ or $\tau^+(\mathbf{s}) < \tau_L$. In addition, the identity $\mathcal{F}(\tau_L, \mathbf{s}_L) = 0$ expresses that either $\tau^-(\mathbf{s}_L) = \tau_L$ or $\tau^+(\mathbf{s}_L) = \tau_L$. But Lemma 5.7 ensures that $\bar\tau(\mathbf{s}) < \tau_L$ for all $\mathbf{s} \in \bar{\mathcal{D}}$. This concludes the proof. □

Equipped with these two lemmas, the proof of Proposition 5.4 is essentially completed: the required inequality $\bar\Lambda(\mathbf{a}) > \Lambda_0(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{S}_+^N$ will be deduced from the derivation of the set $\mathcal{K}$ we propose hereafter. We will need the following technical result.

Lemma 5.9. *For all* $\mathbf{s} \in \Gamma$, $\mathcal{H}(\bar\tau(\mathbf{s}), \mathbf{s}) > 0$.

This statement actually indicates that there is no critical point on Γ.

Proof. To shorten the notation, let us introduce

$$\epsilon(\tau, \mathbf{s}) = \sum_{i=1}^N \epsilon_i(\tau, s_i),$$

and consider the auxiliary function $\psi \in \mathcal{C}^1(\Gamma, \mathbb{R})$ defined for all $\mathbf{s} \in \Gamma$ by

$$\psi(\mathbf{s}) = \epsilon(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}) + H_L(\bar{\tau}(\mathbf{s}) - \tau_L) - \frac{m^2}{2}(\bar{\tau}(\mathbf{s})^2 - \tau_L^2).$$

Here we have set

$$H_L = m^2 \tau_L + p(\tau_L, \mathbf{s}_L),$$

so that $\mathcal{H}(\tau, \mathbf{s})$ recasts as

$$\mathcal{H}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = \psi(\mathbf{s}) + \epsilon(\tau_L, \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}_L).$$

Arguing about the identity $\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = 0$ valid for all $\mathbf{s} \in \Gamma$ (see Lemma 5.7.(ii)), we have

$$H_L = p(\bar{\tau}(\mathbf{s}), \mathbf{s}) + m^2 \bar{\tau}(\mathbf{s}), m^2(\bar{\tau}(\mathbf{s}) - \tau_L) = p(\tau_L, \mathbf{s}_L) - p(\bar{\tau}(\mathbf{s}), \mathbf{s}),$$

which gives successively for all $\mathbf{s} \in \Gamma$:

$$\begin{aligned} \psi(\mathbf{s}) = & \quad \epsilon(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}) + (\bar{\tau}(\mathbf{s}) - \tau_L)\left(p(\bar{\tau}(\mathbf{s}), \mathbf{s}) + \frac{m^2}{2}(\bar{\tau}(\mathbf{s}) - \tau_L)\right), \\ = & \quad \epsilon(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}) + (\bar{\tau}(\mathbf{s}) - \tau_L)\left(p(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \frac{1}{2}(p(\bar{\tau}(\mathbf{s}), \mathbf{s}) - p(\tau_L, s_L))\right). \end{aligned}$$

Moreover, the two identities $\mathcal{F}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = 0$ and $\frac{\partial \mathcal{F}}{\partial \tau}(\bar{\tau}, \mathbf{s}) = 0$ valid for all $\mathbf{s} \in \Gamma$ are easily seen to give for the $\mathbf{s}$ under consideration:

$$p(\bar{\tau}(\mathbf{s}), \mathbf{s}) - p(\tau_L, \mathbf{s}_L) = (\bar{\tau}(\mathbf{s}) - \tau_L)\frac{\partial p}{\partial \tau}(\bar{\tau}(\mathbf{s}), \mathbf{s}).$$

Consequently, for all $\mathbf{s} \in \Gamma$

$$\psi(\mathbf{s}) = \epsilon(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}) + (\bar{\tau}(\mathbf{s}) - \tau_L)p(\bar{\tau}(\mathbf{s}), \mathbf{s}) - \frac{1}{2}(\bar{\tau}(\mathbf{s}) - \tau_L)^2 \frac{\partial p}{\partial \tau}(\bar{\tau}(\mathbf{s}), \mathbf{s}).$$

To conclude, we show that

$$\mathcal{H}(\bar{\tau}(\mathbf{s}), \mathbf{s}) = \theta(\bar{\tau}(\mathbf{s}), \mathbf{s}) + \epsilon(\tau_L, \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}_L) > 0. \tag{5.23}$$

Since $\frac{\partial \epsilon_i}{\partial s_i}(\tau_L, s_i) = T_i(\tau_L, s_i) > 0$ then $\epsilon(\tau_L, \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}_L) > 0$ for all $s \in \Gamma$ since $\mathbf{s}_L \notin \Gamma$. Indeed, observe that Lemma 5.8 implies that equality to zero holds iff $\mathbf{s} = \mathbf{s}_L$ but $\mathbf{s}_L \notin \Gamma$.

To show (5.23), we study the following auxiliary function $\Psi \in \mathcal{C}^1\left(\mathbb{R}_+^*, \mathbb{R}\right)$, setting for fixed $s \in \Gamma$:

$$\Psi(\tau) = \epsilon(\tau, \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}) + (\tau - \tau_L)p(\tau, \mathbf{s}) - \frac{1}{2}(\tau - \tau_L)^2 \frac{\partial p}{\partial \tau}(\tau, \mathbf{s}).$$

Easy calculations give

$$\frac{\partial \Psi}{\partial \tau}(\tau) = -\frac{1}{2}(\tau - \tau_L)^2 \frac{\partial^2 p}{\partial \tau^2}(\tau, \mathbf{s}) \leq 0,$$

with $\Psi(\tau_L) = 0$. Consequently, $\Psi(\tau) > 0$ for all $\tau < \tau_L$. Since $\bar{\tau}(\mathbf{s}) < \tau_L$ (see Lemma 5.7) then $\Psi(\bar{\tau}(\mathbf{s})) > 0$ for all $\mathbf{s} \in \Gamma$. We have thus obtained the required inequality : $\mathcal{H}(\bar{\tau}(\mathbf{s}), \mathbf{s}) > 0$. □

Proof of Proposition 5.5. We first establish the required properties of $\mathcal{H}$ related to the branch of solutions τ^+. Arguing about the identity $\tau^+(\mathbf{s}) = \bar{\tau}(\mathbf{s})$ for all $\mathbf{s} \in \Gamma$, the technical Lemma 5.9 allows to restrict ourselves to $\mathbf{s} \in \bar{\mathcal{D}} \backslash \Gamma$ where τ^+ is continuously differentiable. For such $\mathbf{s}$, the identity $\mathcal{F}(\tau^+(\mathbf{s}), \mathbf{s}) = 0$ re-expresses equivalently:

$$m^2\big(\tau^+(\mathbf{s}) - \tau_L\big) = \big(p(\tau_L, \mathbf{s}_L) - p(\tau^+(\mathbf{s}), \mathbf{s})\big)\,,. \tag{5.24}$$

Let us evaluate $\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s})$ as follows:

$$\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) = \epsilon(\tau^+(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}_L) + (\tau^+(\mathbf{s}) - \tau_L)\left(p(\tau^+(\mathbf{s}), s) + \frac{m^2}{2}(\tau^+(\mathbf{s}) - \tau_L)\right)$$

where $\epsilon(\tau, \mathbf{s}) = \sum_{i=1}^{N} \epsilon_i(\tau, s_i)$. Using (5.24), we then obtain

$$\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) = \epsilon(\tau^+(\mathbf{s}), \mathbf{s}) - \epsilon(\tau_L, \mathbf{s}_L) - \frac{1}{2m^2}\left(p^2(\tau^+(\mathbf{s}), \mathbf{s}) - p^2(\tau_L, \mathbf{s}_L)\right).$$

Let us introduce the auxiliary function $\Theta : \mathbb{R}_+^* \times \bar{\mathcal{D}} \to \mathbb{R}$ by setting

$$\Theta(\tau, \mathbf{s}) = \epsilon(\tau, \mathbf{s}) - \frac{1}{2m^2} p^2(\tau, \mathbf{s}),$$

so that for all $\mathbf{s} \in \bar{\mathcal{D}}$:

$$\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) = \Theta(\tau^+(\mathbf{s}), \mathbf{s}) - \Theta(\tau_L, \mathbf{s}_L), \tag{5.25}$$

with $\mathbf{s} \mapsto \Theta(\tau^+(\mathbf{s}), \mathbf{s}) \in \mathcal{C}^1(\bar{\mathcal{D}}\backslash\Gamma, \mathbb{R})$. Since $\tau^+(\mathbf{s}_L) = \tau_L$ by Lemmas 5.8 and 5.25 reads equivalently:

$$\mathcal{H}(\tau^+(\mathbf{s}), \mathbf{s}) = \Theta(\tau^+(\mathbf{s}), \mathbf{s}) - \Theta(\tau^+(\mathbf{s}_L), \mathbf{s}_L).$$

Moreover, we have for all $\mathbf{s} \in \bar{\mathcal{D}}\backslash\Gamma$:

$$\begin{aligned}
\frac{\partial}{\partial s_i}\theta(\tau^+(\mathbf{s}), \mathbf{s}) &= \frac{\partial \tau^+}{\partial s_i}(\mathbf{s})\frac{\partial \mathcal{H}}{\partial \tau}(\tau^+(\mathbf{s}), \mathbf{s}) + \frac{\partial \mathcal{H}}{\partial s_i}(\tau^+(\mathbf{s}), \mathbf{s}), \\
&= -\frac{\partial \tau^+}{\partial s_i}(\mathbf{s})\mathcal{F}(\tau^+(\mathbf{s}), \mathbf{s}) + \frac{\partial \epsilon_i}{\partial s_i}(\tau^+(\mathbf{s}), \mathbf{s}) \\
&= \frac{\partial \epsilon_i}{\partial s_i}(\tau^+(\mathbf{s}), \mathbf{s}) = T_i(\tau^+(\mathbf{s}), \mathbf{s}) > 0, \qquad (5.26)
\end{aligned}$$

where we have used the identity $\mathcal{F}(\tau^+(\mathbf{s}), \mathbf{s}) = 0$. Consequently, we deduce that

$$\theta(\tau^+(\mathbf{s}), \mathbf{s}) - \theta(\tau^+(\mathbf{s}_L), \mathbf{s}_L) \geq 0, \ \forall \mathbf{s} \in \bar{\mathcal{D}}\backslash\Gamma$$

with equality to zero iff $\mathbf{s} = \mathbf{s}_L$ (see Lemma 5.8). Combining the previous steps with Lemma 5.9 gives the required properties (i) and (ii).

We now derive the remaining properties of $\mathcal{H}$ related to τ^-. Observe that the technical Lemma 5.9 immediately gives

$$\mathcal{H}(\tau^-(\mathbf{s}), \mathbf{s}) > 0, \quad \mathbf{s} \in \Gamma, \tag{5.27}$$

since $\tau^-(\mathbf{s}) = \bar{\tau}(\mathbf{s})$ for the $\mathbf{s}$ under consideration. We can now obtain the following estimate

$$\mathcal{H}(\tau^-(\mathbf{s}_L), \mathbf{s}_L) < 0.$$

To that purpose, let us introduce the following auxiliary function $\psi : \mathbb{R}_+^* \to \mathbb{R}$ setting:

$$\psi(\tau) = \mathcal{H}(\tau, \mathbf{s}_L).$$

Since $\psi'(\tau) = -\mathcal{F}(\tau, \mathbf{s}_L)$ for all $\tau > 0$, Lemma 5.8 is easily seen to imply that ψ strictly increases in $\Big(\tau^-(\mathbf{s}_L), \tau^+(\mathbf{s}_L)\Big)$ with $\mathcal{H}(\tau^+(\mathbf{s}_L), \mathbf{s}_L) = 0$ as we have just established. This yields inequality (5).

To conclude the proof, we follow exactly the same steps as those developed in the proof of Lemma 5.7 devoted to the derivation of the subset $\mathcal{K} \in S^+(\mathbf{s}_L)$.

We introducing the following auxiliary function defined by

$$\Phi(\lambda, \mathbf{a}) = \mathcal{H}\big(\tau^-(\mathbf{s}_L + \lambda\mathbf{a}), \mathbf{s}_L + \lambda\mathbf{a}\big), \qquad \mathbf{a} \in \mathcal{S}_+^N, \quad \lambda \in [0, \bar{\Lambda}(\mathbf{a})[.$$

Note that this function is continuously differentiable on its domain of definition since, in view of Lemma 5.8, the function $(\tau, \lambda) \mapsto \tau^-(\mathbf{s}_L + \lambda\mathbf{a})$ is differentiable.

Straightforward calculations then give

$$\begin{aligned}\frac{\partial\Phi}{\partial\lambda}(\lambda,\mathbf{a}) =& -\mathcal{F}(\tau^-(\mathbf{s}_L+\lambda\mathbf{a}),\mathbf{s}_L+\lambda\mathbf{a})\left(\sum_{1\le i\le N}\frac{\partial\tau^-}{\partial s_i}a_i\right)\\ &+\sum_{1\le i\le N}T_i\left(\tau^-(\mathbf{s}_L+\lambda\mathbf{a}),\mathbf{s}_L+\lambda\mathbf{a}\right)a_i.\end{aligned} \tag{5.28}$$

But the following identity $\mathcal{F}(\tau^-(\mathbf{s}_L+\lambda\mathbf{a}),\mathbf{s}_L+\lambda\mathbf{a})=0$ holds true by definition for all $\mathbf{a}\in\mathcal{S}_+^N$ and $\lambda\in[0,\bar\Lambda(\mathbf{a})]$ so that (5.28) reduces to

$$\frac{\partial\Phi}{\partial\lambda}(\lambda,\mathbf{a})=\sum_{1\le i\le N}T_i(\tau^-(\mathbf{s}_L+\lambda\mathbf{a}),\mathbf{s}_L+\lambda\mathbf{a})a_i>0.$$

Arguing about the inequalities (5.27) and (5), the implicit function theorem implies the existence of a map $\Lambda_0\in\mathcal{C}^1(\mathcal{S}_+^N,\mathbb{R}_+^*)$ with the following properties:

$$\Phi(\Lambda_0(\mathbf{a}),\mathbf{a})=0,\qquad \mathbf{a}\in\mathcal{S}_+^N,$$

together with $\Phi(\lambda,\mathbf{a})<0$ for all $\lambda\in[0,\Lambda_0(\mathbf{a})[$ and $\Phi(\lambda,\mathbf{a})>0$ for all $\lambda\in\,]\Lambda_0(\mathbf{a}),\bar\Lambda(\mathbf{a})]$. This concludes the proof of Proposition 5.5. □

Proof of Proposition 5.6. Arguing about the identity $\partial_\tau\mathcal{H}(\tau,\mathbf{s})=-\mathcal{F}(\tau,\mathbf{s})$ valid for all $(\tau,\mathbf{s})\in\mathbb{R}_+^*\times\mathcal{K}$, Lemma 5.8 immediately implies that the smooth map $\tau\mapsto\mathcal{H}(\tau,\mathbf{s})$, $\mathbf{s}$ being fixed in $\mathcal{K}$, strictly decreases in $]0,\tau^-(s)[$ and $]\tau^+(s),+\infty[$ while it strictly increases in $]\tau^-(\mathbf{s}),\tau^+(\mathbf{s})[$ with the following limits $\lim_{\tau\to0^+}\mathcal{H}(\tau,\mathbf{s})=+\infty$ and $\lim_{\tau\to\infty}\mathcal{H}(\tau,\mathbf{s})=-\infty$ in view of the asymptotic conditions 4.3. In addition, for all $\mathbf{s}\in\mathcal{K}$, we infer from Proposition 5.5 that $\mathcal{H}(\tau^-(\mathbf{s}),\mathbf{s})<0$ and $\mathcal{H}(\tau^+(\mathbf{s}),\mathbf{s})>0$. These observations allow the definition of three maps, namely $\check{\mathcal{T}}$, $\mathcal{T}$, $\hat{\mathcal{T}}:\mathcal{K}\to\mathbb{R}_+^*$ with the following properties:

$$\mathcal{H}(\check{\mathcal{T}}(\mathbf{s}),\mathbf{s})=\mathcal{H}(\mathcal{T}(\mathbf{s}),\mathbf{s})=\mathcal{H}(\hat{\mathcal{T}}(\mathbf{s}),\mathbf{s})=0,\qquad \mathbf{s}\in\mathcal{K},$$

together with

$$0<\check{\mathcal{T}}(\mathbf{s})<\tau^-(\mathbf{s})<\mathcal{T}(\mathbf{s})<\tau^+(\mathbf{s})<\hat{\mathcal{T}}(\mathbf{s}),\qquad \mathbf{s}\in\mathcal{K}.$$

Next, using the notation introduced in the proof of Lemma 5.7, we first compute for all $\mathbf{a}\in\mathcal{S}_+^N$:

$$\lim_{\lambda\to0^+}\check{\mathcal{T}}(\mathbf{s}_L+\lambda\mathbf{a})<\lim_{\lambda\to0^+}\mathcal{T}(\mathbf{s}_L+\lambda\mathbf{a})=\lim_{\lambda\to0^+}\hat{\mathcal{T}}(\mathbf{s}_L+\lambda\mathbf{a})=\tau^+(\mathbf{s}_L),$$

since $\mathcal{H}(\tau^-(\mathbf{s}_L),\mathbf{s}_L)<\mathcal{H}(\tau^+(\mathbf{s}_L),\mathbf{s}_L)=0$ in view of (iii) and (i) in Proposition 5.5. In the same way, we get

$$\begin{aligned}\lim_{\lambda\to\Lambda_0(\mathbf{a})}\check{\mathcal{T}}(\mathbf{s}_L+\lambda\mathbf{a})&=\lim_{\lambda\to\Lambda_0(\mathbf{a})}\mathcal{T}(\mathbf{s}_L+\lambda\mathbf{a})\\ &=\tau^-(\mathbf{s}_L+\Lambda_0(\mathbf{a})\mathbf{a})<\lim_{\lambda\to\Lambda_0(\mathbf{a})}\hat{\mathcal{T}}(\mathbf{s}_L+\lambda\mathbf{a}),\end{aligned}$$

since $\mathcal{H}(\tau^-(\mathbf{s}),\mathbf{s})=0<\mathcal{H}(\tau^+(\mathbf{s}),\mathbf{s})$ for all $\mathbf{s}\in\mathcal{C}$, in view of Proposition 5.5. To conclude, we have to establish the smoothness properties that were put forward in Proposition 5.6. Because of the monotonicity properties of $\tau\mapsto\mathcal{H}(\tau,\mathbf{s})$ we have just established for all $\mathbf{s}\in\mathcal{K}$, all the three maps are obviously in $\mathcal{C}^1\big(\mathcal{K},\mathbb{R}_+^*\big)\cap\mathcal{C}^0\big(\bar{\mathcal{K}},\mathbb{R}_+^*\big)$ thanks to the implicit function theorem. This concludes the proof of Proposition 5.6. □

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