

NONLINEAR PROJECTION METHODS FOR MULTI-ENTROPIES NAVIER-STOKES SYSTEMS

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ABSTRACT. This paper is devoted to the numerical approximation of the compressible Navier-Stokes equations with several independent entropies. Various models for complex compressible materials typically enter the proposed framework. The striking novelty over the usual Navier-Stokes equations stems from the generic impossibility to recast equivalently the present system in full conservation form. Classical finite volume methods are shown to grossly fail in the capture of viscous shock solutions that are of primary interest in the present work. To enforce for validity a set of generalized jump conditions we introduce, we propose a systematic and effective correction procedure, the so-called nonlinear projection method, and prove that it preserves all the stability properties satisfied by suitable Godunov type methods. Numerical experiments assess the relevance of the method when exhibiting approximate solutions in close agreement with exact solutions.

1. INTRODUCTION

The present work treats the numerical approximation of the solutions of the Navier-Stokes equations for a compressible fluid modelled by N , $N \geq 2$, independent pressures. By independent, we mean that the pressures come with their own specific entropy and must be thus governed by N independent entropy balance equations. Then the sum of all the pressures yields the total pressure in the fluid.

Despite that such PDEs models are seen hereafter to exhibit several close relationships with the usual Navier-Stokes equations, the fundamental discrepancy stays in the lack of an admissible change of variables that recasts the present $(N+2) \times (N+2)$ system in full conservation form. Indeed, the three classical conservation laws ruling density, momentum and total energy must be supplemented by $(N-1)$ of the entropy balance equations that govern the N pressures. However, none of these entropy balance equations boils down to a conservation law without restrictive modelling assumptions to be put on the N viscosity laws. These equations generally involve non conservative products that account for dissipative phenomena: namely the entropy dissipation rates.

Such systems can be understood as a natural extension of the classical Navier-Stokes equations, i.e. equipped with a single entropy balance equation, in that they actually occur in several distinct settings from the physics. They arise for instance in plasma physics where the electrons temperature must be distinguished from the mixture temperature of the other (heavy) species (see [6] for instance). After Berthon [1], they are also recognized under that form within the frame of the so-called “two transport” equations models for turbulent compressible flows where the averaged thermodynamic temperature must be distinguished from the specific

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turbulent kinetic energy k . The comprehensive study [1] actually shows that the (k, ϵ) , (k, ω) or (k, l) models of prominent practical importance enter the present framework (see [18, 19], [20], [24] for a derivation from the physics). An important issue is that all these models are usually addressed with the assumption of a very large Reynolds number. From a practical point of view, the Reynolds number is typically larger than 10^6 .

The numerical capture of the viscous shock layers coming with the Navier-Stokes extended system, is the heart of the present work. Since the Reynolds numbers of interest are large, these layers display the character of a shock wave in that they differ from their end states only in a small interval of rapid transition. Hence for mesh refinements of practical interest, the associated discrete profiles stay largely under resolved (see figure 1 for the negative consequence within the frame of a classical splitting numerical method we briefly described hereafter). Our purpose is actually to properly capture the two end states, $\mathbf{u}_L$ and $\mathbf{u}_R$, of a given shock layer together with its relevant speed of propagation σ without resolving the viscous layer itself. The correct prediction of these end states should be achieved independently of the mesh refinement as far as possible.

It is quite well-known that such an issue does not raise special difficulties within the standard frame of the Navier-Stokes equations, e.g. in conservation form. Indeed, the triples $(\sigma; \mathbf{u}_L, \mathbf{u}_R)$ are solutions of the classical Rankine-Hugoniot relations (see Gilbarg [10]) and the celebrated finite volume methods in conservation form readily ensure their suitable capture (see Lax-Wendroff [16]). At the very core of this success, is that the end states just solve the usual jump conditions that are obviously free from the underlying viscous regularization: *i.e.* the precise definition of (say) the right end state $\mathbf{u}_R = \mathbf{u}_R(\sigma; \mathbf{u}_L)$ stays free from the exact modelling of the diffusive phenomena. For this reason and loosely speaking, the numerical capture of this end state stays fairly free from the exact form of the numerical dissipation artificially induced by a scheme in conservation form.

The situation is however completely different in the setting of the extended Navier-Stokes system we tackle. Its non conservation form makes this time the exact definition of the end state $\mathbf{u}_R(\sigma; \mathbf{u}_L)$ to heavily depend on the precise shape of the diffusive tensor, since there does no longer exist a full set of classical jump conditions. The missing information in the jump is primarily dictated by the underlying viscous regularization. The reader can consult at this stage the Figures 2 for convincing numerical illustrations reporting the strong sensitiveness of exact end states $\mathbf{u}_R(\sigma; \mathbf{u}_L)$ to the ratio of viscosities entering the present extended Navier-Stokes model. Let us underline that such a dependence stays at the basis of recent works devoted to hyperbolic systems involving non conservative products (see Lefloch [17], Dal Maso-LeFloch-Murat [8], Raviart-Sainsaulieu [21]). When rephrased in the setting of numerical methods, this sensitivity translates into a heavy dependence of the approximate solutions to the underlying artificial viscosity. Loosely speaking, expecting a good agreement with exact solutions urges the numerical viscosity matrix entering the discrete method (see Tadmor [26]) to fit in some sense the exact diffusive tensor involved in the PDE's. Such a vague sense has received a theoretical foundation in a work by Hou-LeFloch [12] (see also [4] for a complementary standpoint) and is actually the starting point of some numerical methods as devised by Karni [13], for instance. In this work, fitness is measured via a modified equations approach according to which unsuited truncation errors are forced to cancel up to a given order. Motivated by Hou and LeFloch [12], we propose a powerful and complementary approach based on the analysis of the discrete entropy dissipation rates coming with a given numerical method. We refer the reader to Tadmor [26] for a precise link between numerical viscosity matrix and discrete entropy rate. We exhibit in the present setting underlying generalized jump conditions, in sufficient number, expressing a set of $(N - 1)$ key balance properties

between each of the N entropy dissipation rates. When added to the three usual jump relations for density, momentum and total energy, they yield a complete set of conditions but relating $(N+3)$ quantities in their jump for only $(N+2)$ independent variables. The additional quantity is nothing but a Lax mathematical entropy for the extended Navier-Stokes system. As we prove hereafter, the refinement free capture of the end states of viscous shock layers requires the exact preservation of this complete set of generalized jump conditions. At the discrete level, we have to face the non standard and difficult issue of controlling a prescribed dissipation rate in the additional Lax entropy. In order to illustrate the negative consequence of a failure in the above requirement, let us consider the numerical results displayed in figure 1. These are obtained using a classical splitting method briefly derived in the forthcoming numerical section. This problem enters the present setting for

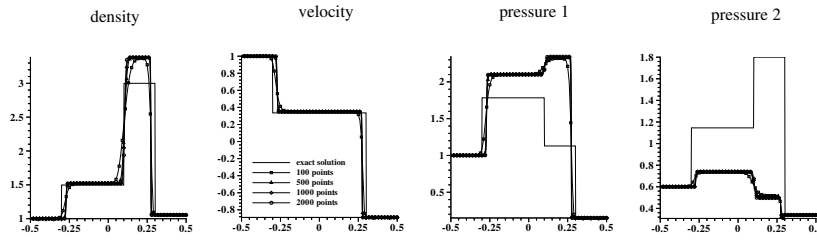


FIGURE 1. Typical failure of classical finite volume methods in the capture of viscous shock layers

extended Navier-Stokes equations with $N = 2$ when choosing two strictly positive viscosity coefficients. The discrete solutions seem to coincide with a given function. However such a function exhibits unacceptable errors with the exact solution plotted in solid lines. Let us underline that the two extreme discontinuities in the approximate solutions formally coincide with two viscous shock layers. In agreement with our main purpose, these layers are under-resolved for the mesh refinements under consideration.

For the sake of simplicity in the notations, we focus ourselves in the sequel to the case of two independent specific entropies. The nonlinear projection method we propose will be seen to easily extend to additional independent entropies.

The format of the present paper is as follows. The second section describes the extended Navier-Stokes system, paying attention to the main properties of its travelling wave solutions (i.e. viscous shock layers). A close characterization of the triples $(\sigma; \mathbf{u}_L, \mathbf{u}_R)$ will be proposed on the ground of a complete set of generalized jump relations, encoding the small scale effects taking place within viscous shock layers. The third section is devoted to the numerical approximation of solutions of the model with stiff viscous shocks for mesh refinements of practical interest. Such solutions are thus by definition under resolved. Classical splitting schemes built from suitable Godunov type methods are shown to fail in satisfying the set of generalized jump conditions. The error analysis suggests then the introduction of a correction step, the so-called nonlinear projection procedure, to enforce for validity these generalized jump conditions. This new procedure achieves this requirement fairly independently of the mesh refinement. The desirable stability properties of relevant Godunov methods are shown to be preserved by this procedure. Finally in the last section, several numerical results are displayed, intending to illustrate the benefits of the proposed nonlinear projection.

2. THE MATHEMATICAL MODEL

We consider a gas with density $\rho > 0$ and velocity u which is modelled by 2 general pressure laws $p_1(\tau, \epsilon_1)$ and $p_2(\tau, \epsilon_2)$, where $\tau = 1/\rho$ is the specific volume and $\epsilon_j > 0$ ($j = 1, 2$) denote two independent specific internal energies. Each pressure law $p_j(\tau, \epsilon_j)$ is assumed to obey the second law of the thermodynamics: namely, there exists a specific entropy $s_j(\tau, \epsilon_j)$, $s_j : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$, ($j = 1, 2$) satisfying for some temperature $T_j(\tau, \epsilon_j) > 0$:

$$(2.1) \quad -T_j ds_j = d\epsilon_j + p_j d\tau, \quad j = 1, 2,$$

with the following properties:

$$(2.2) \quad \text{the mapping } (\tau, \epsilon_j) \rightarrow s_j(\tau, \epsilon_j) \text{ is strictly convex,}$$

and

$$(2.3) \quad \frac{\partial s_j}{\partial \tau}(\tau, \epsilon_j) = -\frac{p_j(\tau, \epsilon_j)}{T_j(\tau, \epsilon_j)} < 0, \quad \frac{\partial s_j}{\partial \epsilon_j}(\tau, \epsilon_j) = -\frac{1}{T_j(\tau, \epsilon_j)} < 0, \quad j = 1, 2.$$

Next, each of the specific entropies is asked without restriction to meet the following asymptotic conditions for any given fixed $\tau > 0$:

$$(2.4) \quad \lim_{\epsilon_j \rightarrow 0+} s_j(\tau, \epsilon_j) = +\infty, \quad \lim_{\epsilon_j \rightarrow +\infty} s_j(\tau, \epsilon_j) = -\infty, \quad j = 1, 2.$$

To avoid unnecessary technical details, we consider pressure laws with the following standard non-linearity property:

$$(2.5) \quad \text{the function } \tau \rightarrow p_j(\tau, s_j), \quad s_j \text{ being fixed, is strictly convex, } \quad j = 1, 2,$$

where (τ, s_1, s_2) are chosen as admissible thermodynamic variables (see (2.3)).

The 4×4 system of PDE's that governs the fluid model under consideration writes:

$$(2.6) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1(\tau, \epsilon_1) + p_2(\tau, \epsilon_2)) = \partial_x ((\mu_1 + \mu_2) \partial_x u), \\ \partial_t \rho \epsilon_j + \partial_x \rho \epsilon_j u + p_j(\tau, \epsilon_j) \partial_x u = \mu_j (\partial_x u)^2, \quad j = 1, 2; \end{cases}$$

and is associated with the following natural phase space:

$$(2.7) \quad \Omega_{\mathbf{u}} = \{\mathbf{u} := (\rho, \rho u, \rho \epsilon_1, \rho \epsilon_2)^T \in \mathbb{R}^4 \mid \rho > 0, \rho u \in \mathbb{R}, \rho \epsilon_j > 0, j = 1, 2\}.$$

To shorten the notations, the system (2.6) is given the following condensed form:

$$(2.8) \quad \partial_t \mathbf{u} + \mathcal{A}(\mathbf{u}) \partial_x \mathbf{u} = \mathcal{E}(\mathbf{u}, \partial_{xx}^2 \mathbf{u}), \quad x \in \mathbb{R}, \quad t > 0.$$

This convection-diffusion system can be understood as an extension of the standard compressible Navier-Stokes equations when considering an additional PDE for governing an additional pressure. Owing to the classical thermodynamic assumptions (2.1)–(2.2), the first order underlying system in (2.6) is easily seen to be hyperbolic over the phase space $\Omega_{\mathbf{u}}$ with eigenvalues $\lambda_1(\mathbf{u}) = u - c(\mathbf{u})$, $\lambda_2(\mathbf{u}) = \lambda_3(\mathbf{u}) = u$ and $\lambda_4(\mathbf{u}) = u + c(\mathbf{u})$. Here, the sound speed $c(\mathbf{u})$ follows from $c^2(\mathbf{u}) = c_1^2(\mathbf{u}) + c_2^2(\mathbf{u})$ with $c_j^2(\mathbf{u}) = \partial_\rho p_j(\rho, s_j)_{s_j} > 0$ (see (2.2)–(2.3)). Invoking the convexity property (2.5), the two extreme fields are seen to be genuinely nonlinear while the two intermediate ones are linearly degenerate. Considering the diffusive effects, we focus hereafter our attention on the interplay of the viscosity coefficients and we neglect the Fourier laws. Depending on the closure relations for the viscosities μ_1, μ_2 and the pressure laws p_1, p_2 , several distinct models from the physics enter the present framework. The reader is referred to the Introduction for examples motivated by complex compressible materials and to the work by Berthon [1] for a comprehensive analysis of compressible turbulence models.

In most of the present paper, the viscosity coefficients are assumed to be fixed positive constants for the sake of simplicity in the discussion. The reader is referred to section 3.2.3 for the case of varying coefficients.

In close symmetry with the classical Navier-Stokes equations, the smooth solutions of system (2.6) obey the following additional non trivial scalar equations:

Lemma 2.1. *Smooth solutions $\mathbf{u} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \Omega_{\mathbf{u}}$ of (2.6) satisfy the following conservation law:*

$$(2.9) \quad \partial_t \rho E(\mathbf{u}) + \partial_x (\rho E(\mathbf{u}) + p_1(\tau, \epsilon_1) + p_2(\tau, \epsilon_2))u = \partial_x ((\mu_1 + \mu_2)u \partial_x u),$$

where the total energy ρE is defined by:

$$(2.10) \quad \rho E(\mathbf{u}) = \frac{(\rho u)^2}{2\rho} + \rho \epsilon_1 + \rho \epsilon_2.$$

These smooth solutions satisfy in addition the following balance equations:

$$(2.11) \quad \partial_t \rho s_j(\mathbf{u}) + \partial_x \rho s_j(\mathbf{u})u = -\frac{\mu_j}{T_j}(\partial_x u)^2, \quad j = 1, 2.$$

Consequently, smooth solutions of (2.6) obey:

$$(2.12) \quad \mu_2 T_1 (\partial_t \rho s_1(\mathbf{u}) + \partial_x \rho s_1(\mathbf{u})u) - \mu_1 T_2 (\partial_t \rho s_2(\mathbf{u}) + \partial_x \rho s_2(\mathbf{u})u) = 0.$$

The proof of this easy result is left to the reader but note in passing that either ρE , ρs_1 or ρs_2 may serve as independent variables in place of the specific internal energies (see again (2.3)). At the expense of assuming that one of the two viscosities identically vanishes, say μ_2 , then the admissible change of variable $(\rho, \rho u, \rho E, \rho s_2)$ clearly gives rise from (2.9) and (2.11) with $j = 2$ to a system in conservation form for governing the smooth solutions of (2.6). But, the two viscosity coefficients must generally be prescribed with strictly positive values according to the physics. The key point is then that the three balance equations (2.9) and (2.11) can be actually proved to be the only non trivial additional equations for smooth solutions [1] (up to some standard nonlinear transforms in s_1 and s_2). Despite of several close relationships with the usual Navier-Stokes system, the very discrepancy stays in the lack of four non trivial conservation laws for governing four independent variables. Without restrictive modelling assumptions (see below), none of the equations (2.11) and (2.12) boils down to a conservation law. As a consequence, the PDE's model (2.6) cannot recast, generally speaking, in full conservation form.

Our purpose here is to highlight after the pioneering works by LeFloch [17], Raviart-Sainsaulieu [21] and Sainsaulieu [23] that the non conservation form met by (2.6) makes the end states of viscous shock layers to intrinsically depend on the shape of the diffusive operator: here namely on the ratio of the viscosity coefficients μ_1 and μ_2 . In order to assess this issue, let us focus our attention on the non standard balance equation (2.12) when temporarily adopting the next restrictive modeling assumptions on the viscosity coefficients:

Lemma 2.2. *Let μ_1^0 and μ_2^0 denote two positive constants and assume that $\mu_1 = \mu_1^0 T_1$, $\mu_2 = \mu_2^0 T_2$. Then (2.12) coincides with the following conservation law:*

$$(2.13) \quad \partial_t \rho (\mu_2^0 s_1 - \mu_1^0 s_2) + \partial_x \rho (\mu_2^0 s_1 - \mu_1^0 s_2) u = 0.$$

The above assumptions therefore allow for an additional non trivial conservation law encoded in the non standard balance equation (2.12). The associated Rankine-Hugoniot condition clearly indicates that the end states of the viscous shock layers under consideration actually depend on the ratio μ_2^0/μ_1^0 . The dependence we have just pointed out is numerically illustrated in figure 2 in a more general setting in which the system (2.6) does not admit an equivalent full conservation form. For a given left end state $\mathbf{u}_L$ and a given velocity σ , the required right end states $\mathbf{u}_R$ are defined when solving numerically the nonlinear ODE's system governing travelling wave solutions (see next section) for various ratios of the viscosities. Furthermore, (2.12) continue to play a major role in a general setting since they encode a generalized jump condition which turns out to play a central role for our numerical purpose.

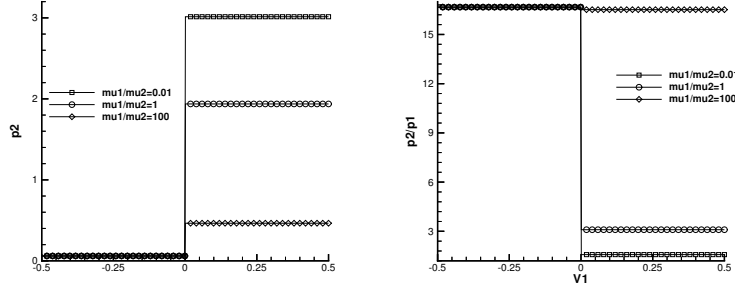


FIGURE 2. Right end state $\mathbf{u}_R$ as a function of μ_1/μ_2 for a given $(\sigma, \mathbf{u}_L)$.

2.1. Travelling wave solutions and jump relations. In this section, we derive generalized jump relations that are needed to characterize the triple $(\sigma; \mathbf{u}_L, \mathbf{u}_R)$ associated with a given viscous shock layer. To that end, let us first recall that a travelling wave solution of (2.6) is a smooth solution in the form $\mathbf{u}(x, t) = \hat{\mathbf{u}}(x - \sigma t)$, $\sigma \in \mathbb{R}$, with

$$(2.14) \quad \lim_{\xi \rightarrow -\infty} \hat{\mathbf{u}}(\xi) = \mathbf{u}_L, \quad \lim_{\xi \rightarrow +\infty} \hat{\mathbf{u}}(\xi) = \mathbf{u}_R, \quad \xi = x - \sigma t,$$

where $(\mathbf{u}_L, \mathbf{u}_R) \in \Omega_{\mathbf{u}}^2$. Besides (2.14), $\hat{\mathbf{u}}(\xi)$ must satisfies the following nonlinear ODE's system:

$$(2.15) \quad -\sigma d_\xi \hat{\mathbf{u}} + \mathcal{A}(\hat{\mathbf{u}}) d_\xi^2 \hat{\mathbf{u}} = \mathcal{E}(\hat{\mathbf{u}}, d_\xi^2 \hat{\mathbf{u}}), \quad \xi \in \mathbb{R},$$

where we have used the condensed form (2.8) of system (2.6). Berthon-Coquel [2] have proved global existence and uniqueness of smooth travelling wave solutions of (2.6) for fairly general (nonlinear) viscosity functions and pressure laws.

Let us then observe after LeFloch [17], Raviart-Sainsaulieu [21], that the end states of the integral curve solution of (2.14), (2.15) does not depend on the amplitude of the diffusion tensor modelled in (2.15) but just on its shape. To that end, let us introduce the function

$$\mathbf{u}_\delta(x, t) = \hat{\mathbf{u}}\left(\frac{x - \sigma t}{\delta}\right),$$

where $\delta > 0$ denotes a positive rescaling parameter. For any given fixed $\delta > 0$, $\mathbf{u}_\delta(x, t)$ turns out to be a travelling wave solution of (2.6) but for the rescaled viscosities μ_1^δ and μ_2^δ defined by

$$(2.16) \quad \mu_1^\delta = \delta \mu_1, \quad \mu_2^\delta = \delta \mu_2.$$

Here, δ plays nothing but the role of the inverse Reynolds number. The travelling wave $\mathbf{u}_\delta(x, t)$ obviously achieves the same endpoints $\mathbf{u}_L$ and $\mathbf{u}_R$ and propagates with speed σ .

As a consequence, the following results hold true (see [3] for a proof and extensions to a more general setting):

Theorem 2.3. *Assume that μ_1 and μ_2 denote positive constants. Then the triple $(\sigma, \mathbf{u}_L, \mathbf{u}_R)$ associated with the resulting dissipative tensor necessarily obeys the jump relations:*

$$(2.17) \quad \begin{cases} -\sigma[\rho] + [\rho u] = 0, \\ -\sigma[\rho u] + [\rho u^2 + p_1 + p_2] = 0, \\ -\sigma[\rho E] + [(\rho E + p_1 + p_2)u] = 0, \end{cases}$$

and necessarily satisfies the two entropy inequalities:

$$(2.18) \quad \begin{cases} -\sigma[\rho s_1] + [\rho s_1 u] < 0, \\ -\sigma[\rho s_2] + [\rho s_2 u] < 0. \end{cases}$$

In order to specify (2.18), let us consider for any given $\delta > 0$, $\mathbf{u}_\delta$ a rescaled travelling wave. Let us then define from (2.16) and with clear notations:

$$(2.19) \quad \int_{\mathbb{R}} (\widehat{T_1})_\delta (\partial_t (\widehat{\rho s_1})_\delta + \partial_x (\widehat{\rho s_1 u})_\delta) (\xi) d\xi = -\delta \mu_1 \int_{\mathbb{R}} (d_\xi u_\delta)^2 d\xi = -\mu_1 \int_{\mathbb{R}} (d_\xi u)^2 d\xi \equiv \zeta_1(\sigma; \mathbf{u}_L, \mathbf{u}_R),$$

$$(2.20) \quad \int_{\mathbb{R}} (\widehat{T_2})_\delta (\partial_t (\widehat{\rho s_2})_\delta + \partial_x (\widehat{\rho s_2 u})_\delta) (\xi) d\xi = \zeta_2(\sigma; \mathbf{u}_L, \mathbf{u}_R).$$

Then the dissipation rates ζ_1 and ζ_2 entering the above two entropy inequalities are bounded and do not depend on the rescaling parameter $\delta > 0$ but only depend on the ratio of the viscosities through the following identity:

$$(2.21) \quad \mu_2 \zeta_1(\sigma; \mathbf{u}_L, \mathbf{u}_R) - \mu_1 \zeta_2(\sigma; \mathbf{u}_L, \mathbf{u}_R) = 0.$$

Rephrasing the above result, $(\sigma; \mathbf{u}_L, \mathbf{u}_R)$ is entirely characterized by the classical jump relations (2.17) and the generalized jump condition (2.21). Despite that (2.21) cannot be given generally speaking a close form of expression, it seems convenient to re-express it equivalently when introducing two suitable averaged temperatures as follows:

$$\begin{aligned} \overline{T_1} \left\{ -\sigma[\rho s_1] + [\rho s_1 u] \right\} &= \zeta_1(\sigma; \mathbf{u}_L, \mathbf{u}_R), \\ \overline{T_2} \left\{ -\sigma[\rho s_2] + [\rho s_2 u] \right\} &= \zeta_2(\sigma; \mathbf{u}_L, \mathbf{u}_R), \end{aligned}$$

so that (2.21) now reads:

$$(2.22) \quad \mu_2 \overline{T_1} \left\{ -\sigma[\rho s_1] + [\rho s_1 u] \right\} - \mu_1 \overline{T_2} \left\{ -\sigma[\rho s_2] + [\rho s_2 u] \right\} = 0.$$

This form of the generalized jump condition obviously finds its root in the companion PDE (2.12).

Remark 2.4. In view of the jump relations (2.17) and (2.22), one of the two thermodynamic entropies, either ρs_1 or ρs_2 , must be obviously understood as a non-linear function of the four remaining independent variables; respectively $\rho s_1 := \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)$ or $\rho s_2 := \{\rho s_2\}(\rho, \rho u, \rho E, \rho s_1)$.

Notice after [17] and [21], that the family of travelling wave solutions $\{\hat{\mathbf{u}}\}_{\delta>0}$, for a given triple $(\sigma; \mathbf{u}_L, \mathbf{u}_R)$, can be seen to converge as δ goes to zero in the L^1_{loc} strong topology to the following step function:

$$(2.23) \quad \mathbf{u}_0(x, t) = \begin{cases} \mathbf{u}_L, & x < \sigma t, \\ \mathbf{u}_R, & x > \sigma t. \end{cases}$$

Let us underline that this so-called shock-solution satisfies by construction the (generalized) jump conditions (2.17) and (2.22). Note from (2.16) that numerically speaking, capturing viscous shock layers in the regime of large Reynolds, *i.e.* for vanishing δ , means nothing but capturing the underlying shock solution (2.23).

2.2. Equivalent formulations and Convexity properties. Here the extended Navier-Stokes system (2.6) is given two equivalent formulations for smooth solutions that allow for highly desirable convexity properties in the analysis of the forthcoming numerical methods. Indeed the feasibility of a stable method for properly capturing the sharp travelling wave solutions of (2.6) in the regime of large Reynolds numbers will essentially result from convexity arguments. The convexity properties of central importance in the sequel are stated right now:

Lemma 2.5. *Under the thermodynamic assumptions (2.1)–(2.3), the mappings $(\tau, s_j) \rightarrow \epsilon_j(\tau, s_j)$ are well-defined and strictly convex. In addition, the mappings:*

$$(2.24) \quad (\rho, \rho u, \rho E, \rho s_2) \rightarrow \{\rho s_1\} \left(\frac{1}{\rho}, \frac{\rho E - \frac{(\rho u)^2}{2\rho} - \rho \epsilon_2 \left(\frac{1}{\rho}, \frac{\rho s_2}{\rho} \right)}{\rho} \right)$$

$$(2.25) \quad (\rho, \rho u, \rho E, \rho s_1) \rightarrow \{\rho s_2\} \left(\frac{1}{\rho}, \frac{\rho E - \frac{(\rho u)^2}{2\rho} - \rho \epsilon_1 \left(\frac{1}{\rho}, \frac{\rho s_1}{\rho} \right)}{\rho} \right)$$

are strictly convex.

The proof is postponed at the end of the present section.

To take advantage of these convexity results, the first equivalent system for smooth solutions we consider is obtained when qualifying $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)^T$ as the unknown and it thus writes:

$$(2.26) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = \partial_x ((\mu_1 + \mu_2) \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = \partial_x ((\mu_1 + \mu_2) u \partial_x u), \\ \partial_t \rho s_2 + \partial_x \rho s_2 u = -\frac{\mu_2}{T_2} (\partial_x u)^2. \end{cases}$$

The associated natural phase space reads:

$$(2.27) \quad \Omega_{\mathbf{v}} = \left\{ \mathbf{v} \in \mathbb{R}^4; \quad \rho > 0, \rho u \in \mathbb{R}, \right. \\ \left. \rho E - \frac{(\rho u)^2}{2\rho} - \rho \epsilon_2 \left(\frac{1}{\rho}, \frac{\rho s_2}{\rho} \right) > 0, \rho s_2 \in \mathbb{R} \right\}.$$

Here the pair $(\rho s_1, \rho s_1 u) : \Omega_{\mathbf{v}} \times \Omega_{\mathbf{v}} \rightarrow \mathbb{R}^2$ with $\mathbf{v} \rightarrow \{\rho s_1\}(\mathbf{v})$ strictly convex, can be understood as a Lax entropy pair but for which, smooth solutions of (2.26) must satisfy without additional conditions the following scalar law with a prescribed rate of entropy dissipation (see Lemma 2.1):

$$(2.28) \quad \partial_t \{\rho s_1\}(\mathbf{v}) + \partial_x \{\rho s_1 u\}(\mathbf{v}) = -\frac{\mu_1}{T_1} (\partial_x u)^2.$$

Some desirable stability properties will be inferred from (2.28) (thus symmetrically from (2.26) despite that ρs_2 and $\{\rho s_1\}(\mathbf{v})$ do not strictly speaking play the same role), just after the description of the second equivalent system we now want to promote. The key balance equation (2.12) is explicitly involved in, so as to consider:

$$(2.29a) \quad \partial_t \rho + \partial_x \rho u = 0, \quad x \in \mathbb{R}, t > 0,$$

$$(2.29b) \quad \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = \partial_x ((\mu_1 + \mu_2) \partial_x u),$$

$$(2.29c) \quad \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = \partial_x ((\mu_1 + \mu_2) u \partial_x u),$$

$$(2.29d) \quad \mu_1 T_2 \left\{ \partial_t \rho s_2 + \partial_x \rho s_2 u \right\} - \mu_2 T_1 \left\{ \partial_t \rho s_1 + \partial_x \rho s_1 u \right\} = 0,$$

so that ρs_1 and ρs_2 are kept in symmetry. The very motivation for this non classical formulation obviously stems from the fact that (2.29d) is intrinsically associated with the generalized jump relation (2.22). With this respect, (2.29) will play a central role in the forthcoming numerical method. Since the above system involves five partial derivatives in time for only four independent unknowns, either the total energy ρE or one of the two specific entropies, ρs_1 or ρs_2 , must be understood as a nonlinear function of the four remaining independent variables: namely either $\rho E := \{\rho E\}(\rho, \rho u, \rho s_1, \rho s_2)$ or $\rho s_1 := \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)$, or $\rho s_2 := \{\rho s_2\}(\rho, \rho u, \rho E, \rho s_1)$. To fix the ideas, let $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)$ play *without restriction* the role of the unknown. The underlying first order system in (2.29) then takes the somewhat

unusual form:

$$\begin{aligned}\partial_t \rho + \partial_x \rho u &= 0, & x \in \mathbb{R}, \ t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) &= 0, \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u &= 0, \\ \mu_1 T_2(\mathbf{v}) \left\{ \partial_t \rho s_2 + \partial_x \rho s_2 u \right\} - \mu_2 T_1(\mathbf{v}) \left\{ \partial_t \{\rho s_1\}(\mathbf{v}) + \partial_x \{\rho s_1 u\}(\mathbf{v}) \right\} &= 0,\end{aligned}$$

but actually enters the basic theory for nonlinear hyperbolic systems. Indeed, the linearization of (2.30) at any given constant state $\mathbf{v}_0 \in \Omega_{\mathbf{v}}$ reads with clear notations:

$$(2.30) \quad \mathcal{A}(\mathbf{v}_0) \partial_t \mathbf{v} + \mathcal{B}(\mathbf{v}_0) \partial_x \mathbf{v} = 0,$$

where direct calculations show that $\det(\mathcal{A}(\mathbf{v}_0)) = \mu_1 T_2^0 - \mu_2 T_1^0 \partial_{\rho s_2} \{\rho s_1\}(\mathbf{v}_0) = (\mu_1 + \mu_2) T_2^0 > 0$. Hence, $\mathcal{A}(\mathbf{v}_0)$ is invertible for any given $\mathbf{v}_0 \in \Omega_{\mathbf{v}}$ and the usual L^2 stability analysis applies: of course, (2.30) is a nonlinear hyperbolic system but in non conservation form (see LeFloch [17] for other examples). The next statement yields in particular the expected equivalence of (2.29) and (2.26) but also gives rise to *a priori* estimates of importance in the sequel:

Lemma 2.6. *Smooth solutions $\mathbf{v} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \Omega_{\mathbf{v}}$ of (2.29) obey the following Lax entropy balance equations with a prescribed rate of entropy dissipation:*

$$(2.31) \quad \partial_t \{\rho s_j\}(\mathbf{v}) + \partial_x \{\rho s_j u\}(\mathbf{v}) = -\frac{\mu_j}{T_j} (\partial_x u)^2, \quad j = 1, 2.$$

In addition, the next entropy-like inequalities hold true for any given smooth increasing function $\phi_j : \mathbb{R} \rightarrow \mathbb{R}_+$:

$$(2.32) \quad \partial_t \{\rho \phi_j(s_j)\}(\mathbf{v}) + \partial_x \{\rho \phi_j(s_j) u\}(\mathbf{v}) \leq 0, \quad j = 1, 2.$$

The interest in the inequalities (2.32) stems from the resulting formal *a priori* estimates:

$$(2.33) \quad 0 \leq \int_{\mathbb{R}} \{\rho \phi_j(s_j)\}(\mathbf{v}(x, t)) dx \leq \int_{\mathbb{R}} \{\rho \phi_j(s_j)\}(\mathbf{v}_0(x)) dx,$$

where $\mathbf{v}_0 : \mathbb{R} \rightarrow \Omega_{\mathbf{v}}$ denotes some initial data for the Cauchy problem (2.29) (or equivalently (2.26)). Using related estimates, Tadmor [25] has proved the next important maximum principle on the specific entropies (see [25] for a refined version):

$$(2.34) \quad s_j(\mathbf{v}(x, t)) \leq \max_{y \in \mathbb{R}} (s_j(\mathbf{v}_0(y))), \quad j = 1, 2, \quad a.e. x \in \mathbb{R}, \ t > 0,$$

when privileging $\phi_j(s_j) = \max(0, s_j - \max_{y \in \mathbb{R}} (s_j(\mathbf{v}_0(y))))$. Most of the inequalities (2.32) and in particular their consequence (2.34) will be recovered in the numerical method we develop hereafter. These desirable stability estimates will in particular allow us to prove that the discrete method preserves the positivity of the internal energies $\epsilon_j, j = 1, 2$.

Proof. The equations (2.29a)–(2.29c) can be seen to yield from the second law of the thermodynamics (2.1) the next identity:

$$-T_1 \{\partial_t \rho s_1 + \partial_x \rho s_1 u\} - T_2 \{\partial_t \rho s_2 + \partial_x \rho s_2 u\} = (\mu_1 + \mu_2) (\partial_x u)^2,$$

and (2.31) follows when invoking (2.29d). Inequalities (2.32) are then recovered from:

$$\rho \{\partial_t \phi_j(s_j) + u \partial_x \phi_j(s_j)\} = -\frac{\mu_j \phi_j'(s_j)}{T_j} (\partial_x u)^2 \leq 0, \quad j = 1, 2.$$

□

Let us conclude the present section when establishing the convexity lemma 2.5. The proof readily follows from the technical results given below. The following first statement is proved in Godlewski-Raviart [11], chapter 2.

Lemma 2.7. *Let the index $j \in \{1, 2\}$ be fixed. Under the thermodynamic assumptions (2.2)–(2.3), the following assertions are equivalent:*

- (i) *The mapping $(\tau, \epsilon_j) \rightarrow s_j(\tau, \epsilon_j)$ is strictly convex.*
- (ii) *The mapping $(\tau, s_j) \rightarrow \epsilon_j(\tau, s_j)$ is strictly convex.*
- (iii) *The mapping $(\rho, \rho s_j) \rightarrow \rho \epsilon_j(\frac{1}{\rho}, \frac{\rho s_j}{\rho})$ is strictly convex.*

This classical result is easily seen to imply:

Lemma 2.8. *The mapping*

$$(2.35) \quad (\rho, \rho u, \rho s_1, \rho s_2) \rightarrow \frac{(\rho u)^2}{2\rho} + \rho \epsilon_1 \left(\frac{1}{\rho}, \frac{\rho s_1}{\rho} \right) + \rho \epsilon_2 \left(\frac{1}{\rho}, \frac{\rho s_2}{\rho} \right),$$

is strictly convex.

Equipped with these results, we are now in a position to establish the convexity Lemma 2.5.

Proof of Lemma 2.5. Let us restrict our attention to the mapping $\mathbf{v} \rightarrow \{\rho s_1\}(\mathbf{v})$ since the companion entropy $\{\rho s_2\}$ plays a symmetric role. The strict convexity of the mapping $\mathbf{w} := (\rho, \rho u, \rho s_1, \rho s_2) \rightarrow \{\rho E\}(\mathbf{w})$ (see Lemma 2.8) is known to be equivalent to the validity of the next inequality for any given pair $(\mathbf{w}_L, \mathbf{w}_R) \in \Omega_{\mathbf{w}}^2$:

$$\{\rho E\}(\mathbf{w}_R) - \{\rho E\}(\mathbf{w}_L) > \nabla_{\mathbf{w}} \{\rho E\}(\mathbf{w}_L) \cdot (\mathbf{w}_R - \mathbf{w}_L).$$

This one is seen to recast as follows:

$$(2.36) \quad -\partial_{\rho s_1} \{\rho E\}(\mathbf{w}_L)((\rho s_1)_R - (\rho s_1)_L) > \\ \partial_{\rho} \{\rho E\}(\mathbf{w}_L)(\rho_R - \rho_L) + \partial_{\rho u} \{\rho E\}(\mathbf{w}_L)((\rho u)_R - (\rho u)_L) + \\ \partial_{\rho s_2} \{\rho E\}(\mathbf{w}_L)((\rho s_2)_R - (\rho s_2)_L) - (\{\rho E\}(\mathbf{w}_R) - \{\rho E\}(\mathbf{w}_L))$$

Since $\partial_{\rho s_1} \{\rho E\}(\mathbf{w}) = \partial_{s_1} \epsilon_1(\tau, s_1) = -T_1 < 0$ in view of (2.3), the inverse mapping $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)^T \rightarrow \{\rho s_1\}(\mathbf{v})$ is well-defined and the classical chain rule gives:

$$\partial_{\rho E} \{\rho s_1\}(\mathbf{v}) = \frac{1}{\partial_{\rho s_1} \{\rho E\}}(\mathbf{w}(\mathbf{v})), \quad \partial_X \{\rho s_1\}(\mathbf{v}) = -\frac{\partial_X \{\rho E\}}{\partial_{\rho s_1} \{\rho E\}}(\mathbf{w}(\mathbf{v})),$$

where X denotes either $\rho, \rho u$ or ρs_2 . Since $-\partial_{\rho s_1} \{\rho E\}(\mathbf{w}) > 0$, we thus arrive from (2.36) at the following inequality, valid for all pairs $(\mathbf{v}_L, \mathbf{v}_R) \in \Omega_{\mathbf{v}}^2$:

$$\{\rho s_1\}(\mathbf{v}_R) - \{\rho s_1\}(\mathbf{v}_L) > \nabla_{\mathbf{v}} \{\rho s_1\}(\mathbf{v}_L) \cdot (\mathbf{v}_R - \mathbf{v}_L),$$

which is nothing but the required result. $\square$

3. GODUNOV METHODS WITH NONLINEAR PROJECTIONS

This section is devoted to the numerical approximation of the solutions of the convection-diffusion system (2.6) in the regime of a large Reynolds number with a special emphasis put on the satisfaction of the generalized jump condition (2.22) at the discrete level. As pointed out in the previous sections, several equivalent formulations of (2.6) do coexist but of course, they are all associated with the same generalized jump conditions.

As proved below, such an equivalence property, valid for exact solutions, turns out to be lost in general at the discrete level within the frame of classical numerical methods: namely methods built from a consistent operator splitting into the underlying hyperbolic system and its viscous perturbation. For instance, such seemingly well-grounded methods when derived from the formulation (2.26), *i.e* in the unknown $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)$, and next from the companion form expressed in the $(\rho, \rho u, \rho E, \rho s_1)$ state variable, systematically produce grossly different approximate solutions. These in addition exhibit unacceptable errors with the exact solution. As established below, errors in the usual approach essentially find their roots in the

numerical approximation of the underlying hyperbolic system. Roughly speaking, if suitable finite difference formulae for approximating the second order operator can be actually devised so as to preserve the generalized jump conditions in (2.29), the use of stable approximate Riemann solvers inevitably results in a violation of these relations. The core of this breakdown rests with the averaging procedure of neighbouring approximate Riemann solutions over each computational cell. Essentially because of the Jensen inequality, the L^2 -averaging induces a too large amount of artificial entropy dissipation which overtakes the one prescribed in the exact PDE's. Exemplifying this assertion, the Lax entropy $\{\rho s_1\}(\mathbf{v})$ turns out to be over-dissipated when $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)$ serves as the unknown and so does conversely $\{\rho s_2\}(\mathbf{w})$ when invoking $\mathbf{w} = (\rho, \rho u, \rho E, \rho s_1)$ as the state variable. In both cases, $\rho s_1(x, t)$ and $\rho s_2(x, t)$ cannot be kept in balance according to (2.12). This flaw just translates into a violation of the required generalized jump condition (2.22) coming with (2.12) and we again refer the reader to the results displayed in Figure 1 for an illustration of the heavily negative consequences.

To bypass this failure, we introduce below a nonlinear projection procedure. Involved as an additional and last step to any given relevant splitting method, this procedure provides a simple and effective correction in enforcing the two entropies ρs_1 and ρs_2 to be kept in balance in their evolution in time according to (2.12). As strongly attested by the numerical experiments, discrete solutions obtained by the nonlinear projection procedure achieve a far much better agreement with exact solutions.

The key issue in the feasibility of the nonlinear correction step (existence and uniqueness of a solution of a scalar nonlinear equation in each cell) surprisingly stays in the property that one of the two entropies, namely the Lax entropy, has been over-dissipated in the splitting step. In other words, the correction can only take place provided that a stable (and thus dissipative) approximate Riemann solver has been involved in the prediction step. In addition, the nonlinear correction procedure is proved to inherit all the stability properties from the underlying classical approximate Riemann solver: namely positivity properties, a full set of discrete entropy inequalities and a maximum principle for the specific entropies.

The format of this section is dictated by the steps entering the prediction-correction method we propose for approximating the solutions of system (2.29). First, a convenient splitting method is derived on the ground of the equivalent form (2.26) to serve as a building block in the prediction step. Its main properties are briefly discussed with a central attention paid on the violation of the key balance equation (2.29d). Then the nonlinear projection procedure is introduced and the stability properties met by this correction step are analysed.

3.1. Godunov type methods and L^2 projections. This prediction step makes use of the equivalent form (2.26), *i.e.* we privilege $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)^T$ to be the state variable, so as to take advantage of the convexity property (2.24) stated in Lemma 2.5. Let Δt and Δx respectively denote the time and space increments, chosen to be constant for simplicity. The numerical approximate solution, $\mathbf{v}_{\Delta x} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \Omega_{\mathbf{v}}$, is as usual supposed to be piecewise constant and we set using classical notations:

$$(3.1) \quad \mathbf{v}_{\Delta x}(x, t) = \mathbf{v}_i^n, \quad (x, t) \in (x_{i-1/2}, x_{i+1/2}) \times (t^n, t^{n+1}), \quad i \in \mathbb{Z}, n \in \mathbb{N}.$$

3.1.1. First step: Underlying first order system ($t^n \rightarrow t^{n+1,=}$). Assume that the discrete solution $\mathbf{v}_{\Delta x}(x, t^n)$ is known at the time level t^n . According to the standard splitting strategy, this one is first partially updated when solving for small times

$t \in (0, \Delta t)$ the following Cauchy problem:

$$(3.2) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = 0, \\ \partial_t \rho s_2 + \partial_x \rho s_2 u = 0, \end{cases}$$

with $\mathbf{v}_h(x, t^n)$ as initial data. For the sake of uniqueness, weak solutions of the above nonlinear hyperbolic system are classically selected according to the Lax entropy inequality (see Lemma 2.5):

$$(3.3) \quad \partial_t \{\rho s_1\}(\mathbf{v}) + \partial_x \{\rho s_1 u\}(\mathbf{v}) \leq 0.$$

For convenience in the notations, the system (3.2) is given the clear condensed form:

$$(3.4) \quad \partial_t \mathbf{v} + \partial_x \mathbf{g}(\mathbf{v}) = 0.$$

Under the CFL like restriction:

$$(3.5) \quad \frac{\Delta t}{\Delta x} \max_{\mathbf{v}} |\lambda_i(\mathbf{v})| \leq \frac{1}{2},$$

to be met for all the $\mathbf{v}$ under consideration, the exact solution of (3.2)–(3.3) is classically made of the superposition of the sequence of non interacting Riemann solutions emerging at each interface $x_{i+1/2}$, which we denote by $\omega(\cdot; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)$. This exact solution is then averaged over each cell to yield the following piecewise constant partial update:

$$(3.6) \quad \mathbf{v}_i^{n+1,=} = \mathbf{v}_i^n - \frac{\Delta t}{\Delta x} \{ \mathbf{g}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) - \mathbf{g}(\omega(0^+; \mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \}, \quad i \in \mathbb{Z}.$$

Next in view of the convexity Lemma (2.5), classical considerations (see for instance [11]) ensures that under the CFL restriction (3.5), the update $(\rho s_1)_i^{n+1,=} \equiv \{\rho s_1\}(\mathbf{v}_i^{n+1,=})$ obeys as a consequence of the L^2 -averaging procedure in (3.6):

$$(3.7) \quad \begin{aligned} (\rho s_1)_i^{n+1,=} - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \{ \{\rho s_1\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) - \{\rho s_1\}(\omega(0^+; \mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \} \\ \leq 0. \end{aligned}$$

This inequality is, in general, strict because of the entropy satisfying shock solutions in (3.2)–(3.3) but also in view of the well-known Jensen inequality (see [11] for the details):

$$\begin{aligned} \{\rho s_1\}(\mathbf{v}_i^{n+1,=}) &= \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)(x, t^{n+1,=}) dx - \\ &\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \left((\mathbf{v}(x, t^{n+1,=}) - \mathbf{v}_i^{n+1,=}), \{\overline{\nabla^2 \rho s_1}\}_i^{n+1,=} (\mathbf{v}(x, t^{n+1,=}) - \mathbf{v}_i^{n+1,=}) \right) dx, \end{aligned}$$

where we have set

$$\{\overline{\nabla^2 \rho s_1}\}_i^{n+1,=} = \int_0^1 \{ \nabla^2 \rho s_1 \} (s \mathbf{v}(x, t^{n+1,=}) + (1-s) \mathbf{v}_i^{n+1,=}) s(1-s) ds$$

to obtain

$$(3.8) \quad \{\rho s_1\}(\mathbf{v}_i^{n+1,=}) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)(x, t^{n+1,=}) dx.$$

This inequality is strict as soon as $\mathbf{v}_i^{n+1,=} \neq \mathbf{v}(x, t^{n+1,=})$ for $x \in (x_{i-1/2}, x_{i+1/2})$. Obviously, the (strict) inequality (3.7) met by $(\rho s_1)_i^{n+1,=}$ must be compared with the updating formula for $(\rho s_2)_i^{n+1,=}$ in (3.6):

$$(3.9) \quad \begin{aligned} (\rho s_2)_i^{n+1,=} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \{ \{\rho s_2\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) - \{\rho s_2\}(\omega(0^+; \mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \} \\ = 0 \end{aligned}$$

where the equality just reflects the fact that s_2 is simply advected by the flow, contrarily to s_1 . This discrepancy in the rates of entropy dissipation, strictly negative versus zero, obviously results in a failure for satisfying the expected balance equation (2.12) at the discrete level. It is worth analysing further the reported discrepancy. To that end, let us recall the next estimate (see Coquel-LeFloch [5])

$$\begin{aligned} & \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \left((\mathbf{v}(x, t^{n+1,=}) - \mathbf{v}_i^{n+1,=}), \{\overline{\nabla^2 \rho s_1}\}_i^{n+1,=} (\mathbf{v}(x, t^{n+1,=}) - \mathbf{v}_i^{n+1,=}) \right) dx \\ &= O(1) \left(\|\mathbf{v}_{i+1}^n - \mathbf{v}_i^n\|^2 + \|\mathbf{v}_i^n - \mathbf{v}_{i-1}^n\|^2 \right), \end{aligned}$$

where $O(1)$ is related to the convexity modulus of $\{\nabla^2 \rho s_1\}(\mathbf{v})$. Therefore in the region of smoothness of the solution, the above term is small but achieves large negative values across shock discontinuities. Notice that this error term clearly increases with the strength of the shocks.

Of course, this negative result cannot be bypassed when using instead of the exact Godunov scheme a relevant entropy satisfying approximate Riemann solver (see Hou-LeFloch [11] in the scalar case). Nevertheless, a discrete inequality for the Lax entropy $\{\rho s_1\}(\mathbf{v})$ like (3.7) is actually required for the forthcoming nonlinear correction step to take place.

With this respect, we end this paragraph when proving the desirable stability estimates satisfied by (3.6) that will be preserved in the nonlinear projection procedure. A discrete entropy inequality for $\{\rho s_1\}_i^{n+1,=}$, distinct from (3.7), enters the following statement and will be motivated just hereafter:

Proposition 3.1. *Under the CFL condition (3.5), the Lax entropy pair $(\rho s_1, \rho s_1 u)$ obeys the following discrete entropy inequality:*

$$(3.10) \quad \{\rho s_1\}(\mathbf{v}_i^{n+1,=}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\} = \mathcal{E}_i^n \leq 0,$$

where the numerical entropy flux $\{g_{\rho s_1 u}\}_{i+1/2}^n$ differs from $\{\rho s_1 u\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n))$ and is defined in full symmetry with the Larrourou's flux function [14] $\{g_{\rho s_2 u}\}_{i+1/2}^n = \{\rho s_2 u\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n))$ in (3.9) by the following formula with $j = 1, 2$:

$$(3.11) \quad \{g_{\rho s_j u}\}_{i+1/2}^n = (\rho u)(w(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) \times \begin{cases} (s_j)_i^n & \text{if } (\rho u)(w(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) > 0, \\ (s_j)_{i+1}^n & \text{otherwise.} \end{cases}$$

Next, the following maximum principles on the specific entropies s_1 and s_2 are satisfied (see Tadmor [25]):

$$(3.12) \quad \{s_1\}(\mathbf{v}_i^{n+1,=}) \leq \max(\{s_1\}(\mathbf{v}_{i-1}^n), \{s_1\}(\mathbf{v}_i^n), \{s_1\}(\mathbf{v}_{i+1}^n)),$$

$$(3.13) \quad (s_2)_i^{n+1,=} \leq \max((s_2)_{i-1}^n, (s_2)_i^n, (s_2)_{i+1}^n),$$

away from vacuum. Consequently, both the energies $(\epsilon_1)_i^{n+1,=}$ and $(\epsilon_2)_i^{n+1,=}$ stay positive.

The interest in considering a numerical entropy flux for ρs_1 in full symmetry with the numerical flux entering the update for ρs_2 stems from the need of preserving the exact symmetry in the role played by ρs_1 and ρs_2 in the forthcoming discrete version of the balance equation (2.29d). In addition, discrete analogues of the entropy-like inequalities (2.32) will easily follow from the choice (3.11). The proof detailed below asserts that the discrete entropy dissipation rate $\mathcal{E}_i^n$ in (3.10), is in general strictly negative.

Proof. Let us first establish the unusual discrete entropy inequality (3.10)–(3.11) when closely investigating the property of $\{s_1\}(\omega(x/t; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n))$ in each of the Riemann solutions. In view of the non-linearity assumption (2.5) satisfied by the

two pressure laws $p_j(\tau, s_j)$, the self-similar solution $\omega(x/t; \mathbf{v}_L, \mathbf{v}_R)$ of the Cauchy problem (2.26) with initial data $\mathbf{v}_0(x) = \mathbf{v}_L$, $x < 0$, $\mathbf{v}_R$, $x > 0$ can be seen to be made of at most four constant states, denoted in the natural ordering by $\mathbf{v}_L, \mathbf{v}_L^*, \mathbf{v}_R^*, \mathbf{v}_R$, and separated by three simple waves. In the PDE model (2.26), the specific entropy $\{s_1\}(\mathbf{v})$ is easily seen to be a Riemann invariant for the two extreme genuinely nonlinear fields: this quantity therefore stays constant across the associated rarefaction waves. Considering a discontinuity propagating with speed σ and separating the states $\mathbf{v}^-, \mathbf{v}^+$, the entropy inequality (3.3) is seen to re-express as follows:

$$(3.14) \quad m\{s_1(\mathbf{v}^+) - s_1(\mathbf{v}^-)\} \leq 0, \quad m = \rho^-(u^- - \sigma) = \rho^+(u^+ - \sigma),$$

with strict inequality in the case of the two extreme genuinely nonlinear fields. Then, classical considerations in the usual gas dynamic setting (see [11]) directly extend to the present model (2.26) to prove that the relative mass flux m in (3.14) respectively satisfies $m > 0$ for a 1-shock solution, $m = 0$ across a contact discontinuity and $m < 0$ concerning a 3-shock. The above considerations for smooth and discontinuous elementary waves clearly yield:

$$s_1(\mathbf{v}_L) \geq s_1(\mathbf{v}_L^*), \quad s_1(\mathbf{v}_R^*) \leq s_1(\mathbf{v}_R),$$

and thus suffice to prove that:

$$(3.15) \quad \{s_1\}(\omega(x/t; \mathbf{v}_L, \mathbf{v}_R)) \leq A(x/t; \mathbf{v}_L, \mathbf{v}_R) \equiv \begin{cases} s_1(\mathbf{v}_L) & \text{if } x/t < u^*, \\ s_1(\mathbf{v}_R) & \text{otherwise,} \end{cases}$$

where $u^* = u_L^* = u_R^*$ denotes the speed of the intermediate contact discontinuity. Next, observe that the self-similar function $A(x/t; \mathbf{v}_L, \mathbf{v}_R)$ entering the right-hand side of (3.15) is nothing but a solution of the following transport equation:

$$\partial_t A + u(\omega(x/t; \mathbf{v}_L, \mathbf{v}_R)) \partial_x A = 0,$$

and hence, also solves:

$$(3.16) \quad \partial_t \rho A + \partial_x \rho A u = 0.$$

Equipped with this result, we propose to supplement the system (3.2) with the decoupled PDE (3.16) when prescribing at time t^n , $A(x, t^n) = (s_1)_i^n$, with $x \in (x_{i-1/2}, x_{i+1/2})$, so that (3.15) clearly yields under the CFL condition (3.5):

$$(3.17) \quad \begin{aligned} \rho_i^{n+1=} \times \{s_1\}(\mathbf{v})_i^{n+1=} &\leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)(x, t^{n+1,=}) dx \\ &\leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{\rho A\}(x, t^{n+1,=}) dx, \end{aligned}$$

where the first line is just the Jensen inequality (3.8). But since $(\rho A)(x, t)$ solves (3.16), calculations due to Larrouturou [14] immediately give:

$$(3.18) \quad \begin{aligned} \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{\rho A\}(x, t^{n+1,=}) dx &\equiv \rho_i^{n+1=} \times A_i^{n+1=} = \\ &(\rho A)_i^n - \frac{\Delta t}{\Delta x} \{ \{\rho A\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) - \{\rho A\}(\omega(0^+; \mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \}, \end{aligned}$$

where by construction $(\rho A)_i^n = (\rho s_1)_i^n$ and $\{\rho A\}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) = \{g_{\rho s_1 u}\}_{i+1/2}^n$ as defined in (3.11). The required discrete entropy inequality (3.10) thus follows from (3.17).

Next, the maximum principle reported in (3.12) is recovered here as a by-product of (3.17)–(3.18) (see also [25] for a related approach) arguing that the advected quantity A obeys under the CFL condition (3.5) and away from vacuum (see [14] for the details):

$$A_i^{n+1=} \leq \max(A_{i-1}^n, A_i^n, A_{i+1}^n), \quad A_i^n = (s_1)_i^n, \quad i \in \mathbb{Z}.$$

The companion inequality (3.13) follows from the above inequality with s_2 in place of A since s_2 is also purely advected by the flow. To conclude, notice that the monotonicity property $\partial_{\epsilon_j} s_j(\tau, \epsilon_j) = -T_j < 0$ inferred from (2.3) ensures in view of the maximum principles (3.12)–(3.13) that the specific energies ϵ_j , $j = 1, 2$ are kept positive. $\square$

3.1.2. *Second step: the second order operator* ($t^{n+1,=} \rightarrow t^{n+1,-}$). The partial update $\mathbf{v}_h(x, t^{n+1,=})$ (3.6) is evolved to the next time level $t^{n+1,-}$ when solving for $t \in (0, \Delta t)$ the Cauchy problem:

$$(3.19) \quad \begin{cases} \partial_t \rho = 0, \\ \partial_t \rho u = \partial_x((\mu_1 + \mu_2) \partial_x u), \\ \partial_t \rho E = \partial_x((\mu_1 + \mu_2) u \partial_x u), \\ \partial_t \rho s_2 = -\frac{\mu_2}{T_2} (\partial_x u)^2, \end{cases}$$

with $\mathbf{v}_h(x, t^{n+1,=})$ as initial data. To that end, to keep the CFL restriction (3.5) unchanged, we consider a time implicit scheme defined for $i \in \mathbb{Z}$ by:

$$(3.20a) \quad \rho_i^{n+1,-} = \rho_i^{n+1,=},$$

$$(3.20b) \quad (\rho u)_i^{n+1,-} = (\rho u)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1 + \mu_2) \partial_x u)}_i^{n+1,-},$$

$$(3.20c) \quad (\rho E)_i^{n+1,-} = (\rho E)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1 + \mu_2) u \partial_x u)}_i^{n+1,-},$$

$$(3.20d) \quad (\rho s_2)_i^{n+1,-} = (\rho s_2)_i^{n+1,=} - \Delta t \overline{\frac{\mu_2}{T_2} (\partial_x u)^2}_i^{n+1,-},$$

with the following consistent finite difference formulae built from the time averaging $M^n u_i = (u_i^{n+1,-} + u_i^{n+1,=})/2$, $i \in \mathbb{Z}$:

$$(3.21a) \quad \overline{\partial_x((\mu_1 + \mu_2) \partial_x u)}_i^{n+1,-} = \frac{\mu_1 + \mu_2}{\Delta x^2} (M^n u_{i+1} - 2M^n u_i + M^n u_{i-1}),$$

$$(3.21b) \quad \overline{\partial_x(\mu_1 + \mu_2) u \partial_x u}_i^{n+1,-} = \frac{\mu_1 + \mu_2}{2\Delta x^2} ((M^n u_{i+1})^2 - 2(M^n u_i)^2 + (M^n u_{i-1})^2),$$

$$(3.21c) \quad \overline{\frac{\mu_2}{T_2} (\partial_x u)^2}_i^{n+1,-} = \frac{\mu_2}{2\overline{T_2}_i^{n+1,-} \Delta x^2} ((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2).$$

In (3.21c), the averaged temperature $\overline{T_2}_i^{n+1,-}$ receives the following definition:

$$(\overline{T_2}_i^{n+1,-}) = \begin{cases} T_2 \left(\frac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,-} \right), & \text{if } (s_2)_i^{n+1,-} = (s_2)_i^{n+1,=}, \\ -\frac{\epsilon_2 \left(\frac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,-} \right) - \epsilon_2 \left(\frac{1}{\rho_i^{n+1,=}}, (s_2)_i^{n+1,=} \right)}{(s_2)_i^{n+1,-} - (s_2)_i^{n+1,=}}, & \text{otherwise} \end{cases}$$

Let us stress that the first two discrete operators (3.21a),(3.21b) preserve by construction the conservation property to be met by the unknowns ρu and ρE while as required, the last operator (3.21c) always achieves a positive sign. Besides, the benefit of these formulae is twofold as pointed out below.

On the one hand, since the density is kept unchanged in this second step, the implicit equations (3.20b) for defining $\{u_i^{n+1,-}\}_{i \in \mathbb{Z}}$ are actually completely decoupled from (3.20c) and (3.20d). Therefore solving (3.20) just amounts in practice to solve a linear problem in the unknown $M^n u$ with a positive definite symmetric matrix (away from vacuum). Then, the total energy $(\rho E)_i^{n+1,-}$ can be readily point-wisely computed. Next considering the update of the specific entropy s_2 , the definition of the averaged temperature $\overline{T_2}_i^{n+1,-}$ (3.22) implies that $(s_2)_i^{n+1,-}$, i being fixed,

solves in each cell:

$$(3.23) \quad \epsilon_2 \left(\frac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,-} \right) = \epsilon_2 \left(\frac{1}{\rho_i^{n+1,=}}, (s_2)_i^{n+1,=} \right) + \frac{\mu_2 \Delta t}{2 \rho_i^{n+1,=} \Delta x^2} \left((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2 \right) > 0.$$

The strict monotonicity property $\partial_{s_2} \epsilon_2(\tau, s_2) < 0$ inferred from (2.3) ensures the existence and uniqueness of the expected solution.

On the other hand, direct calculations based on the point-wise update (3.20c) for $(\rho E)_i^{n+1,-}$, the representation formula (3.23) for $(\epsilon_2)_i^{n+1,-}$ and an exact estimate for the kinetic energy $\rho^{n+1,=}(u_i^{n+1,-} - u_i^{n+1,=}) \times (u_i^{n+1,-} + u_i^{n+1,=})/2$ easily deduced from (3.20b)–(3.21b), yield in view of the total energy definition (2.1) the following update for $(\epsilon_1)_i^{n+1,-}$:

$$(\epsilon_1)_i^{n+1,-} = (\epsilon_1)_i^{n+1,=} + \frac{\mu_1 \Delta t}{2 \rho_i^{n+1,=} \Delta x^2} \left((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2 \right).$$

The next identity can be thus derived in exact symmetry with (3.20d)–(3.21c):

$$(3.24) \quad \begin{aligned} \{\rho s_1\}(\mathbf{v}_i^{n+1,-}) &= \{\rho s_1\}(\mathbf{v}_i^{n+1,=}) \\ &- \frac{\mu_1 \Delta t}{2 \Delta x^2 \overline{T}_{1,i}^{n+1,-}} \left((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2 \right), \end{aligned}$$

when defining the averaged temperature $\overline{T}_{1,i}^{n+1,-}$ in full symmetry with $\overline{T}_{2,i}^{n+1,-}$ in (3.22). As a direct consequence of the reported exact symmetry, we get:

$$(3.25) \quad \begin{aligned} \mu_2 \overline{T}_{1,i}^{n+1,-} \{(\rho s_1)_i^{n+1,-} - (\rho s_1)_i^{n+1,=}\} - \\ \mu_1 \overline{T}_{2,i}^{n+1,-} \{(\rho s_2)_i^{n+1,-} - (\rho s_2)_i^{n+1,=}\} = 0. \end{aligned}$$

This identity just expresses that in the second step devoted to the diffusive operator, the two entropies ρs_1 and ρs_2 can be kept in balance according to (2.12). This was precisely the goal of the finite difference scheme we have just described. Let us underline that other finite difference formulae are indeed possible but to our knowledge, such formulae systematically produce (strictly) negative error terms in the discrete entropy balance equation (3.24) for ρs_1 and thus ruin the exact balance in (3.25).

3.1.3. Stability estimates of the classical method with L^2 projection. This paragraph gathers the stability estimates achieved by the splitting method we have described for approximating the solutions of the equivalent system (2.26). Such estimates will allow for the nonlinear correction procedure to take place in the next section and in turn, the desirable ones will be inherited. Using the notations introduced in the previous sections, the next result directly follows from Lemma 3.1 and the identities (3.20d)–(3.21c), (3.24):

Lemma 3.2. *Under the CFL condition (3.5), the unknown ρs_2 updates according to:*

$$(3.26) \quad \begin{aligned} (\rho s_2)_i^{n+1,-} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_2 u}\}_{i+1/2}^n - \{g_{\rho s_2 u}\}_{i-1/2}^n \right\} \\ + \Delta t \frac{\mu_2}{T_2} (\partial_x u)_i^{n+1,-} = 0 \end{aligned}$$

while the Lax entropy pair $(\rho s_1, \rho s_1 u)$ obeys the discrete entropy inequality:

$$(3.27) \quad \begin{aligned} \{\rho s_1\}(\mathbf{v}_i^{n+1,-}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\} \\ + \Delta t \frac{\mu_1}{T_1} (\partial_x u)_i^{n+1,-} = \mathcal{E}_i^n \leq 0. \end{aligned}$$

The relations (3.26) and (3.27) immediately yield the following discrete analogue of (2.12):

$$(3.28) \quad \begin{aligned} \mu_2 \overline{T_1}_i^{n+1,-} \left(\{\rho s_1\}(\mathbf{v}_i^{n+1,-}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\} \right) - \\ \mu_1 \overline{T_2}_i^{n+1,-} \left((\rho s_2)_i^{n+1,-} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_2 u}\}_{i+1/2}^n - \{g_{\rho s_2 u}\}_{i-1/2}^n \right\} \right) = \\ \mu_2 \overline{T_1}_i^{n+1,-} \times \mathcal{E}_i^n \neq 0. \end{aligned}$$

which means nothing but a violation of the generalized jump condition (2.22). The nonlinear projection procedure precisely intends to cure this failure.

Remark 3.3. The proposed procedure, based on the variable $(\rho, \rho u, \rho E, \rho s_2)$ does not allow to keep in balance ρs_2 and $\rho s_1(\mathbf{v})$ according to the jump relation (2.22). To enforce for validity this jump relation, one should be tempted to derive a related algorithm but based on the variable $(\rho, \rho u, \rho s_1, \rho s_2)$. With this choice, the total energy ρE must be understood as a function $\{\rho E\}(\mathbf{w})$ of $\mathbf{w}$. The lemma 2.8 ensures that this function is strictly convex in $\mathbf{w}$. As a consequence, one would get at the end of the first step devoted to the approximation of the hyperbolic problem:

$$\begin{aligned} \{\rho E\}(\mathbf{w}_i^{n+1,=}) - \{\rho E\}(\mathbf{w}_i^n) + \frac{\Delta t}{\Delta x} \left(\{(\rho E + p_1 + p_2)u\}(\mathbf{w}(0^+; \mathbf{w}_i^n, \mathbf{w}_{i+1}^n)) - \right. \\ \left. \{(\rho E + p_1 + p_2)u\}(\mathbf{w}(0^+; \mathbf{w}_{i-1}^n, \mathbf{w}_i^n)) \right) \leq 0, \end{aligned}$$

with strict inequality in general. In other words, the total energy is not conserved. A correction procedure to restore the conservation of ρE is actually possible (see [1]) and it is close to the technique we describe in the $\mathbf{v}$ variable.

To assess the feasibility and the relevance of the forthcoming nonlinear correction procedure, the following stability estimates in the classical splitting method with L^2 projection are in order:

Theorem 3.4. *Under the CFL restriction (3.5), the following discrete entropy inequalities are satisfied:*

$$(3.29) \quad \begin{aligned} \{\rho \phi_j(s_j)\}(\mathbf{v}_i^{n+1,-}) - (\rho \phi_j(s_j))_i^n + \\ \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho \phi_j(s_j)u}\}_{i+1/2}^n - \{g_{\rho \phi_j(s_j)u}\}_{i-1/2}^n \right\} \leq 0, \quad j = 1, 2, \end{aligned}$$

for any given increasing and convex smooth function $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$. The numerical entropy flux function entering (3.29) is given by the Larrouturnou's formula (3.11). Next, the following maximum principles on the specific entropies s_1 and s_2 hold true away from vacuum:

$$(3.30) \quad (s_j)_i^{n+1,-} \leq \max((s_j)_{i-1}^n, (s_j)_i^n, (s_j)_{i+1}^n), \quad j = 1, 2.$$

As a consequence, the internal energies $(\epsilon_1)_i^{n+1,-}$ and $(\epsilon_2)_i^{n+1,-}$ are kept positive.

The inequalities (3.29) are of course discrete versions of the entropy-like inequalities (2.32) encoding in particular the maximum principles stated in (2.34) and recovered in (3.30).

Proof of theorem 3.4. Let us first establish (3.29). To that purpose, let us write from the discrete entropy inequalities stated in Lemma 3.2 with $j \in \{1, 2\}$ fixed:

$$(3.31) \quad \rho_i^{n+1,-} \times \{s_j\}(\mathbf{v}_i^{n+1,-}) \leq \rho_i^n \times (s_j)_i^n - \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_j u}\}_{i+1/2}^n - \{g_{\rho s_j u}\}_{i-1/2}^n \right\},$$

Next, observe after Larrouturou [14] that the formula for $\{g_{\rho s_j u}\}_{i+1/2}^n$ reads equivalently:

$$\{g_{\rho s_j u}\}_{i+1/2}^n = \rho_{i+1/2}^n \times \{(u_{i+1/2}^n)_+ (s_j)_i + (u_{i+1/2}^n)_- (s_j)_{i+1}\},$$

where to shorten the notations, we have set

$$\rho_{i+1/2}^n = \rho(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) \quad \text{and} \quad u_{i+1/2}^n = u(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)).$$

So that (3.31) can be rewritten (see [14] for the details):

$$(3.32) \quad \rho_i^{n+1,-} \times \{s_j\}(\mathbf{v}_i^{n+1,-}) \leq \rho_i^{n+1,-} \times \left\{ (s_j)_i^n - \frac{\Delta t}{\Delta x} (v_{i+1/2}^n)^- ((s_j)_{i+1}^n - (s_j)_i^n) - \frac{\Delta t}{\Delta x} (v_{i-1/2}^n)^+ ((s_j)_i^n - (s_j)_{i-1}^n) \right\},$$

where we have introduced

$$\begin{aligned} (v_{i+1/2}^n)^- &= \rho_{i+1/2}^n \times (u_{i+1/2}^n)_- / \rho_i^{n+1,-} \leq 0, \\ (v_{i-1/2}^n)^+ &= \rho_{i-1/2}^n \times (u_{i-1/2}^n)_+ / \rho_i^{n+1,-} \geq 0. \end{aligned}$$

Hence, (3.32) yields away from vacuum:

$$(3.33) \quad \begin{aligned} \{s_j\}(\mathbf{v}_i^{n+1,-}) &\leq \left(1 - \frac{\Delta t}{\Delta x} ((v_{i-1/2}^n)^+ - (v_{i+1/2}^n)^-)\right) \times (s_j)_i^n \\ &\quad - \frac{\Delta t}{\Delta x} (v_{i+1/2}^n)^- \times (s_j)_{i+1}^n + \frac{\Delta t}{\Delta x} (v_{i-1/2}^n)^+ \times (s_j)_{i-1}^n, \end{aligned}$$

where after Larrouturou, the right hand is nothing but a convex decomposition of $((s_j)_{i-1}^n, (s_j)_i^n, (s_j)_{i+1}^n)$ under the CFL restriction (3.5) (see again [14] for the details). We immediately infer from this convex decomposition the next inequality:

$$(3.34) \quad \begin{aligned} \phi_j(s_j(\mathbf{v}_i^{n+1,-})) &\leq \left(1 - \frac{\Delta t}{\Delta x} ((v_{i-1/2}^n)^+ - (v_{i+1/2}^n)^-)\right) \times \phi_j((s_j)_i^n) \\ &\quad - \frac{\Delta t}{\Delta x} (v_{i+1/2}^n)^- \times \phi_j((s_j)_{i+1}^n) + \frac{\Delta t}{\Delta x} (v_{i-1/2}^n)^+ \times \phi_j((s_j)_{i-1}^n), \end{aligned}$$

where we have successively used the increasing property and then the convexity property of ϕ_j . It now suffices to multiply the inequality (3.34) by $\rho_i^{n+1,-}$ and to apply backward each of the above steps from (3.33) to (3.31) to get the expected inequality (3.29). The validity of the maximum principles stated in (3.30) simply follows from the convex decomposition put forward in (3.33). The required positivity of the specific energies is obtained using the same arguments as those developed in the proof of Proposition 3.1. $\square$

3.2. Nonlinear projection methods. According to the discrete balance equation (3.28), standard finite volume methods induce a too large numerical rate of entropy dissipation for ρs_1 in comparison with the entropy dissipation rate of ρs_2 that in turn precludes the satisfaction of (2.22). Here, we propose to add as an additional step to classical methods based on the L^2 projection, a nonlinear correction procedure, the so-called nonlinear projection step, which purpose is precisely to restore at the discrete level the correct balance between the two entropy-dissipation rates when enforcing (2.22) for validity. The overall method will be understood as consistent method for approximating the solutions of the unusual equivalent formulation (2.29). The correction procedure is first grounded when proving existence and uniqueness of a solution in each of the local nonlinear problems to be solved

(one per cell). Then several stability estimates are derived. The relevance of this fairly simple technique will be further assessed by the numerical experiments.

3.2.1. *Third step: Nonlinear projection* ($t^{n+1,-} \rightarrow t^{n+1}$). In order to preserve the required conservation properties, let us define:

$$(3.35) \quad \rho_i^{n+1} = \rho_i^{n+1,-}, \quad (\rho u)_i^{n+1} = (\rho u)_i^{n+1,-}, \quad (\rho E)_i^{n+1} = (\rho E)_i^{n+1,-}.$$

Then to enforce for validity the generalized jump condition (2.21) at the discrete level, we propose to seek for $(\rho s_2)_i^{n+1}$ as a solution of:

$$(3.36) \quad \mu_2 \overline{T_1}_i^{n+1,-} \left(\{\rho s_1\}(\mathbf{v}_i^{n+1}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\} \right) - \\ \mu_1 \overline{T_2}_i^{n+1,-} \left((\rho s_2)_i^{n+1} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_2 u}\}_{i+1/2}^n - \{g_{\rho s_2 u}\}_{i-1/2}^n \right\} \right) = 0,$$

where in view of (3.35), one must read

$$(3.37) \quad \{\rho s_1\}(\mathbf{v}_i^{n+1}) = \{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}; (\rho s_2)_i^{n+1}),$$

i.e. as a nonlinear function of solely $(\rho s_2)_i^{n+1}$. Of course, as soon as $(\rho s_2)_i^{n+1}$ does exist, then $(\rho s_1)_i^{n+1}$ updates according to (3.37).

The scalar nonlinear problem (3.36) is built in each cell from the numerical entropy fluxes and the averaged temperatures respectively involved in the first and second steps. The discrete equation (3.36) thus gives rise to a consistent discretisation of (2.29d), provided that it can be uniquely solved in $(\rho s_2)_i^{n+1}$. This is the matter of the following statement which proof heavily relies on the property of the (approximate) Riemann solver in the first step to be entropy dissipative with respect to the Lax entropy pair $(\rho s_1, \rho s_1 u)$:

Theorem 3.5. *Under the thermodynamic assumptions (2.2)–(2.4) and the CFL condition (3.5), the nonlinear projection step (3.36) built from step 1 and step 2 is uniquely solvable in $(\rho s_2)_i^{n+1}$ for each $i \in \mathbb{Z}$. In addition, the following stability estimates are in order:*

$$(3.38) \quad \{\rho s_1\}(\mathbf{v}_i^{n+1}) \geq \{\rho s_1\}(\mathbf{v}_i^{n+1,-}) \quad \text{and} \quad (\rho s_2)_i^{n+1} \leq (\rho s_2)_i^{n+1,-}.$$

The proof of this result is postponed at the end of the present section. Let us underline that the Theorem 3.5 stays valid as soon as any given entropy satisfying approximate Riemann solver is involved in the first step in place of the Godunov method.

Remark 3.6. The above existence result allows us to consider $\mathbf{w}_i^{n+1} = (\rho_i^{n+1}, (\rho u)_i^{n+1}, (\rho E)_i^{n+1}, (\rho s_1)_i^{n+1})$ with $(\rho s_1)_i^{n+1}$ defined by (3.37). So that when understanding $\{\rho s_2\}$ as a function of $\mathbf{w}$, $(\rho s_1)_i^{n+1}$ is seen to be the unique solution of the symmetric version of (3.36):

$$(3.39) \quad \mu_2 \overline{T_1}_i^{n+1,-} \left((\rho s_1)_i^{n+1} - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{\rho s_1 u\}_{i+1/2}^n - \{\rho s_1 u\}_{i-1/2}^n \right\} \right) - \\ \mu_1 \overline{T_2}_i^{n+1,-} \left(\{\rho s_2\}(\mathbf{w}_i^{n+1}) - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{\rho s_2 u\}_{i+1/2}^n - \{\rho s_2 u\}_{i-1/2}^n \right\} \right) = 0.$$

In other words, the two entropies $(\rho s_1)_i^{n+1}$ and $(\rho s_2)_i^{n+1}$ are kept in exact symmetry. Furthermore due to the use of the Larouturou formula (3.11) for defining the numerical entropy fluxes, it can be seen that working out the splitting method when privileging the unknown $\mathbf{w}$ in place of $\mathbf{v}$ gives rise to exactly (3.39) and thus (3.36). In this sense, the nonlinear projection procedure we propose restores at the discrete level the equivalence in the different formulations.

The next statement asserts that the nonlinear projection procedure (3.36) preserves the desirable stability properties of the prediction step. The estimates (3.38), essentially inferred from the use of an entropy satisfying approximate Riemann solver in the first step, will turn out to be responsible for this central issue.

Theorem 3.7. *Under the CFL restriction (3.5), the following discrete entropy inequalities are satisfied:*

$$(3.40) \quad \{\rho\phi(s_j)\}(\mathbf{v}_i^{n+1}) - (\rho\phi(s_j))_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho\phi_j(s_j)u}\}_{i+1/2}^n - \{g_{\rho\phi_j(s_j)u}\}_{i-1/2}^n \right\} \leq 0,$$

for any given increasing and convex function $\phi_j : \mathbb{R} \rightarrow \mathbb{R}$. Next, the following maximum principles on the specific entropies s_1 and s_2 are met :

$$(3.41) \quad \begin{cases} (s_1)_i^{n+1} \leq \max((s_1)_{i-1}^n, (s_1)_i^n, (s_1)_{i+1}^n), \\ (s_2)_i^{n+1} \leq \max((s_2)_{i-1}^n, (s_2)_i^n, (s_2)_{i+1}^n). \end{cases}$$

Both the specific internal energies $(\epsilon_1)_i^{n+1}$ and $(\epsilon_2)_i^{n+1}$ are thus kept positive.

We now turn proving the Theorem 3.5.

Proof. To shorten the notations, let us introduce:

$$(3.42) \quad (\rho s_1)_i^{n+1,*} = (\rho s_1)_i^n - \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\},$$

$$(3.43) \quad C_i^{n+1,-} = \frac{\mu_2 \overline{T_1}_i^{n+1,-}}{\mu_1 \overline{T_2}_i^{n+1,-}} > 0,$$

so that (3.36) recasts as follows:

$$(3.44) \quad (\rho s_2)_i^{n+1} - (\rho s_2)_i^{n+1,*} = C_i^{n+1,-} \left(\{\rho s_1\}(\mathbf{v}_i^{n+1}) - (\rho s_1)_i^{n+1,*} \right).$$

It turns out to be convenient to solve (3.44) in the auxiliary unknown $\mathcal{M}_i^{n+1}$ defined by the linear change of variable $C_i^{n+1,-} \mathcal{M}_i^{n+1} = (\rho s_2)_i^{n+1} - (\rho s_2)_i^{n+1,*}$: namely to look for $\mathcal{M}_i^{n+1}$ solution of:

$$(3.45) \quad \begin{aligned} \mathcal{M}_i^{n+1} &= \{\rho s_1\}(\rho_i^{n+1-}, (\rho u)_i^{n+1-}, (\rho E)_i^{n+1-}, C_i^{n+1,-} \mathcal{M}_i^{n+1} + (\rho s_2)_i^{n+1,*}) \\ &- (\rho s_1)_i^{n+1,*}. \end{aligned}$$

Existence and uniqueness of such a solution now follows from a direct study of the next function:

$$(3.46) \quad \begin{aligned} \theta(m) &= \{\rho s_1\}(\rho_i^{n+1-}, (\rho u)_i^{n+1-}, (\rho E)_i^{n+1-}, C_i^{n+1,-} m + (\rho s_2)_i^{n+1,*}) \\ &- (\rho s_1)_i^{n+1,*} - m. \end{aligned}$$

Notice that (3.46) reads equivalently with little abuse in the notations:

$$(3.47) \quad \theta(m) = \rho_i^{n+1-} \times \{s_1\}(\tau_i^{n+1-}, \{\epsilon_1\}(\tau_i^{n+1-}, m)) - (\rho s_1)_i^{n+1,*} - m,$$

where

$$(3.48) \quad \begin{aligned} \{\epsilon_1\}(\tau_i^{n+1-}, m) &= (E)_i^{n+1-} - \frac{(u_i^{n+1-})^2}{2} \\ &- \epsilon_2(\tau_i^{n+1-}, C_i^{n+1,-} \tau_i^{n+1-} m + (s_2)_i^{n+1,*}). \end{aligned}$$

First, note that in view of the asymptotic assumption (2.4), s_2 runs from $-\infty$ to $+\infty$ so that the m -variable in (3.46) naturally runs over $\mathbb{R}$. Then, again in view of (2.4), $\lim_{m \rightarrow -\infty} \epsilon_2 = 0+$ yields from (3.48):

$$\lim_{m \rightarrow -\infty} \{\epsilon_1\}(\tau_i^{n+1-}, m) = (E)_i^{n+1-} - \frac{(u_i^{n+1-})^2}{2} > \{\epsilon_1\}(\tau_i^{n+1-}, 0) > 0$$

since $\{\epsilon_j\}(\tau_i^{n+1,-}, 0) = (\epsilon_j)_i^{n+1,-} > 0$ with $j = 1, 2$ as stated in Theorem 3.4. The above finite and positive limit in turn ensures from (3.47) that:

$$\lim_{m \rightarrow -\infty} \theta(m) = +\infty.$$

We now turn proving that $\theta(0) \leq 0$. This will prove that there exists at least one solution $\mathcal{M}_i^{n+1} \leq 0$ of $\theta(m) = 0$. Let us indeed compute:

$$(3.49) \quad \theta(0) = \{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho s_2)_i^{n+1,-}) - (\rho s_1)_i^{n+1,*},$$

and assume at this stage that:

$$(3.50) \quad \begin{aligned} &\{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho s_2)_i^{n+1,-}) \leq \\ &\{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho s_2)_i^{n+1,-}). \end{aligned}$$

This inequality will be derived shortly but is immediately seen to imply from (3.49) that:

$$(3.51) \quad \begin{aligned} \theta(0) &\leq \{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho s_2)_i^{n+1,-}) \\ &\quad - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_1 u}\}_{i+1/2}^n - \{g_{\rho s_1 u}\}_{i-1/2}^n \right\}, \end{aligned}$$

where $(\rho s_1)_i^{n+1,*}$ has been given its definition (3.42). The expected conclusion $\theta(0) \leq 0$ is then a direct consequence of the discrete entropy inequality (3.27) satisfied by the pair $(\rho s_1, \rho s_1 u)$ under the CFL condition (3.5). At last, in view of $\partial_{s_2}\{s_1\}(\mathbf{v}) = -T_2/T_1 < 0$, the smooth function $\theta(m)$ is actually strictly decreasing:

$$(3.52) \quad \begin{aligned} \frac{d\theta(m)}{dm} &= -1 \\ -C_i^{n+1,-} \left\{ \frac{T_2}{T_1} \right\} (\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, C_i^{n+1,-} m + (\rho s_2)_i^{n+1,-}) &< 0, \end{aligned}$$

and this guarantees the uniqueness of a solution $\mathcal{M}_i^{n+1}$ of (3.45). Let us conclude to the existence and uniqueness of a solution of (3.45) with $\mathcal{M}_i^{n+1} \leq 0$ when proving the estimate (3.50). To that end, observe that $(\rho, \rho u, \rho E)$ being fixed, the function $(\rho s_2) \rightarrow \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)$ is strictly decreasing in view of $\partial_{s_2}\{s_1\}(\mathbf{v}) = -T_2/T_1 < 0$. Next, the discrete entropy inequality (3.26) ensures that $(\rho s_2)_i^{n+1,-} \leq (\rho s_2)_i^{n+1,-}$ and the required result directly follows from the above monotonicity property.

Let us now turn deriving the second estimate entering (3.38). To that purpose, let us introduce:

$$m_i^{n+1,-} = -\frac{\mu_1 \Delta t}{2\Delta x^2 \overline{T_1}_i^{n+1,-}} ((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2) \leq 0,$$

in order to first rewrite from (3.20d):

$$(3.53) \quad (\rho s_2)_i^{n+1,-} = (\rho s_2)_i^{n+1,-} + C_i^{n+1,-} m_i^{n+1,-}.$$

and then to re-express the discrete entropy inequality (3.27) under the form:

$$(3.54) \quad \begin{aligned} &\{\rho s_1\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho s_2)_i^{n+1,-} + C_i^{n+1,-} m_i^{n+1,-}) - (\rho s_1)_i^{n+1,*} \\ &= \mathcal{E}_i^n + m_i^{n+1,-}, \end{aligned}$$

where $(\rho s_1)_i^{n+1,*}$ is given in (3.42) and $\mathcal{E}_i^n \leq 0$ denotes the numerical entropy dissipation rate put forward in Lemma 3.1. By the definition (3.46) of the function $\theta(m)$, (3.54) just recasts as:

$$(3.55) \quad \theta(m_i^{n+1,-}) = \mathcal{E}_i^n \leq 0.$$

The function $m \rightarrow \theta(m)$ being strictly decreasing (see (3.52)) with the property that $\theta(\mathcal{M}_i^{n+1}) = 0$, the above estimate ensures:

$$m_i^{n+1,-} \geq \mathcal{M}_i^{n+1}.$$

The second inequality in (3.38) thus follows since we deduce the required estimate:

$$(\rho s_2)_i^{n+1} - (\rho s_2)_i^{n+1,-} = C_i^{n+1,-} (\mathcal{M}_i^{n+1} - m_i^{n+1,-}) \leq 0.$$

in view of the representation formulae (3.53) and:

$$(\rho s_2)_i^{n+1} - (\rho s_2)_i^{n+1,-} = C_i^{n+1,-} \mathcal{M}_i^{n+1}.$$

But this estimate now allows us to deduce the first inequality in (3.38), namely:

$$\begin{aligned} \{\rho s_1\}(\rho_i^{n+1-}, (\rho u)_i^{n+1-}, (\rho E)_i^{n+1-}, (\rho s_2)_i^{n+1-}) \leq \\ \{\rho s_1\}(\rho_i^{n+1-}, (\rho u)_i^{n+1-}, (\rho E)_i^{n+1-}, (\rho s_2)_i^{n+1}), \end{aligned}$$

since as already claimed, the function $(\rho s_2) \rightarrow \{\rho s_1\}(\rho, \rho u, \rho E, \rho s_2)$ is strictly decreasing. This completes the proof. $\square$

The next technical result allows to prove the stability estimates stated in Theorem 3.7 when adopting exactly the same steps as those developed in the course of the proof of Theorem 3.4:

Lemma 3.8. *Under the CFL restriction (3.5), the following inequalities are preserved by the nonlinear projection procedure:*

$$(3.56) \quad \begin{aligned} \{\rho s_j\}(\mathbf{v}_i^{n+1}) &\leq \{\rho s_j\}(\mathbf{v}_i^n) \\ &\quad - \frac{\Delta t}{\Delta x} \left\{ \{g_{\rho s_j u}\}_{i+1/2}^n - \{g_{\rho s_j u}\}_{i-1/2}^n \right\}, \quad j = 1, 2, \end{aligned}$$

where the numerical entropy flux function are of the Larrouturou's type (3.11).

The proof heavily makes use of the second estimate in (3.38), Theorem 3.5:

$$(3.57) \quad (\rho s_2)_i^{n+1} \leq (\rho s_2)_i^{n+1,-}.$$

Let us underline that its validity essentially stems from the non positive entropy dissipation rate $\mathcal{E}_i^n$ entering the estimate (3.55). This observation again highlights the interest in involving a relevant entropy satisfying Riemann solver in the first step of the method.

Proof of lemma 3.8. The required inequality (3.56) with $j = 2$ then immediately follows from the estimate (3.57) and (3.29) in Theorem 3.4. It in turn implies the companion inequality with $j = 1$ just by the definition (3.36) of the nonlinear projection procedure. $\square$

3.2.2. Consistency with two asymptotic cases. This paragraph briefly highlights the consistency of the nonlinear projection method with two systems in full conservation form that are asymptotically recovered in the regime of a vanishing ratio of viscosities: namely either μ_1/μ_2 or μ_2/μ_1 goes to zero but with $\mu_1 + \mu_2 \geq \mu_0 > 0$. These two regimes are of practical importance in all the turbulence models (see [20] and the references therein). The equivalent system (2.29) indeed strongly suggests that in such a limit, the PDE's under consideration should meet a purely conservative formulation. The following statement gives support to this observation (see [3] for a proof):

Theorem 3.9. *In the limit μ_2 goes to zero, $\mu_1 > 0$ being fixed, travelling wave solutions of the system (2.6) converge uniformly over $\mathbb{R}$ to travelling wave solutions of the following system of conservation laws:*

$$(3.58) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = \partial_x (\mu_1 \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = \partial_x (\mu_1 u \partial_x u), \\ \partial_t \rho s_2 + \partial_x \rho s_2 u = 0, \end{cases}$$

with $\partial_t \rho s_1(\mathbf{v}) + \partial_x \rho s_1(\mathbf{v}) u \leq 0$.

Conversely, if μ_1 goes to zero, $\mu_2 > 0$ being fixed, travelling wave solutions of the system (2.6) converge uniformly over $\mathbb{R}$ to travelling wave solutions of the next system of conservation laws:

$$(3.59) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = \partial_x (\mu_2 \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = \partial_x (\mu_2 u \partial_x u), \\ \partial_t \rho s_1 + \partial_x \rho s_1 u = 0, \end{cases}$$

with $\partial_t \rho s_2(\mathbf{w}) + \partial_x \rho s_2(\mathbf{w}) u \leq 0$.

Let us observe that the nonlinear projection step (3.36) is by construction consistent with the above two asymptotic systems in full conservation form. Indeed, in the regime of a vanishing viscosity μ_2 , the algebraic relation (3.36) clearly gives the required asymptotic identities (see (3.58)):

$$\begin{aligned} (\rho s_2)_i^{n+1} &= (\rho s_2)_i^n - \frac{\Delta t}{\Delta x} \left\{ \{\rho s_2 u\}_{i+1/2}^n - \{\rho s_2 u\}_{i-1/2}^n \right\}, \\ (\rho \phi(s_1))_i^{n+1} - (\rho \phi(s_1))_i^n &+ \frac{\Delta t}{\Delta x} \left\{ \{\rho \phi(s_1) u\}_{i+1/2}^n - \{\rho \phi(s_1) u\}_{i-1/2}^n \right\} \leq 0. \end{aligned}$$

Conversely, the regime $\mu_1 \rightarrow 0$ also gives the expected asymptotic behaviour in view of the symmetric form (3.39) of (3.36).

3.2.3. About variable viscosities. In practical applications, the viscosity laws are smooth functions of the state variable (see [20] for examples). All the above considerations are seen to extend to this framework when adopting in the second step of the method the following finite difference formulae respectively for $\overline{\partial_x((\mu_1 + \mu_2)\partial_x u)}_i^{n+1,-}$, $\overline{\partial_x((\mu_1 + \mu_2)u\partial_x u)}_i^{n+1,-}$ and $\overline{\frac{\mu_2}{T_2}(\partial_x u)^2}_i^{n+1,-}$:

$$\begin{aligned} &\frac{1}{\Delta x^2} \left((\mu_T)_{i+1/2}^n (M^n u_{i+1} - M^n u_i) - (\mu_T)_{i-1/2}^n (M^n u_i - M^n u_{i-1}) \right), \\ &\frac{1}{2\Delta x^2} \left((\mu_T)_{i+1/2}^n ((M^n u_{i+1})^2 - (M^n u_i)^2) - (\mu_T)_{i-1/2}^n ((M^n u_i)^2 - (M^n u_{i-1})^2) \right), \\ &\frac{1}{2\Delta x^2 \overline{T_2}_i^{n+1,-}} \left((\mu_2)_{i+1/2}^n (M^n u_{i+1} - M^n u_i)^2 + (\mu_2)_{i-1/2}^n (M^n u_i - M^n u_{i-1})^2 \right), \end{aligned}$$

with $(\mu_T)_{i+1/2}^n = (\mu_1)_{i+1/2}^n + (\mu_2)_{i+1/2}^n$ and $\overline{T_2}_i^{n+1,-}$ given by (3.22). The relevant extension of the nonlinear projection equation (3.36) then reads:

$$\begin{aligned} &\overline{\mu_2}_i^{n+1,-} \overline{T_1}_i^{n+1,-} \left(\{\rho s_1\}(\mathbf{v}_i^{n+1}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{\rho s_1 u\}_{i+1/2}^n - \{\rho s_1 u\}_{i-1/2}^n \right\} \right) - \\ &\overline{\mu_1}_i^{n+1,-} \overline{T_2}_i^{n+1,-} \left((\rho s_2)_i^{n+1} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{\rho s_2 u\}_{i+1/2}^n - \{\rho s_2 u\}_{i-1/2}^n \right\} \right) = 0, \end{aligned}$$

where for $j = 1, 2$ the averaged viscosities receive the next definitions:

$$\overline{\mu_j}_i^{n+1,-} = \begin{cases} \frac{(\mu_j)_{i+1/2}^n + (\mu_j)_{i-1/2}^n}{2}, & \text{if } M^n u_{i+1} = M^n u_i = M^n u_{i-1}, \\ \frac{(\mu_j)_{i+1/2}^n (M^n u_{i+1} - M^n u_i)^2 + (\mu_j)_{i-1/2}^n (M^n u_i - M^n u_{i-1})^2}{(M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2}, & \text{otherwise.} \end{cases}$$

4. THE METHOD FOR A GENERAL ENTROPY SATISFYING NUMERICAL FLUX FUNCTION

The numerical method is useful in that the nonlinear projection step may be used as a final corrector step to one's favourite underlying stable scheme, and preserves its stability properties. We briefly summarize the proposed method for a general entropy satisfying numerical flux function.

4.1. First step: Underlying first order system ($t^n \rightarrow t^{n+1,=}$). Let be given a consistent 2-points numerical flux function $\mathcal{F} : \Omega_{\mathbf{v}} \times \Omega_{\mathbf{v}} \rightarrow \mathbb{R}^4$ for the numerical approximation of the weak solutions of the hyperbolic problem (3.2) so as to define for all $i \in \mathbb{Z}$:

$$\begin{aligned}\rho_i^{n+1,=} &= \rho_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_\rho(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}_\rho(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)), \\ (\rho u)_i^{n+1,=} &= (\rho u)_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{\rho u}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}_{\rho u}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)), \\ (\rho E)_i^{n+1,=} &= (\rho E)_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{\rho E}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}_{\rho E}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)), \\ (\rho s_2)_i^{n+1,=} &= (\rho s_2)_i^n - \frac{\Delta t}{\Delta x} (\mathcal{F}_{\rho s_2}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}_{\rho s_2}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)).\end{aligned}$$

The finite volume method is assumed to be entropy satisfying; namely there exists a consistent 2-points numerical entropy flux function $\mathcal{F}_{\rho s_1} : \Omega_{\mathbf{v}} \times \Omega_{\mathbf{v}} \rightarrow \mathbb{R}$ such that, for all $i \in \mathbb{Z}$, the following inequality is valid under some CFL condition:

$$\{\rho s_1\}(\mathbf{v}_i^{n+1,=}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} (\mathcal{F}_{\rho s_1}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}_{\rho s_1}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \leq 0.$$

4.2. Second step: Second order operator ($t^{n+1,=} \rightarrow t^{n+1,-}$).

$$\begin{aligned}\rho_i^{n+1,-} &= \rho_i^{n+1,=}, \\ (\rho u)_i^{n+1,-} &= (\rho u)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1 + \mu_2)\partial_x u)}_i^{n+1,-}, \\ (\rho E)_i^{n+1,-} &= (\rho E)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1 + \mu_2)u\partial_x u)}_i^{n+1,-}, \\ (\rho s_2)_i^{n+1,-} &= (\rho s_2)_i^{n+1,=} - \Delta t \frac{\mu_2}{T_2} \overline{(\partial_x u)^2}_i^{n+1,-},\end{aligned}$$

with the definitions introduced in (3.21) and (3.22).

The above two steps are referred as to a L^2 -projection method.

4.3. Third step: Nonlinear projection ($t^{n+1,-} \rightarrow t^{n+1}$). We keep unchanged

$$\rho_i^{n+1} = \rho_i^{n+1,-}, \quad (\rho u)_i^{n+1} = (\rho u)_i^{n+1,-}, \quad (\rho E)_i^{n+1} = (\rho E)_i^{n+1,-},$$

and we solve in $(\rho s_2)_i^{n+1}$ the following equation:

$$\begin{aligned}\mu_2 \overline{T_1}_i^{n+1,-} \left(\{\rho s_1\}(\mathbf{v}_i^{n+1}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x} \left\{ (\mathcal{F}_{\rho s_1})_{i+1/2}^n - (\mathcal{F}_{\rho s_1})_{i-1/2}^n \right\} \right) - \\ \mu_1 \overline{T_2}_i^{n+1,-} \left((\rho s_2)_i^{n+1} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x} \left\{ (\mathcal{F}_{\rho s_2})_{i+1/2}^n - (\mathcal{F}_{\rho s_2})_{i-1/2}^n \right\} \right) = 0,\end{aligned}$$

where $(\mathcal{F}_{\rho s_j})_{i+1/2}^n = \mathcal{F}_{\rho s_j}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)$, $j = 1, 2$.

This completes the description of the nonlinear projection method.

5. NUMERICAL EXPERIMENTS

The ability of the discussed discrete methods, namely the usual splitting scheme with L^2 projection and the new nonlinear projection method, in the capture of stiff viscous shock layers for (2.6) is evaluated when testing their sensitivity in the prediction of the end states with respect to the mesh refinement. The pressure laws under consideration are those of two polytropic gases with distinct constant adiabatic exponent while the viscosities are taken to be positive constants. For the sake of comparison, the popular approach for approximating the solutions of the extended Navier-Stokes equations, briefly discussed below, is included in the benchmarks.

5.1. The popular approach for solving (2.6). Falling within the frame of splitting methods, the usual (if not systematic) approach for solving (2.6) consists in treating all the non conservative products as being “source terms” so as to consider in a first step the following system in conservation form:

$$(5.1) \quad \begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = 0, \\ \partial_t \rho \epsilon_2 + \partial_x \rho \epsilon_2 u = 0, \end{cases}$$

while in a next step, the “source term” $-p_2 \partial_x u$ is to be given some *ad hoc* finite difference approximation. We refer for instance the reader to [15] and [20] for the details. Let us underline the work by Forestier-Herard-Louis [9] pointing out, on a theoretical ground, the violation of the positivity requirement for the internal energies. These authors have shown how to restore this positivity property when keeping the non conservative product in (5.1) while defining it for shock solutions thanks to the Volpert product: $0.5 (p_2^- + p_2^+) \times (u^+ - u^-)$, *i.e.* regardless of the driving ratio of the viscosities.

5.2. Numerical results. The initial data of the Cauchy problems to be solved are made of two constant states, the discontinuity being located at $x = 0$. In agreement with real life situations, the Reynolds number is set to 10^6 in all our experiments. The exact solutions of the associated Cauchy problem (2.6) therefore correspond to a stiff regularization of some Riemann solutions for the underlying hyperbolic system. The two extreme waves in our benchmarks will always correspond to compression waves. Thus, the exact solutions under consideration can be thought of being made of two sharp travelling wave solutions separated by a contact wave. In the reference solution, displayed in solid line hereafter, the end points of the two extreme waves are determined when integrating the ODE’s system (2.15) governing the exact travelling wave solutions, for a prescribed state $\mathbf{v}$ in the past and a given velocity σ . To that end, we heavily use the property that the density $\rho(\xi)$ strictly increases with $\xi = x - \sigma t$ along the shock layer (see [1]). This allows to re-express the ODE system (2.15) in the new space like variable ρ in a compact interval. We refer the reader to Berthon [1] for the technical details. Discrete solutions are systematically compared with this reference solution.

All the calculations described below have been performed according to the following strategy. An exact Roe type linearization for system (3.2) yields an approximate Riemann solver to solve the first step (3.1.1) (see [6, 2] for the required formulae). Successive grids refinements, ranging from 100 to 2000 cells, are considered. The CFL number is fixed at the constant value 0.5.

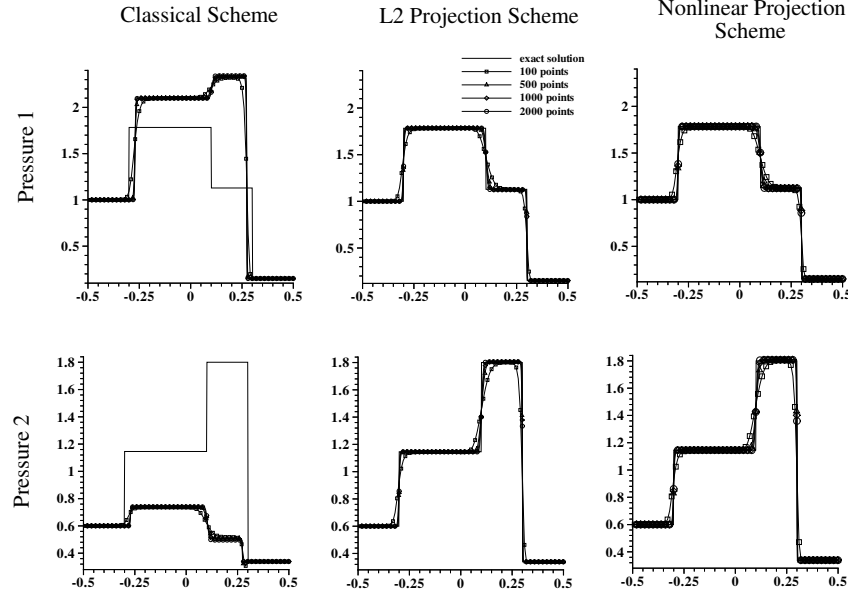
Three test cases, labelled from A to C, are addressed. They are directly motivated by the three distinct regimes that underlie the fluid model under consideration and that are dictated by the amplitude of the viscosity ratio μ_2/μ_1 . Namely, they involve a viscosity ratio which when compared to unity, is successively small, close and large.

The associated initial data are defined in table 1.

The theorem 3.9 indicates that the solutions of benchmarks A and C are asymptotically close to solutions of systems of conservation laws. Namely, case A makes the entropy ρs_2 to be merely advected by the flow while conversely, ρs_1 inherits such a property in problem C. By contrast, the solution of problem B stays far away from these two asymptotic regimes. All the figures assess that the popular numerical strategy (5.1) grossly fails to properly restore the correct end states in all the three regimes. Considering the L^2 projection method built from (2.29), the discrete solutions agree with the exact ones only in case A, as expected since here

Test	γ_1	γ_2	μ_2/μ_1	ρ	u	p_1	p_2
A	1.4	1.6	0.01	1	1	1	0.6
				1.05518	-0.88895	0.15031	0.33869
B, D	1.4	1.6	1	1	1	1	0.6
				1.92678	-1.25451	2.74247	2.09561
C	1.4	1.6	100	1	1	1	0.6
				1.01595	-0.93415	0.24142	0.15528

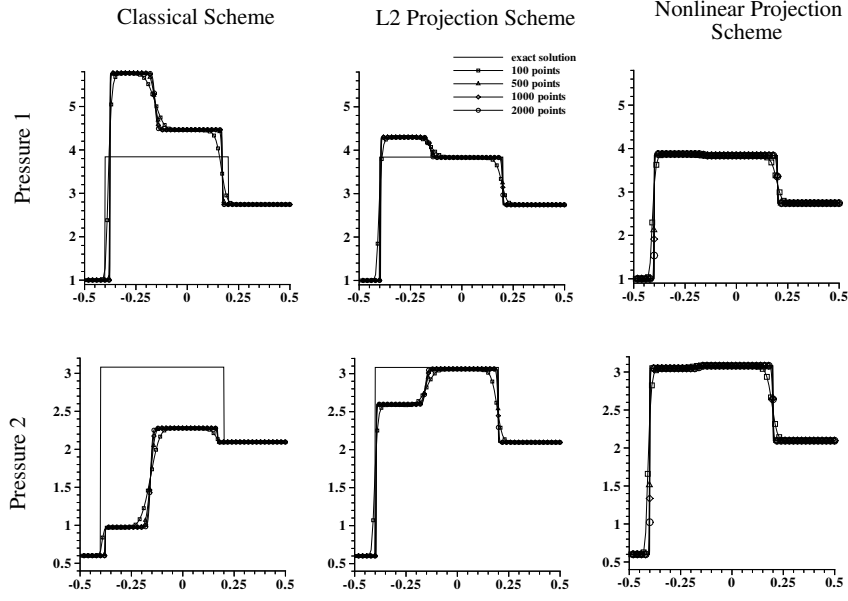
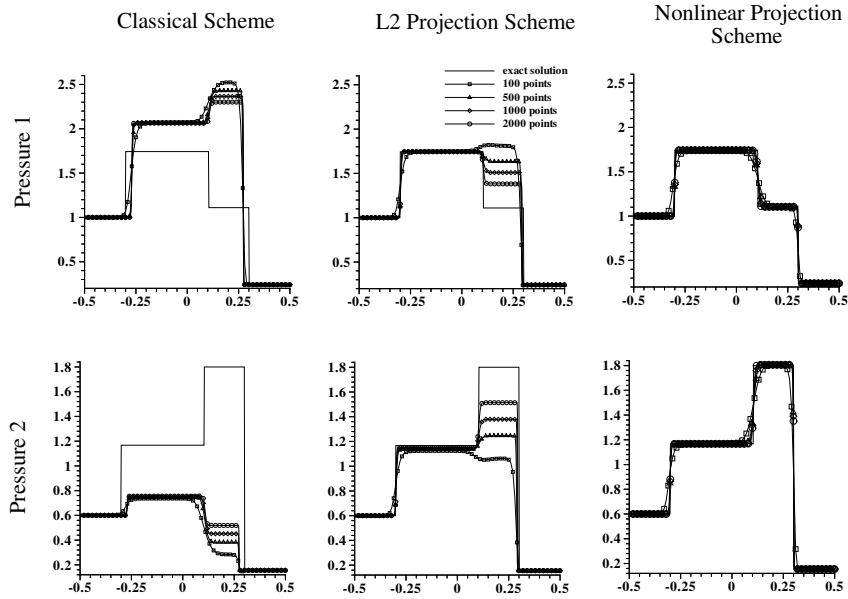
TABLE 1

FIGURE 3. Problem A: $\mu_2/\mu_1 \ll 1$

ρs_2 is close to be a conservative variable. Such a property no longer holds for problems B and C and the discrete solutions exhibit large errors with the reference ones. Considering the nonlinear projection method, it produces approximate solutions in fairly good (if not excellent) agreement with the reference solutions while staying almost insensitive to the mesh refinement in the three investigated regimes.

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FIGURE 4. Problem B: $\mu_2/\mu_1 = 1$ FIGURE 5. Problem C: $\mu_2/\mu_1 \gg 1$

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