

Nonlinear projection methods for multi-entropies Navier-Stokes systems

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Abstract. This paper is devoted to the numerical approximation of the compressible Navier-Stokes equations with several independent temperatures. Several models derived in plasma physics or in turbulence typically enter the proposed framework. The striking novelty over the usual Navier-Stokes equations stems from the impossibility to recast equivalently the present system in full conservation form. Classical finite volume methods are shown to grossly fail in the capture of travelling wave solutions that are of primary interest in the present work. We propose a systematic and effective procedure, the so-called nonlinear projection operator, for correcting the errors while preserving all the stability properties satisfied by suitable Godunov type methods. This nonlinear procedure is exemplified when coupled with an exact Roe solver and the resulting method is shown numerically to yield approximate solutions in close agreement with exact solutions.

1 Introduction

The present work treats the numerical approximation of the solutions of the Navier-Stokes equations for a compressible fluid modelled by N , $N \geq 2$, independent temperatures. By independent, we mean that each of the temperatures comes with its own specific entropy so that smooth solutions of the system under consideration obey simultaneously N independent entropy balance equations. N independent pressure laws can be then defined and the sum of all the pressures yields the total pressure.

Despite that such fluid models are seen below to exhibit several close relationships with the usual Navier-Stokes system, the fundamental discrepancy stays in the lack of an admissible change of variables that recasts the governing equations in full conservation form. Indeed, the three conservation laws ruling density, momentum and total energy must be supplemented by $(N - 1)$ of the entropy balance equations that govern the N temperatures. However and without restrictive modelling assumptions to be put on the viscosities and the conductivities, none of the N entropy balance equations boils down to a conservation law. These equations generally involve non

conservative products that account for dissipative phenomena : namely the entropy dissipation rates. The governing PDE's system can be thus given the following abstract form :

$$\partial_t \mathbf{v} + \partial_x f(\mathbf{v}) = \mathcal{E}(\mathbf{v}, \partial_{xx}^2 \mathbf{v}), \quad \mathbf{v} \in \mathbb{R}^{N+2}, \quad x \in \mathbb{R}, \quad t > 0, \quad (1)$$

where the right hand side involves diffusive terms in conservation form as well as the entropy dissipation rates in non conservation form. Here smooth solutions of (1) satisfy without condition the following additional and non trivial balance equation (namely the N^{th} entropy balance equation which has not been addressed yet) :

$$\partial_t U(\mathbf{v}) + \partial_x F(\mathbf{v}) = \mathcal{E}_N(\mathbf{v}, \partial_{xx}^2 \mathbf{v}), \quad (2)$$

where the right hand side is directly inferred from the dissipative phenomena modelled in (1).

Such systems can be understood as a natural extension of the classical Navier-Stokes equations, i.e. equipped with a single entropy balance equation, in that they actually occur in several distinct physical settings. They arise for instance in plasma physics where the electrons temperature must be distinguished from the mixture temperature of the other (heavy) species. They can be also recognized under that form within the frame of the so-called “two transport” equations models for turbulent compressible flows where the averaged thermodynamic temperature must be distinguished from the specific turbulent kinetic energy k (or for short turbulent temperature). All these models are addressed below with the assumption of a large Reynolds number.

The numerical capture of the viscous shock layers coming with (1), is of primary importance in the present work. Since the Reynolds numbers of interest are large, these layers display the character of a shock wave in that they differ from their end states only in a small interval of rapid transition. Hence for mesh refinements of practical interest, the associated discrete profiles stay largely under resolved. Our purpose is actually to correctly capture the two end states, $\mathbf{v}_L$ and $\mathbf{v}_R$, of a given shock layer together with its relevant speed of propagation σ without resolving sharply the viscous layer itself.

It is quite well-known that such an issue does not raise special difficulties within the standard frame of the Navier-Stokes equations, e.g. in conservation form. Indeed, the triples $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ are solutions of the classical Rankine-Hugoniot relations and finite volume methods in conservation form readily ensure their suitable capture. At the very core of this success, is the independence property of (say) the right end state $\mathbf{v}_R = \mathbf{v}_R(\sigma; \mathbf{v}_L)$ with respect to the precise modelling of the diffusive phenomena (see for instance Gilbarg [9]). Put in other words, the definition of $\mathbf{v}_R(\sigma; \mathbf{v}_L)$ stays completely free from the resulting entropy dissipation rate and in this sense, its numerical capture stays also free from the rate of numerical dissipation coming with a given scheme in conservation form.

The situation turns out to be completely different in the setting of the extended Navier-Stokes equations (1). Its non conservation form makes this time the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ to heavily depend on the precise shape of the diffusive tensor. In deep contrast with the setting of systems in conservation form, such a dependence stays at the basis of recent works devoted to hyperbolic systems involving non conservative products (see Lefloch [14], Dal Maso-LeFloch-Murat [7], Raviart-Sainsaulieu [16]) and it will be numerically illustrated below (cf. Figures 2). When rephrased in the setting of numerical methods, this dependence implies in turn that the numerical viscous form of the method must fit the exact diffusive tensor in (1). Such a requirement has received a theoretical foundation in a work by Hou-LeFloch [11] and is actually the starting point of some numerical methods as devised for instance by Karni [12]. In this work, fitness is measured via a modified equations approach according to which unsuited truncature errors are forced to cancel up to a given order but such a correction turns out to be insufficient (see [11] for proofs and numerical illustrations and also [5] for an enlightening presentation). In this work, we propose another approach based on the analysis of the discrete dissipation rates of a given Lax entropy. As pointed out below, it turns out that the end states of under resolved viscous shock layers of (1) require for their correct capture to prescribe explicitly the Lax entropy dissipation rate $\mathcal{E}_N(\mathbf{v}, \partial_{xx}^2 \mathbf{v})$ in (2). This requirement will be explained below but notice from now on that it asks the numerical methods to satisfy in turn an imposed rate of entropy dissipation. This non standard issue is precisely the main motivation of the present work.

In order to illustrate the negative consequence of a failure in the above requirement, let us consider the numerical results displayed in figure 1. These are obtained using a standard and consistent finite volume method which is briefly derived in the forthcoming numerical section. This problem enters the

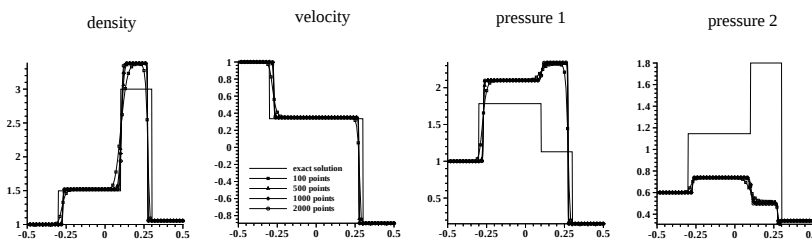


Fig. 1. Typical failure of classical finite volume methods in the capture of viscous shock layers

present setting for extended Navier-Stokes equations with two independent specific entropies when choosing two strictly positive viscosities and two identically zero conductivities. The discrete solutions stay virtually non sensitive with respect to the mesh refinements (from 100 points up to 2000 points) and clearly seem to coincide with a given function. However such a “limit” function exhibits large errors with respect to the exact solution plotted in solid lines. Let us underline that the above result does not contradict the celebrated Lax-Wendroff convergence theorem because the numerical method under consideration cannot be put in conservation form since the system (1) to be tackled cannot.

For the sake of simplicity in the notations, we focus ourselves in the sequel to the case of two independent specific entropies. Governing PDE's will be furthermore addressed using only one space variable. Let us underline that most of the results we state below extend in a straightforward way to higher space dimensions, taking advantage of the rotational invariance of the equations. In the same way, the numerical method we propose will be seen to easily extend to additional independent entropies.

The format of the present paper is as follows. The first section describes the convective-diffusive system under consideration with a special emphasis put on the analysis of its travelling wave solutions (i.e. viscous shock layers). A close characterization of the triples $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ will be proposed on the basis of generalized jump relations. These relations are given some convenient forms that in turn will be shown to encode the small scale effects taking place within viscous shock layers. The second section is devoted to the numerical approximation of the travelling wave solutions for mesh refinements of practical interest. Such solutions are thus by definition under resolved. We first underline that despite relevant Godunov type methods actually enjoy several stability properties, their corresponding numerical rate of entropy dissipation stays always smaller than the required one in (2) and as a result, the discrepancies with the exact solutions can only unendly amplify with time. This error analysis will then suggest the introduction of a nonlinear projection step that enforce the validity of the generalized jump conditions at the discrete level. Such a nonlinear projection may be understood as a systematic way to correct standard Godunov type methods in order to achieve a much better agreement between exact and discrete solutions. Furthermore, approximate solutions performed thanks to the nonlinear method satisfy the required positivity preserving properties in addition to several (nonlinear) stability requirements. Several numerical results are displayed, intending to illustrate the benefits of the proposed nonlinear projection. Most of the proofs of the forthcoming statements are omitted and we refer the reader to the companion papers [1], [2] and [3] for the details.

2 Mathematical model

We consider a gas with density ρ and velocity u , which is modelled by two independent pressure laws p and p_τ , associated with two constant adiabatic exponents $\gamma > 1$ and $\gamma_\tau > 1$. The system of PDE's that governs such a fluid model writes :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x ((\mu + \mu_\tau) \partial_x u), \\ \partial_t p + \partial_x p u + (\gamma - 1) p \partial_x u = (\gamma - 1) \left(\mu (\partial_x u)^2 + \partial_x (\kappa \partial_x T) \right), \\ \partial_t p_\tau + \partial_x p_\tau u + (\gamma_\tau - 1) p_\tau \partial_x u = (\gamma_\tau - 1) \left(\mu_\tau (\partial_x u)^2 + \partial_x (\kappa_\tau \partial_x T_\tau) \right), \end{cases} \quad (3)$$

where the involved temperatures respectively read $T = p/\rho$ and $T_\tau = p_\tau/\rho$. This convective-diffusive system can be understood as an extension of the standard Navier-Stokes equations when considering an additional PDE for governing an additional pressure. Depending on the closure relations for the viscosities μ , μ_τ and the thermal conductivities κ and κ_τ , several distinct physical models enter the present framework. Let us quote for instance plasma models (see [6] for instance) but also turbulence models (see [1] and below). In this section, all these transport coefficients are assumed to be fixed positive constants for the sake of simplicity in the discussion. The reader is referred to section 3 for the case of varying coefficients.

Similarly to the classical Navier-Stokes equations, the smooth solutions of system (3) obey additional governing equations as we now state :

Lemma 2.1 *Smooth solutions of (3) satisfy the following conservation law:*

$$\partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x ((\mu + \mu_\tau) u \partial_x u) + \partial_x (\kappa \partial_x T) + \partial_x (\kappa_\tau \partial_x T_\tau), \quad (4)$$

where the total energy ρE is defined by :

$$\rho E = \frac{(\rho u)^2}{2\rho} + \frac{p}{\gamma - 1} + \frac{p_\tau}{\gamma_\tau - 1}. \quad (5)$$

Smooth solutions satisfy in addition the following balance equations :

$$\partial_t \rho \log(s) + \partial_x \rho \log(s) u = \frac{\gamma - 1}{T} \left(\mu (\partial_x u)^2 + \partial_x (\kappa \partial_x T) \right), \quad (6)$$

$$\partial_t \rho \log(s_\tau) + \partial_x \rho \log(s_\tau) u = \frac{\gamma_\tau - 1}{T_\tau} \left(\mu_\tau (\partial_x u)^2 + \partial_x (\kappa_\tau \partial_x T_\tau) \right), \quad (7)$$

where the specific entropies are respectively given by

$$s := \frac{p}{\rho^\gamma}, \quad s_\tau := \frac{p_\tau}{\rho^{\gamma_\tau}}. \quad (8)$$

Consequently, smooth solutions of (3) obey :

$$\begin{aligned} \mu_\tau \frac{T}{\gamma-1} (\partial_t \rho \log(s) + \partial_x \rho \log(s)u) - \mu \frac{T_\tau}{\gamma_\tau-1} (\partial_t \rho \log(s_\tau) + \partial_x \rho \log(s_\tau)u) \\ = \partial_x (\mu_\tau \kappa \partial_x T - \mu \kappa_\tau \partial_x T_\tau), \end{aligned} \quad (9)$$

where the right hand side follows under the assumption of two constant viscosities μ and μ_τ .

The three balance equations (4), (6) and (7) can be proved to be the only non trivial additional equations for smooth solutions (up to some standard non-linear tranforms in s and s_τ). As a consequence and besides several close relationships with the usual Navier-Stokes system (see Berthon [1] for a deeper insight), the very discrepancy stays in the lack of four non trivial conservation laws. Indeed and without restrictive modelling assumptions (see below), none of the equations (6), (7) and (9) boils down to a conservation law. As a consequence, (3) cannot be recast, generally speaking, in full conservation form.

Our purpose here is to highlight after the pioneering works by LeFloch [14], Raviart-Sainsaulieu [16] and Sainsaulieu [18] that the non conservation form met by (3) makes the end states of viscous shock layers to intrinsically depend on the closure relations for the transport coefficients μ , μ_τ and κ , κ_τ . Under the assumption of two constant viscosities, such a dependence more precisely occurs in term of the ratio of μ and μ_τ . In order to assess this issue, let us focus our attention on the non standard balance equation (9). Its straightforward derivation reflects a cancellation property. Namely the entropy balance equations (6) and (7) are not independent but actually evolve proportionally to the ratio of the two viscosities μ and μ_τ . Indeed and at least formally, (9) yields once integrated with respect to the space variable :

$$\begin{aligned} \frac{\mu_\tau}{\mu + \mu_\tau} \left\{ \int_{\mathbb{R}} \frac{T}{\gamma-1} (\partial_t \rho \log(s) + \partial_x \rho \log(s)u) dx \right\} - \\ \frac{\mu}{\mu + \mu_\tau} \left\{ \int_{\mathbb{R}} \frac{T_\tau}{\gamma_\tau-1} (\partial_t \rho \log(s_\tau) + \partial_x \rho \log(s_\tau)u) dx \right\} = 0, \end{aligned} \quad (10)$$

so that the evolution in time of the two entropies must be kept in balance according to the ratio of the two viscosities. Let us underline that by contrast with (6) and (7) where entropy dissipation rates are actually independently imposed, the weighted equation (9) exhibits a compared rate of both the entropy dissipations.

To further assess this claim, let us adopt temporarily some restrictive modelling assumptions on both the viscosities and conductivities coefficients. Noticing that (9) reexpresses equivalently as :

$$\begin{aligned} \mu_\tau \frac{\rho^{\gamma-1}}{\gamma-1} (\partial_t \rho s + \partial_x \rho s u) - \mu \frac{\rho^{\gamma_\tau-1}}{\gamma_\tau-1} (\partial_t \rho s_\tau + \partial_x \rho s_\tau u) = \\ \partial_x (\mu_\tau \kappa \partial_x T - \mu \kappa_\tau \partial_x T_\tau). \end{aligned} \quad (11)$$

the following result easily follows :

Lemma 2.2 (i) Assume that μ and μ_τ are two given positive constants and $\kappa = \kappa_\tau = 0$. If moreover $\gamma = \gamma_\tau$, then (11) reduces to the following conservation law :

$$\partial_t \rho (\mu_\tau s - \mu s_\tau) + \partial_x \rho (\mu_\tau s - \mu s_\tau) u = 0. \quad (12)$$

(ii) Let μ_0 and $\mu_{\tau 0}$ denote two positive constants and assume that $\mu = \mu_0 T$, $\mu_\tau = \mu_{\tau 0} T_\tau$ with $\kappa = \kappa_\tau = 0$. Then (9) coincides with the following conservation law :

$$\begin{aligned} \partial_t \rho \left(\frac{\mu_{\tau 0}}{\gamma - 1} \log(s) - \frac{\mu_0}{\gamma_\tau - 1} \log(s_\tau) \right) + \\ \partial_x \rho \left(\frac{\mu_{\tau 0}}{\gamma - 1} \log(s) - \frac{\mu_0}{\gamma_\tau - 1} \log(s_\tau) \right) u = 0. \end{aligned} \quad (13)$$

These restrictive assumptions therefore allow for additional non trivial conservation laws that are encoded in the non standard balance equation (9) or its equivalent form (11). When considering their associated Rankine-Hugoniot condition, they clearly indicate that the end states of the viscous shock layers under consideration actually depend on the ratio of both viscosities. The above restrictive assumptions will no longer be adopted in the sequel. The dependence we have just pointed out is numerically illustrated in figure 2 in a more general setting in which the system (3) does not admit an equivalent full conservation form. For a given left end state $\mathbf{v}_L$ and a given velocity σ , the

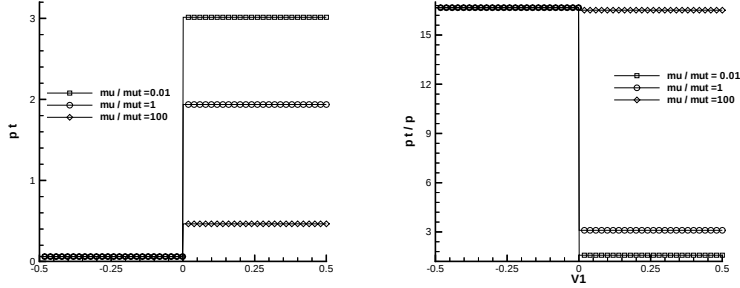


Fig. 2. Right end state $\mathbf{v}_R$ as a function of μ/μ_τ for a given $(\sigma, \mathbf{v}_L)$.

required right end states $\mathbf{v}_R$ are defined when solving numerically the nonlinear EDO's system governing travelling wave solutions (see next section) for various ratios of the viscosities. Furthermore and in a sense described below, (9) or (11) continue to play a major role in a general setting since they encode

a generalized jump condition which turns out to play a central role for our numerical purpose.

2.1 Travelling wave solutions and jump relations

In this section, we derive generalized jump relations that are needed to characterize the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ associated with a given viscous shock layer. In that aim, let us first recall that a travelling wave solution of (3) is a particular solution in the form $\mathbf{v}(x, t) = \widehat{\mathbf{v}}(x - \sigma t)$ with

$$\lim_{\xi \rightarrow -\infty} \widehat{\mathbf{v}}(\xi) = \mathbf{v}_L, \quad \lim_{\xi \rightarrow +\infty} \widehat{\mathbf{v}}(\xi) = \mathbf{v}_R, \quad \xi = x - \sigma t, \quad (14)$$

where the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ is prescribed. In addition to (14), it thus satisfies the following nonlinear EDO's system :

$$-\sigma d_\xi \widehat{\mathbf{v}} + d_\xi f(\widehat{\mathbf{v}}) = \mathcal{E}(\widehat{\mathbf{v}}, d_\xi^2 \widehat{\mathbf{v}}), \quad \xi \in \mathbb{R}, \quad (15)$$

where we have used the abstract form (1) of system (3). Neglecting the conductivity coefficients κ and κ_τ in (3), Berthon-Coquel [2] have proved global existence and uniqueness of smooth travelling wave solutions of (3) for general (nonlinear) viscosity functions.

Our purpose is to highlight, after LeFloch [14], Raviart-Sainsaulieu [16], that the end states of integral curve solution of (14), (15) does not depend on the amplitude of the diffusive tensor modelled in (15) but just on its shape. Put in other words, the end states as well as the velocity σ do not depend on the Reynolds number under consideration. Indeed let us introduce, after [14] and [16], the function

$$\mathbf{v}_\delta(x, t) = \widehat{\mathbf{v}}\left(\frac{x - \sigma t}{\delta}\right),$$

where $\delta > 0$ denotes a positive rescaling parameter. For all $\delta > 0$, $\mathbf{v}_\delta(x, t)$ turns out to be a travelling wave solution of (3) but for the viscosities μ^δ , μ_τ^δ and the conductivities κ^δ , κ_τ^δ defined by

$$\mu^\delta = \delta\mu, \quad \mu_\tau^\delta = \delta\mu_\tau, \quad \kappa^\delta = \delta\kappa, \quad \kappa_\tau^\delta = \delta\kappa_\tau.$$

The function $\mathbf{v}_\delta(x, t)$ is obviously associated with the same triple $(\sigma, \mathbf{v}_L, \mathbf{v}_R)$ for all values of the rescaling parameter $\delta > 0$: the expected independence property with respect to the Reynolds number thus follows. As a consequence, the following results hold true (see [3] for a proof and extensions to a more general setting):

Theorem 2.3 *Assume that μ , μ_τ , κ and κ_τ denote positive constants. Then the triple $(\sigma, \mathbf{v}_L, \mathbf{v}_R)$ associated with the resulting dissipative tensor necessarily obeys the jump relations :*

$$-\sigma[\rho] + [\rho u] = 0, \quad (16.i)$$

$$-\sigma[\rho u] + [\rho u^2 + p + p_\tau] = 0, \quad (16.ii)$$

$$-\sigma[\rho E] + [(\rho E + p + p_\tau)u] = 0, \quad (16.iii)$$

and necessarily satisfies the two entropy inequalities :

$$-\sigma[\rho s] + [\rho s u] > 0, \quad (17.i)$$

$$-\sigma[\rho s_\tau] + [\rho s_\tau u] > 0. \quad (17.ii)$$

In order to specify (17), let us consider for all $\delta > 0$, $\mathbf{v}_\delta$ a rescaled travelling wave for the same triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$. Let us then define with clear notations:

$$\int_{\mathbb{R}} \frac{\widehat{\rho}_\delta^{\gamma-1}}{\gamma-1} (\partial_t(\widehat{\rho}s)_\delta + \partial_x(\widehat{\rho}s u)_\delta)(\xi) d\xi = \zeta, \quad (18)$$

$$\int_{\mathbb{R}} \frac{\widehat{\rho}_\delta^{\gamma_\tau-1}}{\gamma_\tau-1} (\partial_t(\widehat{\rho}s_\tau)_\delta + \partial_x(\widehat{\rho}s_\tau u)_\delta)(\xi) d\xi = \zeta_\tau. \quad (19)$$

Then the total masses ζ and ζ_τ of the two entropy inequalities are bounded and do not depend on the rescaling parameter $\delta > 0$ but only depend on the ratio of the viscosities through the following identity:

$$\mu_\tau \zeta(\sigma; \mathbf{v}_L, \mathbf{v}_R) - \mu \zeta_\tau(\sigma; \mathbf{v}_L, \mathbf{v}_R) = 0. \quad (20)$$

Rephrasing the above result, the triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$ is entirely characterized by the classical jump relations (16.i), (16.ii), (16.iii) and the generalized jump condition (20). It can be seen that when reexpressed for a travelling wave solution, the averaged balance equation (10) exactly coincides with the generalized jump relation (20). Despite that (20) cannot be given generally speaking a close form of expression, it seems convenient to reexpress it equivalently when introducing two suitable averages as follows :

$$\overline{\frac{\rho^{\gamma-1}}{\gamma-1}} \left\{ -\sigma[\rho s] + [\rho s u] \right\} = \zeta(\sigma; \mathbf{v}_L, \mathbf{v}_R), \quad (21.i)$$

$$\overline{\frac{\rho^{\gamma_\tau-1}}{\gamma_\tau-1}} \left\{ -\sigma[\rho s_\tau] + [\rho s_\tau u] \right\} = \zeta_\tau(\sigma; \mathbf{v}_L, \mathbf{v}_R), \quad (21.ii)$$

so that (20) now reads :

$$\begin{aligned} & \frac{\mu_\tau/\mu}{1 + \mu_\tau/\mu} \overline{\frac{\rho^{\gamma-1}}{\gamma-1}} \left\{ -\sigma[\rho s] + [\rho s u] \right\} \\ & - \frac{\mu/\mu_\tau}{1 + \mu/\mu_\tau} \overline{\frac{\rho^{\gamma_\tau-1}}{\gamma_\tau-1}} \left\{ -\sigma[\rho s_\tau] + [\rho s_\tau u] \right\} = 0. \end{aligned} \quad (22)$$

Remark 2.4 In view of the jump relations (16) and (22), one of the two thermodynamic entropies, either ρs or ρs_τ , must be obviously understood as a nonlinear function of the four remaining independent variables $(\rho, \rho u, \rho E, \cdot)$.

Notice after [14] and [16], that the family of travelling wave solutions $\{\hat{\mathbf{v}}\}_{\delta>0}$, for a given triple $(\sigma; \mathbf{v}_L, \mathbf{v}_R)$, can be seen to converge as δ goes to zero in the L^1_{loc} strong topology to the following step function :

$$\mathbf{v}_0(x, t) = \begin{cases} \mathbf{v}_L, & x < \sigma t, \\ \mathbf{v}_R, & x > \sigma t. \end{cases} \quad (23)$$

Let us underline that this so-called shock-solution satisfies by construction the (generalized) jump conditions (16), (22). Numerically speaking, under resolved viscous shock layers are nothing else but an approximation, relevant or not, of the underlying shock solution (23).

The relation (22) clearly suggests that the system (3) admits two equivalent limit systems in full conservation form when the ratio of viscosities μ/μ_τ or μ_τ/μ goes to zero. Since the jump relations are explicitly known in the two limit cases, the following statement allows for asymptotic expansions of the generalized jump condition (20) respectively in $\frac{\mu_\tau}{\mu}$ and $\frac{\mu}{\mu_\tau}$. More precisely we have (see [3] for a proof and detailed expansions):

Theorem 2.5 *Let us assume that $\kappa = \kappa_\tau = 0$.*

When the ratio $\frac{\mu_\tau}{\mu}$ goes to zero with μ set at a fixed positive constant, travelling wave solutions of the system (3) converge uniformly over $\mathbb{R}$ to travelling wave solutions of the following system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x (\mu \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x (\mu u \partial_x u), \\ \partial_t \rho s_\tau + \partial_x \rho s_\tau u = 0, \\ \partial_t \rho s + \partial_x \rho s u \leq 0. \end{cases} \quad (24)$$

Conversely, if $\frac{\mu}{\mu_\tau}$ goes to zero with μ_τ set at a fixed positive constant, travelling wave solutions of the system (3) converge uniformly over $\mathbb{R}$ to travelling wave solutions of the following system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x (\mu_\tau \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x (\mu_\tau u \partial_x u), \\ \partial_t \rho s + \partial_x \rho s u = 0, \\ \partial_t \rho s_\tau + \partial_x \rho s_\tau u \leq 0. \end{cases} \quad (25)$$

2.2 Equivalent formulations and Convexity properties

Here the extended Navier-Stokes system (3) is given two equivalent formulations for smooth solutions that allow for some convexity properties of importance in the forthcoming numerical derivations. Indeed several stability

properties will be inherited from convexity. In this way, the equivalent formulations involve nonlinear versions of the specific entropies s_τ and s_τ , namely $S = g(s)$ and $S_\tau = h(s_\tau)$, so that (say) the entropy ρS , is strictly convex when understood as a function of the independent variables $(\rho, \rho u, \rho E, \rho S_\tau)$. To enforce the convexity property, both the functions g and h are assumed to obey (see Berthon [1] and Berthon-Coquel [4] for a proof):

f is a strictly decreasing and strictly convex $\mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ function such that for $\alpha = \max(\gamma, \gamma_\tau)$

$$f''(x) > \frac{\alpha - 1}{\alpha} (-f'(x)), \quad \text{for all } x > 0. \quad (26)$$

Notice that the above requirements exclude the “natural” choice $g = h = Id$.

Equipped with these relevant nonlinear transforms, the first equivalent system for smooth solutions we consider is obtained when choosing (for instance) $(\rho, \rho u, \rho E, \rho S_\tau)$ as independent variables and thus writes :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x ((\mu + \mu_\tau) \partial_x u), \\ \partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x ((\mu + \mu_\tau) u \partial_x u) + \partial_x (\kappa \partial_x T) + \partial_x (\kappa_\tau \partial_x T_\tau), \\ \partial_t \rho S_\tau + \partial_x \rho S_\tau u = \frac{\gamma_\tau - 1}{\rho^{\gamma_\tau - 1}} h'(h^{-1}(S_\tau)) (\mu_\tau (\partial_x u)^2 + \partial_x (\kappa_\tau \partial_x T_\tau)). \end{cases} \quad (27)$$

Here the pair $(\rho S, \rho S u)$ must be understood, under the assumption (26), as a Lax entropy pair but that must satisfy without additional conditions for smooth solutions the following imposed rate of entropy dissipation :

$$\partial_t \rho S + \partial_x \rho S u = \frac{\gamma - 1}{\rho^\gamma} g'(g(S)) (\mu (\partial_x u)^2 + \partial_x (\kappa \partial_x T)). \quad (28)$$

We next propose a second equivalent system in which a straightforward extension of the balance equation (11) is explicitly involved. Let us indeed consider :

$$\partial_t \rho + \partial_x \rho u = 0, \quad x \in \mathbb{R}, t > 0, \quad (29.i)$$

$$\partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x ((\mu + \mu_\tau) \partial_x u), \quad (29.ii)$$

$$\partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x ((\mu + \mu_\tau) u \partial_x u) + \partial_x (\kappa \partial_x T) + \partial_x (\kappa_\tau \partial_x T_\tau), \quad (29.iii)$$

$$\begin{aligned} & \mu \left(\frac{\rho^{\gamma_\tau - 1}}{\gamma_\tau - 1} \frac{1}{h'(h^{-1}(S_\tau))} (\partial_t \rho S_\tau + \partial_x \rho S_\tau u) \right) - \\ & \mu_\tau \left(\frac{\rho^{\gamma - 1}}{\gamma - 1} \frac{1}{g'(g^{-1}(S))} (\partial_t \rho S + \partial_x \rho S u) \right) = \\ & \partial_x (\mu_\tau \kappa \partial_x T - \mu \kappa_\tau \partial_x T_\tau). \end{aligned} \quad (29.iv)$$

Since the above system involves five partial derivatives in time for only four independent variables, one of the two Lax entropies, either ρS or ρS_τ must be understood as a nonlinear function of the four remaining independent

variables $(\rho, \rho u, \rho E, \cdot)$. Let us underline that (29.iv) is actually equivalent to the non standard balance equation (11) and is thus intrinsically associated with the jump relation (20) (see Berthon-Coquel [4] for the details).

3 On a physical example

In this section, our purpose is to briefly exemplify the proposed framework on the basis of a crude model for turbulent compressible flows. Such a model is derived when assuming a constant turbulent mixing lenght. Let us underline that more sophisticated models, the so-called “two transport” equations models, also enter the present framework but the required material falls beyond the scope of the present paper. The reader is referred to the companion works [1], [4] for the details.

Under the assumption of a large Reynolds number, the PDE’s system we are interested in is usually written under the following (unsuited) form :

$$\partial_t \rho + \partial_x \rho u = 0, \quad x \in \mathbb{R}, t > 0, \quad (30.i)$$

$$\partial_t \rho u + \partial_x (\rho u^2 + p + \frac{2}{3} \rho k) = \partial_x ((\mu + \mu_\tau) \partial_x u), \quad (30.ii)$$

$$\partial_t \rho E + \partial_x (\rho E + p + \frac{2}{3} \rho k) u = \partial_x ((\mu + \mu_\tau) u \partial_x u) + \partial_x (\kappa \partial_x T) + \partial_x (\kappa_\tau \partial_x T_\tau), \quad (30.iii)$$

$$\partial_t \rho k + \partial_x \rho k u = \left\{ -\frac{2}{3} \rho k \partial_x u + \mu_\tau (\partial_x u)^2 \right\} + \partial_x (\kappa_\tau \partial_x T_\tau) - \rho \epsilon, \quad (30.iv)$$

where ρk denotes the so-called turbulent kinetic energy. Here the laminar viscosity $\mu := \mu(T)$ obeys the standard Sutherland law while the turbulent viscosity μ_τ is modelled by :

$$\mu_\tau = C_\mu L \rho \sqrt{k}, \quad (31)$$

where L , the so-called turbulent mixing lenght, is assumed to be a fixed positive real number and C_μ denotes some positive constant of the model. At last the relaxation term $\rho \epsilon$ in (30) follows from the definition:

$$\epsilon = \frac{k^{3/2}}{L}. \quad (32)$$

Introducing the following convenient notation :

$$p_\tau = (\gamma_\tau - 1) \rho k, \quad \gamma_\tau = \frac{5}{3}. \quad (33)$$

the system (30) readily finds the following form :

$$\partial_t \rho + \partial_x \rho u = 0, \quad x \in \mathbb{R}, t > 0,$$

$$\partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = \partial_x ((\mu + \mu_\tau) \partial_x u),$$

$$\partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = \partial_x ((\mu + \mu_\tau) u \partial_x u) + \partial_x (\kappa \partial_x T) + \partial_x (\kappa_\tau \partial_x T_\tau),$$

$$\partial_t p_\tau + \partial_x p_\tau u + (\gamma_\tau - 1) p_\tau \partial_x u = (\gamma_\tau - 1) \left\{ \mu_\tau (\partial_x u) + \partial_x (\kappa_\tau \partial_x T_\tau) - \rho \epsilon \right\},$$

when redistributing suitably the products in non conservation form in (30). The above system is nothing else but the system (3) up to some relaxation term. Let us underline that both the viscosities functions nonlinearly depends on the unknown and that the non standard balance equation (11) extends to :

$$\begin{aligned} & \mu_\tau \left\{ \frac{\rho^{\gamma-1}}{\gamma-1} (\partial_t \rho s + \partial_x \rho s u) - \partial_x (\kappa \partial_x T) \right\} - \\ & \mu \left\{ \frac{\rho^{\gamma_\tau-1}}{\gamma_\tau-1} (\partial_t \rho s_\tau + \partial_x \rho s_\tau u) - \partial_x (\kappa_\tau \partial_x T_\tau) \right\} = 0. \end{aligned} \quad (35)$$

A straightforward extension of the generalized jump condition (20) thus easily follows when suitably redefining both (18) and (19) according to (35).

4 Godunov methods with nonlinear projections

This section is devoted to the numerical approximation of the smooth solutions of the convective-diffusive system (3) with a special emphasis put on the satisfaction at the discrete level of the generalized jump condition (22). As pointed out in the previous sections, several equivalent forms for smooth solutions exist and by construction they are all associated with the same generalized jump conditions.

As proved below, such an equivalence principle, valid for exact solutions, turns out to be lost in general at the discrete level within the frame of classical finite volume methods. Indeed such methods when derived for one equivalent form and another systematically produce grossly different approximate solutions that exhibit large errors with the exact solution. As pointed out below, errors find essentially their roots in the numerical approximation of the first order extracted system. Roughly speaking, if suitable finite difference formulae for approximating the second order operator can be actually devised so that they yield no further error, on the contrary the errors induced by classical approximate Riemann solvers cannot be avoided. Referred as to L^2 projection errors in the sequel, these are indeed shown below to result from the averaging procedure of neighbouring approximate Riemann solutions over each computational cell. Essentially due to the Jensen inequality, linear averagings induce a too large numerical entropy dissipation rate in comparison with the one to be prescribed. This in turn forbids the right hand sides in (6) and (7) to be kept in balance at the discrete level according to (10). This can only result at the discrete level in a violation of the required generalized jump condition (29.iv).

To enforce the validity of (29.iv), we introduce below a nonlinear projection procedure. Involved as an additional and last step to any given standard finite volume method, this procedure provides a systematic and effective correction in redistributing the L^2 projection errors between the two numerical

entropy dissipation rates in (6) and (7) in order to keep in balance their evolution in time according to (29.iv). As a benefit, discrete solutions obtained by nonlinear projection actually achieve a much better agreement with exact solutions.

This nonlinear correction procedure is proved in addition to preserve all the stability properties met by the underlying classical Approximate Riemann solver, namely and for the relevant Riemann solvers : positivity properties, a full set of discrete entropy inequalities and a maximum principle for both the specific entropies.

The numerical method we propose relies on a splitting of operators. It is made of three distinct steps. The first two steps coincide with a standard finite volume method for the system (27) and makes use of a given approximate Riemann solver. The third step details the nonlinear projection operator. We then conclude in presenting the required discretization of the initial data.

For the sake of conciseness in the forthcoming developments, we do not address the discretization of the involved Fourier laws. We thus let $\kappa = \kappa_\tau = 0$ and we refer the reader to the companion paper [4] for the required discrete formulae.

4.1 Godunov type methods and L^2 projections

Let Δt and Δx respectively denote the time and space increments, chosen to be constant without restriction. The numerical approximate solution, $\mathbf{v}_{\Delta x} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \Omega$, is as usual supposed to be piecewise constant and we set using classical notations :

$$\mathbf{v}_{\Delta x}(x, t) = \mathbf{v}_i^n, \quad (x, t) \in (x_{i-1/2}, x_{i+1/2}) \times (t^n, t^{n+1}), \quad i \in Z, n \in N. \quad (36)$$

First step : Extracted first order system ($t^n \rightarrow t^{n+1}, =$)

Assume that the discrete solution $\mathbf{v}_h(x, t^n)$ is known at the time level t^n . In order to evolve it in time, we propose to solve as a first step the following Cauchy problem :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = 0, \\ \partial_t \rho S_\tau + \partial_x \rho S_\tau u = 0, \end{cases} \quad (37)$$

when prescribing the initial data to $\mathbf{v}_h(x, t^n)$. Weak solutions of the above hyperbolic systems are asked to satisfy the following Lax entropy inequality

$$\partial_t \rho S + \partial_x \rho S u \leq 0, \quad (38)$$

to rule out unphysical solutions. For convenience in the discussion, the problem (37), (38) is solved exactly. Then under the following CFL like condition :

$$\frac{\Delta t}{\Delta x} \max |\lambda_i(\mathbf{v})| \leq \frac{1}{2}, \quad (39)$$

the solution is classically made of neighbouring and non interacting elementary Riemann solutions. This solution is then classically averaged over each cell. Let denote by $g : \Omega \times \Omega \rightarrow \mathbb{R}^4$ the associated Lipschitz continuous numerical flux function. Setting $g_{i+1/2}^n = g(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n)$, the updated solution then reads :

$$\mathbf{v}_i^{n+1,=} = \mathbf{v}_i^n - \frac{\Delta t}{\Delta x} \{g_{i+1/2}^n - g_{i-1/2}^n\}, \quad i \in Z. \quad (40)$$

The following result easily follows using standard arguments (see for instance Godlewski-Raviart [10]) :

Lemma 4.1

Under the CFL condition (39), the unknown ρS_τ updates according to the following identity :

$$(\rho S_\tau)_i^{n+1,=} - (\rho S_\tau)_i^n + \frac{\Delta t}{\Delta x} \{ \{ \rho S_\tau u \}_{i+1/2}^n - \{ \rho S_\tau u \}_{i-1/2}^n \} = 0, \quad (41)$$

while the Lax entropy pair $(\rho S, \rho S u)$ obeys the following discrete entropy inequality :

$$\{ \rho S \}(\mathbf{v}_i^{n+1,=}) - (\rho S)_i^n + \frac{\Delta t}{\Delta x} \{ \{ \rho S u \}_{i+1/2}^n - \{ \rho S u \}_{i-1/2}^n \} = \mathcal{E}_i^n \leq 0 \quad (42)$$

We refer the reader to the companion paper [4] for a precise definition of the numerical flux functions in (41) and (42). Let us underline that the numerical rate of entropy dissipation in (42) is generally strictly negative. Indeed the L^2 projection of the exact solution of (37), (38) cannot in general preserve ρS since by the well-known Jensen inequality, we have :

$$\{ \rho S \}(\mathbf{v}_i^{n+1,=}) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \{ \rho S \}(\rho, \rho u, \rho E, \rho S_\tau)(x, t^{n+1,=}) dx. \quad (43)$$

By contrast, the specific entropy S_τ is simply advected by the flow according to system (37) and is thus preserved at the L^2 projection step. This discrepancy in the rates of entropy dissipation, strictly negative versus zero, results in a failure for satisfying the expected balance equation (11) at the discrete level. Notice that this negative result cannot be bypassed when using instead of the exact Godunov scheme a relevant entropy satisfying approximate Riemann solver (see [10] for examples).

Second step : Diffusive operator ($t^{n+1,=} \rightarrow t^{n+1,-}$)

The discrete solution $\mathbf{v}_h(x, t^{n+1,=})$ is next evolved in time to the date $t^{n+1,-}$ when solving with the initial data $\mathbf{v}_h(x, t^{n+1,=})$:

$$\partial_t \rho = 0, \quad (44.i)$$

$$\partial_t \rho u = \partial_x ((\mu + \mu_\tau) \partial_x u), \quad (44.ii)$$

$$\partial_t \rho E = \partial_x ((\mu + \mu_\tau) u \partial_x u), \quad (44.iii)$$

$$\partial_t \rho S_\tau = \frac{\gamma_\tau - 1}{\rho^{\gamma_\tau - 1}} h'(h^{-1}(S_\tau)) \mu_\tau (\partial_x u)^2. \quad (44.iv)$$

In that aim, we suggest to adopt the following implicate finite difference scheme :

$$\rho_i^{n+1,-} = \rho_i^{n+1,=}, \quad (45.i)$$

$$(\rho u)_i^{n+1,-} = (\rho u)_i^{n+1,=} + \Delta t \overline{\partial_x ((\mu + \mu_\tau) \partial_x u)_i}^{n+1,-}, \quad (45.ii)$$

$$(\rho E)_i^{n+1,-} = (\rho E)_i^{n+1,=} + \Delta t \overline{\partial_x ((\mu + \mu_\tau) u \partial_x u)_i}^{n+1,-}, \quad (45.iii)$$

$$(\rho S_\tau)_i^{n+1,-} = (\rho S_\tau)_i^{n+1,=} + \Delta t \overline{\frac{\gamma_\tau - 1}{\rho^{\gamma_\tau - 1}} h'(h^{-1}(S_\tau)) \mu_\tau (\partial_x u)^2}_i^{n+1,-}, \quad (45.iv)$$

where we have set :

$$\overline{\partial_x ((\mu + \mu_\tau) \partial_x u)_i}^{n+1,-} = \frac{\mu + \mu_\tau}{\Delta x^2} (M^n u_{i+1} - 2M^n u_i + M^n u_{i-1}), \quad (46.i)$$

$$\overline{\partial_x (\mu + \mu_\tau) u \partial_x u}_i^{n+1,-} = \frac{\mu + \mu_\tau}{2\Delta x^2} ((M^n u_{i+1})^2 - 2(M^n u_i)^2 + (M^n u_{i-1})^2), \quad (46.ii)$$

$$\begin{aligned} \overline{\frac{\gamma_\tau - 1}{\rho^{\gamma_\tau - 1}} h'(h^{-1}(S_\tau)) \mu_\tau (\partial_x u)^2}_i^{n+1,-} &= \frac{\gamma_\tau - 1}{(\rho_i^{n+1,-})^{\gamma_\tau - 1}} \overline{h'(h^{-1}(S_\tau))}_i^{n+1,-} \times \\ &\quad \frac{\mu_\tau}{2\Delta x^2} ((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2), \end{aligned} \quad (46.iii)$$

together with

$$\overline{h'(h^{-1}(S_\tau))}_i^{n+1,-} = \begin{cases} \frac{S_{\tau i}^{n+1,-} - S_{\tau i}^{n+1,=}}{h^{-1}(S_{\tau i}^{n+1,-}) - h^{-1}(S_{\tau i}^{n+1,=})}, & \text{if } S_{\tau i}^{n+1,-} \neq S_{\tau i}^{n+1,=} \\ h'(h^{-1}(S_{\tau i}^{n+1,-})), & \text{otherwise.} \end{cases} \quad (47)$$

In (46), M^n denotes a time averaging operator given by :

$$M^n X = \frac{X^{n+1,-} + X^{n+1,=}}{2} \quad (48)$$

Let us underline that the first two finite difference operators, (46.i), (46.ii), preserve by construction the conservation property to be satisfied by the unknowns ρu and ρE while as expected, the last operator always achieves the signe of $h'(h^{-1}(S_\tau))$.

Besides and as pointed out below, the benefit of these formulae is twofold.

In the one hand, since the density is kept constant during this second step, the implicit equations (45.ii) stay completely decoupled from the others, namely (45.iii) and (45.iv).

Therefore solving (45) just amounts in practice to invert a positive definite symmetric matrix for the unknown $M^n u$. $(\rho E)_i^{n+1,-}$ can be then evaluated. Turning considering the last unknown $(\rho S_\tau)_i^{n+1,-} = h(s_{\tau_i}^{n+1,-})$, it turns out that by construction $s_{\tau_i}^{n+1,-}$ explicetely reads :

$$s_{\tau_i}^{n+1,-} = s_{\tau_i}^{n+1,=} + \frac{\gamma_\tau - 1}{(\rho_i^{n+1,-})^{\gamma_\tau}} \frac{\mu_\tau \Delta t}{2\Delta x^2} ((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2). \quad (49)$$

In the second hand, straightforward calculations yield from the above formulae the following identity :

$$\{s\}(\mathbf{v}_i^{n+1,-}) = s_i^{n+1,=} + \frac{\gamma_\tau - 1}{(\rho_i^{n+1,-})^{\gamma_\tau}} \frac{\mu_\tau \Delta t}{2\Delta x^2} ((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2). \quad (50)$$

Hence during this second step, the proposed finite difference operators allow for preserving at the discrete level the expected balance between the two rates of entropy dissipation. Indeed, we easily get from (49) and (50) :

$$\begin{aligned} & \mu_\tau \frac{(\rho_i^{n+1,-})^{\gamma-1}}{\gamma-1} \{(\rho s)_i^{n+1,-} - (\rho s)_i^{n+1,=}\} \\ & - \mu \frac{(\rho_i^{n+1,-})^{\gamma_\tau-1}}{\gamma_\tau-1} \{(\rho s_\tau)_i^{n+1,-} - (\rho s_\tau)_i^{n+1,=}\} = 0. \end{aligned} \quad (51)$$

Let us underline that other finite difference formulae are actually possible but up to our knowledge, such formulae systematically produce (strictly) negative errors in the discrete entropy balance equation for ρS .

Summary of the first two steps : Classical L^2 projection methods.

The above two steps yield a standard finite volume method for approximating the solutions of (27). Such a method will be referred in the sequel as to a classical L^2 projection method. The properties of interest are stated below :

Lemma 4.2

Under the CFL condition (39), the variable ρS_τ satisfies :

$$\begin{aligned} & (\rho S_\tau)_i^{n+1,-} - (\rho S_\tau)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{ \rho S_\tau u \}_{i+1/2}^n - \{ \rho S_\tau u \}_{i-1/2}^n \right\} = \\ & \frac{\Delta t}{\rho^{\gamma_\tau-1}} \overline{h'(h^{-1}(S_\tau)) \mu_\tau (\partial_x u)^2}^{n+1,-}_i \leq 0, \end{aligned} \quad (52)$$

while simultaneously, the Lax entropy pair $(\rho S, \rho S u)$ satisfies the discrete entropy inequality :

$$\begin{aligned} \{\rho S\}(\mathbf{v}_i^{n+1,-}) - (\rho S)_i^n + \frac{\Delta t}{\Delta x} \left\{ \{\rho S u\}_{i+1/2}^n - \{\rho S u\}_{i-1/2}^n \right\} = \\ \mathcal{E}_{c_i}^n + \Delta t \overline{\frac{\gamma-1}{\rho^{\gamma-1}} g'(g^{-1}(S)) \mu (\partial_x u)^2}_i^{n+1,-} \leq 0, \end{aligned} \quad (53)$$

where the last term obeys a definition similar to (46.iii).

Besides, the classical L^2 projection method can be shown to satisfy several desirable stability properties : namely it is positivity preserving and obeys discrete maximum principles for both specific entropies S and S_τ (see Tadmor [19] for the setting of 3×3 Euler equations). Nevertheless, (52) and (53) are easily seen to yield the following discrete form for (11) :

$$\begin{aligned} \mu_\tau \left\{ \overline{\frac{\gamma-1}{\rho^{\gamma-1}} g'(g^{-1}(S))} \right\}^{-1} \left\{ \{\rho S\}(\mathbf{v}_i^{n+1,-}) - (\rho S)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S u\}_{i+1/2}^n \right\} - \\ \mu \left\{ \overline{\frac{\gamma_\tau-1}{\rho^{\gamma_\tau-1}} h'(h^{-1}(S_\tau))} \right\}^{-1} \left\{ (\rho S_\tau)_i^{n+1,-} - (\rho S_\tau)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S_\tau u\}_{i+1/2}^n \right\} = \\ \mu_\tau \left\{ \overline{\frac{\gamma-1}{\rho^{\gamma-1}} g'(g^{-1}(S))} \right\}^{-1} \times \mathcal{E}_{c_i}^n \neq 0. \end{aligned} \quad (54)$$

This strongly suggests that the classical L^2 projection method can only fail in satisfying the generalized jump condition (22). The reader is referred to [1] for a rigorous proof and to the numerical results below for an illustration of the negative consequences of such a failure.

4.2 Nonlinear projection methods.

According to the discrete balance equation (54), standard finite volume methods induce a too large numerical rate of entropy dissipation for ρS in comparison with that of ρS_τ that in turn precludes the satisfaction of (22). Here, we propose to add as an additional step to classical L^2 projection methods a nonlinear procedure, the so-called nonlinear projection step, which purpose is precisely to correct the former errors. Indeed, the aim of the third step we propose is to redistribute the errors between the two rates of entropy dissipation in order to keep them in balance according to (22). Let us underline that the numerical procedure derived below inherits by construction all the desirable stability properties satisfied by relevant approximate Riemann solvers.

Third step : Nonlinear projection ($t^{n+1,-} \rightarrow t^{n+1}$)

In order to preserve the required conservation properties, let us define :

$$\rho_i^{n+1} = \rho_i^{n+1,-}, \quad (\rho u)_i^{n+1} = (\rho u)_i^{n+1,-}, \quad (\rho E)_i^{n+1} = (\rho E)_i^{n+1,-}. \quad (55)$$

Then to enforce the validity of the generalized jump condition at the discrete level (20), we propose to seek for $(\rho S_\tau)_i^{n+1}$ as a solution of :

$$\begin{aligned} & \mu_\tau \frac{\overline{\rho^{\gamma-1}}}{\gamma-1} \frac{1}{g'(g^{-1}(S))_i} \times \left(\{\rho S\}(\rho_i^{n+1,-}, (\rho u)_i^{n+1,-}, (\rho E)_i^{n+1,-}, (\rho S_\tau)_i^{n+1}) \right. \\ & \quad \left. - (\rho S)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S u\}_{i+1/2}^n \right) - \\ & \mu \frac{\overline{\rho^{\gamma_\tau-1}}}{\gamma_\tau-1} \frac{1}{h'(h^{-1}(S_\tau))_i} \left((\rho S_\tau)_i^{n+1} - (\rho S_\tau)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S_\tau u\}_{i+1/2}^n \right) = 0. \end{aligned} \quad (56)$$

The above nonlinear problem in the unknown $(\rho S_\tau)_i^{n+1}$ can be shown to admit a unique nonnegative solution as soon as the approximate Riemann solver involved in the first step (4.1) obeys discrete entropy inequalities for the Lax pair $(\rho S, \rho S u)$. This in turn uniquely defines $(\rho S)_i^{n+1}$ according to :

$$(\rho S)_i^{n+1} = \{\rho S\}(\rho_i^{n+1}, (\rho u)_i^{n+1}, (\rho E)_i^{n+1}, (\rho S_\tau)_i^{n+1}).$$

Let us furthermore underline that when understanding $\{\rho S_\tau\}$ as a function of the unknowns $\mathbf{w}_i^{n+1} = (\rho_i^{n+1}, (\rho u)_i^{n+1}, (\rho E)_i^{n+1}, (\rho S)_i^{n+1})$, $(\rho S)_i^{n+1}$ can be easily seen to be the unique solution of the symmetric version of (56) :

$$\begin{aligned} & \mu_\tau \frac{\overline{\rho^{\gamma-1}}}{\gamma-1} \frac{1}{g'(g^{-1}(S))_i} \left((\rho S)_i^{n+1} - (\rho S)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S u\}_{i+1/2}^n \right) - \\ & \mu \frac{\overline{\rho^{\gamma_\tau-1}}}{\gamma_\tau-1} \frac{1}{h'(h^{-1}(S_\tau))_i} \left(\{\rho S_\tau\}(\mathbf{w}_i^{n+1}) - (\rho S_\tau)_i^n + \frac{\Delta t}{\Delta x} \Delta \{\rho S_\tau u\}_{i+1/2}^n \right) = 0. \end{aligned} \quad (57)$$

In other words, the two entropies $(\rho S)_i^{n+1}$ and $(\rho S_\tau)_i^{n+1}$ now play a symmetric role. Numerical methods based on (56) are referred in the sequel as to nonlinear L^2 projection methods. The nonlinear projection step (56) allows to prove in addition the following stability results :

Theorem 4.3

Under the CFL restriction (39), the following discrete entropy inequalities are satisfied :

$$\begin{aligned} & \{\rho \phi(S)\}(\rho_i^{n+1}, (\rho u)_i^{n+1}, (\rho E)_i^{n+1}, (\rho S_\tau)_i^{n+1}) - \\ & \quad (\rho \phi(S))_i^n + \frac{\Delta t}{\Delta x} \Delta_i \{\rho \phi(S) u\}_{i+1/2}^n \leq 0, \end{aligned} \quad (58.i)$$

$$\begin{aligned} & \{\rho \psi(S_\tau)\}(\rho_i^{n+1}, (\rho u)_i^{n+1}, (\rho E)_i^{n+1}, (\rho S_\tau)_i^{n+1}) - \\ & \quad (\rho \psi(S_\tau))_i^n + \frac{\Delta t}{\Delta x} \Delta_i \{\rho \psi(S_\tau) u\}_{i+1/2}^n \leq 0, \end{aligned} \quad (58.ii)$$

for all strictly decreasing and convex functions ϕ and ψ . The following maximum principles for the specific entropies S and S_τ are met :

$$S_i^{n+1} \leq \max(S_{i-1}^n, S_i^n, S_{i+1}^n), \quad (59.i)$$

$$S_{\tau i}^{n+1} \leq \max(S_{\tau i-1}^n, S_{\tau i}^n, S_{\tau i+1}^n). \quad (59.ii)$$

Since the two functions g and h are strictly decreasing under the assumption (26), both the pressures p_i^{n+1} and $p_{\tau i}^{n+1}$ stay positive as soon as the density ρ^{n+1} is positive.

The proof of the above statement is detailed in [1]. Let us conclude the present paragraph in highlighting that the nonlinear projection step (63) is by construction consistent with the two limit cases in full conservation form we have put forward in Theorem 2.5. Indeed, sending formally (for instance) the viscosity μ_τ to zero, makes (63) degenerate into the following expected relations (see (24)):

$$\begin{aligned} (\rho S_\tau)_i^{n+1} &= (\rho S_\tau)_i^n - \frac{\Delta t}{\Delta x} \Delta_i \{ \rho S_\tau u \}_{i+1/2}^n, \\ (\rho \phi(S))_i^{n+1} - (\rho \phi(S))_i^n + \frac{\Delta t}{\Delta x} \Delta_i \{ \rho \phi(S) u \}_{i+1/2}^n &\leq 0. \end{aligned} \quad (60)$$

Similarly and since in view of (57), the nonlinear projection does not break the symmetry in the roles played by ρS and ρS_τ , we get from (63) when sending formally to zero the viscosity μ , the expected relations (see (25)) :

$$\begin{aligned} (\rho S)_i^{n+1} &= (\rho S)_i^n - \frac{\Delta t}{\Delta x} \Delta_i \{ \rho S u \}_{i+1/2}^n, \\ (\rho \psi(S_\tau))_i^{n+1} - (\rho \psi(S_\tau))_i^n + \frac{\Delta t}{\Delta x} \Delta_i \{ \rho \psi(S_\tau) u \}_{i+1/2}^n &\leq 0. \end{aligned} \quad (61)$$

In the sequel, the nonlinear projection operator (56) is coupled with an exact Roe linearization for the system (3) as proposed in [6].

4.3 Discretization of the initial data

For the sake of completeness, we conclude the description of the method we propose in detailing the projection onto piecewise constant functions of the initial data $(\rho_0(x), (\rho u)_0(x), p_0(x), p_{\tau 0}(x))$ for system (3).

L^2 projection step

Let us first define from the initial data the following three real valued functions $\{\rho E\}_0(x)$, $\{\rho S\}_0(x)$ and $\{\rho S_\tau\}_0(x)$. Let us then introduce with clear notations the following five L^2 projections :

$$\rho_i^{0,-} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \rho_0(x) dx, \quad (\rho u)_i^{0,-} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} (\rho u)_0(x) dx,$$

with similar definitions for $(\rho E)_i^{0,-}$, $(\rho S)_i^{0,-}$ and $(\rho S_\tau)_i^{0,-}$.

The averages $\rho_i^{0,-}$, $(\rho u)_i^{0,-}$ and $(\rho E)_i^{0,-}$ being fixed, it is obviously impossible to simultaneously prescribe both $(\rho S)_i^{0,-}$ and $(\rho S_\tau)_i^{0,-}$ without introducing some ambiguity in the definition of the required pressures $p_i^{0,-}$ and $p_{\tau i}^{0,-}$. It indeed suffices to note that the convexity of the mapping $\{\rho S\}$ implies by the Jensen inequality the following (generally strict) inequality :

$$\{\rho S\} \left(\rho_i^{0,-}, (\rho u)_i^{0,-}, (\rho E)_i^{0,-}, (\rho S_\tau)_i^{0,-} \right) \leq \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} (\rho S)_0(x) dx = (\rho S)_i^{0,-}.$$

The next and last step aims at restoring uniqueness in the definition of both the pressures.

Nonlinear projection step In order to preserve the required conservation of density, momentum and total energy, let us set for all $i \in Z$:

$$\rho_i^0 = \rho_i^{0,-}, \quad (\rho u)_i^0 = (\rho u)_i^{0,-}, \quad (\rho E)_i^0 = (\rho E)_i^{0,-}. \quad (62)$$

Then let us redefine $(\rho S_\tau)_i^0$ as the solution of the following nonlinear equation :

$$\begin{aligned} \overline{\frac{(\rho)^{\gamma-1}}{\gamma-1} \frac{1}{g'(g^{-1}(S))_i}}^{0,-} \left(\{\rho S\} \left(\rho_i^0, (\rho u)_i^0, (\rho E)_i^0, (\rho S_\tau)_i^0 \right) - (\rho S)_i^{0,-} \right) - \\ \mu \overline{\frac{(\rho)^{\gamma_\tau-1}}{\gamma_\tau-1} \frac{1}{h'(h^{-1}(S_\tau))_i}}^{0,-} \left((\rho S_\tau)_i^0 - (\rho S_\tau)_i^{0,-} \right) = 0, \end{aligned} \quad (63)$$

where the involved two averages are given by the consistent definitions :

$$\overline{\frac{(\rho)^{\gamma-1}}{\gamma-1} \frac{1}{g'(g^{-1}(S))_i}}^{0,-} = \frac{(\rho_i^{0,-})^{\gamma-1}}{\gamma-1} \frac{1}{g'(g^{-1}(S_i^{0,-}))}, \quad (64)$$

together with a similar definition for $\overline{\frac{(\rho)^{\gamma_\tau-1}}{\gamma_\tau-1} \frac{1}{h'(h^{-1}(S_\tau))_i}}^{0,-}$. The present second step clearly corresponds to the nonlinear projection we have described in the above section. Let us conclude with the following statement :

Theorem 4.4

The nonlinear projection step (63) uniquely determines the pressures p_i^0 , $p_{\tau i}^0$ and preserves their positivity as soon as the initial density $\rho_0(x)$ is positive.

4.4 The usual approach for solving (3)

For the sake of comparison, we end the present section when briefly recalling the most usual (if not systematic) approach for approximating the solutions of systems in non conservation form like (3). According to this approach, all the non conservative products are rejected to the right hand side of the governing equations and are treated as “source terms”. When applied to (27), the first step (4.1) is therefore concerned with the following first order extracted system in conservation form :

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, & x \in \mathbb{R}, t > 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p + p_\tau) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p + p_\tau) u = 0, \\ \partial_t p_\tau + \partial_x p_\tau u = 0, \end{cases} \quad (65)$$

while in a next step, the “source term” $-(\gamma_\tau - 1)p_\tau \partial_x u$ is to be given some *ad hoc* finite difference approximation. We refer for instance the reader to [15] and [13] concerning the details. The (severe) drawbacks in the resulting numerical schemes are illustrated below. Let us furthermore underline after Forestier-Herard-Louis [8] that exact Riemann solutions of (65) all preserve the positivity of the total pressure (far away from vacuum) but do not necessarily keep non negative the partial pressure p . By contrast, both the pressures p and p_τ in (3) can be shown to stay positive (again far away from vacuum). Such schemes are referred in the sequel as to classical methods. In what follows, Riemann solutions of (65) are approximated using one of the most widely used method : namely the Roe scheme [17].

5 Numerical results

The ability of the three discussed schemes in the capture of viscous shock layers for (3) is evaluated when testing their sensitivity in the prediction of the end states with respect to the mesh refinement. The initial data of the Cauchy problems to be solved are made of two constant states, the discontinuity being located at $x = 0$. The associated exact solutions thus correspond to smooth regularizations of Riemann solutions for the underlying first order system extracted from (3). They are thus made up generally speaking of the juxtaposition of travelling wave solutions and “rarefaction” solutions. Discrete solutions are systematically compared with the exact solutions obtained when integrating the EDO’s system for governing travelling wave solutions.

All the calculations described below have been performed according to the following strategy. An exact Roe type linearization for system (37) yields an approximate Riemann solver to solve the first step (4.1) (see [6], [1] for the detailed formulae). Successive grids refinements, ranging from 100 to 2000 cells, are considered. The CFL number is fixed at the constant value 0.5.

Four test cases, labelled from A to D, are addressed. Problems A, B and C are directly motivated by the three distinct regimes that underly the flow

model under consideration and that are dictated by the amplitude of the viscosity ratio μ_τ/μ . Namely, they involve a viscosity ratio which is successively small, close and large with respect to unity. Here, both the viscosities μ and μ_τ are assumed to be fixed positive constants. The reader is referred to Berthon [1] for the setting of varying viscosities. The problem D treats the case of non zero heat conductivities. Problem D treats the full model (3) when considering non zero heat conductivities.

In all the benchmarks discussed below, the Reynolds number is set at the constant value $Rey = 10^5$. For the benchmark D, the Prandtl numbers are respectively set to $Pr = 0.72$ and $Pr_\tau = 0.9$. The associated initial data are defined in table 1. Test case D admits the same initial data as problem B.

Test	γ	γ_τ	μ_τ/μ	ρ	u	p	p_τ
A	1.4	1.6	0.01	1	1	1	0.6
				1.05518	-0.88895	0.15031	0.33869
B, D	1.4	1.6	1	1	1	1	0.6
				1.92678	-1.25451	2.74247	2.09561
C	1.4	1.6	100	1	1	1	0.6
				1.01595	-0.93415	0.24142	0.15528

Table 1.

Test cases A, B and C. The Theorem (2.5) indicates that the benchmarks A and C are both asymptotically close to two limit systems in conservation form. Namely, case A makes the entropy ρS_τ to be asymptotically driven by a conservation law while ρS inherits such a property for problem C. By contrast, the solution of problem B stays far away from these two limit situations. All the figures assess that the usual numerical strategy (4.4) grossly fails to properly restore the correct end states in the three regimes. Turning considering the L^2 projection method, the discrete solutions agree with the exact ones only in case A as expected since ρS_τ is close to be a conservative variable. Such a property no longer holds for problems B and C and consequently large errors occur. These two schemes furthermore suffer from a dramatic sensitivity with respect to mesh refinements for problem C in that discrete solutions do not seem to converge to a given limit function even for the finest proposed grids. By contrast and concerning benchmarks A and B, the discrete solutions stay non sensitive with respect to the mesh refinement but the “limit” function does not coincide with the expected exact solution.

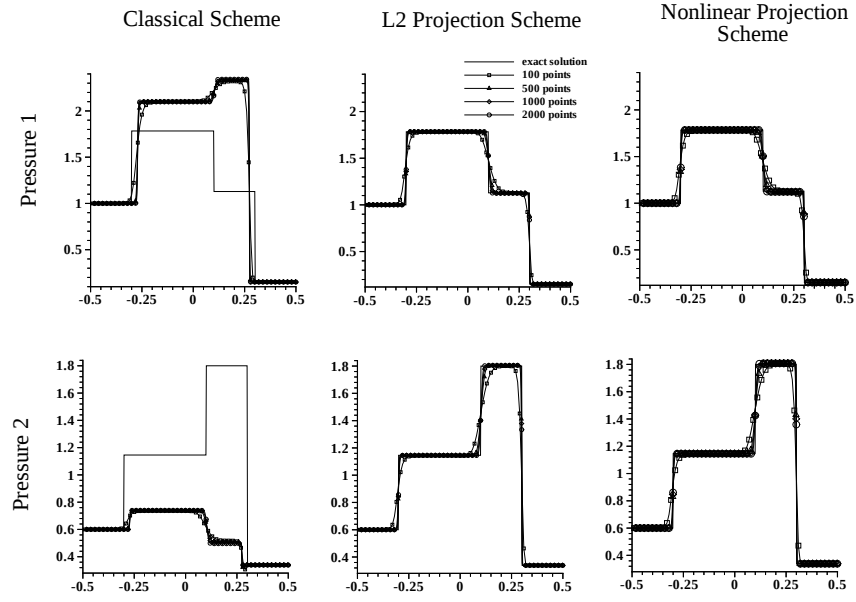


Fig. 3. Problem A : $\mu_\tau/\mu \ll 1$

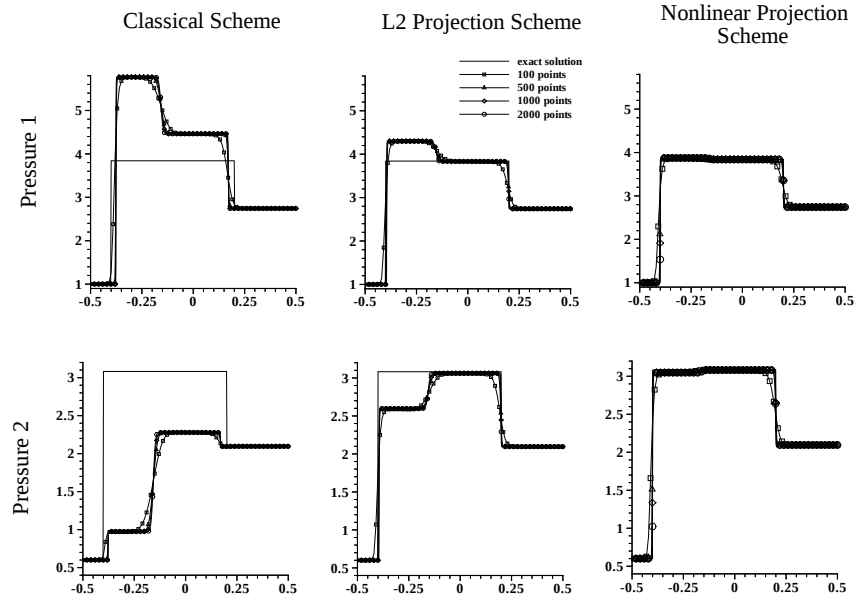
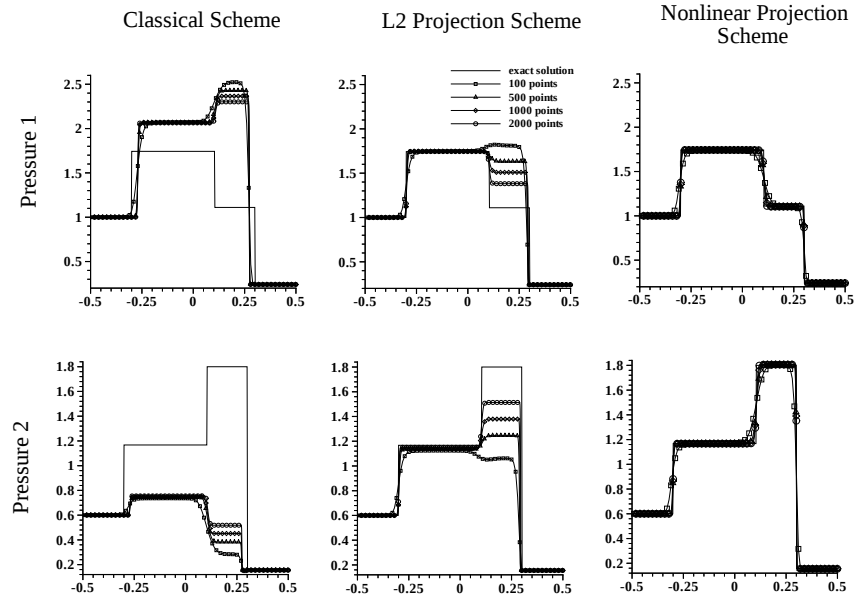
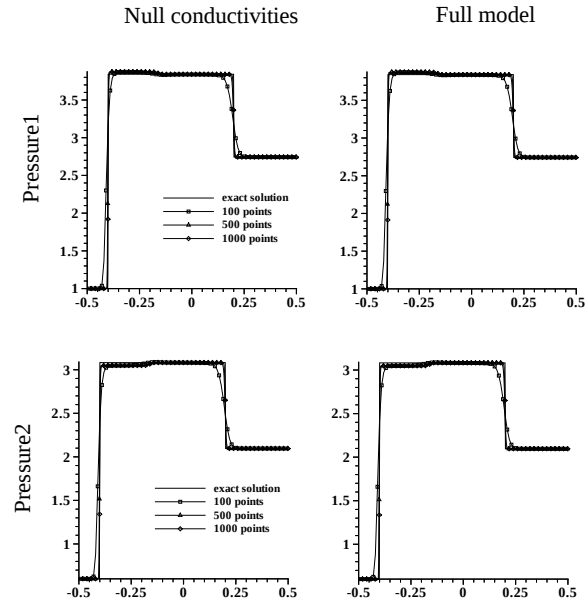


Fig. 4. Problem B : $\mu_\tau/\mu = 1$

Fig. 5. Problem C : $\mu_\tau/\mu \gg 1$ Fig. 6. Problem D : $\mu_\tau/\mu = 1$

Turning considering the nonlinear L^2 projection method, it produces approximate solutions that achieve a fairly good (if not excellent) agreement with the exact solutions while staying almost non-sensitive with Δx in the three investigated regimes.

Test case D. The present problem studies the sensitivity of the nonlinear projection method with respect to heat conductivities. The choice of the initial data (namely the one of problem B) makes the solution to stay far away from two fully conservative limit cases. In this sense, the problem to be solved is the more difficult. The numerical results displayed in figure 6 clearly indicates that the discrete solutions again stay non-sensitive with respect to mesh refinements in the presence of Fourier laws. This illustrates the benefits of the nonlinear projection method we have proposed.

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