AN EFFICIENT SPLITTING TECHNIQUE FOR TWO-LAYER SHALLOW-WATER MODEL

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Abstract. We consider the numerical approximation of the weak solutions of the two-layer shallow-water equations. The model under consideration is made of two usual one-layer shallow-water model coupled by non-conservative products. Because of the non-conservative products of the system, the usual numerical methods have to consider the full model, while uncoupled approaches turn out to be unstable. Of course, uncoupled numerical techniques, just involving finite volume schemes for the basic shallow-water equations, are very attractive since they are very easy to implement and they are costless. Recently, a stable layer splitting was introduced [14]. In the same spirit, we exhibit new splitting techniques, which are proved to be well-balanced and non-negative preserving. The main benefit issuing from the here derived uncoupled methods is the ability to correctly approximate the solution of very severe benchmarks.

1. Introduction.

The present work concerns the numerical approximation of the two-layer shallow-water system [18, 24, 1, 17, 23, 48]. This system of partial differential equations models the flow evolution over non-flat topography of two superposed layers of immiscible fluids. Here, each fluid comes with its own positive constant density:

$$0 < \rho_1 < \rho_2. \tag{1.1}$$

The model under consideration reads:

$$\begin{cases} \partial_t h_1 + \partial_x(h_1 u_1) = 0, \\ \partial_t(h_1 u_1) + \partial_x(h_1 u_1^2 + \frac{g}{2} h_1^2) = -g h_1 \partial_x(h_2 + z), \\ \partial_t h_2 + \partial_x(h_2 u_2) = 0, \\ \partial_t(h_2 u_2) + \partial_x(h_2 u_2^2 + \frac{g}{2} h_2^2) = -g h_2 \partial_x(r h_1 + z), \end{cases} \tag{1.2}$$

where $h_i \geq 0$ and $u_i \in \mathbb{R}$ respectively represent the positive layer thickness and the horizontal velocity associated with the ith layer. According to the density order relation (1.1) considered, we have that $i = 1$ coincides with the upper layer while $i = 2$ denotes the lower layer (see Figure 1.1).

Concerning the parameters of the model, $g > 0$ denotes the gravity constant and we have introduced the density ratio as follows:

$$r = \frac{\rho_2}{\rho_2} \quad \in (0, 1). \tag{1.3}$$

The given smooth function $z := z(x)$ designates the topography.

For the sake of simplicity in notation, let us denote the layer state vector as follows:

$$w_1 = \begin{pmatrix} h_1 \\ h_1 u_1 \end{pmatrix} \qquad \text{and} \qquad w_2 = \begin{pmatrix} h_2 \\ h_2 u_2 \end{pmatrix}$$

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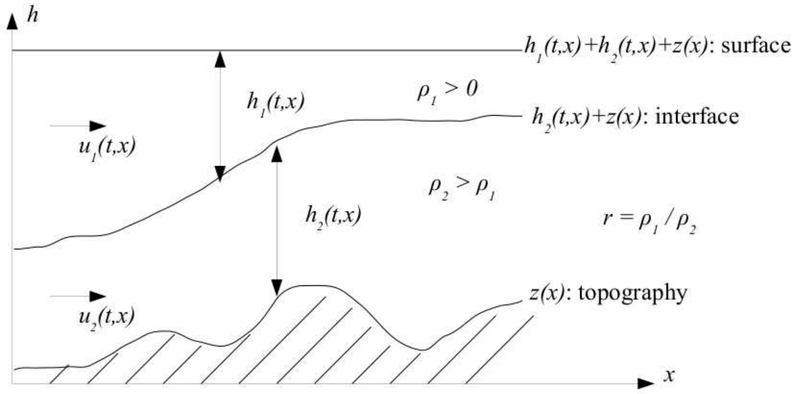

Fig. 1.1: Example of topography and two layers of water

We introduce the layer admissible space given by

$$\Omega = \left\{ w = {}^t(h, hu);\ h \geq 0 \right\}, \tag{1.4}$$

so that $w_i \in \Omega$ for $i = 1$ or 2.

In addition, it is convenient to consider the flux function $f : \Omega \to \mathbb{R}^2$ and the topography source function $S_z : \Omega \to \mathbb{R}^2$ coming from the well-known shallow-water model. The functions f and S_z are thus respectively given by

$$f(w) = \begin{pmatrix} hu \\ hu^2 + g\dfrac{h^2}{2} \end{pmatrix} \qquad \text{and} \qquad S_z(w) = \begin{pmatrix} 0 \\ gh\partial_x z \end{pmatrix}. \tag{1.5}$$

Next, by introducing the coupling function $\mathcal{C} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$ defined as follows:

$$\mathcal{C}(h_\alpha, h_\beta) = \begin{pmatrix} 0 \\ gh_\alpha \partial_x h_\beta \end{pmatrix},$$

the two-layer shallow-water model easily rewrites

$$\begin{cases} \partial_t w_1 + \partial_x f(w_1) = -S_z(w_1) - \mathcal{C}(h_1, h_2), \\ \partial_t w_2 + \partial_x f(w_2) = -S_z(w_2) - r\mathcal{C}(h_2, h_1). \end{cases} \tag{1.6}$$

We notice that if the coupling function uniformly vanishes, *i.e.* $\mathcal{C} \equiv 0$, each layer is governed by the standard shallow-water system:

$$\partial_t w + \partial_x f(w) = -S_z(w). \tag{1.7}$$

As a consequence, it seems very attractive to propose numerical splitting strategies. Unfortunately and as expected, the coupling term $\mathcal{C}$ is of prime importance and particular attention must be paid on its discretization in order to not destroy the very complex structure of the system (1.2).

The main discrepancy with the usual shallow-water system (1.7) is the non-conservative products $h_1\partial_x h_2$ and $h_2\partial_x h_1$ involved within both momentum equations, which make that the two-layer shallow-water model (1.2) cannot be written in a conservative form. In fact, the model satisfies an addition total momentum conservation law given by

$$\partial_t(\rho_1 h_1 u_1 + \rho_2 h_2 u_2) + \partial_x \left(\rho_1(h_1 u_1^2 + \frac{g}{2}h_1^2) + \rho_2(h_2 u_2^2 + \frac{g}{2}h_2^2) + \rho_1 g h_1 h_2\right) = \\ -(\rho_1 h_1 + \rho_2 h_2) g \partial_x z, \tag{1.8}$$

which together with both mass conservation equations means that we lack one conservative equation.

After [28, 20, 10, 9, 8], the lack of conservation form associated with (1.2) makes the Rankine-Hugoniot relations not reachable. Hence, a possible discontinuous solution is not well-characterized and several solutions may exists. From a general framework, we refer the reader to the work by dal Maso, LeFloch and Murat [28] where a possible rigorous definition of the non-conservative products is introduced (see also [26] for a related work). Within this framework, weak solutions for several non-conservative systems find suitable definitions (for instance, see [10, 9, 8, 8]). Concerning the two-layer shallow-water system (1.2), several criteria to select shock solutions have been recently suggested [41, 44, 39], but the problem stays open. We do not consider here this delicate problem with drastic numerical consequences (see [2, 38, 20]). Put in other words, we do not enforce any criteria to uniquely define the discontinuous solutions.

In fact, the problem of the shock definition comes with a second difficulty of the two-layer shallow-water model. Indeed, the algebra associated with the system (1.2) remains unknown. Works to understand the eigenstructure of (1.2) were recently attempted (for instance, see [16] and references therein). Let us rewrite the system (1.2) under the following quasi-linear form:

$$\partial_t U + A(U)\partial_x U = \Phi, \tag{1.9}$$

where we have set

$$U = \begin{pmatrix} h_1 \\ u_1 \\ h_2 \\ u_2 \end{pmatrix}, \ A(U) = \begin{pmatrix} u_1 & h_1 & 0 & 0 \\ g & u_1 & g & 0 \\ 0 & 0 & u_2 & h_2 \\ rg & 0 & g & u_2 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 0 \\ -g\partial_x z \\ 0 \\ -g\partial_x z \end{pmatrix}$$

The eigenvalues of the matrix $A(U)$ are the roots of the characteristic polynomial given by

$$P(\lambda) = ((u_1 - \lambda)^2 - gh_1)((u_2 - \lambda)^2 - gh_2) - rg^2 h_1 h_2.$$

These roots cannot be explicitly evaluated. However, as long as $r \ll 1$, the four eigenvalues are real and (1.9) turns out to be hyperbolic. In this case, the behavior of (1.9) is near to uncoupled one-layer shallow-water models. The eigenvalues are close to

$$u_j \pm \sqrt{gh_j}, \qquad j = 1, 2.$$

The behavior of system (1.2) is very distinct as soon as the density ration r is in a neighborhood of one. In [51], it is stated that, for $r \approx 1$, the first-order approximations of the eigenvalues are given by

$$\lambda_{\text{ext}}^{\pm} = \frac{u_1 h_1 + u_2 h_2}{h_1 + h_2} \pm (g(h_1 + h_2))^{1/2}, \tag{1.10}$$

$$\lambda_{\text{int}}^{\pm} = \frac{u_1 h_2 + u_2 h_1}{h_1 + h_2} \pm \left(\left(1 - \frac{\rho_1}{\rho_2}\right) g \frac{h_1 h_2}{h_1 + h_2} \left[1 - \frac{(u_1 - u_2)^2}{(1-r)\, g(h_1 + h_2)}\right] \right)^{1/2}. \tag{1.11}$$

The two external eigenvalues, $\lambda_{\text{ext}}^{\pm}$, are related to the barotropic components of the flow while the two internal eigenvalues, $\lambda_{\text{int}}^{\pm}$, are associated to the baroclinic components.

Clearly, the system (1.2) is not unconditionally hyperbolic. Indeed, the eigenvalues of the system may become complex as soon as we have

$$(u_1 - u_2)^2 > (1-r) g(h_1 + h_2). \tag{1.12}$$

However, it is agreed that the elliptic region has a repulsive role, and if the solution approaches this region, it is rapidly evacuated to its boundary. So, even if complex eigenvalues exist, the system is nevertheless expected to be well-posed for strong enough nonlinearity.

To conclude this brief presentation of the two-layer shallow-water model (1.2), the derivation of relevant numerical schemes is very challenging. During the last decade, several numerical strategies have been proposed. The path-consistent scheme [47] is certainly one of the most considered in the literature. The reader is also referred to [5, 6, 29] where variants are detailed. After [20] (see also [38, 2]) these numerical procedures are well-known *to fail* when approximating discontinuous solutions prescribed by the dal Maso, LeFloch and Murat Path Theory. However, these schemes are known to give good results in general, even when applied to physical situations. We shall consider here the resulting numerical solutions given by these schemes as reference solutions. Nonetheless, path-consistent (like) schemes are naturally based on the full coupled system (1.2) and may turn to be sophisticated. Indeed, system (1.2), or equivalently system (1.6), seems *near* two independent one-layer shallow-water models. Of course, after the above developments concerning the non-conservation form and the unconditionally hyperbolicity, the *near* property is clearly wrong. As a consequence, it seems absolutely impossible to consider some decoupling approach to derive numerical procedures. However, a scheme coming from one-layer shallow-water scheme to each layer independently is very attractive because of both simplicity and low cost. After [18], such decoupling approaches lead to unstable schemes. Though, in a recent work by Bouchut and Morales [14], a time splitting scheme, involving a natural decoupling of the system (1.2), is derived and is proved to be entropy stable. The resulting simulations are interesting but some improvements are necessary as soon as the density ratio is close to one.

After [14], we here propose to derive a new splitting strategy based on a suitable extension of the recent hydrostatic upwind reconstruction introduced in [11, 12]. The scheme we obtain in this paper is free from the definition of the numerical flux function associated with the one-layer shallow-water flux (1.5). Moreover, the scheme is proved to be water height non-negative preserving and lake at rest preserving. We are here not able to establish entropy preserving property, as given in ([14]), but the performed numerical experiments illustrate the efficiency of the suggested numerical procedure. In addition, the behavior of the scheme with a density ratio near to one is in a very good agreement to the numerical reference solutions.

The paper is organized as follows. In the next section, we recall the steady states of interest; namely the so-called lake at rest. These steady states are considered to introduce an equivalent reformulation of (1.2). This new reformulation, introduced in [11, 12] for a one-layer shallow-water model, is of prime importance in the derivation of the finite volume method given in Section 3. Section 4 is devoted to establish the main properties of the obtained scheme. We prove that the method is well-balanced and water height non-negative preserving. The next section concerns a second-order extension by considering MUSCL techniques. In the last section, Section 6, we display numerous numerical experiments, which attest the efficiency of the proposed method. A short conclusion achieves the paper.

2. Steady states and reformulation of (1.2). The steady states of hydrostatic models are of prime interest. After the work by Greenberg and LeRoux [36] (see also [34, 35]), the efficiency of a numerical scheme is evaluated by its ability to exactly preserve the steady states (or at least a class of steady states). Such schemes are said well-balanced. From now on, after [11, 12] (see also [15, 45, 3, 4]), we emphasize that a relevant way to get a well-balanced scheme is to introduce, in a sense to be prescribed, the reached steady states within the finite volume method.

Considering the two-layer shallow-water system (1.2), the smooth steady states are governed by the following system:

$$\begin{aligned}
&\partial_x h_1 u_1 = 0,\\
&\partial_x \left(h_1 u_1^2 + g\frac{h_1^2}{2} \right) + g h_1 \partial_x (h_2 + z) = 0,\\
&\partial_x h_2 u_2 = 0,\\
&\partial_x \left(h_2 u_2^2 + g\frac{h_2^2}{2} \right) + g h_2 \partial_x (r h_1 + z) = 0.
\end{aligned} \tag{2.1}$$

We easily get the characterization of the steady states as follows:

$$\begin{cases}
h_1 u_1 = \text{cst},\\
\dfrac{1}{2} u_1^2 + g(h_1 + h_2 + z) = \text{cst},\\
h_2 u_2 = \text{cst},\\
\dfrac{1}{2} u_2^2 + g(r h_1 + h_2 + z) = \text{cst}.
\end{cases} \tag{2.2}$$

In fact, it is very difficult to restore all the steady states (for instance, see [22, 50, 49, 46, 14] in the framework of the one-layer shallow-water model). As a consequence, we usually impose the derived scheme to exactly restore the class of the rest steady states. They are defined by vanishing the velocities; the so-called lake at rest which is given by

$$\begin{cases}
u_1 = u_2 = 0\\
h_1 + h_2 + z = \text{cst}\\
r h_1 + h_2 + z = \text{cst}
\end{cases} \tag{2.3}$$

After [11, 12], in order to enforce the lake at rest to be preserved by the scheme, we suggest to reformulate the system (1.2) by introducing the characteristic water height involved by the lake at rest (2.3). As a consequence, we set

$$H_1 = h_1 + h_2 + z \qquad \text{and} \qquad H_2 = r h_1 + h_2 + z, \tag{2.4}$$

which correspond to the free surface for the first layer and the *apparent* free surface for the second layer. Since the model is designed up to a topography translation, in the whole domain of computation we systematically impose

$$H_1 > \varepsilon \qquad \text{and} \qquad H_2 > \varepsilon, \tag{2.5}$$

where $\varepsilon > 0$ denotes an arbitrary small constant.

We also need to introduce water height fractions given by

$$X_1 = \frac{h_1}{H_1} \qquad \text{and} \qquad X_2 = \frac{h_2}{H_2}. \tag{2.6}$$

Next, we propose to reformulate the flux functions and the source terms involved in (1.2), or equivalently (1.6), by plugging the new free surfaces, H_1 and H_2, and the new water height fraction, X_1 and X_2. First, an easy computation gives

$$h_1^2 = X_1 H_1 (H_1 - h_2 - z) \qquad \text{and} \qquad h_2^2 = X_2 H_2 (H_2 - r h_1 - z),$$

so that the system (1.2) immediately reads

$$\begin{cases} \partial_t h_1 + \partial_x (X_1 H_1 u_1) = 0, \\ \partial_t (h_1 u_1) + \partial_x \left(X_1 (H_1 u_1^2 + \frac{g}{2} H_1^2) - \frac{g}{2} h_1 (h_2 + z) \right) = -g h_1 \partial_x (h_2 + z), \\ \partial_t h_2 + \partial_x (X_2 H_2 u_2) = 0, \\ \partial_t (h_2 u_2) + \partial_x \left(X_2 (H_2 u_2^2 + \frac{g}{2} H_2^2) - \frac{g}{2} h_2 (r h_1 + z) \right) = -g h_2 \partial_x (r h_1 + z). \end{cases} \tag{2.7}$$

Now, since we have

$$h_2 + z = H_1 - h_1 \text{ and } r h_1 + z = H_2 - h_2,$$

we are able to rewrite system (2.7), or equivalently (1.2), in the following form:

$$\begin{cases} \partial_t w_1 + \partial_x (X_1 f(W_1)) = S(H_1, h_1), \\ \partial_t w_2 + \partial_x (X_2 f(W_2)) = S(H_2, h_2), \end{cases} \tag{2.8}$$

where we have set

$$W_j = {}^t(H_j, H_j u_j) \qquad j = 1, 2,$$

and

$$S(H, h) = \begin{pmatrix} 0 \\ \frac{g}{2} \partial_x (h(H - h)) - g h \partial_x (H - h) \end{pmatrix} \tag{2.9}$$

It is worth noticing that, introducing $H = h + z$, the one-layer shallow-water model (1.7) exactly recast as follows (see [11, 12] for the details):

$$\partial_t w + \partial_x (X f(W)) = S(H, h).$$

As a consequence, we remark that (2.8) can be understood as two *independent* one-layer shallow-water systems. In fact, we underline that the actual coupling in (2.8) is performed via the definition of H_1 and H_2.

3. Finite volume scheme. Based on the writing of (2.8), we now derive a finite volume numerical scheme. As usual, we define a uniform mesh $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ of size $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, and we denote by z_i a constant averaged value of the topography on the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$.

At time t^n, on each cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ we assume known a constant approximation $w_{j,i}^n = {}^t(h_{j,i}^n, h_{j,i}^n u_{j,i}^{n+1})$ of w_j with $j = 1, 2$. In addition, we introduce

$$\begin{cases} H_{1,i}^n = h_{1,i}^n + h_{2,i}^n + z_i \quad \text{and} \quad X_{1,i}^n = \dfrac{h_{1,i}^n}{H_{1,i}^n}, \\ H_{2,i}^n = r h_{1,i}^n + h_{2,i}^n + z_i \quad \text{and} \quad X_{2,i}^n = \dfrac{h_{2,i}^n}{H_{2,i}^n}. \end{cases} \tag{3.1}$$

Now, to obtain the updated state $(w_{j,i}^{n+1})_{j=1,2}$ at time $t^{n+1} = t^n + \Delta t$, we suggest to discretize the equivalent formulation (2.8). To address such an issue, after [11, 12], we adopt a finite volume scheme to approximate the weak solutions of the one-layer homogeneous Saint-Venant model given by (1.7) with a constant topography function (*i.e.* $S_z \equiv 0$). The scheme under consideration reads as follows:

$$(w^{\text{homo}})_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{\Delta x}(w_i^n, w_{i+1}^n) - f_{\Delta x}(w_{i-1}^n, w_i^n)\right), \tag{3.2}$$

where $f_{\Delta x} = {}^t(f_{\Delta x}^h, f_{\Delta x}^{hu}) : \Omega_1 \times \Omega_1 \to \mathbb{R}^2$ stands for the numerical flux function which is assumed to be consistent with the exact flux function:

$$f_{\Delta x}(w, w) = f(w) \quad \text{for all } w \in \Omega. \tag{3.3}$$

Because the hyperbolic system of conservation laws $(1.7)_{\mathrm{S}_z \equiv 0}$ turns out to be very basic, the numerical flux function $f_{\Delta x}$ easily finds an appropriate definition (for instance Lax-Friedrichs schemes [13], HLL and HLLC schemes [37, 53, 52], relaxation schemes [13, 25, 27, 43, 40], VFRoe scheme [31, 32, 30], kinetic schemes [33]).

Moreover, in order to rule out some instabilities, the homogeneous scheme (3.2) is endowed with a CFL like condition given by:

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} |\lambda^{\pm}(w_i^n, w_{i+1}^n)| \leq \frac{1}{2}, \tag{3.4}$$

where $\lambda^{\pm}(w_i^n, w_{i+1}^n)$ denotes some numerical approximation of the external eigenvalues (1.10) of the system (2.8) and associated to the definition of the numerical flux function $f_{\Delta x}$. In addition, we here impose the homogeneous scheme (3.2) to be Ω-preserving; *i.e.* provided $h_i^n \geq 0$ for all $i \in \mathbb{Z}$, we get $(h^{\text{homo}})_i^{n+1} \geq 0$.

Next, after [11, 12], we introduce upwind functions according to the direction of velocity:

$$\bar{H}(W_L, W_R) = \begin{cases} H_L \text{ if } f_{\Delta x}^h(W_L, W_R) > 0, \\ H_R \text{ elsewhere,} \end{cases} \tag{3.5}$$

$$\bar{X}(W_L, W_R, h_L, h_R) = \begin{cases} \frac{h_L}{H_L} \text{ if } f_{\Delta x}^h(W_L, W_R) > 0, \\ \frac{h_R}{H_R} \text{ elsewhere,} \end{cases} \tag{3.6}$$

to suggest the following interface approximation of water height, h_j, water height fraction X_j and *total* water height, H_j, for $j = 1, 2$, as follows:

$$H_{j,i+\frac{1}{2}} = \bar{H}(W_{j,i}^n, W_{j,i+1}^n), \tag{3.7}$$

$$X_{j,i+\frac{1}{2}} = \bar{X}(W_{j,i}^n, W_{j,i+1}^n, w_{j,i}^n, w_{j,i+1}^n), \tag{3.8}$$

$$h_{j,i+\frac{1}{2}} = H_{j,i+\frac{1}{2}} X_{j,i+\frac{1}{2}}. \tag{3.9}$$

Using these approximations at the interface located at $x_{i+\frac{1}{2}}$, we propose the following approximation of the source term, $S(H_j, h_j)$:

$$\begin{aligned}
-\frac{g}{2}\partial_x h_j(H_j - h_j) + g h_j \partial_x(H_j - h_j), \\
&\simeq -\frac{g}{2}\left(h_{j,i+\frac{1}{2}}(H_{j,i+\frac{1}{2}} - h_{j,i+\frac{1}{2}}) - h_{j,i-\frac{1}{2}}(H_{j,i-\frac{1}{2}} - h_{j,i-\frac{1}{2}})\right), \\
&\quad + \frac{g}{2}(h_{j,i+\frac{1}{2}} + h_{j,i-\frac{1}{2}})\left((H_{j,i+\frac{1}{2}} - h_{j,i+\frac{1}{2}}) - (H_{j,i-\frac{1}{2}} - h_{j,i-\frac{1}{2}})\right), \\
&\simeq -\frac{g}{2}(h_{j,i+\frac{1}{2}} H_{j,i-\frac{1}{2}} - h_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}}), \\
&\simeq -\frac{g}{2} H_{j,i+\frac{1}{2}} H_{j,i-\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}}).
\end{aligned}$$

Now, we derive the following scheme to approximate solutions of (2.8), combining the homogeneous scheme (3.2) for each layer and the previous discretization for the terms $S(H_j, h_j)$:

$$\begin{aligned}
w_{j,i}^{n+1} = w_{j,i}^n &- \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} f_{\Delta x}(W_{j,i}^n, W_{j,i+1}^n) - X_{j,i-\frac{1}{2}} f_{\Delta x}(W_{j,i-1}^n, W_{j,i}^n)) \\
&+ \frac{g}{2}\frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ H_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}}) \end{pmatrix}, \qquad j = 1, 2.
\end{aligned} \tag{3.10}$$

4. Properties. The numerical scheme (3.10) satisfies the two following properties: it preserves steady states at rest, and it preserves the water height of each layer non-negative.

THEOREM 4.1. *Let $(w_{j,i}^n)_{j=1,2,i\in\mathbb{Z}}$ belong to Ω. Assume that the updated states $(w_{j,i}^{n+1})_{j=1,2}$ are defined by the scheme (3.10). Assume that the numerical flux function $f_{\Delta x}$ satisfies the consistency condition (3.3). The two following properties hold:*

(i) Robustness property:
Let us assume that the following CFL like-condition holds:

$$\frac{\Delta t}{\Delta x}\left(\max(0, f_{\Delta x}^h(W_{j,i}^n, W_{j,i+1}^n)) - \min(0, f_{\Delta x}^h(W_{j,i-1}^n, W_{j,i}^n))\right) \le H_{j,i}^n, \tag{4.1}$$

then the scheme (3.10) is non-negative preserving: if we have, for all $i \in \mathbb{Z}$, $h_{1,i}^n \ge 0$ and $h_{2,i}^n \ge 0$ then we get $h_{1,i}^{n+1} \ge 0$ and $h_{2,i}^{n+1} \ge 0$.

(ii) Well-balancing:
For all i in $\mathbb{Z}$, provided $w_{j,i}^n$ satisfies the lake at rest condition (2.3); namely

$$\begin{aligned}
&u_{1,i}^n = 0 \qquad \text{and} \qquad u_{2,i}^n = 0, \\
&h_{1,i}^n + h_{2,i}^n + z_i = H_1 \qquad \text{and} \qquad r h_{1,i}^n + h_{2,i}^n + z_i = H_2,
\end{aligned}$$

where H_1 and H_2 are positive given constant, then the updated state $w_{j,i}^{n+1}$ preserves the lake at rest:

$$\begin{aligned}
&u_{1,i}^{n+1} = 0 \qquad \text{and} \qquad u_{2,i}^{n+1} = 0, \\
&h_{1,i}^{n+1} + h_{2,i}^{n+1} + z_i = H_1 \qquad \text{and} \qquad r h_{1,i}^{n+1} + h_{2,i}^{n+1} + z_i = H_2.
\end{aligned}$$

Proof. To establish (i), because of (3.10), we first notice that both updated water heights read:

$$h_{j,i}^{n+1} = h_{j,i}^{n} - \frac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) - X_{j,i-\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})). \tag{4.2}$$

Involving the definition of $X_{j,i+\frac{1}{2}}$, given by (3.8), we get

$$\begin{aligned} X_{j,i+\frac{1}{2}} f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) =& \frac{1}{2}(X_{j,i}^{n} + X_{j,i+1}^{n}) f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) \\ &- \frac{1}{2}(X_{j,i+1}^{n} - X_{j,i}^{n})|f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n})|. \end{aligned}$$

As a consequence, (4.2) rewrites:

$$h_{j,i}^{n+1} = \alpha_{j,i} X_{j,i-1}^{n} + (\tilde{H}_{j,i}^{n+1} - \alpha_{j,i} - \beta_{j,i}) X_{j,i}^{n} + \beta_{j,i} X_{j,i+1}^{n}$$

where we have set

$$\begin{cases} \alpha_{j,i} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n}) + |f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})|\right) \geq 0, \\ \beta_{j,i} = \frac{1}{2}\frac{\Delta t}{\Delta x}\left(-f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) + |f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n})|\right) \geq 0, \\ \tilde{H}_{j,i}^{n+1} = H_{j,i}^{n} - \frac{\Delta t}{\Delta x}\left(f_{\Delta x}^{h}(W_{j,i}^{n}, W_{j,i+1}^{n}) - f_{\Delta x}^{h}(W_{j,i-1}^{n}, W_{j,i}^{n})\right). \end{cases}$$

Since $X_{j,i}^{n}$ are non-negative for all $i \in \mathbb{Z}$ and $j = 1, 2$, then $h_{j,i}^{n}$ is proved to be non-negative as soon as we establish $\tilde{H}_{j,i}^{n+1} - \alpha_{j,i} - \beta_{j,i} \geq 0$. This condition directly comes from the additional CFL like-restriction (4.1). Hence, both water height $h_{1,i}^{n+1}$ and $h_{2,i}^{n+1}$ are proved to be non-negative.

Concerning (ii), let us first assume that $u_{j,i}^{n} = 0$ and $H_{j,i}^{n} = H_j$, where H_j is constant, $j = 1, 2$. Then we have

$$W_{j,i}^{n} = \begin{pmatrix} H_j \\ 0 \end{pmatrix} =: W_j, \qquad \forall i \in \mathbb{Z}, \quad j = 1, 2.$$

Since $f_{\Delta x}$ is consistent (3.3), we obtain

$$f_{\Delta x}(W_{j,i}^{n}, W_{j,i+1}^{n}) = \begin{pmatrix} 0 \\ \frac{g}{2} H_j^2 \end{pmatrix}, \qquad \forall i \in \mathbb{Z}, \quad j = 1, 2.$$

Plugging this numerical flux evaluation inside the scheme (3.10), with $u_{j,i}^{n} = 0$, we get

$$\begin{cases} h_{j,i}^{n+1} = h_{j,i}^{n}, \\ (hu)_{j,i}^{n+1} = - \dfrac{\Delta t}{\Delta x}(X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}})\dfrac{g}{2} H_j^2 \\ \qquad\qquad + \dfrac{g}{2}\dfrac{\Delta t}{\Delta x} H_{j,i-\frac{1}{2}} H_{j,i+\frac{1}{2}} (X_{j,i+\frac{1}{2}} - X_{j,i-\frac{1}{2}}) \end{cases}$$

From (3.5) and (3.7), we immediately deduce $H_{j,i+\frac{1}{2}} = H_j$ for all $i \in \mathbb{Z}$ and $j = 1, 2$, so that we obtain $(hu)_{j,i}^{n+1} = 0$ for $j = 1$ or 2, to write $u_{j,i}^{n+1} = 0$. The scheme (3.10) is thus proved to be well-balanced. □

5. Second-order MUSCL extension. In order to obtain a better order of accuracy of the scheme (3.10), we now propose to consider a MUSCL extension [54, 7, 13, 42]. To address such an issue, over each cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, we substitute the constant approximation by a linear reconstruction given by

$$\begin{aligned} &h_{j,i}^n(x) = h_{j,i}^n + (x - x_i)s_i^{h_j}, \\ &(hu)_{j,i}^n(x) = h_{j,i}^n u_{j,i}^n + (x - x_i)s_i^{h_j u_j}, \\ &H_{j,i}^n(x) = H_{j,i}^n + (x - x_i)s_i^{H_j}, \end{aligned} \tag{5.1}$$

where s_i^α (with $\alpha = h_j$, $h_j u_j$, H_j) denote slopes to be described.

To enforce robustness requirements, we assume that the reconstruction is positive preserving:

$$\forall x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}), \quad j = 1, 2, \quad h_{j,i}^n(x) \geqslant 0 \text{ and } H_{j,i}^n(x) > 0$$

and constant preserving:

$$s_i^\alpha = 0 \text{ if } \alpha_{i-1} = \alpha_i = \alpha_{i+1}. \tag{5.2}$$

Arguing this linear reconstruction (5.1), we introduce the inner approximation on each side of the interface $x_{i+1/2}$ as follows:

$$w_{j,i+1/2}^{n,-} = \begin{pmatrix} h_{j,i}^n(x_{i+1/2}) \\ (hu)_{j,i}^n(x_{i+1/2}) \end{pmatrix} \quad \text{and} \quad w_{j,i+1/2}^{n,+} = \begin{pmatrix} h_{j,i+1}^n(x_{i+1/2}) \\ (hu)_{j,i+1}^n(x_{i+1/2}) \end{pmatrix} \quad j = 1, 2.$$

Moreover, we introduce inner reconstructions associated with the states vectors W_j as follows:

$$W_{j,i+1/2}^{n,\pm} = \begin{pmatrix} H_{j,i+1/2}^{n,\pm} \\ (Hu)_{j,i+1/2}^{n,\pm} \end{pmatrix} \qquad j = 1, 2,$$

where we have set with $j = 1$, 2:

$$H_{j,i+1/2}^{n,-} = H_{j,i}^n(x_{i+1/2}) \quad \text{and} \quad H_{j,i+1/2}^{n,+} = H_{j,i+1}^n(x_{i+1/2}),$$

$$(Hu)_{j,i+1/2}^{n,\pm} = \begin{cases} H_{j,i+1/2}^{n,\pm} \dfrac{(hu)_{j,i+1/2}^{n,\pm}}{h_{j,i+1/2}^{n,\pm}} & \text{if } h_{j,i+1/2}^{n,\pm} > 0, \\ 0 & \text{if } h_{j,i+1/2}^{n,\pm} = 0. \end{cases}$$

Now, after [13, 7, 42], let recall that the MUSCL scheme based on a linear reconstruction given by (5.1) reads

$$w_{j,i}^{n+1} = \frac{1}{2}\left(w_{j,i}^{n+1,-} + w_{j,i}^{n+1,+}\right), \qquad j = 1, 2, \tag{5.3}$$

where $w_{j,i}^{n+1,-}$ results from the first-order 3-point scheme (3.10) applied on the left half-cell $(x_{i-\frac{1}{2}}, x_i)$ by involving the three states $W_{j,i-1}^{n,+}$, $W_{j,i}^{n,-}$, $W_{j,i}^{n,+}$, and $w_{j,i}^{n+1,+}$ is obtained by the scheme (3.10) on the right half-cell $(x_i, x_{i+\frac{1}{2}})$ with the three inner approximation states $W_{j,i}^{n,-}$, $W_{j,i}^{n,+}$, $W_{j,i+1}^{n,-}$.

For the sake of simplicity in the notations, in order to rewrite (5.3) in a more convenient formulation let us set

$$\bar{H}_{j,i} = \bar{H}(W_{j,i}^{n,-}, W_{j,i}^{n,+}), \tag{5.4}$$

$$\bar{H}_{j,i+\frac{1}{2}} = \bar{H}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}), \tag{5.5}$$

$$\bar{X}_{j,i} = \bar{X}(W_{j,i}^{n,-}, W_{j,i}^{n,+}, h_{j,i}^{n,-}, h_{j,i}^{n,+}), \tag{5.6}$$

$$\bar{X}_{j,i+\frac{1}{2}} = \bar{X}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}, h_{j,i}^{n,+}, h_{j,i+1}^{n,-}), \tag{5.7}$$

$$\tag{5.8}$$

where the functions $\bar{H}$ and $\bar{X}$ are respectively defined by (3.5) and (3.6). Then the updated states $w_{i,j}^{n,\pm}$, for $j = 1, 2$, are given by

$$\begin{aligned} w_{j,i}^{n+1,-} &= w_{j,i}^{n,-} \\ &- \frac{\Delta t}{\Delta x/2}\left(\bar{X}_{j,i} f_{\Delta x}((W_{j,i}^{n,-}, W_{j,i}^{n,+}) - \bar{X}_{j,i-\frac{1}{2}} f_{\Delta x}((W_{j,i-1}^{n,+}, W_{j,i}^{n,-})\right) \\ &\qquad + \frac{g}{2}\frac{\Delta t}{\Delta x/2}\begin{pmatrix} 0 \\ \bar{H}_{j,i-\frac{1}{2}}\bar{H}_{j,i}\left(\bar{X}_{j,i} - \bar{X}_{j,i-\frac{1}{2}}\right)\end{pmatrix} \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} w_{j,i}^{n+1,+} &= w_{j,i}^{n,+} \\ &- \frac{\Delta t}{\Delta x/2}\left(\bar{X}_{j,i+\frac{1}{2}} f_{\Delta x}((W_{j,i}^{n,+}, W_{j,i+1}^{n,-}) - \bar{X}_{j,i} f_{\Delta x}((W_{j,i}^{n,-}, W_{j,i}^{n,+})\right) \\ &\qquad + \frac{g}{2}\frac{\Delta t}{\Delta x/2}\begin{pmatrix} 0 \\ \bar{H}_{j,i}\bar{H}_{j,i+\frac{1}{2}}\left(\bar{X}_{j,i+\frac{1}{2}} - \bar{X}_{j,i}\right)\end{pmatrix}. \end{aligned} \tag{5.10}$$

Involving (5.3), the resulting MUSCL scheme now writes with $j = 1,\ 2$: so that we deduce the MUSCL scheme:

$$\begin{aligned} w_{j,i}^{n+1} &= w_{j,i}^{n} - \frac{\Delta t}{\Delta x}\left(\bar{X}_{j,i+\frac{1}{2}} f_{\Delta x}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}) - \bar{X}_{j,i-\frac{1}{2}} f_{\Delta x}(W_{j,i-1}^{n,+}, W_{j,i}^{n,-})\right) \\ &+ \frac{g}{2}\frac{\Delta t}{\Delta x}\begin{pmatrix} 0 \\ \bar{H}_{j,i}\left(\bar{H}_{j,i-\frac{1}{2}}(\bar{X}_{j,i} - \bar{X}_{j,i-\frac{1}{2}}) + \bar{H}_{j,i+\frac{1}{2}}(\bar{X}_{j,i+\frac{1}{2}} - \bar{X}_{j,i})\right)\end{pmatrix} \end{aligned} \tag{5.11}$$

Now, we state that this second-order accurate MUSCL scheme preserves the well-balancing and the robustness required properties.

THEOREM 5.1. *Let $(w_{j,i}^{n})_{j=1,2,i\in\mathbb{Z}}$ be given in Ω. Assume the updated state $(w_{j,i}^{n+1})_{j=1,2}$ be defined by (5.11). The two following properties hold:*

(i) Robustness property:

Let us assume that the following CFL like conditions hold:

$$\begin{aligned} \frac{\Delta t}{\Delta x/2}\left(\max(0, f_{\Delta x}^{h}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-})) - \min(0, f_{\Delta x}^{h}(W_{j,i}^{n,-}, W_{j,i}^{n,+}))\right) &\le H_{j,i}^{n,+}, \\ \frac{\Delta t}{\Delta x/2}\left(\max(0, f_{\Delta x}^{h}(W_{j,i}^{n,-}, W_{j,i}^{n,+})) - \min(0, f_{\Delta x}^{h}(W_{j,i-1}^{n,+}, W_{j,i}^{n,-}))\right) &\le H_{j,i}^{n,-}, \end{aligned} \tag{5.12}$$

then the scheme (5.11) is non-negative preserving: for all i in $\mathbb{Z}$, if we have $h_{1,i}n \ge 0$ and $h_{2,i}^{n} \ge 0$ then we get $h_{1,i}^{n+1} \ge 0$ and $h_{2,i}^{n+1} \ge 0$.

(ii) Well-balancing:
The scheme (5.11) exactly preserves the lake at rest.

Proof. Concerning the establishment of (i), the updated water heights $h_{1,i}^{n+1}$ and $h_{2,i}^{n+1}$ are proved to be non-negative for all i in $\mathbb{Z}$ as soon as the intermediate water heights $h_{1,i}^{n+1,\pm}$ and $h_{2,i}^{n+1,\pm}$ stay non-negative for all $i \in \mathbb{Z}$. Now, we recall that both $w_{j,i}^{n+1,-}$ and $w_{j,i}^{n+1,+}$, defined by (5.9) and (5.10), are respectively given by the 3-point first-order scheme (3.10) involving the three states $(w_{j,i-1}^{n,+}, w_{j,i}^{n,-}, w_{j,i}^{n,+})$ and $(w_{j,i}^{n,-}, w_{j,i}^{n,+}, w_{j,i+1}^{n,-})$. All the involved states at time t^n belong to Ω. Next, by applying Theorem 4.1 (i), we immediately get the required non-negativeness.

To prove (ii), at time t^n, for all i in $\mathbb{Z}$ and $j = 1,\ 2$, we assume that the states $w_{i,j}^n$ satisfies the lake at rest condition:

$$\begin{aligned} &u_{1,i}^n = u_{2,i}^n = 0, \\ &h_{1,i}^n + h_{2,i}^n + z_i = H_1 \qquad \text{and} \qquad r h_{1,i}^n + h_{2,i}^n + z_i = H_2, \end{aligned}$$

where $H_1 > 0$ and $H_2 > 0$ denote given constants.

Since, after (5.2), the reconstruction (5.1) preserves the constants, we get

$$(hu)_{j,i+1/2}^{n,\pm} = 0 \qquad \text{and} \qquad H_{j,i+1/2}^{n,\pm} = H_j, \qquad j = 1, 2.$$

As a consequence, we obtain $(Hu)_{j,i+1/2}^{n,\pm} = 0$ and thus we have

$$W_{1,i+1/2}^{n,\pm} = \begin{pmatrix} H_1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad W_{2,i+1/2}^{n,\pm} = \begin{pmatrix} H_2 \\ 0 \end{pmatrix}, \qquad i \in \mathbb{Z}.$$

Hence, the numerical flux function involved in (5.11) reads

$$f_{\Delta x}(W_{j,i}^{n,+}, W_{j,i+1}^{n,-}) = \begin{pmatrix} 0 \\ \frac{g}{2} H_j^2 \end{pmatrix}, \qquad \forall i \in \mathbb{Z}, \quad j = 1, 2.$$

Finally, from (5.11) we immediately obtain

$$w_{j,i}^{n+1} = w_{j,i}^n \qquad \forall i \in \mathbb{Z}, \quad j = 1, 2,$$

and the proof is achieved. □

6. Numerical results. We now intend to illustrate the relevance of the proposed numerical scheme. Four numerical tests shall be presented that correspond to relevant test cases considered in [23, 14, 19, 21, 18]. The first one concerns the validity of the scheme for a density ratio of $r = 0.7$. In this numerical experiment, we show the results obtained by involving several numerical flux functions. The three other tests are known to be very challenging since the density ratio is fixed to $r = 0.98$ and they correspond to situations known to be particularly difficult when handled numerically.

All benchmarks have been performed over uniform meshes and adopting the same CFL restriction strategy given by

$$\frac{\Delta t}{\Delta x} \max_{w_i} |\lambda_{ext}^{\pm}(w_i)| \leq C_{CFL},$$

where $\lambda_{ext}^{\pm}$ is given by (1.10) and $\max_{w_i}$ represents the maximum over all approximated states involved within the simulation. As usual, the number C_{CFL} is fixed to 0.5 as long as first order approximations are considered (3.10). For the second order accurate scheme (5.11), after [7] where the number C_{CFL} is prescribed to be smaller than 1/6, we fix $C_{CFL} = 0.1$.

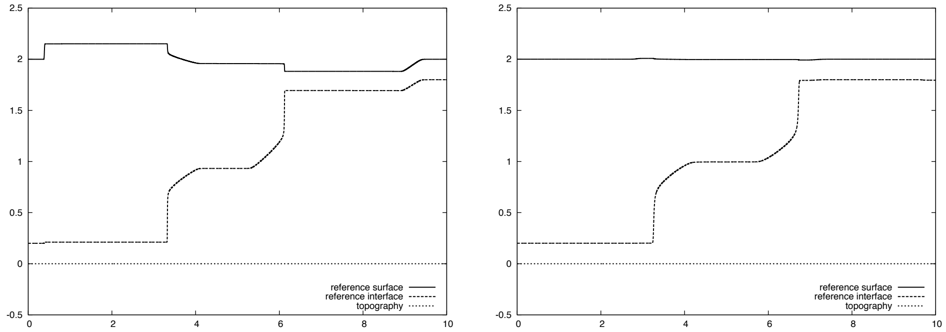

Fig. 6.1: Test 1: reference solutions, r = 0.7 and t=1 (left) and r=0.98 and t=5 (right)

6.1. Tests on flat topography (Test 1 and 2). Both tests involve the same initial data and they differ in the density ratio value. Since the topography is flat, we fix z to an arbitrary positive constant. Concerning the initial data, as given in [14], we adopt

$$\begin{aligned} h_2(t=0,x) &= \begin{cases} 0.2 \text{ if } x < 5 \\ 1.8 \text{ if } x > 5 \end{cases} \\ h_1(t=0,x) &= 2 - h_2(t=0,x) \\ u_1(t=0,x) &= u_2(t=0,x) = 0 \end{aligned} \tag{6.1}$$

Let us underline that this problem does not involve vanishing layers, but the situation will turn distinct with Test 3 and 4.

The boundary conditions are fixed to simulate infinite reservoirs at both boundaries. The two layers must be allowed to go out as freely as possible, with the only constraint $h_1u_1 = -h_2u_2$, as explained in [23]. It is usual for the numerical treatment to introduce a ghost cell at each end of the domain and duplicate the values computed in the neighbor cell, which is to simulate open boundaries (Neumann conditions). Here to simulate $h_1u_1 = -h_2u_2$ at each boundary, we do as follows. At time t^n, let us suppose that the approximations obtained at the first cell are $w^n_{j,1}$, $j = 1, 2$, and values at the left ghost cell are $w_{j,0}$, $j = 1, 2$. If $u_{2,1} = 0$, that is if the lower layer has not reached the left boundary, we duplicate the states $w_{j,1}$ at the ghost cell. In the other case, we put appropriate values depending on the first cell such that $h_{1,0}u_{1,0} = -h_{2,0}u_{2,0}$:

$$\begin{cases} h_{j,0} = h_{j,1} \\ (hu)_{j,0} = (hu)_{j,1} - \frac{(hu)_{1,1}+(hu)_{2,1}}{|(hu)_{1,1}|+|(hu)_{2,1}|}|(hu)_{j,1}| \end{cases} \quad , \text{ for } j = 1, 2$$

A reference solution is performed considering a mesh made of 16384 cells, a first order scheme with a numerical flux function given by the Suliciu relaxation scheme [13], and a number C_{CFL} equal to 0.5. Both reference solutions, respectively involving $r = 07$ and $r = 0.98$, are displayed in Fig. 6.1.

In the first test, we impose $r = 0.7$. We evaluate the behavior of the scheme adopting several numerical flux functions $f_{\Delta x}$ given by VFRoe [31, 32, 30], Lax-Friedrichs (LF) [13], Suliciu relaxation [13], and kinetic [33] schemes. The approximate

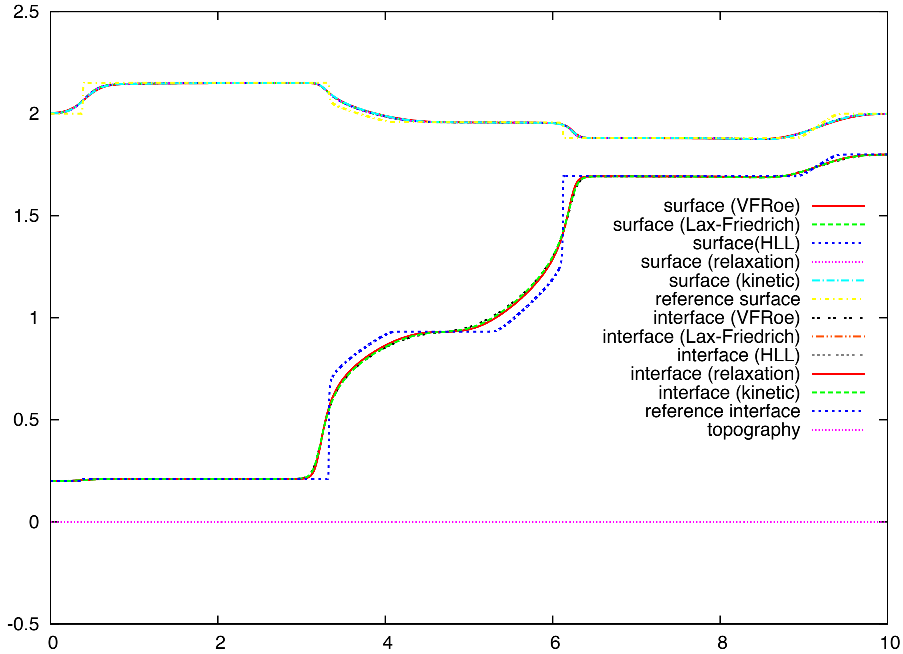

Fig. 6.2: Test 1: r = 0.7, t = 1, ns=512, first-order, C_{CFL} = 0.5, comparison of numerical flux

water heights are displayed in Fig. 6.2 and Fig. 6.3 respectively involving first and second order of accuracy.

In Table 6.1 and Table 6.2, we present the water height and flux discharge error involved by the first order scheme over a fixed mesh made of 512 cells. Clearly, we notice that the behavior of the scheme is not modified by the choice of the numerical flux function. Similarly, Table 6.3 and Table 6.4 exhibit the numerical behavior of the second order scheme. Since both first and second order admit the same behavior independently from the numerical flux function, we now adopt the Suliciu relaxation flux function.

Concerning the second test, we consider the initial data defined by (6.1) but for a density ratio given by $r = 0.98$. The numerical results obtained by considering meshes made of 512 and 2048 cells are displayed in Fig. 6.4. We observe that with 2048 cells, the result with the first-order scheme is close to one obtained with the second-order scheme with 512 cells. Comparing with results presented in [14], we observe that the result is improved. Indeed, in [14] we remark a stationary shock that appears at the interface when the splitting technique was used which does not appear when other coupled schemes are used.

Table 6.5 to Table 6.8, we give both water heights and flux discharge errors versus mesh refinements. Once again, we get a good behavior of the scheme.

6.2. Tests on non-flat bottom (Test 3 and 4). The third numerical experiment is a lock-exchanged test over a bump with $r = 0.98$. The initial data (see Fig.

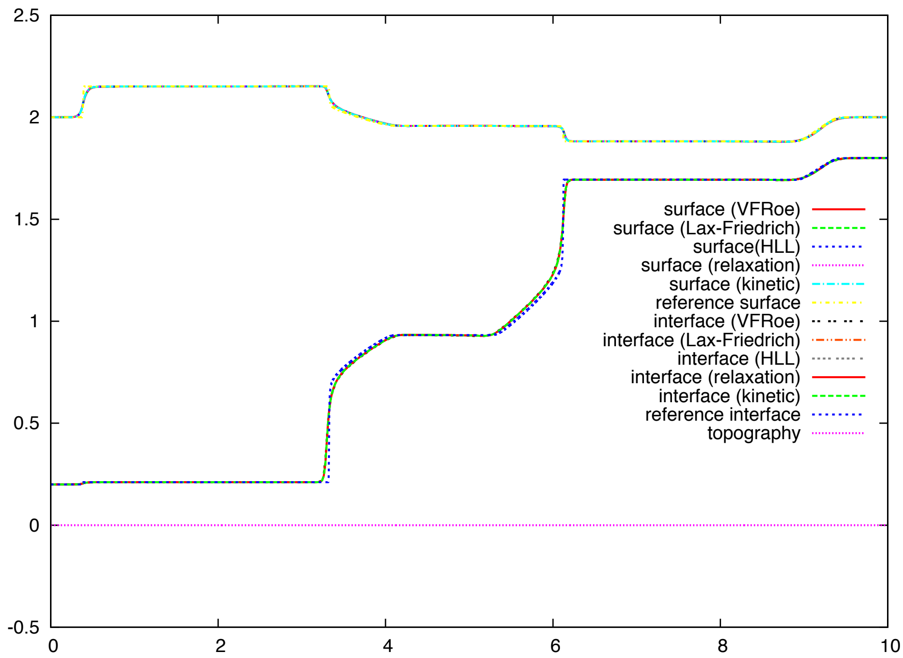

Fig. 6.3: Test 1: r = 0.7, t = 1, ns=512, second-order, C_{CFL} = 0.1, comparison of numerical flux

	$\|h_1 - h_1^{ref}\|_1$	$\|h_1 - h_1^{ref}\|_2$	$\|h_2 - h_2^{ref}\|_1$	$\|h_2 - h_2^{ref}\|_2$
VFRoe	0.29801759	0.19102120	0.25150242	0.16183789
LF	0.25184345	0.17504048	0.21128915	0.14825192
HLL	0.25322178	0.17549789	0.21244211	0.14863172
relaxation	0.25230557	0.17500915	0.21169171	0.14822640
kinetic	0.28619331	0.18699861	0.23981616	0.15820369

Table 6.1: Test 1: r = 0.7, t = 1, ns=512, first-order, C_{CFL} = 0.5, errors on water heights depending on numerical flux

	$\|q_1 - q_1^{ref}\|_1$	$\|q_1 - q_1^{ref}\|_2$	$\|q_2 - q_2^{ref}\|_1$	$\|q_2 - q_2^{ref}\|_2$
VFRoe	4.3384E-1	2.8386E-1	3.4666E-1	2.1992E-1
LF	3.7101E-1	2.6478E-1	2.9196E-1	2.0526E-1
HLL	3.7257E-1	2.6528E-1	2.9320E-1	2.0559E-1
relaxation	3.7167E-1	2.6461E-1	2.9250E-1	2.0502E-1
kinetic	4.1675E-1	2.7900E-1	3.2754E-1	2.1537E-1

Table 6.2: Test 1: r = 0.7, t = 1, ns=512, first-order, C_{CFL} = 0.5, errors on water flux depending on numerical flux

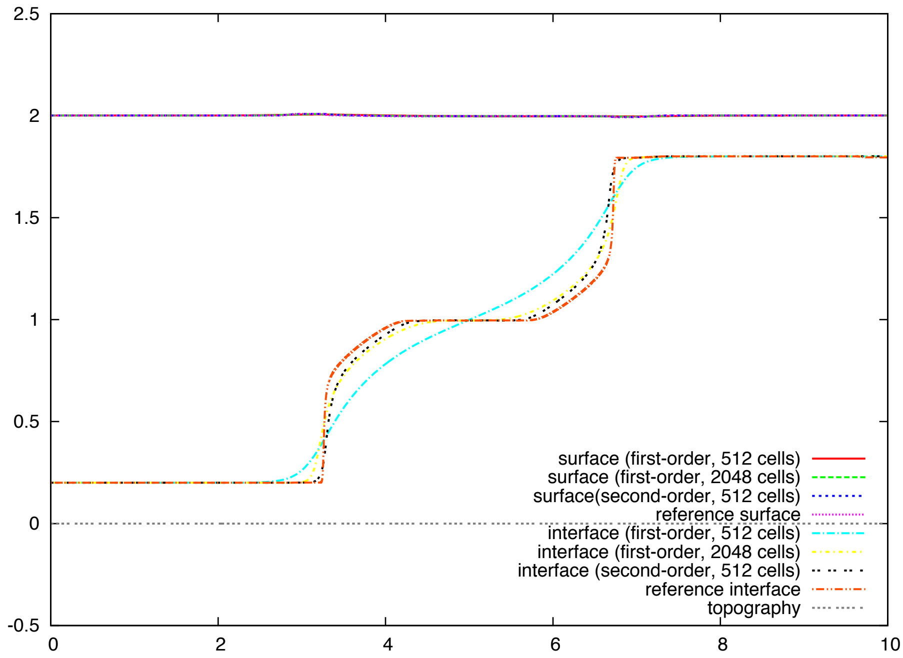

Fig. 6.4: Test 2: r = 0.98, t = 5, ns = 512 or 2048, C_{CFL} = 0.5 (first-order), $C_{CFL} = 0.1$ (second-order)

$$\begin{aligned} h_2(t=0,x) &= \begin{cases} 0.2 \text{ if } x < 5 \\ 1.8 \text{ if } x > 5 \end{cases} \\ h_1(t=0,x) &= 2 - h_2(t=0,x) \\ u_1(t=0,x) &= u_2(t=0,x) = 0 \end{aligned} \tag{6.2}$$

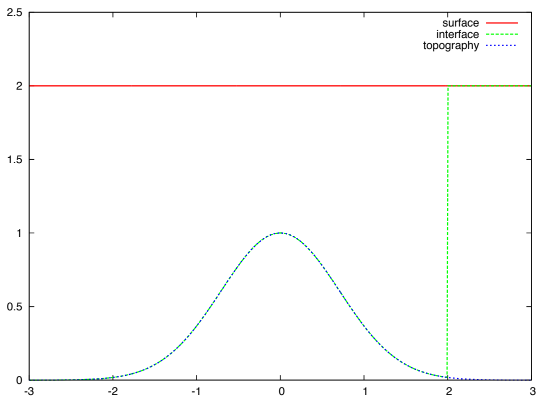

Fig. 6.5: Test 3: initial data

	$\|h_1 - h_1^{ref}\|_1$	$\|h_1 - h_1^{ref}\|_2$	$\|h_2 - h_2^{ref}\|_1$	$\|h_2 - h_2^{ref}\|_2$
VFRoe	7.7892E-2	8.5514E-2	6.6048E-2	7.4517E-2
LF	6.8823E-2	8.0717E-2	5.7646E-2	7.0235E-2
HLL	6.8822E-2	8.0717E-2	5.7646E-2	7.0235E-2
relaxation	6.8822E-2	8.0717E-2	5.7646E-2	7.0235E-2
kinetic	6.8822E-2	8.0717E-2	5.7646E-2	7.0235E-2

Table 6.3: Test 1: r = 0.7, t = 1, ns=512, second-order, $C_{CFL} = 0.1$, errors on water heights depending on numerical flux

	$\|q_1 - q_1^{ref}\|_1$	$\|q_1 - q_1^{ref}\|_2$	$\|q_2 - q_2^{ref}\|_1$	$\|q_2 - q_2^{ref}\|_2$
VFRoe	1.1587E-1	1.3767E-1	8.9009E-2	1.0968E-1
LF	1.0377E-1	1.3026E-1	7.7842E-2	1.0355E-1
HLL	1.0377E-1	1.3026E-1	7.7842E-2	1.0355E-1
relaxation	1.0377E-1	1.3026E-1	7.7842E-2	1.0355E-1
kinetic	1.0377E-1	1.3026E-1	7.7842E-2	1.0355E-1

Table 6.4: Test 1: r = 0.7, t = 1, ns=512, second-order, $C_{CFL} = 0.1$, errors on water flux depending on numerical flux

6.5) is given on the interval $[-3, 3]$ by

$$\begin{aligned} &z(x) = \exp(-x^2), \\ &h_1(t=0,x) + h_2(t=0,x) + z(x) = 2, \\ &h_2(t=0,x) = 0 \quad \text{ if } x < 2, \\ &h_1(t=0,x) = 0 \quad \text{ if } x > 2, \\ &u_1(t=0,x) = u_2(t=0,x) = 0. \end{aligned} \tag{6.3}$$

It is worth noticing that each layer alternatively vanishes. It is a classical test, and the reader is referred, for example, to [19, 21, 18, 23]. We adopt a numerical flux function given by the Suliciu relaxation scheme to compute the approximate solution. Here, a steady state is reached. In Fig. 6.6 to 6.8, we give the results at different step times. Finally, Fig. 6.9 shows the final steady solution and the reference solution obtained by considering a mesh made of 16384 cells. We obtain good agreement between the approximation and the reference solution.

In the last test, a lock-exchanged type experiment, we consider a topography given by

$$z(x) = \begin{cases} (x-0.6)(x+0.6) & \text{ if } x \in (0.6, 0.6), \\ 0 \quad \text{ otherwise.} \end{cases}$$

The initial data is now defined by (see Fig. 6.10)

$$\begin{aligned} &h_1(t=0,x) + h_2(t=0,x) + z(x) = 2, \\ &h_2(t=0,x) = \begin{cases} 1 & \text{ if } x < 0, \\ 0 & \text{ if } x > 0, \end{cases} \\ &u_1(t=0,x) = u_2(t=0,x) = 0. \end{aligned} \tag{6.4}$$

cells	$\|h_1 - h_1^{ref}\|_1$	$\|h_1 - h_1^{ref}\|_2$	$\|h_2 - h_2^{ref}\|_1$	$\|h_2 - h_2^{ref}\|_2$
128	1.3045	7.0910E-1	1.2983	7.0378E-1
256	9.2600E-1	5.0083E-1	9.2027E-1	4.9669E-1
512	5.9017E-1	3.2390E-1	5.8562E-1	3.2079E-1
1024	3.4832E-1	2.0426E-1	3.4410E-1	2.0199E-1
2048	1.9345E-1	1.2942E-1	1.9055E-1	1.2795E-1

Table 6.5: Test 2: r = 0.98, t = 5, first-order, $C_{CFL} = 0.5$, errors on water heights depending on cells number

cells	$\|q_1 - q_1^{ref}\|_1$	$\|q_1 - q_1^{ref}\|_2$	$\|q_2 - q_2^{ref}\|_1$	$\|q_2 - q_2^{ref}\|_2$
128	4.1807E-1	2.1253E-1	4.3917E-1	2.1738E-1
256	2.7284E-1	1.4115E-1	2.7778E-1	1.4248E-1
512	1.6711E-1	9.1425E-2	1.6358E-1	8.9993E-2
1024	1.0056E-1	6.0703E-2	9.4736E-2	5.8338E-2
2048	5.7930E-2	4.0460E-2	5.4391E-2	3.8916E-2

Table 6.6: Test 2: r = 0.98, t = 5, first-order, $C_{CFL} = 0.5$, errors on water flux depending on cells number

The simulation is performed over a mesh made of 500 cells. The expected physical solution is known to converge to a steady test with a shock. We check that our scheme is able to capture such a chock, as we can observe in Fig. 6.11.

7. Conclusion. In the present work, we have derived a very easy splitting scheme to approximate the weak solutions of the two-layer shallow-water model. This splitting technique is very simple to implement and is *cheap* from the computational point of view since it reduces to approximating two *independent* homogeneous shallow-water equations coupled by obvious upwind source terms. Moreover, we have established the robustness and the well-balanced properties satisfied by the derived numerical strategy. In addition, a second order MUSCL extension is proposed and it is proved to preserve the required properties (robustness and well-balancing). Several numerical experiments attest the relevance of the here derived numerical procedure.

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cells	$\Vert h_1 - h_1^{ref}\Vert_1$	$\Vert h_1 - h_1^{ref}\Vert_2$	$\Vert h_2 - h_2^{ref}\Vert_1$	$\Vert h_2 - h_2^{ref}\Vert_2$
128	6.4377E-1	3.9357E-1	6.3871E-1	3.9017E-1
256	3.1239E-1	2.2604E-1	3.0878E-1	2.2379E-1
512	1.4260E-1	1.2700E-1	1.4100E-1	1.2568E-1
1024	6.2036E-2	7.6167E-2	6.1319E-2	7.5370E-2
2048	3.0867E-2	6.1731E-2	3.0720E-2	6.1106E-2

Table 6.7: Test 2: r = 0.98, t = 5, second-order, $C_{CFL} = 0.1$, errors on water heights depending on cells number

cells	$\Vert q_1 - q_1^{ref}\Vert_1$	$\Vert q_1 - q_1^{ref}\Vert_2$	$\Vert q_2 - q_2^{ref}\Vert_1$	$\Vert q_2 - q_2^{ref}\Vert_2$
128	1.8653E-1	1.2077E-1	1.8166E-1	1.1944E-1
256	9.3584E-2	7.1501E-2	8.9968E-2	7.0003E-2
512	4.4369E-2	4.1444E-2	4.3157E-2	4.0257E-2
1024	2.0614E-2	2.5624E-2	2.0950E-2	2.5067E-2
2048	1.2718E-2	2.1088E-2	1.2307E-2	2.0973E-2

Table 6.8: Test 2: r = 0.98, t = 5, second-order, $C_{CFL} = 0.1$, errors on water flux depending on cells number

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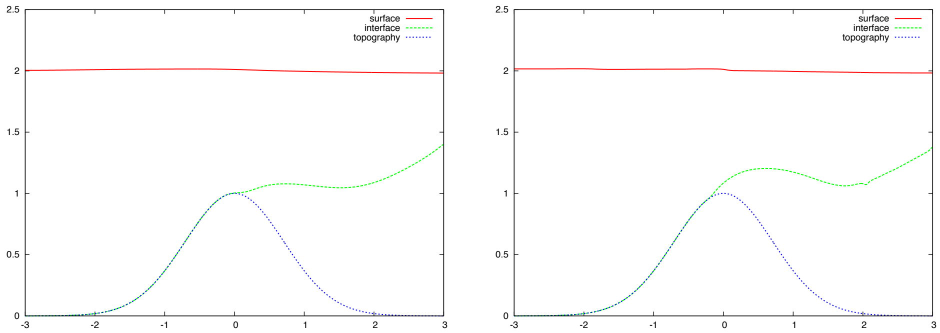

Fig. 6.6: Test 3: t = 5, ns = 500 (left: first-order, $C_{CFL} = 0.5$; right: second-order, $C_{CFL} = 0.1$)

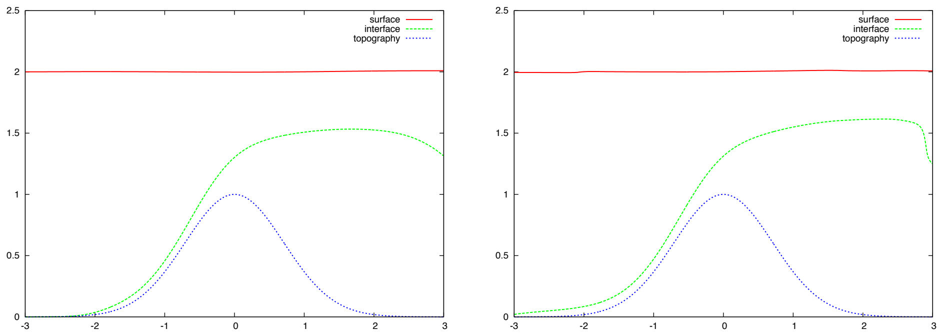

Fig. 6.7: Test 3: t = 10, ns = 500 (left: first-order, $C_{CFL} = 0.5$; right: second-order, $C_{CFL} = 0.1$)

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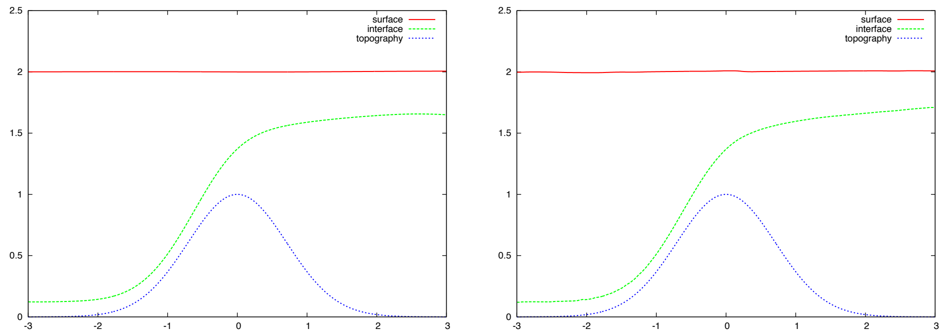

Fig. 6.8: Test 3: t = 20, ns = 500 (left: first-order, $C_{CFL} = 0.5$; right: second-order, $C_{CFL} = 0.1$)

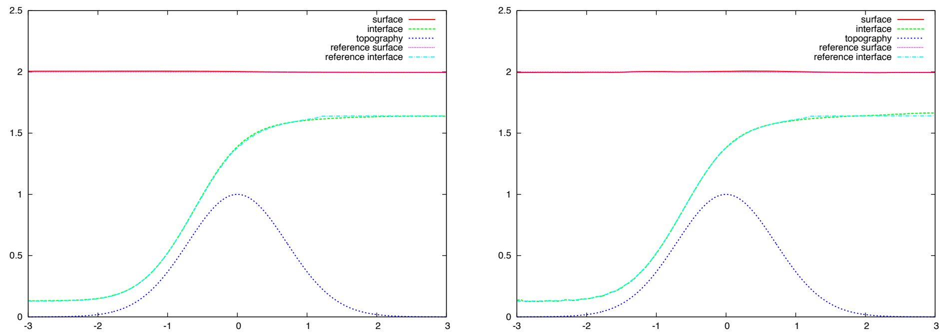

Fig. 6.9: Test 3: t = 40, ns = 500 (left: first-order, $C_{CFL} = 0.5$; right: second-order, $C_{CFL} = 0.08$)

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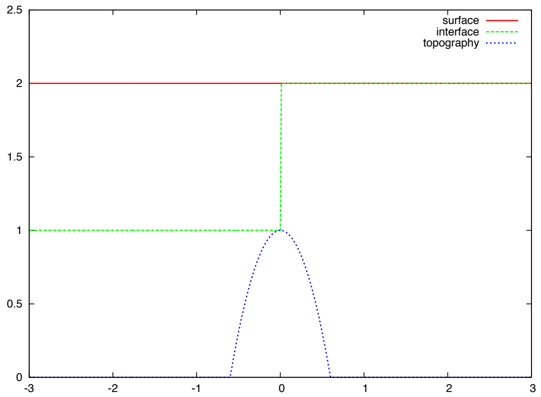

Fig. 6.10: Test 4: Initial data

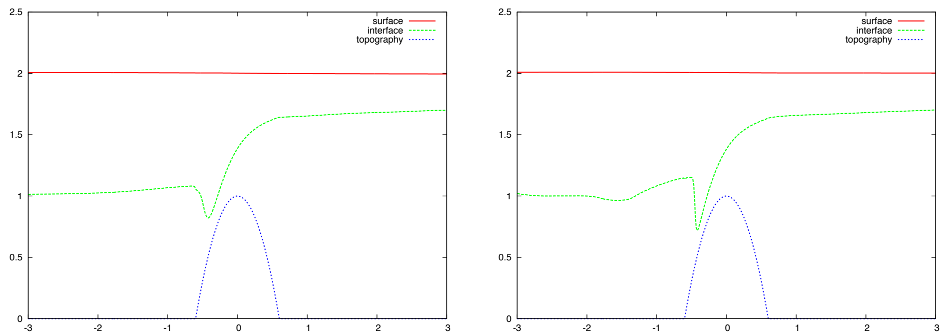

Fig. 6.11: Test 4: Interface with a shock at t = 40, ns=500 (left: first-order, $C_{CFL} = 0.5$; right: second-order, $C_{CFL} = 0.1$)

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