# A fully well-balanced and asymptotic preserving scheme for the shallow-water equations with a generalized Manning friction source term*

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July 21, 2021

**Abstract.** The aim of this paper is to prove the preservation of the diffusive limit by a numerical scheme for the shallow-water equations with a generalized Manning friction source term. This asymptotic behavior coincides with the long time and stiff friction limit. The adopted discretization was initially developed to preserve all the steady states of the model under concern. In this work, a relevant improvement is performed in order to preserve also the diffusive limit of the problem and to exactly capture the moving and non-moving steady solutions. In addition, a second-order time and space extension is detailed. Involving suitable linearizations, the obtained second-order scheme exactly preserves the steady states and the diffusive behavior. Several numerical experiments illustrate the relevance of the designed schemes.

**Keywords.** Finite volume scheme; Asymptotic preserving scheme; Fully well-balanced scheme; Shallow-water equations

**AMS subject classifications.** 65M08; 35Q35

## 1 Introduction

This paper is devoted to the study, in the diffusive limit, of a fully well-balanced scheme discretizing the solutions of the shallow-water equations with Manning-type friction

*Received date, and accepted date (The correct dates will be entered by the editor).

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source term (see [32]), which write:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \dfrac{q^2}{h} + \dfrac{gh^2}{2} \right) = -k|q|qh^{-\eta}, \end{cases} \tag{1.1}$$

where $h(t,x) > 0$ is the water height and $q(t,x) \in \mathbb{R}$ is the water discharge, both depending on the time $t$ and the space $x$. As usual, the water velocity is defined by $u = q/h$ where it is further assumed that the water height $h$ never vanishes. The parameter $g = 9.81 \text{ m.s}^{-2}$ denotes the gravity constant. The parameter $k$, corresponding to the Manning coefficient, is used to determine the intensity of the friction: the higher $k$ is, the more the bottom exerts friction on the water. Moreover, $\eta$ stands for a positive parameter. The reader is referred to [1, 19, 34, 36] where several values of eta are suggested according to the choice of the model.

For the sake of simplicity in the notations, it turns out convenient to introduce the following condensed form of (1.1):

$$\partial_t W + \partial_x F(W) = S(W), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.2}$$

where $W = {}^t(h,q)$ is the vector of the unknowns, $F(W) = {}^t(q, \frac{q^2}{h} + g\frac{h^2}{2})$ is the physical flux and $S(W) = {}^t(0, -kq|q|h^{-\eta})$ is the source term.

## 1.1 Diffusive limit

Here, we are concerned with the behavior of $h$ and $q$ in long time and dominant friction. Such behaviours are of particular interest since long time solutions coincide with solutions of diffusive regimes. During the three last decades, several works studied this kind of mathematical difficulties (see [18, 26, 33, 43] or [15, 16] for instance). We refer the reader to [1] for the specfic case of the shallow-water equations. To exhibit the asymptotic regime, let us introduce a small parameter $\varepsilon$ in order to scale the time $t$ and the friction coefficient $k$ as follows:

$$t \leftarrow t/\varepsilon \quad \text{and} \quad k \leftarrow k/\varepsilon^2. \tag{1.3}$$

Let us emphasize that the friction parameter has to be rescaled with $1/\varepsilon^2$ because of the quadratic term in $q$ into the source term (see [10, 20]). Equipped with the above rescaling, the system (1.1) now reads as follows:

$$\begin{cases} \varepsilon\partial_t h + \partial_x q = 0, \\ \varepsilon\partial_t q + \partial_x \left( \dfrac{q^2}{h} + \dfrac{gh^2}{2} \right) = -\dfrac{k}{\varepsilon^2}|q|qh^{-\eta}. \end{cases} \tag{1.4}$$

To study the behavior of $h$ and $q$ when $\varepsilon$ goes to zero, the following asymptotic expansions are introduced:

$$h = h^0 + \varepsilon h^1 + \dots \quad \text{and} \quad q = q^0 + \varepsilon q^1 + \dots, \tag{1.5}$$

where the term of zero order $h^0$ is also assumed positive. Injecting the expansions (1.5) in (1.4) and letting $\varepsilon$ tend to zero provides:

$$\begin{cases} \partial_x q^0 = 0, \\ -k|q^0|q^0(h^0)^{-\eta} = 0. \end{cases}$$

Since $h^0 > 0$, the second equation of the above system necessarily gives $q^0 = 0$. The expansions (1.5) now read:

$$h = h^0 + \mathcal{O}(\varepsilon) \quad \text{and} \quad q = \varepsilon\left(q^1 + \mathcal{O}(\varepsilon)\right). \tag{1.6}$$

Injecting the expansions (1.6) in (1.4) and letting $\varepsilon$ tend to zero now provides the following limit problem:

$$\begin{cases} \partial_t h^0 + \partial_x q^1 = 0, \\ \partial_x \left(\dfrac{g(h^0)^2}{2}\right) = -k|q^1|q^1(h^0)^{-\eta}. \end{cases} \tag{1.7}$$

The second equation of the above system gives an expression of $q^1$ in terms of $h^0$, called the local equilibrium (see [28]):

$$q^1 = -\text{sign}(\partial_x h^0)\sqrt{\frac{(h^0)^\eta}{k}\left|\partial_x\left(\frac{g(h^0)^2}{2}\right)\right|}. \tag{1.8}$$

Moreover, injecting (1.8) in the first equation of (1.7) gives the following nonlinear diffusion equation satisfied by $h^0$:

$$\partial_t h^0 + \partial_x\left(-\text{sign}(\partial_x h^0)\sqrt{\frac{(h^0)^\eta}{k}\left|\partial_x\left(\frac{g(h^0)^2}{2}\right)\right|}\right) = 0. \tag{1.9}$$

The diffusion equation (1.9) coincides with the non-stationary $p$-laplacian like equation:

$$\partial_t h^0 + \partial_x\left(-|\partial_x h^0|^{p-2}\partial_x h^0\right) = 0, \quad p > 1,$$

for $p = 3/2$ and with the additional factor $\sqrt{\frac{g}{k}}(h^0)^{\frac{\eta+1}{2}}$ in the flux. It is a degenerate parabolic equation that has been widely studied (see [3,25,30,40] for theoretical aspects and [2,6,29] for some numerical studies) and appears in several physical problems as, for instance, in non-Newtonian fluids [17].

## 1.2 Steady states

In addition to the asymptotic behavior satisfied by the solutions of (1.1), in the present work, we are also interested in the steady state solutions. These particular time independent solutions are described by the following system:

$$\begin{cases} \partial_x q = 0, \\ \partial_x\left(\dfrac{q^2}{h} + \dfrac{gh^2}{2}\right) = -kq|q|h^{-\eta}. \end{cases} \tag{1.10}$$

A steady state for (1.1) is thus governed by a uniform discharge $q_0$ and the following equation on the water height:

$$\partial_x\left(\frac{q_0^2}{h} + \frac{gh^2}{2}\right) = -kq_0|q_0|h^{-\eta}.$$

The above equation can be integrated and yields to a nonlinear equation for $h$. A study of these steady states is proposed in [35, 36] where some analysis of subcritical and supercritical solutions are proposed.

### 1.3 Purpose and organization

In the present work, we are concerned with the derivation of a numerical scheme to approximate the weak solutions of (1.1), which, in addition, accurately captures the steady state solutions of (1.10) and the asymptotic regime given by (1.7). From now on, let us recall that a scheme able to capture the steady states is said Well-Balanced while a scheme able to restore the asymptotic regimes is said Asymptotic Preserving. The well-balanced schemes were introduced in [7, 23] in the framework of the shallow-water model with non-flat topography. During the two last decades, numerous works were devoted to the derivation of well-balanced schemes (see [5, 21, 27, 37] for a non-exhaustive list). More recently, in [8, 35], a fully well-balanced Godunov-type scheme was introduced. The originality of these works stays in the incorporation of the source term in the approximate Riemann solver. This approach allows now to consider extensions to more general systems which include nonlinear source terms. In particular, in [36], a fully well-balanced scheme is derived to approximate the weak solutions of (1.1). Such a numerical method exactly captures the steady states governed by (1.10). Now we give some recalls concerning the Asymptotic Preserving behavior, to be satisfied by the numerical scheme. The derivation of a suitable discretization of (1.4) in order to get a consistent approximation of (1.7) in the limit of $\varepsilon$ to zero is essential to ensure an accurate approximation. After the pioneer work by Jin [28], several methods were developed to design a suitable numerical viscosity in order to restore the expected asymptotic diffusive regime (for instance, see [9, 10, 12–14, 22]). In [11], the authors introduced a technique to morph the numerical viscosity into the correct diffusive regime. This approach was extended in [20] in order to deal with the system (1.1) and the associated diffusive regime (1.7).

In order to design a numerical scheme with well-balanced and asymptotic preserving properties, we adopt the fully well-balanced Godunov-type scheme introduced in [36]. The objective is here to propose a relevant extension of this numerical method in order to enrich the scheme with the asymptotic preserving property. To address such an issue, the paper is organized as follows. In the next section, we give the main ingredients, as introduced in [36], to derive a fully well-balanced Godunov-type scheme. In fact, we establish the preservation of the steady states given by (1.10) up to the definition of an average operator. Afterwards, in Section 3, we exhibit relevant choices of the average operator to get the expected asymptotic behavior defined by (1.7). We state that the discretization given by the derived scheme is consistent with the limit problem (1.7) in the diffusive regime. Next, in Section 4, a higher-order in space version of the scheme is proposed. This scheme is built such that the steady states and the diffusive limit stay preserved. Next, in Section 5, we propose a discretization of the limit equation (1.7) in order to compare the discretizations in the asymptotic regime with it. Finally, a numerical assessment of the different features of the proposed schemes is given in Section 6.

## 2 A fully well-balanced Godunov-type scheme

This section is devoted to the presentation of the Godunov-type scheme adopted in [36]. It also allows to introduce notations we will use in the sequel. First, let us introduce a space discretization given by a uniform mesh made of cells $K_i = (x_{i-1/2}, x_{i+1/2})$, $i \in \mathbb{Z}$, of constant size $\Delta x$ and of center $x_i$. Concerning the time discretization, we denote by $\Delta t$ the time increment and we set $t^{n+1} = t^n + \Delta t$ for all $n \in \mathbb{N}$. Next, over each cell $K_i$, at time $t^n$, an approximation of the solution $W$ of (1.1), denoted by $(W_i^n)_{i\in\mathbb{Z}}$, is assumed to be known. At time $t^{n+1} = t^n + \Delta t$, we update this approximation as follows:

$$W_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^{\Delta}\left(x, t^n + \Delta t\right)\, dx, \tag{2.1}$$

where we have set

$$W^{\Delta}\left(x, t^n + t\right) = \widetilde{W}\left(\frac{x - x_{i+\frac{1}{2}}}{t}; W_i^n, W_{i+1}^n\right) \quad \text{if } x \in (x_i, x_{i+1}).$$

In the above update formula, $\widetilde{W}(x/t; W_\mathrm{L}, W_\mathrm{R})$ stands for a relevant approximate Riemann solver given as follows (see Figure 1):

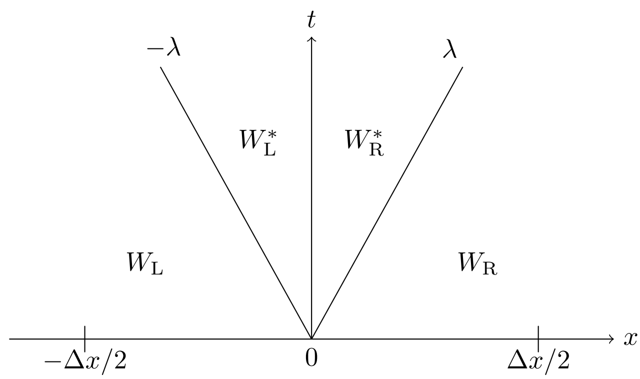



Figure 1: Structure of the approximate Riemann solver

$$\widetilde{W}\left(\frac{x}{t}; W_\mathrm{L}, W_\mathrm{R}\right) = \begin{cases} W_\mathrm{L}, & \text{if } \dfrac{x}{t} \le -\lambda, \\ W_\mathrm{L}^*, & \text{if } -\lambda < \dfrac{x}{t} \le 0, \\ W_\mathrm{R}^*, & \text{if } 0 < \dfrac{x}{t} \le \lambda, \\ W_\mathrm{R}, & \text{if } \lambda < \dfrac{x}{t}. \end{cases} \tag{2.2}$$

The two intermediate states $W_{L,R}^* = {}^t(h_{L,R}^*, q_{L,R}^*)$ have to be determined according to an integral consistency condition (see [8, 24]) given by

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widetilde{W}\left(\frac{x}{\Delta t}; W_\mathrm{L}, W_\mathrm{R}\right)\, dx = \frac{1}{2}\left(W_\mathrm{L} + W_\mathrm{R}\right) - \frac{\Delta t}{\Delta x}\left(F(W_\mathrm{R}) - F(W_\mathrm{L})\right) + \Delta t \overline{S}(W_\mathrm{L}, W_\mathrm{R}), \tag{2.3}$$

where $\overline{S}(W_\mathrm{L}, W_\mathrm{R})$ is a suitable approximation of the source term in (1.2). From now on, let us underline that the above integral consistency is valid as long as the exact wave speeds of (1.1) belong to $(-\lambda, \lambda)$. As a consequence, we impose

$$\lambda = \max\left(\frac{|q_\mathrm{L}|}{h_\mathrm{L}} + \sqrt{g h_\mathrm{L}}, \frac{|q_\mathrm{R}|}{h_\mathrm{R}} + \sqrt{g h_\mathrm{R}}\right). \tag{2.4}$$

In addition, we enforce a CFL condition as follows:

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\lambda_{i+\frac{1}{2}}\leq\frac{1}{2},$$

so that $W^\Delta(x,t^n+t)$ is nothing but a non-interacting juxtaposition of approximate Riemann solvers stated at each interface $x_{i+\frac{1}{2}}$ for all $i\in\mathbb{Z}$. Now, equipped with (2.3), the update state $W_i^{n+1}$, given by (2.1), easily rewrites as follows:

$$\begin{aligned}W_i^{n+1}=W_i^n&-\frac{\Delta t}{\Delta x}\left(F^\Delta(W_i^n,W_{i+1}^n)-F^\Delta(W_{i-1}^n,W_i^n)\right)\\&+\frac{\Delta t}{2}\left(\overline{S}(W_{i-1}^n,W_i^n)+\overline{S}(W_i^n,W_{i+1}^n)\right),\end{aligned}\tag{2.5}$$

where we have set

$$F^\Delta(W_\mathrm{L},W_\mathrm{R})=F(W_\mathrm{L})+\frac{\Delta x}{2\Delta t}W_\mathrm{L}+\frac{\Delta x}{2}\overline{S}(W_\mathrm{L},W_\mathrm{R})-\frac{1}{\Delta t}\int_{-\frac{\Delta x}{2}}^0\widetilde{W}\left(\frac{x}{\Delta t};W_\mathrm{L},W_\mathrm{R}\right)\,dx.$$

In addition to the above numerical scheme formulation, because of the form of the solver (2.2), $W_i^{n+1}$ also rewrites as follows:

$$W_i^{n+1}=W_i^n+\frac{\Delta t}{\Delta x}\left(\lambda_{i+\frac{1}{2}}\left(W_{i+\frac{1}{2}}^{L*}-W_i^n\right)+\lambda_{i-\frac{1}{2}}\left(W_{i-\frac{1}{2}}^{R*}-W_i^n\right)\right),\tag{2.6}$$

where $W_{i+\frac{1}{2}}^{L,R*}=W_{L,R}^*(W_i^n,W_{i+1}^n)$ are the intermediate states of the approximate Riemann solver $\widetilde{W}(x/t;W_i^n,W_{i+1}^n)$.

The description of the scheme is now achieved as soon as both intermediate states, $W_\mathrm{L}^*$ and $W_\mathrm{R}^*$, are relevantly defined. According to [36], we adopt

$$\begin{aligned}h_\mathrm{L}^*&=h_\mathrm{HLL}+\frac{k\overline{q}|\overline{q}|\overline{h^{-\eta}}\Delta x}{2\alpha},\\h_\mathrm{R}^*&=h_\mathrm{HLL}-\frac{k\overline{q}|\overline{q}|\overline{h^{-\eta}}\Delta x}{2\alpha},\\q_\mathrm{L}^*&=q_\mathrm{R}^*=q_\mathrm{HLL}-\frac{k\overline{q}|\overline{q}|\overline{h^{-\eta}}\Delta x}{2\lambda}=:q^*,\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}h_\mathrm{HLL}&=\frac{1}{2}\left(h_\mathrm{R}+h_\mathrm{L}\right)-\frac{1}{2\lambda}\left(q_\mathrm{R}-q_\mathrm{L}\right),\\q_\mathrm{HLL}&=\frac{1}{2}\left(q_\mathrm{R}+q_\mathrm{L}\right)-\frac{1}{2\lambda}\left(\frac{q_\mathrm{R}^2}{h_\mathrm{R}}+g\frac{h_\mathrm{R}^2}{2}-\frac{q_\mathrm{L}^2}{h_\mathrm{L}}-g\frac{h_\mathrm{L}^2}{2}\right).\end{aligned}$$

The parameter $\alpha$ involved in the definition of the intermediate water heights (2.7) is given by:

$$\alpha=-\frac{\overline{q}^2}{h_\mathrm{L}h_\mathrm{R}}+\frac{g}{2}\left(h_\mathrm{R}+h_\mathrm{L}\right).\tag{2.8}$$

As mentioned in [36], the quantity $\alpha$ can be equal to zero in rare transcritical cases, namely when the Froude number is equal to one. The definition (2.7) is then ill posed but this case does not occur in simulations performed. Moreover, discharges considered in the diffusive regime vanish and we have $\alpha > 0$ at the asymptotic limit.

We have skipped all the details of the computations, but after [36], the choice of the intermediate states (2.7) satisfies the integral consistency condition (2.3) when the source term average reads

$$\overline{S} = \begin{pmatrix} 0 \\ -k\overline{q}|\overline{q}|\overline{h^{-\eta}} \end{pmatrix}.$$

In fact, the main difficulty now is to propose a relevant definition of the averages $\overline{q}$ and $\overline{h^{-\eta}}$ such that the steady states are preserved. First, let us set

$$\overline{h^{-\eta}} = \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]} - \frac{\operatorname{sign}(\overline{q})}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]}\frac{[h^{\eta-1}]}{\eta-1}\right), \tag{2.9}$$

with the notation $[X] = X_{\mathrm{R}} - X_{\mathrm{L}}$. From now on, we emphasize that $\overline{h^{-\eta}}$ is nothing but an approximation of $h^{-\eta}$ at the interface. Concerning the average $\overline{q} = \overline{q}(q_{\mathrm{L}}, q_{\mathrm{R}})$, we only impose a consistency condition given by

$$\overline{q}(q,q) = q \quad \forall\, q \in \mathbb{R}. \tag{2.10}$$

According to [36, Lemma 3 page 125], the adopted approximate Riemann solver preserves all the steady states and, as a consequence, the scheme (2.6) is fully well-balanced. We refer the reader to the mentioned paper for the proof.

Now, for the sake of clarity in the forthcoming developments, we present the extended flux formulation of the scheme (2.6), according to (2.5). From (2.6), we immediately have

$$\begin{aligned} h_i^{n+1} &= h_i^n + \frac{\Delta t}{\Delta x}\left(\lambda_{i+\frac12}\left(h^{L*}_{i+\frac12} - h_i^n\right) + \lambda_{i-\frac12}\left(h^{R*}_{i-\frac12} - h_i^n\right)\right),, \\ q_i^{n+1} &= q_i^n + \frac{\Delta t}{\Delta x}\left(\lambda_{i+\frac12}\left(q^{L*}_{i+\frac12} - q_i^n\right) + \lambda_{i-\frac12}\left(q^{R*}_{i-\frac12} - q_i^n\right)\right). \end{aligned}$$

Injecting the definition of the intermediate states (2.7), tedious computations allow to rewrite (2.6) as follows:

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{\Delta t}{2\Delta x}\left(\lambda_{i+\frac12}\left(h_{i+1}^n - h_i^n\right) - \lambda_{i-\frac12}\left(h_i^n - h_{i-1}^n\right)\right) \\ &- \frac{\Delta t}{2\Delta x}\left(q_{i+1}^n - q_{i-1}^n\right) \\ &+ \frac{k\Delta t}{2}\left(\lambda_{i+\frac12}\frac{\overline{q}_{i+\frac12}|\overline{q}_{i+\frac12}|\overline{h^{-\eta}}_{i+\frac12}}{\alpha_{i+\frac12}} - \lambda_{i-\frac12}\frac{\overline{q}_{i-\frac12}|\overline{q}_{i-\frac12}|\overline{h^{-\eta}}_{i-\frac12}}{\alpha_{i-\frac12}}\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} q_i^{n+1} = q_i^n &+ \frac{\Delta t}{2\Delta x}\left(\lambda_{i+\frac12}\left(q_{i+1}^n - q_i^n\right) - \lambda_{i-\frac12}\left(q_i^n - q_{i-1}^n\right)\right) \\ &- \frac{\Delta t}{2\Delta x}\left(\frac{(q_{i+1}^n)^2}{h_{i+1}^n} + g\frac{(h_{i+1}^n)^2}{2} - \frac{(q_{i-1}^n)^2}{h_{i-1}^n} - g\frac{(h_{i-1}^n)^2}{2}\right) \\ &- \frac{k\Delta t}{2}\left(\overline{q}_{i+\frac12}|\overline{q}_{i+\frac12}|\overline{h^{-\eta}}_{i+\frac12} + \overline{q}_{i-\frac12}|\overline{q}_{i-\frac12}|\overline{h^{-\eta}}_{i-\frac12}\right). \end{aligned} \tag{2.12}$$

We immediately notice that the above discretization is consistent with the initial system (1.1). A detailed proof of this consistency, based on the flux formulation of the scheme

(2.5), is established in [36].

It is worth noticing that the derived scheme is free from the definition of the average $\overline{q}$ under the consistency condition (2.10). By adopting additional conditions to be satisfied by $\overline{q}$, we now enforce the expected asymptotic preserving property.

# 3 Preservation of the diffusive limit

By introducing a suitable definition of the discharge average $\overline{q}$ involved in the approximate Riemann solver (2.2), (2.7), we now show that the asymptotic diffusive regime, given by (1.7), can be recovered by the scheme (2.6). In order to model a long time and dominant friction, we introduce the following scaling (see also (1.3)):

$$\Delta t \leftarrow \Delta t/\varepsilon \quad \text{and} \quad k \leftarrow k/\varepsilon^2.$$

Injecting the above rescaling in (2.11)-(2.12), we directly obtain

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{\Delta t}{2\varepsilon\Delta x}\left(\lambda_{i+\frac{1}{2}}\left(h_{i+1}^n - h_i^n\right) - \lambda_{i-\frac{1}{2}}\left(h_i^n - h_{i-1}^n\right)\right) \\ &- \frac{\Delta t}{2\varepsilon\Delta x}\left(q_{i+1}^n - q_{i-1}^n\right) \\ &+ \frac{k\Delta t}{2\varepsilon^3}\left(\lambda_{i+\frac{1}{2}}\frac{\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|\overline{h^{-\eta}}_{i+\frac{1}{2}}}{\alpha_{i+\frac{1}{2}}} - \lambda_{i-\frac{1}{2}}\frac{\overline{q}_{i-\frac{1}{2}}|\overline{q}_{i-\frac{1}{2}}|\overline{h^{-\eta}}_{i-\frac{1}{2}}}{\alpha_{i-\frac{1}{2}}}\right), \end{aligned} \tag{3.1}$$

$$\begin{aligned} q_i^{n+1} = q_i^n &+ \frac{\Delta t}{2\varepsilon\Delta x}\left(\lambda_{i+\frac{1}{2}}\left(q_{i+1}^n - q_i^n\right) - \lambda_{i-\frac{1}{2}}\left(q_i^n - q_{i-1}^n\right)\right) \\ &- \frac{\Delta t}{2\varepsilon\Delta x}\left(\frac{(q_{i+1}^n)^2}{h_{i+1}^n} + g\frac{(h_{i+1}^n)^2}{2} - \frac{(q_{i-1}^n)^2}{h_{i-1}^n} - g\frac{(h_{i-1}^n)^2}{2}\right) \\ &- \frac{k\Delta t}{2\varepsilon^3}\left(\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|\overline{h^{-\eta}}_{i+\frac{1}{2}} + \overline{q}_{i-\frac{1}{2}}|\overline{q}_{i-\frac{1}{2}}|\overline{h^{-\eta}}_{i-\frac{1}{2}}\right), \end{aligned} \tag{3.2}$$

where $\overline{h^{-\eta}}_{i+\frac{1}{2}}$ is now defined by:

$$\overline{h^{-\eta}}_{i+\frac{1}{2}} = \frac{[h^2]_{i+\frac{1}{2}}}{2}\frac{\eta+2}{[h^{\eta+2}]_{i+\frac{1}{2}}} - \frac{\varepsilon^2 \mathrm{sign}(\overline{q}_{i+\frac{1}{2}})}{k\Delta x}\left(\left[\frac{1}{h}\right]_{i+\frac{1}{2}} + \frac{[h^2]_{i+\frac{1}{2}}}{2}\frac{\eta+2}{[h^{\eta+2}]_{i+\frac{1}{2}}}\frac{[h^{\eta-1}]_{i+\frac{1}{2}}}{\eta-1}\right), \tag{3.3}$$

with $[X]_{i+\frac{1}{2}} = X_{i+1}^n - X_i^n$. In addition, $\alpha_{i+\frac{1}{2}} = \alpha(W_i^n, W_{i+1}^n)$ is given by (2.8). The discharge average $\overline{q}_{i+\frac{1}{2}} := \overline{q}(q_i^n, q_{i+1}^n)$ must satisfy (2.10).

Note that the above scheme is restricted by a Courant-Friedrichs-Lewy stability condition depending on $\varepsilon$ of the following form:

$$\Lambda_i^n \frac{\Delta t}{\Delta x} \leq \frac{\varepsilon}{2},$$

where

$$\Lambda_i^n = \max_i\left(\frac{|q_i^n|}{h_i^n} + \sqrt{gh_i^n}\right),$$

at each time step. This condition does not prevent the scheme from being consistent in the limit $\varepsilon \to 0$ as it will be shown in Theorem 3.1.

Now, we introduce the condition required on the average $\overline{q}(q_{\mathrm{L}}, q_{\mathrm{R}})$ in addition to to (2.10) to preserve the diffusive limit:

$$\overline{q}(q_{\mathrm{L}}, q_{\mathrm{R}}) = 0 \Leftrightarrow q_{\mathrm{L}} = q_{\mathrm{R}} = 0. \tag{3.4}$$

For instance, in this work we adopt the following definition, satisfying simultaneously the consistency condition (2.10) and the discharge vanishing restriction (3.4):

$$\overline{q} = \operatorname{sign}(q_{\mathrm{L}} + q_{\mathrm{R}})\sqrt{\frac{1}{2}\,(q_{\mathrm{L}}^2 + q_{\mathrm{R}}^2)}. \tag{3.5}$$

Now, with the suitable definition (3.5) of $\overline{q}$, we establish that the rescaled scheme (3.1)-(3.2) preserves the asymptotic regime governed by (1.7).

**Theorem 3.1.** *Assume the wave speeds $(\lambda_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ to be defined such that:*

$$\lambda_{i+\frac{1}{2}}\alpha_{i-\frac{1}{2}} + \lambda_{i-\frac{1}{2}}\alpha_{i+\frac{1}{2}} \neq 0 \quad \forall i \in \mathbb{Z}. \tag{3.6}$$

*Let us adopt a discharge average $\overline{q}_{i+\frac{1}{2}}$ given by (3.5). Then, when $\varepsilon$ tends to zero, the discretizations given by the scheme (3.1)-(3.2) fulfill*

$$h_i^n = h_i^{n(0)} + \mathcal{O}(\varepsilon) \quad \textit{and} \quad q_i^n = \varepsilon q_i^{n(1)} + \mathcal{O}(\varepsilon^2) \quad \forall i \in \mathbb{Z}, \forall n \in \mathbb{N},$$

$$h_i^{n+1(0)} = h_i^{n(0)} - \frac{\Delta t}{2\Delta x}\left(q_{i+1}^{n(1)} - q_{i-1}^{n(1)}\right) + \mathcal{O}(\Delta t \Delta x), \tag{3.7}$$

$$\frac{\frac{g}{2}(h_{i+1}^{n(0)})^2 - \frac{g}{2}(h_{i-1}^{n(0)})^2}{2\Delta x} = -\frac{k}{2}\left(\widetilde{q|q|}_{i+\frac{1}{2}}^{(2)}\overline{h^{-\eta}}_{i+\frac{1}{2}}^{(0)} + \widetilde{q|q|}_{i-\frac{1}{2}}^{(2)}\overline{h^{-\eta}}_{i-\frac{1}{2}}^{(0)}\right), \tag{3.8}$$

*where*

$$\overline{h^{-\eta}}_{i+\frac{1}{2}}^{(0)} = \frac{(h_{i+1}^{n(0)})^2 - (h_i^{n(0)})^2}{2}\,\frac{\eta+2}{(h_{i+1}^{n(0)})^{\eta+2} - (h_i^{n(0)})^{\eta+2}}, \tag{3.9}$$

$$\widetilde{q|q|}_{i+\frac{1}{2}}^{(2)} = \operatorname{sign}\left(q_i^{n(1)} + q_{i+1}^{n(1)}\right)\frac{1}{2}\left((q_i^{n(1)})^2 + (q_{i+1}^{n(1)})^2\right). \tag{3.10}$$

From the above theorem, the limit scheme (3.7)-(3.8) is consistent with the diffusive limit system (1.7). From now on let us underline that, in all the numerical simulations carried out, the choice (2.4) for the wave speeds $(\lambda_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$ satisfies the assumption (3.6).

*Proof.* The sketch of the proof follows the following lines.

- As usual, we perform a Chapman-Enskog expansion of the terms involved in the scheme with respect to $\varepsilon$.
- We identify the zero and first-order terms in $\varepsilon$ within the scheme.

The zero-order water height is then shown to satisfy the expected asymptotic diffusion equation (3.7) while the zero-order discharge vanishes and its first-order satisfies the local equilibrium (3.8).

- *Zero-order Chapman-Enskog expansions.*

We adopt asymptotic expansions as follows:

$$h_i^n = h_i^{n(0)} + \mathcal{O}(\varepsilon) \quad \text{and} \quad q_i^n = q_i^{n(0)} + \mathcal{O}(\varepsilon), \tag{3.11}$$

where we assume $h_i^{n(0)} > 0$. First, we introduce the following notation for $\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|$:

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|.$$

Injecting the asymptotic expansions (3.11) in the definition (3.5) of $\overline{q}_{i+\frac{1}{2}}$, we can write

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \text{sign}\left(q_i^{n(0)} + q_{i+1}^{n(0)} + \mathcal{O}(\varepsilon)\right)\frac{1}{2}\left((q_i^{n(0)})^2 + (q_{i+1}^{n(0)})^2 + \mathcal{O}(\varepsilon)\right).$$

Moreover, since for $\varepsilon$ small enough we have

$$\text{sign}(q_i^{n(0)} + q_{i+1}^{n(0)} + \mathcal{O}(\varepsilon)) = \begin{cases} \text{sign}(q_i^{n(0)} + q_{i+1}^{n(0)}) & \text{if} \quad q_i^{n(0)} \neq 0 \quad \text{or} \quad q_{i+1}^{n(0)} \neq 0, \\ \text{sign}(\mathcal{O}(\varepsilon)) & \text{if} \quad q_i^{n(0)} = q_{i+1}^{n(0)} = 0, \end{cases}$$

then, setting

$$\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} = \text{sign}\left(q_i^{n(0)} + q_{i+1}^{n(0)}\right)\frac{1}{2}\left((q_i^{n(0)})^2 + (q_{i+1}^{n(0)})^2\right), \tag{3.12}$$

the asymptotic expansion of $\widetilde{q|q|}_{i+\frac{1}{2}}$ for all $i \in \mathbb{Z}$ is

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon). \tag{3.13}$$

Now, injecting the asymptotic expansions (3.11) in the definition (3.3) of $\overline{h^{-\eta}}_{i+\frac{1}{2}}$, we can write

$$\begin{aligned} \overline{h^{-\eta}}_{i+\frac{1}{2}} =& \frac{\eta+2}{2}\frac{(h_{i+1}^{n(0)})^2 - (h_i^{n(0)})^2 + \mathcal{O}(\varepsilon)}{(h_{i+1}^{n(0)})^{\eta+2} - (h_i^{n(0)})^{\eta+2} + \mathcal{O}(\varepsilon)} \\ & - \frac{\varepsilon^2 \text{sign}(\overline{q}_{i+\frac{1}{2}})}{k\Delta x}\left(\frac{1}{h_{i+1}^{n(0)} + \mathcal{O}(\varepsilon)} - \frac{1}{h_i^{n(0)} + \mathcal{O}(\varepsilon)}\right. \\ & \left. + \frac{(\eta+2)}{2(\eta-1)}\frac{\left((h_{i+1}^{n(0)})^2 - (h_i^{n(0)})^2 + \mathcal{O}(\varepsilon)\right)\left((h_{i+1}^{n(0)})^{\eta-1} - (h_i^{n(0)})^{\eta-1} + \mathcal{O}(\varepsilon)\right)}{(h_{i+1}^{n(0)})^{\eta+2} - (h_i^{n(0)})^{\eta+2} + \mathcal{O}(\varepsilon)}\right), \end{aligned}$$

which give the following asymptotic expansion for $\overline{h^{-\eta}}_{i+\frac{1}{2}}$ for all $i \in \mathbb{Z}$

$$\overline{h^{-\eta}}_{i+\frac{1}{2}} = \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon),$$

with $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}$ given by (3.9). Moreover, injecting the asymptotic expansions (3.11) and (3.13) in the definition (2.8) of $\alpha_{i+\frac{1}{2}}$, we can write

$$\alpha_{i+\frac{1}{2}} = -\frac{|\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}}| + \mathcal{O}(\varepsilon)}{h_i^{n(0)}h_{i+1}^{n(0)} + \mathcal{O}(\varepsilon)} + \frac{g}{2}\left(h_i^{n(0)} + h_{i+1}^{n(0)} + \mathcal{O}(\varepsilon)\right), \tag{3.14}$$

and we obtain the following asymptotic expansion for $\alpha_{i+\frac{1}{2}}$ for all $i \in \mathbb{Z}$

$$\alpha_{i+\frac{1}{2}} = \alpha^{(0)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon), \tag{3.15}$$

where

$$\alpha^{(0)}_{i+\frac{1}{2}} = -\frac{(q_i^{n(0)})^2 + (q_{i+1}^{n(0)})^2}{2h_i^{n(0)} h_{i+1}^{n(0)}} + \frac{g}{2}\left(h_i^{n(0)} + h_{i+1}^{n(0)}\right). \tag{3.16}$$

We underline that, as for the definition of $\overline{h^{-\eta}}_{i+\frac{1}{2}}$ given by (3.3), $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}$ is an approximation of $(h^{(0)})^{-\eta}$ at the interface $x_{i+\frac{1}{2}}$. In particular, we have $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} = (h_i^{n(0)})^{-\eta}$ when $h_i^{n(0)} = h_{i+1}^{n(0)}$. We thus have that the quantity $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}$ is positive.

- *Identification of the zero-order discharge.*

Multiplying both equations of the hyperbolic scheme (3.1) and (3.2) by $\varepsilon^3$, we easily get

$$\begin{cases} \mathcal{O}(\varepsilon) = \dfrac{k\Delta t}{2}\left(\lambda_{i+\frac{1}{2}} \dfrac{\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}}{\alpha^{(0)}_{i+\frac{1}{2}}} - \lambda_{i-\frac{1}{2}} \dfrac{\widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}}}{\alpha^{(0)}_{i-\frac{1}{2}}}\right), \\ \mathcal{O}(\varepsilon) = -\dfrac{k\Delta t}{2}\left(\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}}\right), \end{cases}$$

and thus, in the limit of $\varepsilon$ to zero, the two following relations hold:

$$\begin{cases} \lambda_{i+\frac{1}{2}} \dfrac{\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}}{\alpha^{(0)}_{i+\frac{1}{2}}} - \lambda_{i-\frac{1}{2}} \dfrac{\widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}}}{\alpha^{(0)}_{i-\frac{1}{2}}} = 0, \\ \widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}} = 0. \end{cases} \tag{3.17}$$

Combining the two above equations, (3.17) rewrites:

$$\begin{cases} \left(\dfrac{\lambda_{i+\frac{1}{2}}}{\alpha^{(0)}_{i+\frac{1}{2}}} + \dfrac{\lambda_{i-\frac{1}{2}}}{\alpha^{(0)}_{i-\frac{1}{2}}}\right) \widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} = 0, \\ \widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}} = -\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}. \end{cases}$$

Using the assumption (3.6), the above expressions imply:

$$\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} = \widetilde{q|q|}^{(0)}_{i-\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}} = 0.$$

Since $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} > 0$ for all $i \in \mathbb{Z}$, the above equalities thus give $\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}} = 0$ for all $i \in \mathbb{Z}$.

Next, from the definition (3.12) of $\widetilde{q|q|}^{(0)}_{i+\frac{1}{2}}$, we immediately obtain that $q_i^{n(0)} = 0$ for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$.

• *Zero-order water height and first-order discharge expansions.*
As a consequence, the expansions (3.11) now rewrite:

$$h_i^n = h_i^{n(0)} + \mathcal{O}(\varepsilon) \quad \text{and} \quad q_i^n = \varepsilon\left(q_i^{n(1)} + \mathcal{O}(\varepsilon)\right). \tag{3.18}$$

Injecting the new expansions (3.18) in the definition (3.5) of $\overline{q}_{i+\frac{1}{2}}$, the development of $\widetilde{q|q|}_{i+\frac{1}{2}}$ is now given by:

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \operatorname{sign}\left(q_i^{n(1)} + q_{i+1}^{n(1)} + \mathcal{O}(\varepsilon)\right)\frac{\varepsilon^2}{2}\left((q_i^{n(1)})^2 + (q_{i+1}^{n(1)})^2 + \mathcal{O}(\varepsilon)\right).$$

Moreover, since for $\varepsilon$ small enough we have

$$\operatorname{sign}(q_i^{n(1)} + q_{i+1}^{n(1)} + \mathcal{O}(\varepsilon)) = \begin{cases} \operatorname{sign}\left(q_i^{n(1)} + q_{i+1}^{n(1)}\right) & \text{if} \quad q_i^{n(1)} \neq 0 \quad \text{or} \quad q_{i+1}^{n(1)} \neq 0, \\ \operatorname{sign}(\mathcal{O}(\varepsilon)) \quad \text{if} \quad q_i^{n(1)} = q_{i+1}^{n(1)} = 0, \end{cases}$$

then, the asymptotic expansion of $\widetilde{q|q|}_{i+\frac{1}{2}}$ for all $i \in \mathbb{Z}$ is

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \varepsilon^2\left(\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon)\right),$$

with $\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}}$ defined by (3.10). Moreover, the quantity $\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|\overline{h^{-\eta}}_{i+\frac{1}{2}}$ admits the following expansion:

$$\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|\overline{h^{-\eta}}_{i+\frac{1}{2}} = \varepsilon^2\left(\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}}\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon)\right).$$

We underline that the development of $\alpha_{i+\frac{1}{2}}$ is still given by (3.15) but, since $q_i^{n(0)} = 0$ for all $i \in \mathbb{Z}$, the definition (3.16) of $\alpha^{(0)}_{i+\frac{1}{2}}$ now reads:

$$\alpha^{(0)}_{i+\frac{1}{2}} = \frac{g}{2}\left(h_i^{n(0)} + h_{i+1}^{n(0)}\right). \tag{3.19}$$

• *Identification of the diffusion regime.*
Multiplying the second equation of the hyperbolic scheme (3.2) by $\varepsilon$, we obtain:

$$\begin{aligned} \mathcal{O}(\varepsilon) = &- \frac{\Delta t}{2\Delta x}\left(g\frac{(h_{i+1}^{n(0)})^2}{2} - g\frac{(h_{i-1}^{n(0)})^2}{2}\right) \\ &- \frac{k\Delta t}{2}\left(\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}}\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(2)}_{i-\frac{1}{2}}\overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}}\right), \end{aligned}$$

and the limit of this equation when $\varepsilon$ tends to zero is given by:

$$\frac{\frac{g}{2}(h_{i+1}^{n(0)})^2 - \frac{g}{2}(h_{i-1}^{n(0)})^2}{2\Delta x} = -\frac{k}{2}\left(\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}}\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(2)}_{i-\frac{1}{2}}\overline{h^{-\eta}}^{(0)}_{i-\frac{1}{2}}\right).$$

We recognize the expected limit equation (3.8) and the local equilibrium (1.8) is thus preserved. Now the proof is achieved as soon as (3.7) is established.

Multiplying both equations of the hyperbolic scheme (3.1) and (3.2) by $\varepsilon$, we obtain:

$$\left\{\begin{aligned} \mathcal{O}(\varepsilon) = & \frac{\Delta t}{2\Delta x}\left(\lambda_{i+\frac12}\left(h_{i+1}^{n(0)}-h_i^{n(0)}\right)-\lambda_{i-\frac12}\left(h_i^{n(0)}-h_{i-1}^{n(0)}\right)\right)\\ & + \frac{k\Delta t}{2}\left(\lambda_{i+\frac12}\frac{\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}}{\alpha_{i+\frac12}^{(0)}}-\lambda_{i-\frac12}\frac{\widetilde{q|q|}_{i-\frac12}^{(2)}\overline{h^{-\eta}}_{i-\frac12}^{(0)}}{\alpha_{i-\frac12}^{(0)}}\right),\\ \mathcal{O}(\varepsilon) = & -\frac{\Delta t}{2\Delta x}\left(g\frac{(h_{i+1}^{n(0)})^2}{2}-g\frac{(h_{i-1}^{n(0)})^2}{2}\right)\\ & -\frac{k\Delta t}{2}\left(\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}+\widetilde{q|q|}_{i-\frac12}^{(2)}\overline{h^{-\eta}}_{i-\frac12}^{(0)}\right),\end{aligned}\right.$$

to get, in the limit of $\varepsilon$ to zero, the following relations:

$$\left\{\begin{aligned} & \lambda_{i+\frac12}\left(\frac{h_{i+1}^{n(0)}-h_i^{n(0)}}{\Delta x}+k\frac{\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}}{\alpha_{i+\frac12}^{(0)}}\right)\\ = & \lambda_{i-\frac12}\left(\frac{h_{i}^{n(0)}-h_{i-1}^{n(0)}}{\Delta x}+k\frac{\widetilde{q|q|}_{i-\frac12}^{(2)}\overline{h^{-\eta}}_{i-\frac12}^{(0)}}{\alpha_{i-\frac12}^{(0)}}\right),\\ & \frac{\frac g2(h_{i+1}^{n(0)})^2-\frac g2(h_{i-1}^{n(0)})^2}{2\Delta x}=-\frac k2\left(\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}+\widetilde{q|q|}_{i-\frac12}^{(2)}\overline{h^{-\eta}}_{i-\frac12}^{(0)}\right).\end{aligned}\right. \tag{3.20}$$

Using the definition of $\alpha_{i+\frac12}^{(0)}$ given by (3.19), the two equations of the above system can be combined together to write:

$$\left(\frac{\lambda_{i+\frac12}}{\alpha_{i+\frac12}^{(0)}}+\frac{\lambda_{i-\frac12}}{\alpha_{i-\frac12}^{(0)}}\right)\left(\frac g2\frac{(h_{i+1}^{n(0)})^2-(h_i^{n(0)})^2}{\Delta x}+k\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}\right)=0.$$

Since, for all $i\in\mathbb{Z}$, $\alpha_{i+\frac12}^{(0)}$ and $\lambda_{i+\frac12}$ are positive quantities then the above expression rewrites:

$$\frac g2\frac{(h_{i+1}^{n(0)})^2-(h_i^{n(0)})^2}{\Delta x}+k\widetilde{q|q|}_{i+\frac12}^{(2)}\overline{h^{-\eta}}_{i+\frac12}^{(0)}=0 \quad \forall i\in\mathbb{Z}, n\in\mathbb{N}. \tag{3.21}$$

Now, in order to exhibit the expected relation (3.7), we have to study the behavior of the order one of the water height and the order two of the discharge:

$$h_i^n=h_i^{n(0)}+\varepsilon h_i^{n(1)}+\mathcal{O}(\varepsilon^2) \quad \text{and} \quad q_i^n=\varepsilon\left(q_i^{n(1)}+\varepsilon q_i^{n(2)}+\mathcal{O}(\varepsilon^2)\right). \tag{3.22}$$

Injecting the asymptotic expansions (3.22) in the definition (3.5) of $\overline{q}$, we can write

$$\begin{aligned}\widetilde{q|q|}_{i+\frac12} = & \operatorname{sign}\left(q_i^{n(1)}+q_{i+1}^{n(1)}+\varepsilon\left(q_i^{n(2)}+q_{i+1}^{n(2)}\right)+\mathcal{O}(\varepsilon^2)\right)\\ & \times\frac{\varepsilon^2}{2}\left(\left(q_i^{n(1)}+\varepsilon q_i^{n(2)}+\mathcal{O}(\varepsilon^2)\right)^2+\left(q_{i+1}^{n(1)}+\varepsilon q_{i+1}^{n(2)}+\mathcal{O}(\varepsilon^2)\right)^2\right).\end{aligned}$$

After some computations, we obtain the following expansion for $\widetilde{q|q|}_{i+\frac{1}{2}}$

$$\widetilde{q|q|}_{i+\frac{1}{2}} = \varepsilon^2 \left( \widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(3)}_{i+\frac{1}{2}} \varepsilon + \mathcal{O}(\varepsilon^2) \right), \tag{3.23}$$

where $\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}}$ is given by (3.10) and

$$\widetilde{q|q|}^{(3)}_{i+\frac{1}{2}} = \operatorname{sign}\left( q_i^{n(1)} + q_{i+1}^{n(1)} \right) \left( q_i^{n(1)} q_i^{n(2)} + q_{i+1}^{n(1)} q_{i+1}^{n(2)} \right).$$

Moreover, injecting expansions (3.22) in the definition (3.3) of $\overline{h^{-\eta}}_{i+\frac{1}{2}}$, some tedious computations give the following development for $\overline{h^{-\eta}}_{i+\frac{1}{2}}$ :

$$\overline{h^{-\eta}}_{i+\frac{1}{2}} = \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \varepsilon \overline{h^{-\eta}}^{(1)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon^2), \tag{3.24}$$

with $\overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}$ given by (3.9) and with

$$\overline{h^{-\eta}}^{(1)}_{i+\frac{1}{2}} = \begin{cases} \dfrac{\eta+2}{2} \dfrac{(h_{i+1}^{n(0)})^2 - (h_i^{n(0)})^2}{(h_{i+1}^{n(0)})^{\eta+2} - (h_i^{n(0)})^{\eta+2}} \left( 2 \dfrac{h_{i+1}^{n(0)} h_{i+1}^{n(1)} - h_i^{n(0)} h_i^{n(1)}}{(h_{i+1}^{n(0)})^2 - (h_i^{n(0)})^2} \right. \\ \qquad \left. -(\eta+2) \dfrac{(h_{i+1}^{n(0)})^{\eta+1} h_{i+1}^{n(1)} - (h_i^{n(0)})^{\eta+1} h_i^{n(1)}}{(h_{i+1}^{n(0)})^{\eta+2} - (h_i^{n(0)})^{\eta+2}} \right) & \text{if} \quad h_i^{n(0)} \neq h_{i+1}^{n(0)}, \\ -\eta \dfrac{h_i^{n(1)} + h_{i+1}^{n(1)}}{2} (h_i^{n(0)})^{-\eta-1} & \text{if} \quad h_i^{n(0)} = h_{i+1}^{n(0)}. \end{cases}$$

Moreover, injecting the asymptotic expansions (3.22) in the definition (2.8) of $\alpha_{i+\frac{1}{2}}$, we get

$$\alpha_{i+\frac{1}{2}} = \mathcal{O}(\varepsilon^2) + \frac{g}{2} \left( h_i^{n(0)} + \varepsilon h_i^{n(1)} + h_{i+1}^{n(0)} + \varepsilon h_{i+1}^{n(1)} + \mathcal{O}(\varepsilon^2) \right),$$

and we obtain the following expansion for $\alpha_{i+\frac{1}{2}}$:

$$\alpha_{i+\frac{1}{2}} = \alpha^{(0)}_{i+\frac{1}{2}} + \varepsilon \alpha^{(1)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon^2), \tag{3.25}$$

where $\alpha^{(0)}$ is given by (3.19) and

$$\alpha^{(1)}_{i+\frac{1}{2}} = \frac{g}{2} \left( h_i^{n(1)} + h_{i+1}^{n(1)} \right).$$

Now, for the sake of simplicity in the notations, let us introduce

$$\beta_{i+\frac{1}{2}} = \frac{\widetilde{q|q|}_{i+\frac{1}{2}} \overline{h^{-\eta}}_{i+\frac{1}{2}}}{\alpha_{i+\frac{1}{2}}}.$$

Using asymptotic expansions of $\widetilde{q|q|}_{i+\frac{1}{2}}$, $\overline{h^{-\eta}}_{i+\frac{1}{2}}$ and $\alpha_{i+\frac{1}{2}}$ respectively given by (3.23), (3.24) and (3.25),this quantity, involved in the first equation of the scheme (3.1), admits the following expansion:

$$\beta_{i+\frac{1}{2}} = \varepsilon^2 \left( \beta^{(2)}_{i+\frac{1}{2}} + \varepsilon \beta^{(3)}_{i+\frac{1}{2}} + \mathcal{O}(\varepsilon^2) \right), \tag{3.26}$$

where

$$\beta^{(2)}_{i+\frac{1}{2}} = \frac{\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}}}{\alpha^{(0)}_{i+\frac{1}{2}}}, \tag{3.27}$$

$$\beta^{(3)}_{i+\frac{1}{2}} = -\frac{\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} \alpha^{(1)}_{i+\frac{1}{2}}}{(\alpha^{(0)}_{i+\frac{1}{2}})^2} + \frac{\widetilde{q|q|}^{(3)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} + \widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(1)}_{i+\frac{1}{2}}}{\alpha^{(0)}_{i+\frac{1}{2}}}. \tag{3.28}$$

From the definition (3.27) of $\beta^{(2)}$, we have

$$\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} = \alpha^{(0)}_{i+\frac{1}{2}} \beta^{(2)}_{i+\frac{1}{2}},$$

or equivalently, using the definition (3.19) of $\alpha^{(0)}$:

$$\widetilde{q|q|}^{(2)}_{i+\frac{1}{2}} \overline{h^{-\eta}}^{(0)}_{i+\frac{1}{2}} = \frac{g}{2}\left(h_i^{n(0)} + h_{i+1}^{n(0)}\right) \beta^{(2)}_{i+\frac{1}{2}}.$$

The relation (3.21) can then be rewritten as follows

$$\frac{g}{2}\left(h_i^{n(0)} + h_{i+1}^{n(0)}\right)\left(\frac{h_{i+1}^{n(0)} - h_i^{n(0)}}{\Delta x} + k\beta^{(2)}_{i+\frac{1}{2}}\right) = 0 \quad \forall i \in \mathbb{Z}, n \in \mathbb{N},$$

or equivalently, since we assume $h_i^{n(0)} > 0$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$

$$\frac{h_{i+1}^{n(0)} - h_i^{n(0)}}{\Delta x} + k\beta^{(2)}_{i+\frac{1}{2}} = 0 \quad \forall i \in \mathbb{Z}, n \in \mathbb{N}. \tag{3.29}$$

Moreover, using expansions (3.22) and (3.26), the first equation of the scheme (3.1) writes:

$$\begin{aligned} h_i^{n+1(0)} = h_i^{n(0)} &+ \frac{\Delta t}{2\varepsilon}\left(\lambda_{i+\frac{1}{2}}\left(\frac{h_{i+1}^{n(0)} - h_i^{n(0)}}{\Delta x} + k\beta^{(2)}_{i+\frac{1}{2}}\right) - \lambda_{i-\frac{1}{2}}\left(\frac{h_i^{n(0)} - h_{i-1}^{n(0)}}{\Delta x} + k\beta^{(2)}_{i-\frac{1}{2}}\right)\right) \\ &+ \frac{\Delta t}{2\Delta x}\left(\lambda_{i+\frac{1}{2}}\left(h_{i+1}^{n(1)} - h_i^{n(1)}\right) - \lambda_{i-\frac{1}{2}}\left(h_i^{n(1)} - h_{i-1}^{n(1)}\right)\right) \\ &+ \frac{k\Delta t}{2}\left(\lambda_{i+\frac{1}{2}}\beta^{(3)}_{i+\frac{1}{2}} - \lambda_{i-\frac{1}{2}}\beta^{(3)}_{i-\frac{1}{2}}\right) - \frac{\Delta t}{2\Delta x}\left(q_{i+1}^{n(1)} - q_{i-1}^{n(1)}\right) + \mathcal{O}(\varepsilon), \end{aligned} \tag{3.30}$$

where the second term of the right-hand side cancels due to relation (3.29). Moreover, we remark that

$$\begin{aligned} &\frac{\Delta t}{2\Delta x}\left(\lambda_{i+\frac{1}{2}}\left(h_{i+1}^{n(1)} - h_i^{n(1)}\right) - \lambda_{i-\frac{1}{2}}\left(h_i^{n(1)} - h_{i-1}^{n(1)}\right)\right) = \mathcal{O}(\Delta t \Delta x), \\ &\frac{k\Delta t}{2}\left(\lambda_{i+\frac{1}{2}}\beta^{(3)}_{i+\frac{1}{2}} - \lambda_{i-\frac{1}{2}}\beta^{(3)}_{i-\frac{1}{2}}\right) = O(\Delta t \Delta x). \end{aligned} \tag{3.31}$$

Injecting (3.29) and (3.31) in the equation (3.30), we recover the expected asymptotic behavior (3.7) when $\varepsilon$ tends to zero. The proof is achieved. □

It is worth noticing that the limit scheme associated to (3.1)-(3.2) cannot be written in an explicit form . Indeed, some numerical viscosity involved in the limit discretization

(3.7)-(3.8) depends on the order one of the asymptotic expansion of the water height and on the order two of the asymptotic expansion of the discharge, which are uncomputable quantities. However, a discretization of the limit problem (1.7), detailed in Section 6, can be used to illustrate the asymptotic convergence. When $\varepsilon$ tends to zero, an error depending on the space step, representing the unknown numerical viscosity, is thus expected to remain between this limit scheme and the scheme (3.1)-(3.2).

# 4 Second-order fully well-balanced and asymptotic preserving MUSCL-type extension

In this section, we propose a space second-order extension of the scheme (3.1)-(3.2) such that the steady states and the diffusive limit stay preserved. The MUSCL procedure (see [31, 38, 41, 42] for instance) consists in adopting a piecewise linear reconstruction instead of a piecewise constant reconstruction in the Godunov-type scheme. Moreover, in order to achieve the same order of accuracy in time, a two stage Runge-Kutta method is performed.

The MUSCL technique consists in applying the flux discretization of the scheme (3.1)-(3.2) on half-cells as follows:

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{\Delta t}{2\varepsilon\Delta x}\left(\lambda_{i+\frac12}\left(h_{i+1}^{n,-} - h_i^{n,+}\right) - \lambda_{i-\frac12}\left(h_i^{n,-} - h_{i-1}^{n,+}\right)\right) \\ &- \frac{\Delta t}{2\varepsilon\Delta x}\left(q_{i+1}^{n,-} - q_i^{n,-} + q_i^{n,+} - q_{i-1}^{n,+}\right) \\ &+ \frac{k\Delta t}{2\varepsilon^3}\left(\lambda_{i+\frac12}\frac{\overline{q}_{i+\frac12}|\overline{q}_{i+\frac12}|\overline{h^{-\eta}}_{i+\frac12}}{\alpha_{i+\frac12}} - \lambda_{i-\frac12}\frac{\overline{q}_{i-\frac12}|\overline{q}_{i-\frac12}|\overline{h^{-\eta}}_{i-\frac12}}{\alpha_{i-\frac12}}\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} q_i^{n+1} = q_i^n &+ \frac{\Delta t}{2\varepsilon\Delta x}\left(\lambda_{i+\frac12}\left(q_{i+1}^{n,-} - q_i^{n,+}\right) - \lambda_{i-\frac12}\left(q_i^{n,-} - q_{i-1}^{n,+}\right)\right) \\ &- \frac{\Delta t}{2\varepsilon\Delta x}\Bigg(\frac{(q_{i+1}^{n,-})^2}{h_{i+1}^{n,-}} + g\frac{(h_{i+1}^{n,-})^2}{2} + \frac{(q_i^{n,+})^2}{h_i^{n,+}} + g\frac{(h_i^{n,+})^2}{2} \\ &\qquad - \frac{(q_i^{n,-})^2}{h_i^{n,-}} - g\frac{(h_i^{n,-})^2}{2} - \frac{(q_{i-1}^{n,+})^2}{h_{i-1}^{n,+}} - g\frac{(h_{i-1}^{n,+})^2}{2}\Bigg) \\ &- \frac{k\Delta t}{2\varepsilon^3}\left(\overline{q}_{i+\frac12}|\overline{q}_{i+\frac12}|\overline{h^{-\eta}}_{i+\frac12} + \overline{q}_{i-\frac12}|\overline{q}_{i-\frac12}|\overline{h^{-\eta}}_{i-\frac12}\right), \end{aligned} \tag{4.2}$$

where $\overline{h^{-\eta}}_{i+\frac12} := \overline{h^{-\eta}}(W_i^{n,+}, W_{i+1}^{n,-})$, $\overline{q}_{i+\frac12} := \overline{q}(W_i^{n,+}, W_{i+1}^{n,-})$ and $\alpha_{i+\frac12} := \alpha(W_i^{n,+}, W_{i+1}^{n,-})$ are respectively defined according to (3.3), (3.5) and (2.8). The reconstructed states $(W_i^{n,-})_{i\in\mathbb{Z}}$ and $(W_i^{n,+})_{i\in\mathbb{Z}}$ are given by

$$W_i^{n,\pm} = W_i^n \pm \frac{\Delta x}{2}\sigma_i^n,$$

where the limited slopes $(\sigma_i^n)_{i\in\mathbb{Z}}$ are defined by

$$\sigma_i^n = \text{minmod}\left(\frac{W_i^n - W_{i-1}^n}{\Delta x}, \frac{W_{i+1}^n - W_i^n}{\Delta x}\right).$$

We recall the definition of the minmod slope limiter :

$$\operatorname{minmod}(a,b)=\begin{cases}\operatorname{sign}(a)\min(|a|,|b|) & \text{if } ab>0,\\ 0 & \text{else.}\end{cases}$$

However, as detailed in [36], the scheme (4.1)-(4.2) is not fully well-balanced anymore. At the discrepancy with the Mood method adopted by the authors, we here propose to modify the slopes $(\sigma_i^n)_{i\in\mathbb{Z}}$ as follows to recover this essential property:

$$\sigma_i^n=\operatorname{minmod}\left(\frac{W_i^n-W_{i-1}^n}{\Delta x},\frac{W_{i+1}^n-W_i^n}{\Delta x}\right)\ \frac{\frac{1}{2}\left(\phi_{i-\frac{1}{2}}^n+\phi_{i+\frac{1}{2}}^n\right)}{\frac{1}{2}\left(\phi_{i-\frac{1}{2}}^n+\phi_{i+\frac{1}{2}}^n\right)+\Delta x},$$

where $\phi_{i+\frac{1}{2}}^n=0$ if $W_i^n$ and $W_{i+1}^n$ define a steady state. With this additional factor in the slopes, the MUSCL procedure is not effected in steady states and the scheme used is thus the fully well-balanced scheme (3.1)-(3.2). The choice proposed here for $\phi_{i+\frac{1}{2}}^n$ is the following:

$$\phi_{i+\frac{1}{2}}^n=\left\|\begin{pmatrix} q_{i+1}^n-q_i^n \\ \dfrac{(q_{i+1}^n)^2}{h_{i+1}^n}+g\dfrac{(h_{i+1}^n)^2}{2}-\dfrac{(q_i^n)^2}{h_i^n}-g\dfrac{(h_i^n)^2}{2}+\dfrac{k}{\varepsilon^2}\overline{q}_{i+\frac{1}{2}}|\overline{q}_{i+\frac{1}{2}}|\overline{h^{-\eta}}_{i+\frac{1}{2}}\Delta x\end{pmatrix}\right\|_2.$$

As illustrated by the numerical experiments, the unknown limit scheme associated to (3.1)-(3.2) is second-order in space. Then the MUSCL procedure should not be applied in the diffusive regime. To address such an issue, the slopes $(\sigma_i^n)_{i\in\mathbb{Z}}$ are modified once again as follows:

$$\sigma_i^n=\operatorname{minmod}\left(\frac{W_i^n-W_{i-1}^n}{\Delta x},\frac{W_{i+1}^n-W_i^n}{\Delta x}\right)\ \frac{\frac{1}{2}\left(\phi_{i-\frac{1}{2}}^n+\phi_{i+\frac{1}{2}}^n\right)}{\frac{1}{2}\left(\phi_{i-\frac{1}{2}}^n+\phi_{i+\frac{1}{2}}^n\right)+\Delta x}\ \theta_{\Delta x}^{\varepsilon},$$

where $\theta_{\Delta x}^{\varepsilon}$ is a parameter consistent with 1 when $\Delta x$ tends to zero and such that $\lim\limits_{\varepsilon\to 0}\theta_{\Delta x}^{\varepsilon}=0$. The MUSCL procedure is thus not applied in the asymptotic regime and, when $\varepsilon$ tends to zero, the discretizations are obtained with the original asymptotic preserving scheme (3.1)-(3.2). The following choice, satisfying $\lim\limits_{\varepsilon\to 0}\theta_{\Delta x}^{\varepsilon}=0$, is proposed for $\theta_{\Delta x}^{\varepsilon}$:

$$\theta_{\Delta x}^{\varepsilon}=\frac{\varepsilon^2}{\varepsilon^2+\Delta x}. \tag{4.3}$$

# 5 Limit scheme

In order to evidence the correct asymptotic behavior of schemes (3.1)-(3.2) and the second-order one given by (4.1)-(4.2), we need to compare them with a discretization of the limit problem (1.7).

The following implicit discretization of the diffusion equation (1.9) is thus considered:

$$h_i^{n+1}=h_i^n+\frac{\Delta t}{2\Delta x}\sqrt{\frac{g}{k}}\left(f(h_i^{n+1},h_{i+1}^{n+1})-f(h_{i-1}^{n+1},h_i^{n+1})\right), \tag{5.1}$$

where the anti-symmetric function $f$ is defined by:

$$f(h_{\mathrm{L}}, h_{\mathrm{R}}) = \operatorname{sign}(h_{\mathrm{R}} - h_{\mathrm{L}}) \sqrt{(h_{\mathrm{R}}^{\eta} + h_{\mathrm{L}}^{\eta}) \frac{|h_{\mathrm{R}}^2 - h_{\mathrm{L}}^2|}{\Delta x}}. \tag{5.2}$$

The implementation of this scheme requires to find zeros of a nonlinear function. Indeed, if we consider a space domain divided in $N \in \mathbb{N}^*$ cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]_{i=1,\dots,N}$ and zero-flux boundary conditions, the scheme (5.1) can be rewritten as follows:

$$F^n(h^{n+1}) = 0, \tag{5.3}$$

where $F^n \in \mathbb{R}^N$ is defined as follows for $i = 1, \dots, N$:

$$F_i^n(h) = \begin{cases} h_i - h_i^n - \dfrac{\Delta t}{2\Delta x}\sqrt{\dfrac{g}{k}} f(h_i, h_{i+1}) & \text{if} \quad i = 1, \\ h_i - h_i^n - \dfrac{\Delta t}{2\Delta x}\sqrt{\dfrac{g}{k}} \left(f(h_i, h_{i+1}) - f(h_{i-1}, h_i)\right) & \text{if} \quad i = 2, \dots, N-1, \\ h_i - h_i^n + \dfrac{\Delta t}{2\Delta x}\sqrt{\dfrac{g}{k}} f(h_{i-1}, h_i) & \text{if} \quad i = N, \end{cases} \tag{5.4}$$

for $h \in (\mathbb{R}_+^*)^N$. To compute the approximation at time $t^n + \Delta t$, a Newton algorithm could thus be applied on $F^n$. However, the function $f$, defined by (5.2) and involved in the definition (5.4) of $F^n$, is not differentiable at each interface where the water height is constant. A truncation method to overcome this issue has been proposed in [39]. A second technique, involving a modified secant method is here proposed. This iterative algorithm is the following:

$$\begin{cases} h^0 \in (\mathbb{R}_+^*)^N, \\ 2\widetilde{\nabla}_\delta F^n \left(h^{k+1} - h^k\right) = -F^n(h^k) \quad \text{for all } k \in \mathbb{N}, \end{cases} \tag{5.5}$$

where the operator $\widetilde{\nabla}_\delta(F)$, approximating the Jacobian of the function $F : \mathbb{R}^N \to \mathbb{R}^N$ with $0 < \delta \ll 1$, is defined as follows:

$$\left(\widetilde{\nabla}_\delta F(h)\right)_{ij} = \frac{F_i(h + \delta e_j) - F_i(h)}{\delta} \quad \text{for all } i, j = 1, \dots, N, \tag{5.6}$$

with $e_j \in \mathbb{R}^N$ given by $(e_j)_i = \mathbf{1}_{i=j}$ for all $i \in \mathbb{N}$. The coefficient 2, involved in the iterative algorithm (5.5), allows the method to converge.

Newton methods for non-differentiable functions exist and some are proposed in the literature (see [4] for example). We nevertheless decide here to explain on our simple specific case why our method converges. It stems from the two following results which roughly state that for some functions that are non-differentiable on their roots, it is possible to design a modified Newton method for which the resulting sequence obeys a contraction property. It thus provides geometric convergence.

**Theorem 5.1.** *Let us give a function $f \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{\overline{x}\})$, with $\overline{x} \in \mathbb{R}$ such that $f(\overline{x}) = 0$. If we assume that there exists a constant $C_f > 0$ such that*

$$\left(\frac{f}{f'}\right)(x) \underset{x \to \overline{x}}{\sim} C_f \left(x - \overline{x}\right), \tag{5.7}$$

*then there exists a constant* $\mu > 0$ *such that the sequence defined by*

$$\begin{cases} x_0 \in (\overline{x} - \mu, \overline{x} + \mu)\,, \\ x_{n+1} = x_n - \dfrac{1}{C_f} \dfrac{f(x_n)}{f'(x_n)} \quad \textit{for all } n \in \mathbb{N}, \end{cases} \tag{5.8}$$

*converges geometrically towards* $\overline{x}$.

The above statement is a consequence of the following technical result.

**Lemma 5.2.** *Let us give* $g : \mathbb{R} \to \mathbb{R}$ *defined by*

$$g(x) = \begin{cases} x - \dfrac{1}{C_f} \dfrac{f(x)}{f'(x)} & \textit{if } x \neq \overline{x}, \\ \overline{x} & \textit{otherwise}, \end{cases} \tag{5.9}$$

*with* $f$ *satisfying assumptions of Theorem 5.1. Then* $g \in C^0(\mathbb{R})$ *and for all* $\varepsilon > 0$*, there exists* $\mu > 0$ *such that*

$$|g(x) - g(\overline{x})| \leq \varepsilon |x - \overline{x}| \quad \textit{for all } x \in (\overline{x} - \mu, \overline{x} + \mu). \tag{5.10}$$

*Proof.* The function $g$ defined by (5.9) is continuous on $\mathbb{R}$ by hypothesis of Theorem 5.1 on the function $f$. Moreover, assumption (5.7) gives that, for all $\varepsilon > 0$, there exists $\mu > 0$ such that

$$(1 - \varepsilon)|x - \overline{x}| \leq \operatorname{sign}(x - \overline{x}) \frac{1}{C_f} \frac{f(x)}{f'(x)} \leq (1 + \varepsilon)|x - \overline{x}| \quad \text{for all } x \in (\overline{x} - \mu, \overline{x} + \mu).$$

The above expression can be rewritten as follows:

$$\left| x - \frac{1}{C_f} \frac{f(x)}{f'(x)} - \overline{x} \right| \leq \varepsilon |x - \overline{x}| \quad \text{for all } x \in (\overline{x} - \mu, \overline{x} + \mu),$$

and, using the definition of $g$ given by (5.9), the expected estimation (5.10) is recovered. □

*Proof.* **(Proof of Theorem 5.1)** Let us give $0 < \varepsilon < 1$. From Lemma 5.2, there exists $\mu > 0$ such that

$$|g(x) - g(\overline{x})| \leq \varepsilon |x - \overline{x}| \quad \text{for all } x \in (\overline{x} - \mu, \overline{x} + \mu). \tag{5.11}$$

The function $g$ is thus a contraction mapping on $(\overline{x} - \mu, \overline{x} + \mu)$ for the metric $|.|$. Since the sequence $(x_n)_{n \in \mathbb{N}}$ given by (5.8) can be also defined by $x_{n+1} = g(x_n)$, we can apply the Banach fixed-point Theorem to obtain the following inequality:

$$|x_n - \overline{x}| \leq \varepsilon^n \, |x_0 - \overline{x}| \,.$$

The expected geometric convergence is thus established. □

The scalar function associated with $f$ defined by (5.2) is given by $x \mapsto \operatorname{sign}(x)\sqrt{|x|}$ and the constant $C_f$ involved in Theorem 5.1 is equal to 2 for this function. This explains the parameter 2 occurring in the sequence defined by the recurrence relation (5.5). Moreover, we insist here that, for this vectorial sequence, the choice of a secant method instead of a Newton method is purely numeric and the coefficient $\delta$ can be chosen as small as machine error.

Concerning the local equilibrium (1.8), an approximation of the order one of the discharge is computed from (5.1) as follows:

$$q_i^n = -\text{sign}\left(h_{i+1}^n - h_{i-1}^n\right)\sqrt{\frac{(h_i^n)^\eta}{k}\frac{\left|\frac{g}{2}(h_{i+1}^n)^2 - \frac{g}{2}(h_{i-1}^n)^2\right|}{2\Delta x}}. \tag{5.12}$$

# 6 Numerical results

This section is devoted to the numerical illustration of the properties possessed by the two schemes under interest, the first-order in space defined by (3.1)-(3.2) and the second-order given by (4.1)-(4.2). In all the simulations, according to [36], the parameter $\eta$ is set equal to $7/3$.

## 6.1 Well-balanced assessment

A numerical example highlighting the fully well-balanced property of the schemes is here proposed. Following the works [35, 36], a steady state $(h_0(x), q_0)$ is a solution of the following equation:

$$\xi(h; x, h_0, q_0, x_0) = 0, \tag{6.1}$$

where

$$\xi(h; x, h_0, q_0, x_0) = -\frac{q_0^2}{\eta - 1}\left(h^{\eta-1} - h_0^{\eta-1}\right) + \frac{g}{\eta+2}\left(h^{\eta+2} - h_0^{\eta+2}\right) + kq_0|q_0|(x - x_0). \tag{6.2}$$

A study of the above function is proposed in [35] where the authors prove the existence of zero, one or two solutions of the equation (6.1) depending on $h_0$, $q_0$ and $x_0$. To assess the fully well-balanced property, we set ourselves in the case where there are two solutions. We approximate the highest solution, called subcritical solution, using a Newton algorithm on the function $\xi$ with a precision of order $10^{-16}$. Following the test case proposed in [36], the space domain considered is the interval $(0.75, 0.9)$, discretized with $N = 400$ cells of constant size $\Delta x$. We set $h_0 = 0.25$, $q_0 = -\sqrt{g}/8$ and $x_0 = 0.75 - \Delta x$. This stationary solution is displayed on Figure 2.

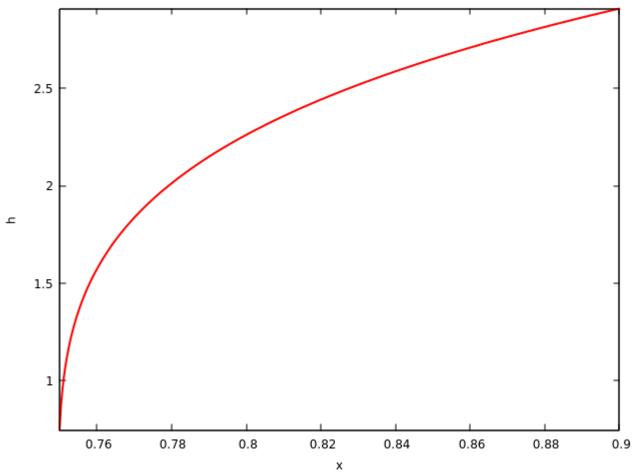
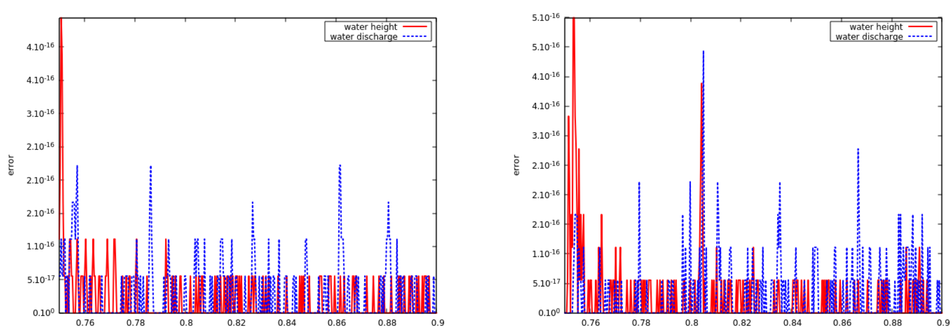



Figure 2: Water height of the subcritical steady state.

Since we are not interested by the diffusive regime here, the parameter $\varepsilon$ is fixed equal to 1. Moreover we set the parameter $k$ equal to 1 and the parameter $\eta$ equal to $7/3$. To assess the fully well-balanced property, we compute the solution obtained with the two schemes (3.1)-(3.2) and (4.1)-(4.2) at time $T = 3$ with $N = 400$ cells and the subcritical steady state as initial condition. Errors between approximate water heights and approximate water discharges with the steady state are displayed on Figure 3. These errors are of order $10^{-16}$, which means that the machine precision is reached for the two schemes. Schemes (3.1)-(3.2) and (4.1)-(4.2) both preserve this subcritical steady state.




a) First-order scheme (3.1)-(3.2).

b) Second-order scheme (4.1)-(4.2).

Figure 3: Error between the discretization and the steady state at time $T = 3$.

## 6.2 Preservation of the diffusive regime

Now we can illustrate the asymptotic preserving property of schemes (3.1)-(3.2) and (4.1)-(4.2). To address this, we compare them with the limit scheme (5.1)-(5.12) described in Section 5.

Two different initial conditions are considered. They are defined on the spatial domain $[-5, 5]$ as follows.

- Continuous initial condition:

$$h_0(x) = \begin{cases} 2 & \text{if} \quad x < -1, \\ \dfrac{1}{2}\left(3 + \sin\left(\dfrac{3\pi x}{2}\right)\right) & \text{if} \quad -1 \leq x < 1, \\ 1 & \text{else,} \end{cases} \quad \text{and} \quad u_0(x) = 0. \tag{6.3}$$

- Discontinuous initial condition:

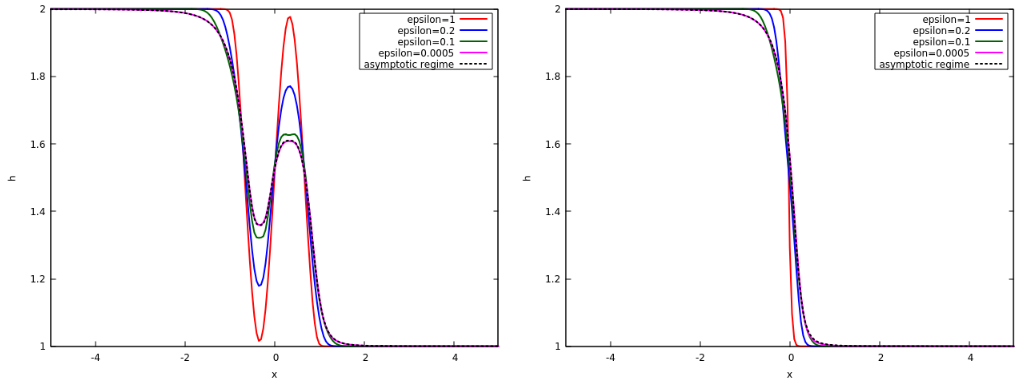
$$h_0(x) = \begin{cases} 2 & \text{if} \quad x < 0, \\ 1 & \text{else,} \end{cases} \quad \text{and} \quad u_0(x) = 0. \tag{6.4}$$

The spatial domain $[-5, 5]$ is discretized with $N \in \mathbb{N}^*$ cells of constant size $\Delta x = \frac{10}{N}$. Boundary conditions used here are zero-flux type boundary conditions :

$$W_0^n = W_1^n \quad \text{and} \quad W_{N+1}^n = W_N^n \quad \text{for all } n \in \mathbb{N}.$$

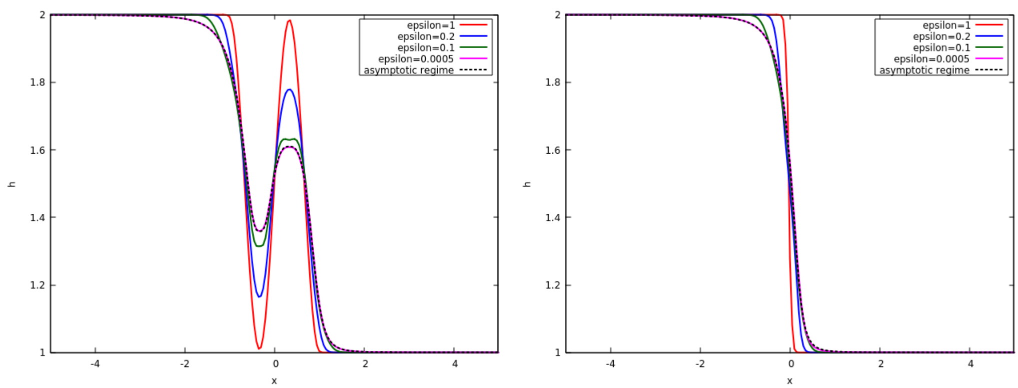
Moreover we set here again the parameter $k$ equal to 1 and the parameter $\eta$ equal to $^7/_3$.

On Figure 4, we compare the approximate water heights obtained at time $T = 0.01$ for $N = 200$ cells with the first-order scheme (3.1)-(3.2) with the approximate diffusive limit (5.1) for different values of $\varepsilon$. On Figure 5, similar results obtained with the second-order scheme (4.1)-(4.2) are displayed. The schemes (3.1)-(3.2) and (4.1)-(4.2) clearly preserve the diffusive limit since we cannot distinguish the water height given by these schemes and the discretization of the asymptotic regime (5.1) for $\varepsilon \leq 0.005$. In the same spirit, we compare on Figures 6 and 7 the order one of the asymptotic expansions of the approximate discharges with the approximate local equilibrium (5.12). The final time is still $T = 0.01$, the number of cells is $N = 200$ cells and the schemes are compared for different values of $\varepsilon$. We underline that, as the order one in the asymptotic expansion of the discharge is considered, the approximate discharge displayed on Figures 6 and 7 coincide with $q_i^n/\varepsilon$ where $q_i^n$ is given by (3.1)-(3.2) or (4.1)-(4.2). These schemes also preserve the local equilibrium since we cannot distinguish the water discharge given by these schemes and the discretization given by (5.12), for $\varepsilon \leq 0.005$.



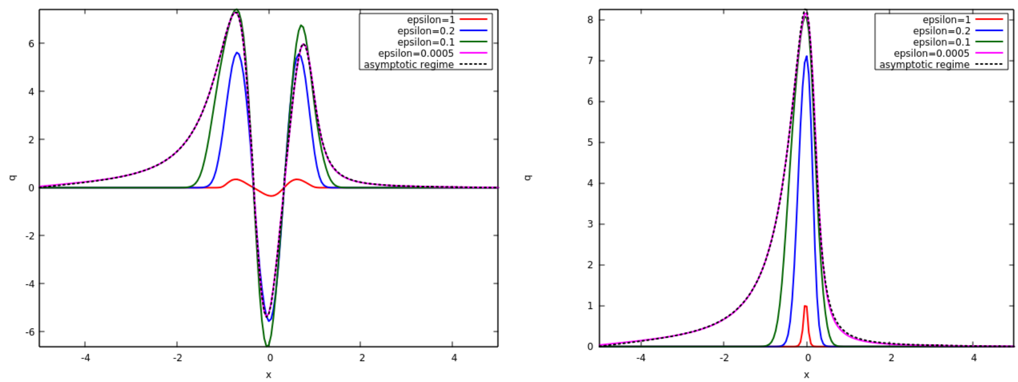

Continuous initial condition Discontinuous initial condition

Figure 4: Water height at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the first-order scheme (3.1)-(3.2).



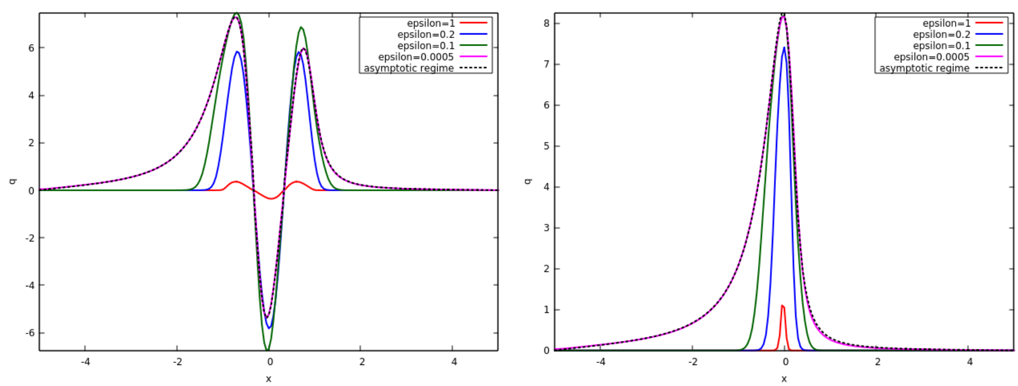

Continuous initial condition Discontinuous initial condition

Figure 5: Water height at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the second-order scheme (4.1)-(4.2).




Continuous initial condition Discontinuous initial condition

Figure 6: Water discharge at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the first-order scheme (3.1)-(3.2).



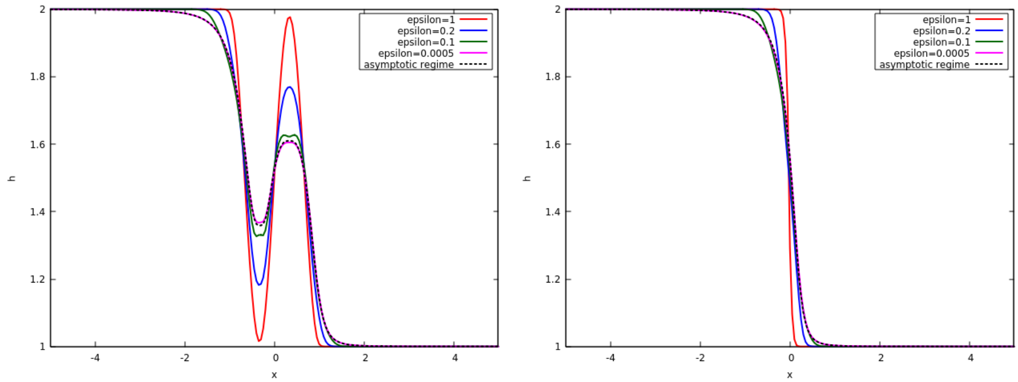

Continuous initial condition Discontinuous initial condition

Figure 7: Water discharge at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the second-order scheme (4.1)-(4.2).

We also display on Figure 8 and 9 the results obtained with the initial scheme proposed in [36], which coincides with the scheme given by (3.1)-(3.2) with the following definition for $\overline{q}$:

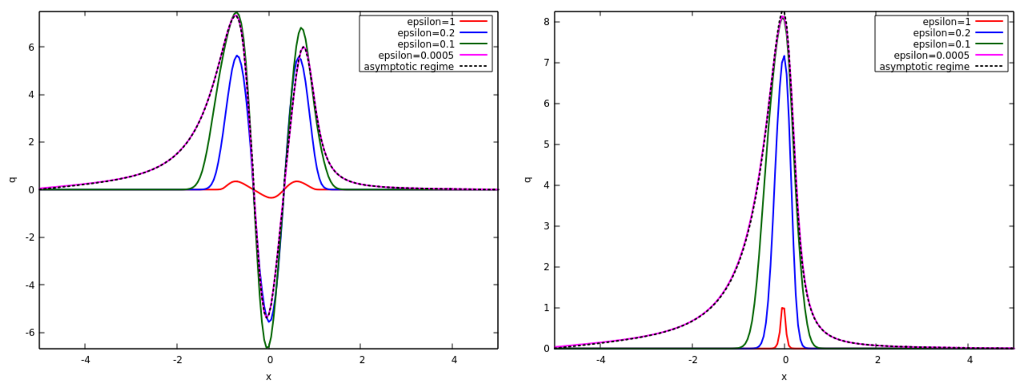
$$\overline{q} = \begin{cases} \mathrm{sign}(q_L + q_R)\dfrac{2|q_L||q_R|}{|q_L| + |q_R|} & \text{if} \quad q_L \neq 0 \text{ and } q_R \neq 0, \\ 0 & \text{else.} \end{cases} \tag{6.5}$$

We can numerically assess the convergence towards the asymptotic regime, even if the above choice for $\overline{q}$ does not satisfy the condition required in Theorem 3.1. This assumption is necessary only to make the proof of Theorem 3.1 easier but is seems that a choice satisfying the consistency relation (2.10) is enough to give the asymptotic preservation.




Continuous initial condition Discontinuous initial condition

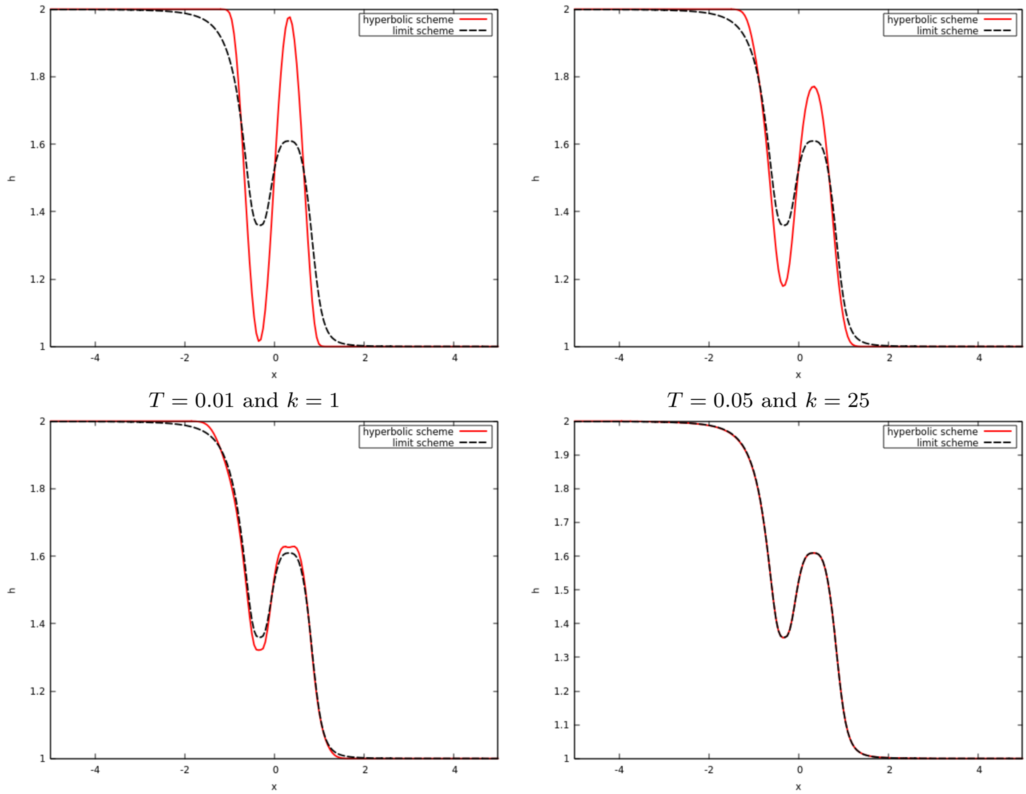
Figure 8: Water height at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the first-order scheme (3.1)-(3.2) with $\overline{q}$ given by (6.5).




Continuous initial condition Discontinuous initial condition

Figure 9: Water discharge at time $T = 0.01$ and $N = 200$ for various values of $\varepsilon$ with the first-order scheme (3.1)-(3.2) with $\overline{q}$ given by (6.5).

We also propose on Figure 10 the illustration of the asymptotic convergence of the first-order scheme (3.1)-(3.2) towards the limit scheme (5.1)-(5.12) when $\varepsilon$ is fixed to 1 and parameters $T$ and $k$ increase. This study is nearly the same than when $\varepsilon$ goes to zero since we consider an increasing of these parameters by setting them respectively equal to $0.01\theta$ and $\theta^2$ with $\theta$ increasing ($\theta = 1,\ 5,\ 10$ and 50 on Figure 10). We only display the water height for the continuous initial condition but similar results than the study with decreasing $\varepsilon$ can be obtained for the water discharge and the discontinuous initial condition.




$T = 0.01$ and $k = 1$ $T = 0.05$ and $k = 25$

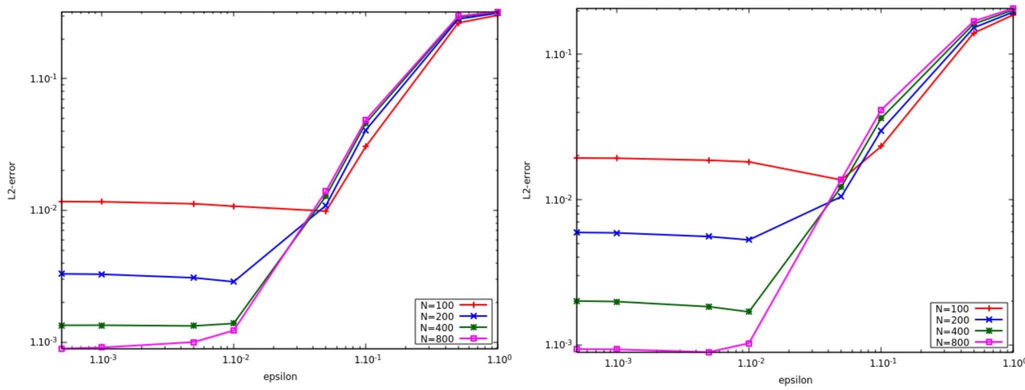
$T = 0.1$ and $k = 100$ $T = 0.5$ and $k = 2500$

Figure 10: Water height for various increasing values of $T$ and $k$ with schemes (3.1)-(3.2) and (5.1)-(5.12) for the continuous initial condition.

We precise that the scheme given by equations (5.1) and (5.12) is not the limit of the scheme (3.1)-(3.2) (neither the limit of (4.1)-(4.2)). Indeed, as detailed in Section 3, this corresponding limit scheme has no algebraic expression. An error depending on $\Delta x$, the space step, should remain between the limit of (3.1)-(3.2) and (5.1)-(5.12) (and also between the limit of (4.1)-(4.2) and (5.1)-(5.12)). This viscosity is not perceptible on Figures 4-7 but we can exhibit it by displaying the $L^2$-error between the approximations of the water height given by the different schemes and the approximate water height given by (5.1). The following formula is thus used to compute the error between $(h_i^n)_{i=1,\dots,N}^{n\in\mathbb{N}}$, the approximate water height given by the scheme considered, and $(\overline{h}_i^n)_{i=1,\dots,N}^{n\in\mathbb{N}}$, the approximate water height given by the limit scheme (5.1):

$$\mathscr{E}_N^{2n}(\varepsilon) = \sqrt{\sum_{i=1}^{N} \left(h_i^n - \overline{h}_i^n\right)^2 \Delta x}.$$

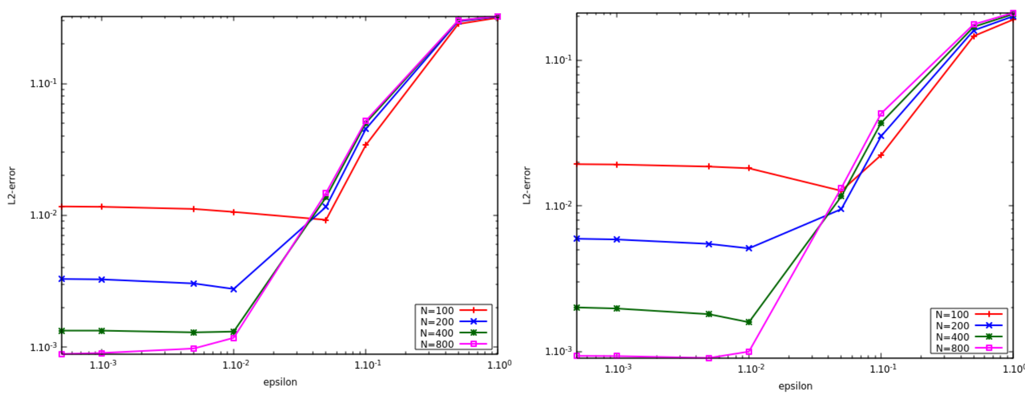
On Figures 11 and 12, this error in $L^2$-norm is displayed with respect to $\varepsilon$ in logarithmic scale, for various values of $N$, at time $T = 0.01$ and for the two initial conditions (6.3) and (6.4), respectively for the scheme (3.1)-(3.2) and (4.1)-(4.2). For each simulation, a plateau can be observed for values of $\varepsilon$ smaller than a threshold. This is the expected result. Indeed this threshold for $\varepsilon$ corresponds to the value where the uncomputable numerical viscosity involved in the limit of the scheme (3.1)-(3.2), depending on $\Delta x$, is larger than the asymptotic error, depending on $\varepsilon$. This also explains why the plateau is located lower if N is larger.




Continuous initial condition

Discontinuous initial condition

Figure 11: $L^2$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the first-order scheme (3.1)-(3.2).




Continuous initial condition

Discontinuous initial condition

Figure 12: $L^2$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the second-order scheme (4.1)-(4.2).

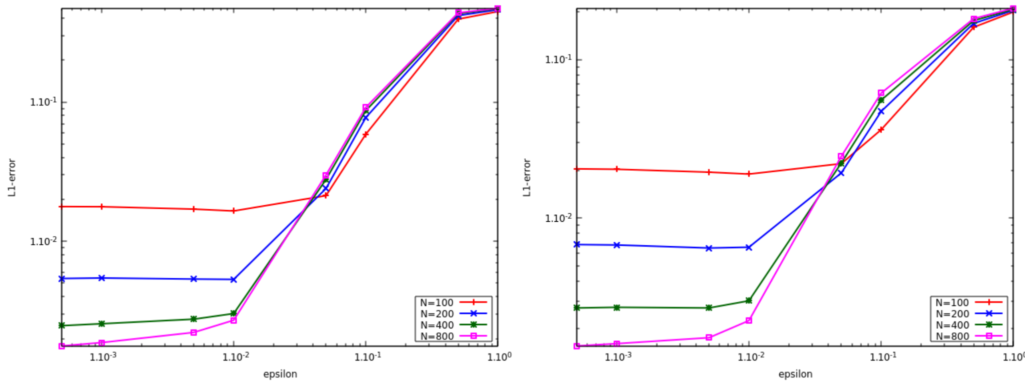
On Figures 13 and 14, the error in norm $L^1$ is displayed, according to the following formula

$$\mathscr{E}_N^{1n}(\varepsilon) = \sum_{i=1}^{N} \left| h_i^n - \overline{h}_i^n \right| \Delta x,$$

and, on Figures 15 and 16, the error in norm $L^\infty$ is displayed, according to the following formula

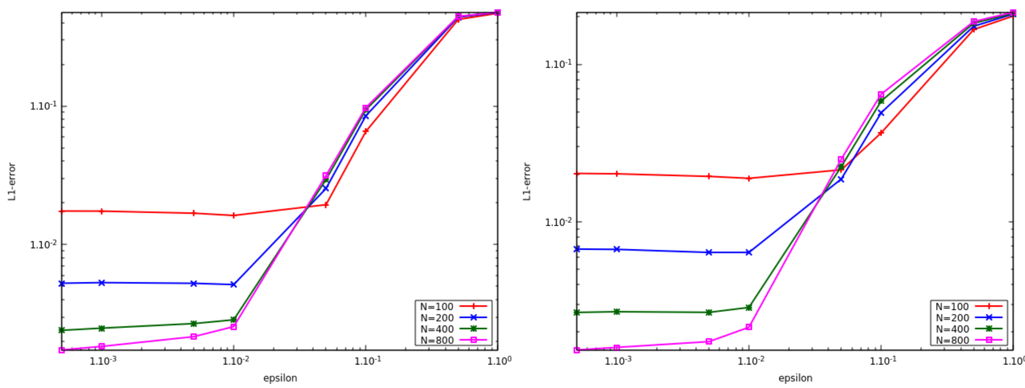
$$\mathscr{E}_N^{\infty n}(\varepsilon) = \max_{i=1,\dots,N} \left| h_i^n - \overline{h}_i^n \right|.$$

We observe that, with these error expressions, the behavior of the error in the asymptotic regime is similar to the one in $L^2$-norm.



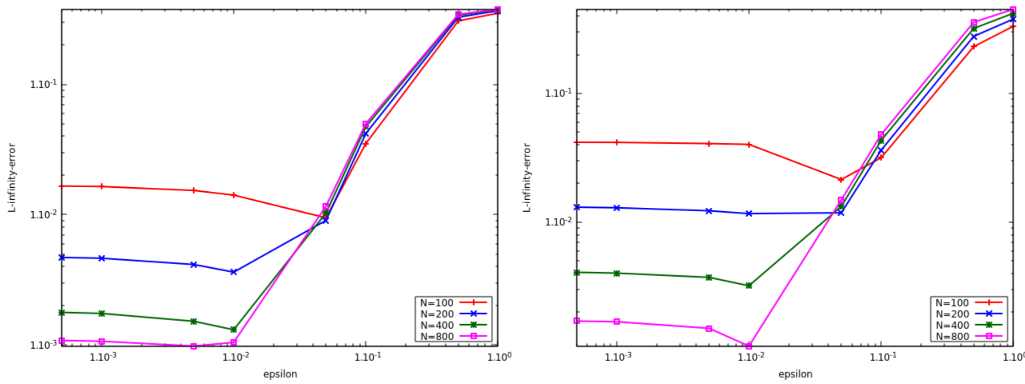

Continuous initial condition    Discontinuous initial condition

Figure 13: $L^1$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the first-order scheme (3.1)-(3.2).



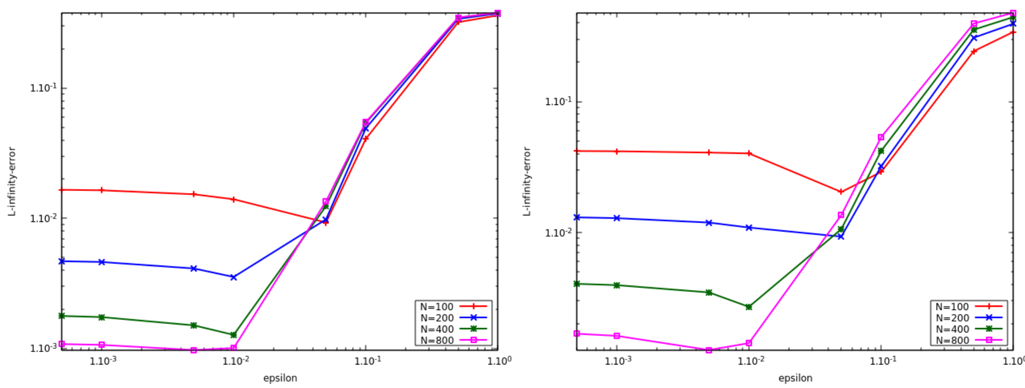

Continuous initial condition    Discontinuous initial condition

Figure 14: $L^1$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the second-order scheme (4.1)-(4.2).




Continuous initial condition Discontinuous initial condition

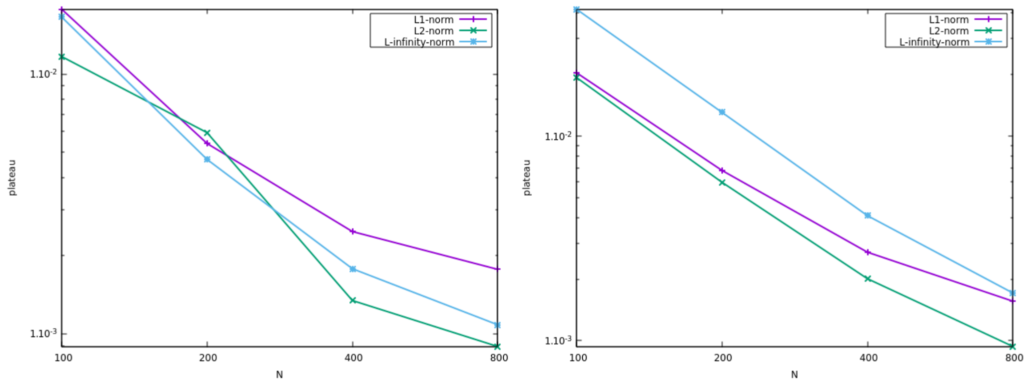
Figure 15: $L^\infty$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the first-order scheme (3.1)-(3.2).




Continuous initial condition Discontinuous initial condition

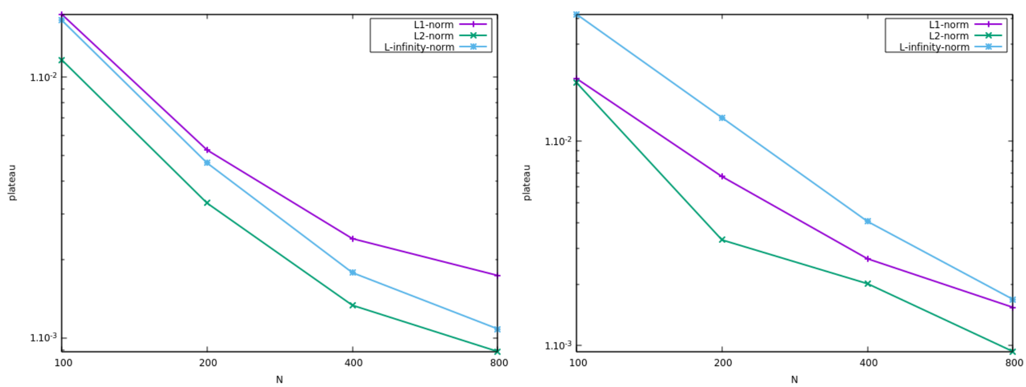
Figure 16: $L^\infty$-error in logarithmic scale at time $T = 0.01$ for different values of $N$ with the second-order scheme (4.1)-(4.2).

Now, we can study the plateau reached by each of the schemes with respect to the number of cells $N$. These results are displayed on Figures 17 and 18 respectively for the first-order scheme (3.1)-(3.2) and the second-order scheme (4.1)-(4.2). We observe that the results with the $L^\infty$-norm are similar for both initial conditions and for the first-order and second-order schemes. With this norm, the difference of the limit of our hyperbolic scheme when $\varepsilon$ goes to zero and the implicit chosen limit scheme seems to be of order $\Delta x^{1.7}$.




Continuous initial condition Discontinuous initial condition

Figure 17: Value of the plateau reached in logarithmic scale at time $T = 0.01$ for different norms with the first-order scheme (3.1)-(3.2).




Continuous initial condition Discontinuous initial condition

Figure 18: Value of the plateau reached in logarithmic scale at time $T = 0.01$ for different norms with the second-order scheme (4.1)-(4.2).

## 6.3 Space accuracy study

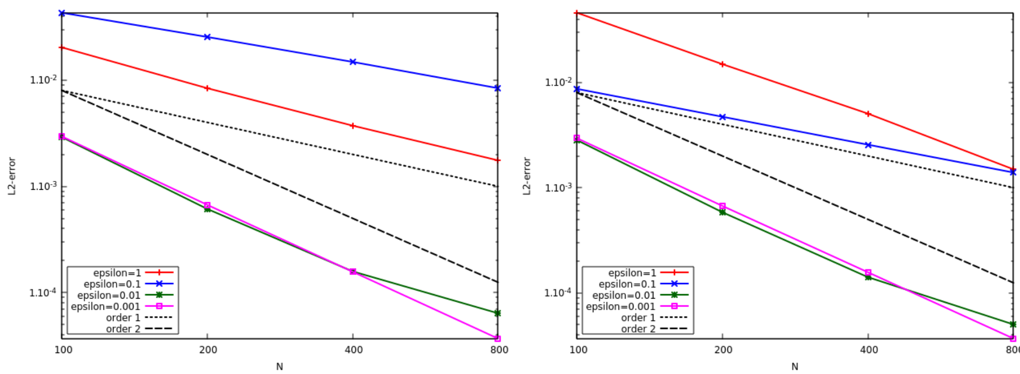
To conclude this paper, a study of the space order of convergence of schemes (3.1)-(3.2) and (4.1)-(4.2) is proposed. The method adopted to compute it is given by the following formula:

$$\mathcal{E}_\varepsilon^n(N) = \sqrt{\sum_{i=1}^{N}\left(\left(h_{2i}^{(2N),n} - h_i^{(N),n}\right)^2 + \left(q_{2i}^{(2N),n} - q_i^{(N),n}\right)^2\right)\Delta x^{(N)}}, \tag{6.6}$$

where $(h_i^{(N),n})_{i=1,\dots,N}^{n\in\mathbb{N}}$ is the approximate water height given by one of the two schemes under concern for a space domain divided into $N$ cells and $(h_i^{(2N),n})_{i=1,\dots,2N}^{n\in\mathbb{N}}$ is the approximate water height given by the same scheme for a space domain divided into

$2N$ cells. Quantities $\Delta x^{(N)}$ and $\Delta x^{(2N)}$ are the corresponding cell sizes.

On Figure 19, the error for various values of $\varepsilon$ and the continuous initial condition (6.3) is displayed with respect to $N$ in logarithmic scale, for each of the two schemes of interest, (3.1)-(3.2) and (4.1)-(4.2). The associated table, giving the numerical space order, is given in Figure 20.




First-order scheme (3.1)-(3.2) Second-order scheme (4.1)-(4.2)

Figure 19: $L^2$-error in logarithmic scale at time $T = 0.01$ for different values of $\varepsilon$ with the continuous initial condition.

| $N$ \ $\varepsilon$ | 1 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|
| 200 | 1.27 | 0.76 | 2.24 | 2.14 |
| 400 | 1.17 | 0.78 | 1.95 | 2.09 |
| 800 | 1.09 | 0.82 | 1.31 | 2.09 |

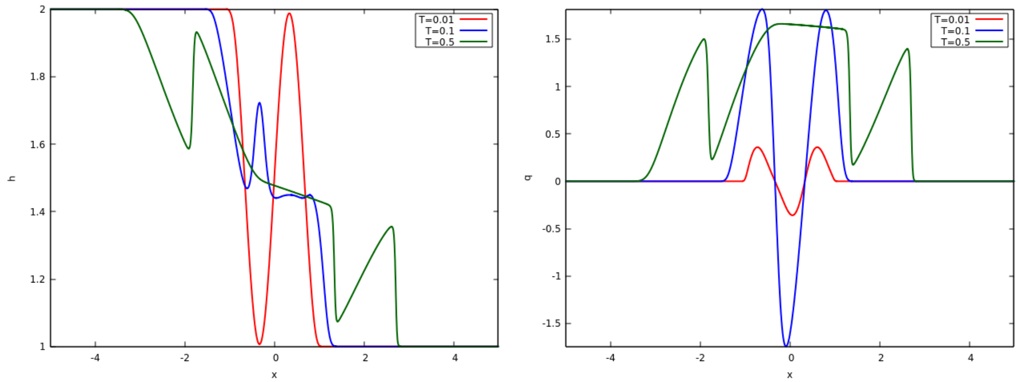
First-order scheme (3.1)-(3.2)

| $N$ \ $\varepsilon$ | 1 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|
| 200 | 1.62 | 0.88 | 2.28 | 2.14 |
| 400 | 1.56 | 0.88 | 2.05 | 2.09 |
| 800 | 1.76 | 0.88 | 1.49 | 2.09 |

Second-order scheme (4.1)-(4.2)

Figure 20: Numerical space order at time $T = 0.01$ for different values of $\varepsilon$ with the continuous initial condition.

We observe that the space error obtained with the scheme (3.1)-(3.2) with $\varepsilon = 1$ is of order 1. Moreover, the order is closer to 2 if $\varepsilon$ is smaller. As mentioned in Section 4, the unknown limit scheme associated with the original first-order scheme (3.1)-(3.2) seems to be of order 2. Concerning the errors obtained with the second-order scheme (4.1)-(4.2), the order is close to 1.6 for $\varepsilon = 1$. The MUSCL procedure performed on the scheme thus permits to increase the space order accuracy of the discretization. We precise here that this order is satisfying regarding the regularity of the solution approximated by schemes. Indeed, even if the initial condition is smooth, shocks can be created in finite time. We illustrate this phenomenon on Figure 21 where we can observe the approximate solution, obtained with the first-order scheme (3.1)-(3.2) with $\varepsilon = 1$ and various values of the final time $T$.

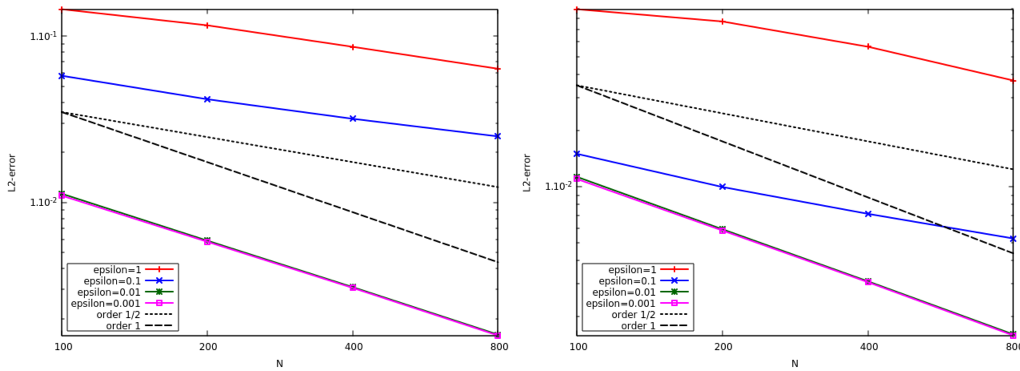



Water height Water discharge

Figure 21: Shocks creation with the scheme (3.1)-(3.2) for the continuous initial condition and various values of $T$.

However, for the second-order scheme (4.1)-(4.2) with the continuous initial condition, the numerical space order decreases towards 0.88 for $\varepsilon = 0.1$, close to the accuracy of the scheme (3.1)-(3.2) for the same value of $\varepsilon$. The MUSCL procedure is thus not able to catch a higher order in every regime in $\varepsilon$, and more specifically in transitory regimes. Indeed, limiters are tailored for the asymptotic limit. However, since the aim of the MUSCL procedure presented here is to increase the order of space accuracy in the hyperbolic regime, with $\varepsilon$ of order 1, these results are still satisfying. Finally, more $\varepsilon$ gets small and more this order gets close to 2, as expected. Indeed, because of the parameter $\theta^{\varepsilon}_{\Delta x}$ in the slopes, the MUSCL procedure is not effected for $\varepsilon$ small and we recover the behavior obtained by the first-order scheme (3.1)-(3.2) displayed on Figure 19.

On Figure 22, the error for various values of $\varepsilon$ and the discontinuous initial condition (6.4) is displayed with respect to $N$ in logarithmic scale, for each of the two schemes of interest, (3.1)-(3.2) and (4.1)-(4.2). The associated table, giving the numerical space order, is given in Figure 23.




First-order scheme (3.1)-(3.2) Second-order scheme (4.1)-(4.2)

Figure 22: $L^2$-error in logarithmic scale at time $T = 0.01$ for different values of $\varepsilon$ with the discontinuous initial condition.

| $N$ \ $\varepsilon$ | 1 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|
| 200 | 0.32 | 0.47 | 0.94 | 0.92 |
| 400 | 0.43 | 0.39 | 0.93 | 0.92 |
| 800 | 0.44 | 0.35 | 0.95 | 0.95 |

First-order scheme (3.1)-(3.2)

| $N$ \ $\varepsilon$ | 1 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|
| 200 | 0.22 | 0.59 | 0.94 | 0.92 |
| 400 | 0.45 | 0.48 | 0.93 | 0.92 |
| 800 | 0.60 | 0.44 | 0.95 | 0.95 |

Second-order scheme (4.1)-(4.2)

Figure 23: Numerical space order at time $T = 0.01$ for different values of $\varepsilon$ with the discontinuous initial condition.

Concerning the discontinuous initial condition, the first-order scheme (3.1)-(3.2) gives a space accuracy of order 0.4 for $\varepsilon = 1$, which seems to converge towards 1 when $\varepsilon$ goes to zero. Moreover, the second-order order scheme (4.1)-(4.2) seems to give a little improvement of the space accuracy for $\varepsilon = 1$. Since this type of procedure is not designed to be applied to such a discontinuous initial condition, these results are satisfying.

# 7 Conclusions

In this paper, we considered the Godunov-type scheme introduced in [36] and improved it to preserve all the steady states of the shallow-water equations with Manning-type friction. We proved the preservation of the diffusive asymptotic regime by this scheme, with a slight modification of the average operator involved in the source term discretization. We insist on the relevance of this result since this scheme is naturally able to preserve the diffusive limit and no additional procedure, modifying the numerical viscosity for instance, has been performed.

A spatial higher-order version of this scheme had then been proposed. Several techniques have been used so that this second scheme is also able to preserve all the steady states and the diffusive limit. In order to assess the asymptotic convergence, we also developed an implicit finite volume scheme for the diffusive limit. Some numerical methods had to be performed to implement this implicit scheme. Finally, many numerical results have been proposed to assess the fully well-balanced and the asymptotic preserving properties and for the spatial accuracy.

**Acknowledgment.** The authors are supported by the MoHyCon project (ANR-17-CE40-0027-01), the MUFFIN project (ANR-19-CE46-0004) and the Centre Henri Lebesgue (ANR-11-LABX-0020-01).

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