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Numerical simulations of some hyperbolic systems

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Contents

1 Introduction

The research tasks detailed in this document are dedicated to the numerical approximation of hyperbolic systems. Two main topics will be discussed. First, hyperbolic systems in non-conservation form are considered. These systems are issued from the multi-entropy model used to derive several models of compressible turbulent flows and/or multi-fluid flows. The second topic of interest concerns the numerical approximation of the solutions of hyperbolic system of conservation laws. Relaxation schemes are proposed to approximate some models and an analysis of the second-order MUSCL schemes is detailed. The majority of these studies were carried out within the framework of a collaboration between Bordeaux 1 University and CEA-CESTA. These works are discribed in four parts.

The first part is devoted to the multi-entropy model and several possible extensions. This model was presented in the authors's PhD thesis. It can be understood as an extension of the Navier-Stokes equations but for a compressible gas modeled by $N \geq 2$ independent entropies. This model plays a crucial role when considering simulation of compressible turbulent flows. In addition, it allows the derivation of models to simulate turbulent multi-fluid flows. The numerical approximation of the solutions of the PDE systems which governs such models raises some difficulties according to the existence of non-conservative products in the model. Indeed, by contrast with systems in conservation form, the viscous shock solutions (traveling wave solutions) are dictated by the shape of the diffusion operator. From a numerical point of view, this implies the need for solving the shock layers using a very fine mesh. Of course, this is out of reach for the usually considered size of grid. To correct this problem, a numerical procedure is introduced: *the nonlinear projection scheme*. This method is based on an additional relation satisfied by the smooth solutions of the system. This relation exhibits correlation between the entropy dissipation rates which cannot independently evolve. The discrete form of this new formula can be understood as a nonlinear projection step to correct the standard L^2-projection scheme. The numerical results obtained attest to the interest for this method. Next, the scheme is extended to several domains of applications. Turbulent multi-fluid flows have been simulated. Extensions to reacting flows have been proposed in the post-doc of D. Reignier. Finally, when considering turbulence applications, the multi-entropy model with relaxation source terms is proved to admit infinitely many distinct solutions under assumptions prescribed by the physics.

The second part deals with the relaxation schemes applied to some systems. These schemes allow to approximate the weak solutions of hyperbolic system of conservation laws. The weak solutions under consideration are approximated by the weak solutions of a relevant first-order system with singular perturbations: the relaxation system. One of the main difficulties arises from the choice of the relaxation model. Indeed, the stability, the robustness and also the accuracy of the numerical methods come with the choice of relaxation model. Let us emphasize that the use of a PDE system associated with the numerical scheme turns out to be a powerful tool and it allows us to establish discrete entropy inequalities and maximum principles. These scheme derivations are proposed to approximate the solutions of the pressureless Euler equations (study proposed in the post-doc of M. Breuss) and the Levermore 10-moment Gaussian model. Another advantage of the relaxation schemes relates to the evaluation of the numerical flux function which turns out to be easy and free from the Jacobian matrix of the flux function. In M. Baudin's PhD thesis, this interest for the method is used to approximate a petroleum flow model. Finally, to perform multi-fluid flow simulations, in B. Braconnier PhD thesis, some extensions of the relaxations schemes are proposed to approximate the weak solutions of hyperbolic systems in non-conservation form.

The third part concerns the derivation of an HLLC scheme and a VF-Roe scheme. First, an HLLC scheme is proposed to approximate the weak solutions of the M_1 model to simulate radiative transfer in two space dimensions. When deriving the scheme, particular attention is paid to enforce all the required stability properties and accuracy. In addition, in the limit of a small parameter to zero, it is proved that the asymptotic behavior of the numerical method coincides with a diffusion regime satisfied by the model. Next, a VF-Roe scheme is extended to approximate the solutions of an hyperbolic system in non-conservation form issuing from a turbulence model.

The last part is dedicated to an analysis of the second-order MUSCL schemes. To increase the order of accuracy of the finite volume schemes, a piecewise linear approximation is considered. The procedure proposed by van Leer consists of introducing a gradient reconstruction to increase the order of accuracy when evaluating the unknowns. These second-order approximations are thus considered to compute the numerical flux functions. This procedure is widely used in the literature but few results of stability have been established. A new technique of slope limitation is introduced and a relevant CFL restriction is considered to enforce all the required stability properties. Next, these results are extended to a second-order time and space scheme: the MUSCL-Hancock scheme. Finally, the MUSCL schemes are studied for 2D unstructured meshes. Once again, involving a relevant CFL restriction and a suitable gradient reconstruction limiter, the robustness of the scheme can be ensured.

2 Multi-entropy model and applications [2, 3, 4, 8, 12, 13, 15]

Non-conservative system in the form:

$$\partial_t \mathbf{w} + \mathbf{A}(\mathbf{w})\partial_x \mathbf{w} = 0, \tag{1}$$

involves several theoretical and numerical difficulties. Indeed, as soon as the matrix $\mathbf{A}(\mathbf{w})$ does not coincide with a Jacobian matrix $\nabla \mathbf{f}$ associated with a relevant flux function $\mathbf{f}(\mathbf{w})$, the non-conservative product $\mathbf{A}(\mathbf{w})\partial_x \mathbf{w}$ forbids the usual weak formulation of the system. As a consequence, the standard approaches to deal with hyperbolic system of conservation laws cannot be applied. In addition, once again because of the non-conservative products, the classical numerical methods turn out to be unsuitable.

The non-conservative product involves mathematical ambiguities as soon as the solution admits discontinuities. In the present analysis, the considered hyperbolic systems may admit two distinct discontinuities: shock waves associated with genuinely non-linear fields and contact discontinuities associated with linearly degenerate fields. With some abuses in the notations, we assume that the non-conservative products are well-defined when considering contact discontinuities. Such an assumption will be satisfied by all the physical applications under interest.

Let us consider shock waves to emphasize that the non-conservative products cannot be uniquely defined (see LeFLoch [86, 87], Dal Maso-LeFloch-Murat [58], Colombeau-LeRoux [54]). According to the theory developed by LeFloch [88] (see also Raviart-Sainsaulieu [103]) to deal with hyperbolic system in non-conservation form, it is necessary to extend the model in order to consider additional dissipation terms. Hence, a parabolic regularization is assumed:

$$\partial_t \mathbf{w}^\epsilon + \mathbf{A}(\mathbf{w}^\epsilon)\partial_x \mathbf{w}^\epsilon = \epsilon \mathbf{R}(\mathbf{w}^\epsilon, \partial_{xx}\mathbf{w}^\epsilon). \tag{2}$$

After recent works [72, 73], *kinetic relations* are used to characterized the shock waves solutions of the system (1). From a numerical point of view, the approximation of the parabolic regularization turns out to be one of the main difficulties. Indeed, we will show that the characterization of the viscous shock layers intrinsically depends on the choice of the dissipation operator.

To illustrate our purpose, let us consider the following hyperbolic system in non-conservation form:

$$\begin{cases} \partial_t u + u\partial_x(u+v) = 0, \\ \partial_t v + v\partial_x(u+v) = 0, \end{cases}$$

where the state space is defined by $\Omega = \{\mathbf{w} = (u, v) \in \mathbb{R}^2; u + v > 0\}$. This simplified model comes from a shallow water bifluid model (see Castro et al. [48]). After [15], the following parabolic regularization is introduced:

$$\begin{cases} \partial_t u^\epsilon + u^\epsilon \partial_x (u^\epsilon + v^\epsilon) = \epsilon \delta_1 \partial_{xx}(u^\epsilon + v^\epsilon), & \delta_1 > 0, \\ \partial_t v^\epsilon + v^\epsilon \partial_x (u^\epsilon + v^\epsilon) = \epsilon \delta_2 \partial_{xx}(u^\epsilon + v^\epsilon), & \delta_2 > 0. \end{cases}$$

The traveling wave solutions are then considered. They are smooth bounded solutions in the form $\mathbf{w}(x,t) = \tilde{\mathbf{w}}(x - \sigma t)$ where $\sigma \in \mathbb{R}$ denotes the propagation speed of the wave. In addition, $\tilde{\mathbf{w}}$ verifies: $\lim_{\xi \to \pm\infty} \tilde{\mathbf{w}}(\xi) = \mathbf{w}_\pm$ where $\mathbf{w}_\pm \in \Omega$. Let us introduce a relevant scaling to set $\tilde{\mathbf{w}}_\epsilon(\xi) = \tilde{\mathbf{w}}(\xi/\epsilon)$. In the limit of ϵ to zero, the sequence $\mathbf{w}_\epsilon$ tends strongly $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ to the following discontinuous function:

$$\mathbf{w}_0(x,t) = \begin{cases} \mathbf{w}_- & \text{if } x < \sigma t, \\ \mathbf{w}_+ & \text{if } x > \sigma t. \end{cases}$$

After [109], the function $\mathbf{w}_0$ defines a weak solution of (1) compatible with the considered parabolic regularization.

In this example, for given $\mathbf{w}_-$ and σ, let us note that the state $\mathbf{w}_+$ is given by the following formulas:

$$\begin{aligned} v_+ &= \frac{\delta_2}{\delta_1 + \delta_2}(2\sigma - u_- - v_-) + \frac{\delta_1 v_- - \delta_2 u_-}{\delta_1 + \delta_2} \mathrm{e}^{2 - 2(u_- + v_-)/\sigma}, \\ u_+ &= 2\sigma - u_- - (v_- + v_+). \end{aligned}$$

It is clear that the value of $\mathbf{w}_+$ depends on the choice of the parameter δ_1/δ_2. The numerical approximation of the discontinuous solution $\mathbf{w}_0$ is very difficult and it needs, a priori, a very fine approximation of the viscous shock layers. The reader is referred to the works of Hou-LeFloch [75] or Colombeau-Noussaïr-Perrot [55] where several numerical experiments are proposed (see also Karni [79] where non-conservative schemes are detailed).

The model under consideration in the present study can be understood as an extension of Navier-Stokes equations where the gas is modeled by $N \geq 2$ independent pressures p_i. Each pressure is associated with a specific entropy s_i and an internal energy e_i. For the sake of simplicity, we fix $N = 2$. The model is thus governed by the following system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p_1(\tau, e_1) + p_2(\tau, e_2)\right) = \partial_x \left((\mu_1 + \mu_2)\partial_x u\right), \\ \partial_t \rho e_j + \partial_x \rho e_j u + p_j(\tau, e_j)\partial_x u = \mu_j \left(\partial_x u\right)^2, \quad j = 1, 2, \end{cases} \tag{3}$$

with $\tau = 1/\rho$. The state space is defined as follows:

$$\Omega = \{\mathbf{w} := (\rho, \rho u, \rho e_1, \rho e_2)^T \in \mathbb{R}^4; \rho > 0,\ \rho u \in \mathbb{R},\ \rho e_1 > 0,\ \rho e_2 > 0\}.$$

We assume that each pressure p_j satisfies the second law of thermodynamics. As a consequence, a specific entropy s_j and a temperature T_j exist such that

$$-T_j\ ds_j = de_j + p_j\ d\tau, \quad j = 1, 2.$$

The function $(\tau, e_j) \to s_j(\tau, e_j)$, $1 \le j \le 2$, is assumed to be convex. Moreover, following the standard mono-pressure gas approach, with $j = 1, 2$ we impose:

$$\frac{\partial s_j}{\partial \tau}(\tau, e_j) = -\frac{p_j(\tau, e_j)}{T_j(\tau, e_j)} < 0, \quad \frac{\partial s_j}{\partial e_j}(\tau, e_j) = -\frac{1}{T_j(\tau, e_j)} < 0,$$
$$\lim_{e_j \to 0+} s_j(\tau, e_j) = +\infty, \quad \lim_{e_j \to +\infty} s_j(\tau, e_j) = -\infty.$$

The system (3) cannot be recast in conservation form, excepted involving very restrictive assumptions. However, an additional conservation law is satisfied by the smooth solutions:

$$\partial_t \rho E(\mathbf{w}) + \partial_x(\rho E(\mathbf{w}) + p_1(\tau, e_1) + p_2(\tau, e_2))u = \partial_x((\mu_1 + \mu_2)u\partial_x u).$$

This additional conservation law governs the evolution of the total energy ρE:

$$\rho E(\mathbf{w}) = \frac{(\rho u)^2}{2\rho} + \rho e_1 + \rho e_2.$$

Let us note that entropy inequalities can be established:

$$\partial_t \rho s_j(\mathbf{w}) + \partial_x \rho s_j(\mathbf{w}) u = -\frac{\mu_j}{T_j}(\partial_x u)^2 \le 0, \quad j = 1, 2. \tag{4}$$

As a consequence, the smooth solutions of (3) satisfy the following additional identity:

$$\mu_2 T_1 \left(\partial_t \rho s_1(\mathbf{w}) + \partial_x \rho s_1(\mathbf{w}) u\right) - \mu_1 T_2 \left(\partial_t \rho s_2(\mathbf{w}) + \partial_x \rho s_2(\mathbf{w}) u\right) = 0. \tag{5}$$

The above relation clearly shows that the entropy dissipation rates do not independently evolve. This relation will play a crucial role to propose a relevant numerical scheme to approximate the solutions of (3).

To conclude this short presentation of the multi-entropy model, let us emphasize that the existence and uniqueness of the traveling solutions can be established.

2.1 Nonlinear projection scheme [2, 12]

This work was done during the author's PhD thesis. The article [12] details the article [2] published in the honor of Phil Roe's 60^{th} birthday.

A numerical method is proposed to approximate the solutions of (3). In the present work, we do not consider sharp approximations of the parabolic regularization. The main purpose is to approximate the viscous shock layers but for coarse meshes. The proposed numerical scheme is based on the following equivalent system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p_1 + p_2) = \partial_x\left((\mu_1 + \mu_2)\partial_x u\right), \\ \partial_t \rho E + \partial_x(\rho E + p_1 + p_2)u = \partial_x\left((\mu_1 + \mu_2)u\partial_x u\right), \\ \mu_1 T_2 \Big\{\partial_t \rho s_2 + \partial_x \rho s_2 u\Big\} - \mu_2 T_1 \Big\{\partial_t \rho s_1 + \partial_x \rho s_1 u\Big\} = 0. \end{cases}$$

For the sake of simplicity in the notations, we set $\mathbf{v} = (\rho, \rho u, \rho E, \rho s_2)$. Let us emphasize that the specific entropy s_1 is a function of $\mathbf{v}$.

We consider an uniform mesh where Δx denotes the constant space increments. As usual, the solution at time t^n is approximated by a piecewise constant function $\mathbf{v}^h(x, t^n) = \mathbf{v}_i^n$ with $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. To evolve this approximation in time, a three step numerical method is adopted.

The first step ($t^n \to t^{n+1,=}$) is devoted to approximate the solutions of the following hyperbolic system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p_1 + p_2) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p_1 + p_2) u = 0, \\ \partial_t \rho s_2 + \partial_x \rho s_2 u = 0, \end{cases} \tag{6}$$

supplemented by the following entropy inequality:

$$\partial_t \{\rho s_1\}(\mathbf{v}) + \partial_x \{\rho s_1 u\}(\mathbf{v}) \leq 0.$$

A finite volume scheme is derived. The considered scheme must satisfy discrete entropy inequalities and the maximum principle of the specific entropy (see Tadmor [115]). Several numerical methods preserve such properties, relaxation schemes for instance (Bouchut [37]). Here, we adopt the Godunov method (see Godlewski-Raviart [67], Leveque [85], Toro [116]):

$$\mathbf{v}_i^{n+1,=} = \mathbf{v}_i^n - \frac{\Delta t}{\Delta x} \big\{ \mathbf{f}(\omega(0^+; \mathbf{v}_i^n, \mathbf{v}_{i+1}^n)) - \mathbf{f}(\omega(0^+; \mathbf{v}_{i-1}^n, \mathbf{v}_i^n)) \big\},$$

where $\mathbf{f}$ denotes the flux function of (6) and $\omega(.; \mathbf{v}_L, \mathbf{v}_R)$ is the exact solution of the Riemann problem associated with the system (6). The time step Δt is restricted according to the following CFL condition:

$$\frac{\Delta t}{\Delta x} \max |\lambda_j(\mathbf{v})| \leq \frac{1}{2}, \tag{7}$$

for all the states $\mathbf{v}$ under consideration. We have set λ_j the eigenvalues of the Jacobian matrix associated with the flux function $\mathbf{f}$.

In the second step ($t^{n+1,=} \to t^{n+1,-}$), the second order operator is approximated. The following system is thus considered:

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t \rho u = \partial_x ((\mu_1 + \mu_2) \partial_x u), \\ \partial_t \rho E = \partial_x ((\mu_1 + \mu_2) u \partial_x u), \\ \partial_t \rho s_2 = -\frac{\mu_2}{T_2} (\partial_x u)^2, \end{cases}$$

where the initial data is given by the approximation $\mathbf{v}_i^{n+1,=}$ issuing from the first step. The following formulas are proposed:

$$\begin{aligned}
\rho_i^{n+1,-} &= \rho_i^{n+1,=},\\
(\rho u)_i^{n+1,-} &= (\rho u)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1+\mu_2)\partial_x u)}_i^{n+1,-},\\
(\rho E)_i^{n+1,-} &= (\rho E)_i^{n+1,=} + \Delta t \overline{\partial_x((\mu_1+\mu_2)u\partial_x u)}_i^{n+1,-},\\
(\rho s_2)_i^{n+1,-} &= (\rho s_2)_i^{n+1,=} - \Delta t \overline{\frac{\mu_2}{T_2}(\partial_x u)^2}_i^{n+1,-},
\end{aligned}$$

where the discrete form of the diffusion operator is given as follows:

$$\begin{aligned}
\overline{\partial_x((\mu_1+\mu_2)\partial_x u)}_i^{n+1,-} &= \frac{\mu_1+\mu_2}{\Delta x^2}\left(M^n u_{i+1} - 2M^n u_i + M^n u_{i-1}\right),\\
\overline{\partial_x(\mu_1+\mu_2)u\partial_x u}_i^{n+1,-} &= \frac{\mu_1+\mu_2}{2\Delta x^2}\left((M^n u_{i+1})^2 - 2(M^n u_i)^2 + (M^n u_{i-1})^2\right),\\
\overline{\frac{\mu_2}{T_2}(\partial_x u)^2}_i^{n+1,-} &= \frac{\mu_2}{2\overline{T_2}_i^{n+1,-}\Delta x^2}\left((M^n u_{i+1} - M^n u_i)^2 + (M^n u_i - M^n u_{i-1})^2\right),
\end{aligned} \tag{8}$$

with $M^n u_i = (u_i^{n+1,-} + u_i^{n+1,=})/2$. Next, we adopt the following formula for the discrete temperature:

$$\overline{T_2}_i^{n+1,-} = \begin{cases} T_2\left(\dfrac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,-}\right), \text{ if } (s_2)_i^{n+1,-} = (s_2)_i^{n+1,=},\\ -\dfrac{\epsilon_2\left(\frac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,-}\right) - \epsilon_2\left(\frac{1}{\rho_i^{n+1,-}}, (s_2)_i^{n+1,=}\right)}{(s_2)_i^{n+1,-} - (s_2)_i^{n+1,=}}, \text{ otherwise.}\end{cases} \tag{9}$$

Here, the viscosity functions μ_1 and μ_2 have been assumed to be constant. The extension to a viscosity defined as a function of the unknowns does not involve difficulties.

At the end of the second step, let us stress that the stability properties imposed at the end of the first step (discrete entropy inequalities and maximum principle satisfied by the specific entropies) are preserved.

This two step scheme gives the L^2-projection scheme. Unfortunately, the numerical results obtained with this scheme are disappointing. As a consequence, we add a third step to correct the approximated solution.

The purpose of the nonlinear projection step ($t^{n+1,-} \to t^{n+1}$) is to enforce the entropy balance law (5) at the discrete level. During this step, the conservative variables are preserved:

$$\rho_i^{n+1} = \rho_i^{n+1,-}, \quad (\rho u)_i^{n+1} = (\rho u)_i^{n+1,-}, \quad (\rho E)_i^{n+1} = (\rho E)_i^{n+1,-}.$$

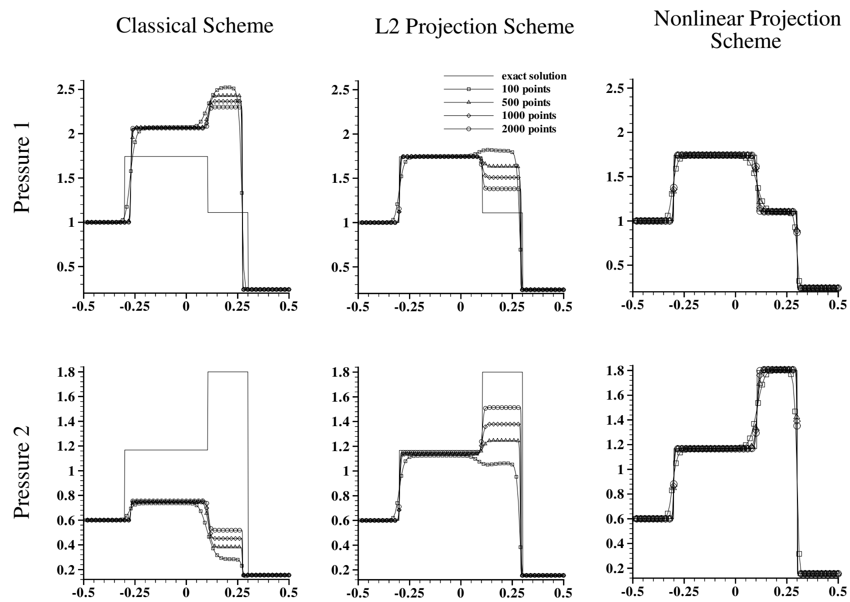

Figure 1: Comparison between a classical scheme, the L^2-projection scheme and the L^2-nonlinear projection scheme to approximate viscous shock layers.

The variable $(\rho s_2)_i^{n+1}$ is corrected to enforce the discrete form of (5). To access such an issue, $(\rho s_2)_i^{n+1}$ solves the following equation:

$$\mu_2 \overline{T_1}_i^{n+1,-}\left(\{\rho s_1\}(\mathbf{v}_i^{n+1}) - (\rho s_1)_i^n + \frac{\Delta t}{\Delta x}\Big\{\{\rho s_1 u\}_{i+1/2}^n - \{\rho s_1 u\}_{i-1/2}^n\Big\}\right) - \tag{10}$$
$$\mu_1 \overline{T_2}_i^{n+1,-}\left((\rho s_2)_i^{n+1} - (\rho s_2)_i^n + \frac{\Delta t}{\Delta x}\Big\{\{\rho s_2 u\}_{i+1/2}^n - \{\rho s_2 u\}_{i-1/2}^n\Big\}\right) = 0,$$

where $\{\rho s_2 u\}_{i+\frac{1}{2}}^n$ is the first step numerical flux function associated with the variable ρs_2 while $\{\rho s_1 u\}_{i+\frac{1}{2}}^n$ denotes the numerical entropy flux function involved in the first step. The existence and uniqueness of $(\rho s_2)_i^{n+1}$ solving (10) can be established.

The nonlinear projection scheme, made of the three steps, preserves all the expected stability properties: discrete entropy inequalities and a maximum principle on the specific entropies.

To illustrate the method, the numerical results obtained with the L^2-nonlinear projection scheme are displayed figure 1. Let us note that the numerical results are compared with the exact solution. This exact solution has been obtained when considering a numerical integration of the traveling wave solutions of (1).

2.2 Multi-fluid flow applications [8]

Now, the multi-entropy model (1) is extended to simulate multi-fluid flows. After [26, 28, 77, 78, 25, 81], where such flows are envisaged, we consider a flow made of n_p components. Each component is associated with a partial density ρ_j, a specific volume fraction v_j, a mass fraction Y_j such that $\sum_j Y_j = 1$. The internal energy of each component is denoted $e_j := e_j(v_j, p_j)$ where p_j is the pressure of the component j. We introduce μ_j the viscosity of the component j and we set

$$\tilde{\mu}_j = \mu_j Y_j \quad \text{and} \quad \mu = \sum_{j=1}^{n_p} \tilde{\mu}_j.$$

Hence, we introduce the following model:

$$\begin{cases} \partial_t Y_j + u \partial_x Y_j = 0, \\ \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p) = \partial_x (\mu \partial_x u), \\ \partial_t (\rho_j e_j) + \partial_x (\rho_j e_j u) + p_j \partial_x u = \tilde{\mu}_j (\partial_x u)^2, \quad 1 \le j \le n_p, \end{cases} \tag{11}$$

where $p = \sum_j p_j$ and where the total density is defined by $1/\rho = \sum_j Y_j v_j$. Each component is assumed to satisfy a perfect gas law. As a consequence, a specific entropy $s_j = \log(\rho^{\gamma_j}/p_j)$ can be associated with each gas where γ_j denotes the adiabatic coefficient of the component j. The system (11) is then rewritten in the following form:

$$\begin{cases} \partial_t \rho Y_j + \partial_x \rho Y_j u = 0, \\ \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = \partial_x (\mu \partial_x u), \\ \partial_t E + \partial_x (E + p) u = \partial_x (\mu u \partial_x u), \\ \frac{\beta_{n_p}}{T_{n_p}} \left\{ \partial_t \rho s_j + \partial_x \rho s_j u \right\} - \frac{\beta_j}{T_j} \left\{ \partial_t \rho s_{n_p} + \partial_x \rho s_{n_p} u \right\} = 0, \quad 1 \le j \le n_p - 1, \end{cases} \tag{12}$$

where $E = \rho u^2/2 + \sum_j \rho_j e_j$ denotes the total energy. In addition, let us remark that the entropy s_{n_p} is a function of the other unknowns and it is given by the following definition:

$$s_{n_p} = -\log \left(\frac{\gamma_{n_p} - 1}{\rho^{\gamma_{n_p}}} \left(E - \rho \frac{u^2}{2} - \sum_{j=1}^{n_p - 1} \frac{\rho^{\gamma_j}}{\gamma_j - 1} \mathrm{e}^{-s_j} \right) \right).$$

The parameters β_j are thus defined by $\beta_j = \mu_j / \sum_j \mu_j$.

It is clear that the system (12) is nothing but the multi-entropy model (1) supplemented by conservation laws to govern the evolution of Y_j. As a consequence, a nonlinear projection scheme is adopted to approximate the solution of the system (12).

One of the main originalities of the model is the possibility of involving arbitrary choice of the viscosity functions. A relevant characterisation of the fluid interfaces can be thus obtained. For example, a bi-fluid flow simulation is now proposed where the viscosity functions are defined as follows:

$$\mu_j = (Y_j T_j)^{m_j} \bar{\mu}_j,$$

where $m_j > 0$ is a fixed constant and $\bar{\mu}_j > 0$ must be chosen (i.e., a constant next to the value of the physical viscosity of the considered gas).

This model is used to simulate a shock-bubble interaction as proposed by Quirk-Karni [102]. Initially, a shock wave propagates into a gas with density $\rho = 1$. The shock interacts with a refrigerant R22 bubble with density $\rho = 3.154$. The obtained numerical results, displayed figure 2, are in accordance with the dynamics of the shock-bubble interaction observed in the physical experiments by Haas-Sturtevant [71].

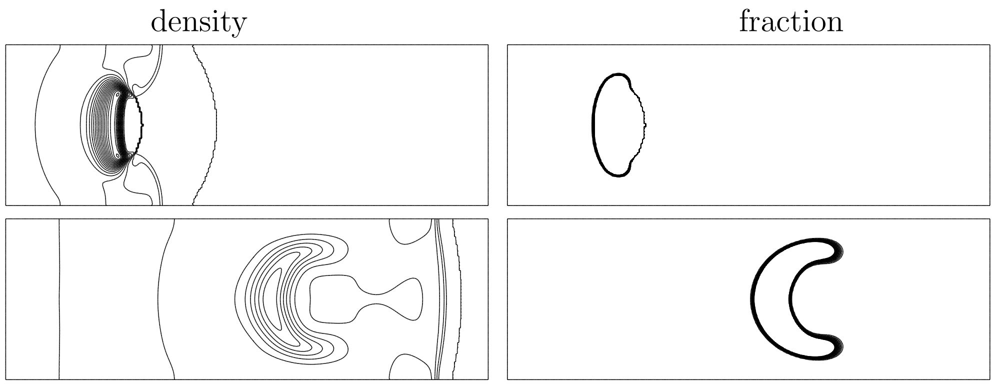

Figure 2: Density and material interface for interaction of shock with velocity 1.22 and a R22 cylindrical bubble [102]. Solutions at time $47\mu s$ and $94\mu s$.

2.3 A multi-fluid turbulent flow [4]

The above multi-fluid model (11) is extended to simulate turbulent flows. The following model is considered:

$$\begin{cases} \partial_t Y_j + u\partial_x Y_j = 0, \\ \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \partial_x((\mu + \mu_\tau)\partial_x u), \\ \partial_t(\rho_j e_j) + \partial_x(\rho_j e_j u) + p_j \partial_x u = \tilde{\mu}_j(\partial_x u)^2 + \rho_j \epsilon, \quad 1 \le j \le n_p, \\ \partial_t \rho k + \partial_x \rho k u + \frac{2}{3}\rho k \partial_x u = \mu_\tau(\partial_x u)^2 - \rho\epsilon, \\ \partial_t \rho\epsilon + \partial_x \rho\epsilon u + \frac{2}{3} C_1 \rho\epsilon\partial_x u = C_1 \frac{\epsilon}{k}\mu_\tau(\partial_x u)^2 - C_2 \rho\frac{\epsilon^2}{k}, \end{cases}$$

where C_1 and C_2 are constants of the model, and $\mu_\tau = C_\mu \rho k^2/\epsilon$ denotes the turbulent viscosity (see Mohammadi-Pironneau [96]). After straightforward computations, this model writes as a multi-entropy model supplemented by source terms:

$$\begin{cases} \partial_t \rho Y_j + \partial_x \rho Y_j u = 0, \\ \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p) = \partial_x(\mu \partial_x u), \\ \partial_t E + \partial_x(E+p)u = \partial_x(\mu u \partial_x u), \\ \frac{\beta_{n_p}}{T_{n_p}} \{\partial_t \rho s_j + \partial_x \rho s_j u\} - \frac{\beta_j}{T_j} \left\{\partial_t \rho s_{n_p} + \partial_x \rho s_{n_p} u\right\} = 0, \quad 1 \leq j \leq n_p - 1, \\ \frac{\mu_{n_p}}{T_{n_p}} \{\partial_t \rho s_\tau + \partial_x \rho s_\tau u\} - Y_{n_p} \mu_\tau \frac{2/3}{\rho^{2/3}} \{\partial_t \rho s_{n_p} + \partial_x \rho s_{n_p} u\} = \frac{\rho\epsilon}{T_{n_p}} \frac{2/3}{\rho^{2/3}} (\mu_{n_p} - Y_{n_p} \mu_\tau), \\ \partial_t \rho \frac{k^{C_1}}{\epsilon} + \partial_x \rho \frac{k^{C_1}}{\epsilon} u = (C_2 - C_1) \rho k^{C_1 - 1}, \end{cases}$$

where the turbulent entropy s_τ is given by $s_\tau = \frac{\frac{2}{3}\rho k}{\rho^{5/3}}$.

Once again, the multi-entropy model is considered but with additional source terms. The nonlinear projection scheme is applied and it is completed when involving a standard explicit discrete form of the source terms.

To illustrate the interest of the numerical method, we simulate a Richtmeyer-Meshkov instability. Initially, two fluids are separated by an oscillating interface. Next, a shock wave interacts with the interface. The numerical results are displayed in figure 3. Both laminar and turbulent model are compared. This numerical experiment shows that the instability grows more quickly when involving turbulence.

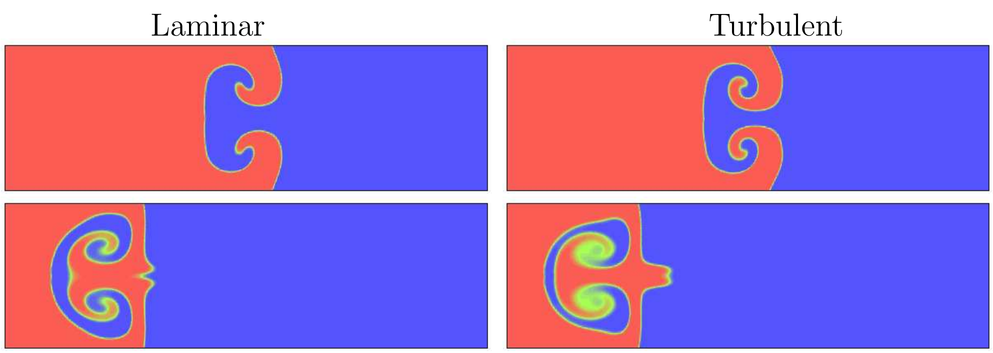

Figure 3: Turbulent behavior of a Richtmeyer-Meshkov instability. Solutions at time $t = 1.05\,ms$ and $t = 2.33\,ms$.

2.4 A combustion model [3]

This work enters the Post-doc of D. Reignier (grant CEA).

In the present work, a combustion turbulent flow model (see Baurle-Girimaji [33]) is considered. With standard notations, the model is governed by the following system:

$$\left\{\begin{array}{l} \partial_t \rho C_j + \partial_x \rho C_j u = \partial_x((\rho D + \rho D_t)\partial_x C_j), \quad 1 \leq j \leq N, \\ \partial_t \rho u + \partial_x(\rho u^2 + p + \frac{2}{3}\rho k) = \partial_x((\mu + \mu_\tau)\partial_x u), \\ \partial_t E + \partial_x(E + p + \frac{2}{3}\rho k)u = \partial_x((\mu + \mu_\tau)u\partial_x u), \\ \partial_t \rho k + \partial_x \rho k u + \frac{2}{3}\rho k \partial_x u = \mu_\tau(\partial_x u)^2, \\ \partial_t \rho\epsilon + \partial_x \rho\epsilon u + \frac{2}{3}C_{\epsilon 1}\rho\epsilon\partial_x u = C_{\epsilon 1}\frac{\epsilon}{k}\mu_\tau(\partial_x u)^2, \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l} \partial_t \rho k_h + \partial_x \rho k_h u = \partial_x(\frac{\lambda_t}{C_p}\partial_x k_h) + 2\frac{\lambda_t}{C_p}(\partial_x p^\star)^2, \\ \partial_t \rho Q + \partial_x \rho Q u = \partial_x(\rho D_t \partial_x Q) + 2\frac{\lambda_t}{S_{ct}}\sum_{1\leq j\leq N}(\partial_x C_j)^2, \end{array}\right. \tag{14}$$

with $p^\star$ given by $p^\star = p + \frac{2}{3}\rho k$ and where the perfect gas law is assumed:

$$p = (\gamma - 1)\left(E - \rho\frac{u^2}{2} - \rho k\right), \quad \gamma \in (1, 3],$$

The quantities C_j denote the mass fraction of each component while k_h denotes the variance of the enthalpy and Q is the variance of the mass fraction. In the present study, the source terms have been omitted; namely the turbulence source terms but also combustion source terms (see Baurle-Girimaji [33] for further details concerning the PDF model). After the above sections devoted to approximate the solutions of the multi-entropy model, our purpose is to derive a numerical method on a coarse grid able to capture shock layers according to the parabolic operator prescribed by the physics.

In order to propose a relevant scheme, let us note that the system (13) is independent of the system (14). In addition, the system (13) enters the framework of the multi-entropy model (see the previous sections). As a consequence, a nonlinear projection scheme is adopted to approximate the solutions of (13).

Concerning the sub-system (14), the transport terms involved in (14) are approximated when considering an usual finite volume method (Godunov scheme for instance). The main difficulties arise from the production terms $(\partial_x p^\star)^2$ and $(\partial_x C_j)^2$. First, let us underline the following identity:

$$(\partial_x C_j)^2 = -\frac{1}{\rho D + \rho D_t}\left\{\left(\partial_t \rho\frac{C_j^2}{2} + \partial_x \rho\frac{C_j^2}{2}u\right) - \partial_x((\rho D + \rho D_t)C_j\partial_x C_j)\right\}.$$

This relation clearly shows that the production of the variable Q is issued from the entropy production $\rho C_j^2/2$. Hence, we write the following evolution law satisfied by Q:

$$\begin{aligned} &(\rho D + \rho D_t)\left\{\partial_t \rho Q + \partial_x \rho Q u - \partial_x(\rho D_t \partial_x Q)\right\} + \\ &2\frac{\lambda_t}{S_{ct}}\sum_{1\leq j\leq N}\left\{\partial_t \rho\frac{C_j^2}{2} + \partial_x \rho\frac{C_j^2}{2}u - \partial_x((\rho D + \rho D_t)C_j\partial_x C_j)\right\} = 0. \end{aligned} \tag{15}$$

This relation must be compared with (5). The nonlinear projection scheme is easily extended to approximate (15).

Next, we establish that the production rate of k_h, given by $(\partial_x p^\star)^2$, and the entropy production rate, $(\partial_x u)^2$, satisfy a formal relation across the shock layers. Indeed, we can prove the existence, across a shock layer, of a function Θ defined as follows:

$$\begin{aligned} (\partial_x p^\star)^2 &= \Theta(\mathbf{w})(\partial_x u)^2, \\ &= \Theta(\mathbf{w})\frac{\rho^{\gamma-1}}{\gamma-1}\frac{1}{\mu}\left\{\partial_t \rho \frac{p}{\rho^\gamma} + \partial_x \rho \frac{p}{\rho^\gamma} u\right\}. \end{aligned} \tag{16}$$

based on this identity, we propose a relevant discrete formula of $(\partial_x p^\star)^2$.

2.5 Multiple solutions multiples for turbulence models [13]

This work was done during the author's PhD thesis.

This analysis concerns the behavior of the traveling wave solutions of the multi-entropy model when considering the relaxation source terms. The model writes as follows:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \qquad x \in \mathbb{R},\ t > 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p_1 + p_2) = \partial_x\left((\mu_1(\rho, T_1) + \mu_2(\rho, T_2))\partial_x u\right), \\ \partial_t p_1 + \partial_x p_1 u + (\gamma_1 - 1)p_1 \partial_x u = (\gamma_1 - 1)\mu_1(\rho, T_1)\left(\partial_x u\right)^2 + (\gamma_1 - 1)\rho\epsilon(T_2), \\ \partial_t p_2 + \partial_x p_2 u + (\gamma_2 - 1)p_2 \partial_x u = (\gamma_2 - 1)\mu_2(\rho, T_2)\left(\partial_x u\right)^2 - (\gamma_2 - 1)\rho\epsilon(T_2). \end{cases} \tag{17}$$

The temperatures are given by $p_i = (\gamma_i - 1)\rho T_i$ where the adiabatic coefficients satisfy $1 < \gamma_1 < \gamma_2$. The source term is assumed to satisfy

$$\begin{aligned} &\epsilon(T_2) > 0 \text{ for all } T_2 > 0 \quad \text{with} \quad \lim_{T_2 \to 0^+} \epsilon(T_2) = 0, \\ &\epsilon \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{R}^+) \quad \text{with} \quad \lim_{T_2 \to 0^+} \frac{d}{dT_2}\epsilon(T_2) = 0. \end{aligned} \tag{18}$$

The role of the source term is to relax the pressure p_2 to zero. These considerations lead us to consider the following natural phase space:

$$\Omega = \left\{\mathbf{w} = {}^t(\rho, \rho u, p_1, p_2) \in \mathbb{R}^4 / \rho > 0, \rho u \in \mathbb{R}, p_1 > 0, p_2 \geq 0\right\}.$$

In addition, the viscosity functions are assumed to satisfy $\mu_1 + \mu_2 > 0$.

Let us note from now on that the present model enters the framework of the one transport equation turbulence model where μ_1 denotes the laminar viscosity and μ_2 is the turbulent viscosity. Let us just mention that T_2 coincides with the specific kinetic turbulent energy usually denoted by k with adiabatic coefficient given by $\gamma_2 = 5/3$. Here the relaxation term $\epsilon(k) = \mathcal{L}k^{3/2}$, $\mathcal{L} > 0$, is the turbulent dissipation rate proposed in the so-called mixing length model where $\mathcal{L}$ denotes the inverse of the mixing length (for instance, see Mohammadi-Pironneau [96] or Hirsh [74] and the references therein).

Here, the viscosity function is assumed to be smooth and belongs to $\mathcal{C}^1(\mathbb{R}_\star^+ \times \mathbb{R}^+, \mathbb{R}^+)$. In the literature, the viscosity μ_2 prescribed by the physics does not satisfy such smoothness property, but the following limit holds:

$$\lim_{y \to 0^+} \mu_2(\rho, y) = 0. \tag{19}$$

In general, the choices of viscosity μ_2 as prescribed in the literature are differentiable over $\mathbb{R}_\star^+ \times \mathbb{R}_\star^+$ but not with $y = 0$. With a fixed α in $[\frac{1}{2}, 1)$, the behavior of μ_2 for $y > 0$ small enough is given by $\mathcal{O}(y^\alpha)$. It turns that such models which fail smoothness, satisfy the following three properties:

$$\lim_{y \to 0^+} \frac{\mu_2(\rho, y)}{y^\alpha} = h(\rho) > 0, \quad \rho > 0, \tag{20}$$

$$(\rho, z) \in (0, +\infty) \times [0, +\infty) \to \Phi(\rho, z) = \mu_2(\rho, z^{\frac{1}{1-\alpha}}) \in \mathbb{R}^+ \text{ is Lipschitz continuous,} \tag{21}$$

$$(\rho, z) \in (0, +\infty) \times [0, +\infty) \to \Psi(\rho, z) = \frac{\mu_2(\rho, z^{\frac{1}{1-\alpha}})}{z^{\frac{\alpha}{1-\alpha}}} \in \mathbb{R}^+ \text{ is Lipschitz continuous.} \tag{22}$$

With such hypothesis satisfied by μ_2, the relaxation source terms is assumed to verify:

$$\begin{aligned} &\epsilon(y) > 0 \quad \text{for all } y > 0 \quad \text{with } \lim_{y \to 0^+} \frac{\epsilon(y)}{y^\alpha} = 0, \\ &y \in \mathbb{R}^+ \to \frac{\epsilon(y)}{y^\alpha} \in \mathbb{R}^+ \quad \text{is Lipschitz continuous.} \end{aligned} \tag{23}$$

Now, we turn considering the traveling wave solutions. Let us recall that such solutions are smooth functions $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^+, \Omega)$ in the form $\mathbf{w}(x,t) = \tilde{\mathbf{w}}(x - \sigma t)$ with $\sigma \in \mathbb{R}$ which satisfy

$$\begin{cases} \lim\limits_{\xi \to -\infty} \tilde{\mathbf{w}}(\xi) = \mathbf{v}_L, \quad \lim\limits_{\xi \to +\infty} \tilde{\mathbf{w}}(\xi) = \mathbf{v}_R, \quad \xi = x - \sigma t, \\ \lim\limits_{\xi \to \pm\infty} \tilde{\mathbf{w}}'(\xi) = \lim\limits_{\xi \to \pm\infty} \tilde{\mathbf{w}}''(\xi) = 0, \end{cases} \tag{24}$$

with $\mathbf{w}_L$ and $\mathbf{w}_R$ in Ω. Let us note that the condition (24) is satisfied if and only if the source term vanishes at infinity. In other words, we have to impose $(p_2)_L = 0$ and $(p_2)_R = 0$. The following statement is thus proved:

Theorem 2.1. *Let $\mathbf{w}_L \in \Omega$ with $(p_2)_L = 0$. Let $\sigma \in \mathbb{R}$ the propagation speed of the traveling wave be prescribed under the Lax condition:*

$$u_L - \sqrt{\frac{\gamma_1 (p_1)_L}{\rho_L}} > \sigma.$$

i) Assume that the viscosities μ_1 and μ_2 belong to $\mathcal{C}^1(\mathbb{R}_\star^+ \times \mathbb{R}^+, \mathbb{R}^+)$ while the relaxation term $\epsilon(T_2)$ obeys (18). Then there exists a unique (up to some translation) traveling wave

solution issuing from $\mathbf{w}_L$ *and arriving at a state* $\mathbf{w}_R \in \Omega$ *with* $(p_2)_R = 0$.
ii) Assume that μ_1 *is smooth and belongs to* $\mathcal{C}^1(\mathbb{R}_\star^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *while* μ_2 *obeys (19)-(22). Assume that* $\epsilon(T_2)$ *satisfies (23). Then there exists infinitely many distinct traveling wave solutions connecting* $\mathbf{w}_L$ *to an end state* $\mathbf{w}_R \in \Omega$ *with* $(p_2)_R = 0$.

For example, the most popular choice is given by $\mu_2(\rho, y) = C\rho y^{\frac{1}{2}}$ while $\epsilon(T_2) = \mathcal{L}y^{3/2}$. The viscosity μ_2 obeys (19)-(22) with $\alpha = \frac{1}{2}$. The turbulence dissipation rate, $\epsilon(T_2)$, satisfies (23) with, once again, $\alpha = \frac{1}{2}$. As a consequence, this standard model admits infinitely many distinct traveling wave solutions.

3 Several applications of the relaxation schemes [5, 7, 9, 16, 22, 24]

The relaxation schemes have been recently introduced by Jin-Xin [76] to approximate the weak solutions of hyperbolic systems of conservation laws (see also the work of Suliciu [113, 114] where similar ideas were used).

Motivated by the works of Liu [94] and Chen-Levermore-Liu [50], we propose to approximate the weak solutions of hyperbolic system of conservation laws, denoted *equilibrium system*, by the weak solutions of a relevant first order system with singular perturbations: *the relaxation model*. Several papers are devoted to such an approach. The reader is referred to Aregba-Natalini [30] and Natalini [97] where this problem is analyzed in the framework of the scalar laws.

Let us stress that the choice of the relaxation model turns out to be crucial. Indeed, the accuracy, the robustness and the stability of the numerical method occur from a relevant relaxation model (see Jin-Xin [76], Bouchut [37], Coquel-Perthame [57]). Some relaxation models allow revisiting several popular finite volume methods as the Lax-Friedrichs scheme or the HLLC scheme (see Bouchut [37] where the relaxation schemes are shown to coincide with some standard schemes). Let us note that to know the relevant first order system (namely the relaxation system) associated with a scheme (namely a relaxation scheme) turns out to be a powerful tool to establish stability properties as discrete entropy inequalities or the numerical invariance principle satisfied by the state space. In the framework of the 3×3 Euler equations, Bouchut [36] and next Chalons [52] (with some extension to the multi-entropy model), prove stability properties satisfied by the relaxation scheme when invoking two distinct approaches. Here, these results are extended to other models.

3.1 The 10-moment model [9, 16]

The communication [16] is an introduction to the article [9].

As long as the local kinetic equilibrium is not satisfied, the Euler equations cannot be considered and alternative model must be derived. Recently, Levermore [92] has generated a hierarchy of moment closure system. The simplest model of this hierarchy is the Euler equations. The second derived model, investigated by Levermore-Morokoff [93], admits 10 equations: *the 10-moment Gaussian closure.*

In one space dimension, the 10-moment model writes as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u_1) = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \partial_t(\rho u_1) + \partial_x(\rho u_1^2 + p_{11}) = 0, \\ \partial_t(\rho u_2) + \partial_x(\rho u_1 u_2 + p_{12}) = 0, \\ \partial_t E_{11} + \partial_x((E_{11} + p_{11})u_1) = 0, \\ \partial_t E_{22} + \partial_x(E_{22}u_1 + p_{12}u_2) = 0, \\ \partial_t E_{12} + \partial_x(E_{12}u_1 + \frac{1}{2}(p_{11}u_2 + p_{12}u_1)) = 0, \end{cases} \tag{25}$$

where the state law reads:

$$E_{ij} = \frac{1}{2}\rho u_i u_j + \frac{1}{2}p_{ij}, \quad 1 \leq i \leq j \leq 2.$$

For the sake of simplicity in the notations, the following abstract form is adopted:

$$\begin{aligned} &\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0, \\ &\mathbf{w} = {}^t(\rho, \rho u_1, \rho u_2, E_{11}, E_{22}, E_{12}). \end{aligned}$$

where $\mathbf{w} \in \Omega$ with

$$\Omega = \left\{ \mathbf{w} \in \mathbb{R}^6;\ \rho > 0,\ (u_1, u_2) \in \mathbb{R}^2,\ p_{11} > 0,\ p_{11}p_{22} - p_{12}^2 > 0 \right\}.$$

The system under consideration is hyperbolic over Ω. Moreover, the weak solutions of (25) satisfy the following entropy inequalities:

$$\begin{aligned} &\partial_t \rho \mathcal{F}(\ln s) + \partial_x \rho \mathcal{F}(\ln s) u_1 = 0, \quad s := s(\mathbf{w}) = \frac{p_{11}}{\rho^3}, \\ &\partial_t \rho \mathcal{G}(\ln \sigma) + \partial_x \rho \mathcal{G}(\ln \sigma) u_1 = 0, \quad \sigma := \sigma(\mathbf{w}) = \frac{p_{11}p_{22} - p_{12}^2}{\rho^4}, \end{aligned}$$

where the functions $\mathcal{F}, \mathcal{G} : \mathbb{R} \to \mathbb{R}$ verify

$$\begin{aligned} &\mathcal{F}'(y) < 0, \quad \frac{\mathcal{F}''(y)}{\mathcal{F}'(y)} < \frac{1}{3}, \quad \forall y \in \mathbb{R}, \\ &\mathcal{G}'(y) < 0, \quad \frac{\mathcal{G}''(y)}{\mathcal{G}'(y)} < \frac{1}{4}, \quad \forall y \in \mathbb{R}, \end{aligned} \tag{26}$$

in order to enforce the functions $\mathbf{w} \to \rho\mathcal{F}(\ln s(\mathbf{w}))$ and $\mathbf{w} \to \rho\mathcal{G}(\ln \sigma(\mathbf{w}))$ to be convex.

We propose to derive a relaxation scheme. The obtained numerical method will be shown to be robust (the numerical invariance of Ω will be preserved) and stable (discrete entropy inequalities will be established). To access such an issue, a relaxation model is introduced where both equilibrium pressures, p_{11} and p_{12}, are approximated by new variables, π_{11} and π_{12}. These new unknowns are next relaxed to the equilibrium pressures. The considered relaxation model reads as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u_1) = 0, \quad t > 0, \ x \in \mathbb{R}, \\ \partial_t(\rho u_1) + \partial_x(\rho u_1^2 + \pi_{11}) = 0, \\ \partial_t(\rho u_2) + \partial_x(\rho u_1 u_2 + \pi_{12}) = 0, \\ \partial_t E_{11} + \partial_x((E_{11} + \pi_{11})u_1) = 0, \\ \partial_t E_{22} + \partial_x(E_{22}u_1 + \pi_{12}u_2) = 0, \\ \partial_t E_{12} + \partial_x(E_{12}u_1 + \dfrac{1}{2}(\pi_{11}u_2 + \pi_{12}u_1)) = 0, \\ \partial_t \rho\pi_{11} + \partial_x(\rho\pi_{11}u_1 + a^2 u_1) = \mu\rho(p_{11} - \pi_{11}), \\ \partial_t \rho\pi_{12} + \partial_x(\rho\pi_{12}u_1 + a^2 u_2) = \mu\rho(p_{12} - \pi_{12}), \end{cases} \tag{27}$$

where the relaxation parameter $a > 0$ must satisfy the well-known sub-characteristic Whitham condition [121]:

$$\frac{a^2}{\rho} > 3p_{11}. \tag{28}$$

In the formal limit of μ to infinity, note that the relaxation system (27) aims to restore the initial equilibrium system (25). Indeed, we obtain the formal equalities $\pi_{11} = p_{11}$ and $\pi_{12} = p_{12}$ as soon as μ tends to infinity. As a consequence, the conservation laws of the momentum (ρu_1 and ρu_2) and the total energy (E_{11}, E_{22} and E_{12}) of the relaxation model (27) give those of the equilibrium system (25).

With clear notations, let us introduce the following abstract form of the system (27):

$$\begin{cases} \partial_t \mathbf{W} + \partial_x \mathbf{F}_a(\mathbf{W}) = \mu\mathcal{R}(\mathbf{W}), \\ \mathbf{W} = {}^t(\rho, \rho u_1, \rho u_2, E_{11}, E_{22}, E_{12}, \rho\pi_{11}, \rho\pi_{12}), \end{cases} \tag{29}$$

where $\mathbf{W}$ belongs to $\mathcal{V} = \{\mathbf{W} \in \mathbb{R}^8; \rho > 0\}$.

The derivation of the relaxation model (27) is dictated by a linear degeneracy of all the fields which makes the associated Riemann problem easily solvable. This property is crucial to derive a relaxation scheme.

Lemma 3.1. *Let be given $a > 0$ and assume $\mu = 0$. The first order system (27) is hyperbolic. It admits $\lambda_1 = u_1 - a/\rho$, $\lambda_2 = u_1$ and $\lambda_3 = u_1 + a/\rho$ as eigenvalues. All the fields are linearly degenerated. In addition, as long as the parameter a satisfies*

$$\begin{aligned} &(u_1)_L - \frac{a}{\rho_L} < u_1^\star < (u_1)_R + \frac{a}{\rho_R}, \\ &u_1^\star = \frac{(u_1)_L + (u_1)_R}{2} + \frac{(\pi_{11})_L - (\pi_{11})_R}{2a}, \end{aligned} \tag{30}$$

where $\mathbf{W}_L \in \mathcal{V}$ *and* $\mathbf{W}_R \in \mathcal{V}$ *are constant states to define the initial data of a Riemann problem:*

$$\mathbf{W}_0(x) = \begin{cases} \mathbf{W}_L & \text{if } x < 0, \\ \mathbf{W}_R & \text{if } x > 0, \end{cases} \tag{31}$$

the solution of (27)-(31) with $\mu = 0$ *is given by*

$$\mathbf{W}(x,t) = \begin{cases} \mathbf{W}_L & \text{if} & & \frac{x}{t} & < \lambda_1(\mathbf{W}_L), \\ \mathbf{W}_1 & \text{if} & \lambda_1(\mathbf{W}_L) < & \frac{x}{t} & < \lambda_2(\mathbf{W}_1), \\ \mathbf{W}_2 & \text{if} & \lambda_2(\mathbf{W}_2) < & \frac{x}{t} & < \lambda_3(\mathbf{W}_R), \\ \mathbf{W}_R & \text{if} & \lambda_3(\mathbf{W}_R) < & \frac{x}{t}, & \end{cases} \tag{32}$$

where $\lambda_2(\mathbf{W}_1) = \lambda_2(\mathbf{W}_2) = u_1^\star$. *With* $1 \le i \le j \le 2$, *we set*

$$\begin{aligned}
u_i^\star &= \frac{(u_i)_L + (u_i)_R}{2} + \frac{(\pi_{1i})_L - (\pi_{1i})_R}{2a}, \\
\pi_{1i}^\star &= \frac{(\pi_{1i})_L + (\pi_{1i})_R}{2} + \frac{a}{2}((u_i)_L - (u_i)_R), \\
\frac{1}{\rho^1} &= \frac{1}{\rho_L} + \frac{u_1^\star - (u_1)_L}{a}, \quad \frac{1}{\rho^2} = \frac{1}{\rho_R} + \frac{(u_1)_R - u_1^\star}{a} \\
e_{ij}^1 &= \left(\frac{(E_{ij})_L}{\rho_L} - \frac{(u_i)_L(u_j)_L}{2}\right) - \frac{1}{2a^2}\left((\pi_{1i})_L(\pi_{1j})_L - \pi_{1i}^\star\pi_{1j}^\star\right), \\
e_{ij}^2 &= \left(\frac{(E_{ij})_R}{\rho_R} - \frac{(u_i)_R(u_j)_R}{2}\right) - \frac{1}{2a^2}\left((\pi_{1i})_R(\pi_{1j})_R - \pi_{1i}^\star\pi_{1j}^\star\right), \\
E_{ij}^1 &= \rho^1\frac{u_i^\star u_j^\star}{2} + \rho^1 e_{ij}^1, \quad E_{ij}^2 = \rho^2\frac{u_i^\star u_j^\star}{2} + \rho^2 e_{ij}^2.
\end{aligned}$$

The constant states $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *in* $\mathcal{V}$ *reads as follows:*

$$\mathbf{W}_i = {}^t(\rho^i, \rho^i u_1^\star, \rho^i u_2^\star, E_{11}^i, E_{22}^i, E_{12}^i, \rho^i\pi_{11}^\star, \rho^i\pi_{12}^\star), \quad 1 \le i \le 2.$$

On the basis of the above relaxation model, the associated numerical scheme is now described when considering an uniform mesh. We note Δx the constant space increment. At time t^n, we assume that a piecewise constant approximate equilibrium solution of (25), denoted $\mathbf{w}^h$, is known where $\mathbf{w}^h$ is defined by:

$$\mathbf{w}^h(x,t^n) = \mathbf{w}_i^n = {}^t\left(\rho_i^n, (\rho u_1)_i^n, (\rho u_2)_i^n, (E_{11})_i^n, (E_{22})_i^n, (E_{12})_i^n\right),$$

for all $x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$. To evolve in time this approximation, a two steps method is adopted.

Evolution: $t^n \to t^{n+1,-}$
For all $0 < t < \Delta t$, the Cauchy problem for (27) with $\mu = 0$ is solved where the equilibrium initial data is given as follows:

$$\begin{aligned}\mathbf{W}^h(x,t^n) &= \mathbf{W}_i^n \\ &= {}^t\left(\rho_i^n, (\rho u_1)_i^n, (\rho u_2)_i^n, (E_{11})_i^n, (E_{22})_i^n, (E_{12})_i^n, (\rho\pi_{11})_i^n, (\rho\pi_{12})_i^n\right),\end{aligned}$$

where $(\pi_{11})_i^n = (p_{11})_i^n$ and $(\pi_{12})_i^n = (p_{12})_i^n$.

Under the CFL like condition

$$\frac{\Delta t}{\Delta x} \max\left(|\lambda_i(\mathbf{W}^h(x,t^n))|\right) \leq \frac{1}{2}, \tag{33}$$

where $(\lambda_i)_{1\leq i\leq 3}$ denote the eigenvalues introduced within lemma 3.1, at time $t^n + \Delta t$ the solution $\mathbf{W}^h$ is made of the juxtaposition of the non-interacting Riemann problem set at the cell interfaces $x_{i+\frac{1}{2}}$. After lemma 3.1, the solution of each Riemann problem is known. Let us note that the relaxation parameter $a_{i+\frac{1}{2}}$ is chosen in order to satisfy the Whitham condition (28) and the order condition (30) at each cell interface $x_{i+\frac{1}{2}}$ with $\mathbf{W}_L = \mathbf{W}_i^n$ and $\mathbf{W}_R = \mathbf{W}_{i+1}^n$. Hence, we set

$$\mathbf{W}_i^{n+1,-} = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{W}^h(x, t^n+\Delta t)dx.$$

Relaxation : $t^{n+1,-} \to t^{n+1}$
The approximation $\mathbf{W}_i^{n+1,-}$ is updated when solving

$$\partial_t \mathbf{W} = \mu\mathcal{R}(\mathbf{W}),$$

with $\mathbf{W}_i^{n+1,-}$ as initial data and in the limit of μ to infinity. Straightforward computations give the following updated state vector:

$$\begin{aligned}\mathbf{w}^{n+1}(x) = {}^t\Big(&\rho_i^{n+1,-}, (\rho u_1)_i^{n+1,-}, (\rho u_2)_i^{n+1,-}, \\ &(E_{11})_i^{n+1,-}, (E_{22})_i^{n+1,-}, (E_{12})_i^{n+1,-}\Big), \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),\end{aligned}$$

with $(\pi_{11})_i^{n+1} = (p_{11})_i^{n+1}$ and $(\pi_{12})_i^{n+1} = (p_{12})_i^{n+1}$.

The obtained relaxation scheme summarizes as follows:

$$\begin{aligned}&\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+\frac{1}{2}}^n - \mathbf{f}_{i-\frac{1}{2}}^n\right), \\ &\mathbf{f}_{i+\frac{1}{2}}^n = \mathbf{F}_{a_{i+\frac{1}{2}}}\left(\mathbf{W}^h(x_{i+\frac{1}{2}}, t^n+\Delta t)\right)|_{[\rho,\rho u_1,\rho u_2,E_{11},E_{22},E_{12}]}.\end{aligned} \tag{34}$$

Let us note that the present scheme is, in fact, a standard conservative 3-points scheme. Indeed, by definition of $\mathbf{W}^h$, the vector $\mathbf{W}^h(x_{i+\frac{1}{2}}, t^n + \Delta t)$ solely dependents on $\mathbf{W}_i^n$ and $\mathbf{W}_{i+1}^n$. But $\mathbf{W}_i^n$ is an equilibrium state which only depends on $\mathbf{w}_i^n$.

Now, the following statement establishes stability properties satisfied by the relaxation scheme.

Theorem 3.2. *Consider the relaxation scheme (34) and assume the CFL condition (33). Assume that the relaxation parameters satisfy the sub-characteristic Whitham condition (28) and the order condition (30).*
i) If $\mathbf{w}_i^n \in \Omega$ *for all* $i \in \mathbb{Z}$ *then* $\mathbf{w}_i^{n+1} \in \Omega$ *for all* $i \in \mathbb{Z}$.
ii) For all functions $\mathcal{F}$ *and* $\mathcal{G}$ *such that (26) holds, the following discrete entropy inequalities are satisfied:*

$$\rho_i^{n+1}\mathcal{F}(\ln s_i^{n+1}) - \rho_i^n \mathcal{F}(\ln s_i^n) + \frac{\Delta t}{\Delta x}\Big(\{\rho\mathcal{F}(\ln s)u_1\}_{i+\frac{1}{2}}^n - \{\rho\mathcal{F}(\ln s)u_1\}_{i-\frac{1}{2}}^n \Big) \leq 0,$$
$$\rho_i^{n+1}\mathcal{G}(\ln \sigma_i^{n+1}) - \rho_i^n \mathcal{G}(\ln \sigma_i^n) + \frac{\Delta t}{\Delta x}\Big(\{\rho\mathcal{G}(\ln \sigma)u_1\}_{i+\frac{1}{2}}^n - \{\rho\mathcal{G}(\ln \sigma)u_1\}_{i-\frac{1}{2}}^n \Big) \leq 0,$$

with

$$s_i^n = \frac{(p_{11})_i^n}{(\rho_i^n)^3}, \quad \sigma_i^n = \frac{(p_{11})_i^n (p_{22})_i^n - ((p_{12})_i^n)^2}{(\rho_i^n)^4}.$$

The numerical entropy flux functions are defined by

$$\{\rho\mathcal{F}(\ln s)u_1\}_{i+\frac{1}{2}}^n = (\mathbf{f}_\rho)_{i+\frac{1}{2}}^n \times \begin{cases} \mathcal{F}(\ln s_i^n) & \text{if } (\mathbf{f}_\rho)_{i+\frac{1}{2}}^n > 0, \\ \mathcal{F}(\ln s_{i+1}^n) & \text{otherwise}, \end{cases}$$
$$\{\rho\mathcal{G}(\ln \sigma)u_1\}_{i+\frac{1}{2}}^n = (\mathbf{f}_\rho)_{i+\frac{1}{2}}^n \times \begin{cases} \mathcal{G}(\ln \sigma_i^n) & \text{if } (\mathbf{f}_\rho)_{i+\frac{1}{2}}^n > 0, \\ \mathcal{G}(\ln \sigma_{i+1}^n) & \text{otherwise}, \end{cases}$$

where $(\mathbf{f}_\rho)_{i+\frac{1}{2}}^n$ *is the first component of the numerical flux function (34).*
iii) The following maximum principle is satisfied (see Tadmor [115]):

$$s_i^{n+1} \geq \min\left(s_{i-1}^n, s_i^n, s_{i+1}^n\right), \quad \sigma_i^{n+1} \geq \min\left(\sigma_{i-1}^n, \sigma_i^n, \sigma_{i+1}^n\right),$$

3.2 The pressureless gas dynamics [7]

This work enters the Post-doc of M. Breuss (grant DFG).

The relaxation scheme is extended to approximate the weak solutions of a weakly hyperbolic system: the pressureless Euler equations (see Bouchut et al. [38, 41, 42]). In one space dimension, the model is governed by the following system:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x \rho u^2 = 0, \end{cases} \tag{35}$$

with $\rho > 0$ and $u \in \mathbb{R}$. Let us emphasize that the system (35) is not hyperbolic but weakly hyperbolic. Indeed, u is a double eigenvalue but the system is not diagonalizable in $\mathbb{R}$. As a consequence, the natural solutions of (35) are not L^2-stable [38, 41, 42, 110]. Such a system enters several plasma simulations [39] or cosmology simulations [122].

We derive a relaxation scheme to approximate the weak solutions of (35). According to the exact solution, we establish a maximum principle and we show that the approximate velocity is TVD. In addition, discrete entropy inequalities are given. A particular attention is paid concerning the vacuum numerical approximation which are natural solution of (35) (for instance, see Bouchut et al. [38, 41, 42] or Chen-Liu [51]).

The following relaxation model is proposed:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + \Pi) = 0, \\ \partial_t \rho \Pi + \partial_x (\rho \Pi u + a^2 u) = -\mu \rho \Pi. \end{array}\right. \tag{36}$$

With $\mu = 0$, the system (36) is hyperbolic over

$$\mathcal{V} = \left\{ \mathbf{W} = (\rho,\, \rho u,\, \rho \Pi) \in \mathbb{R}^3;\, \rho > 0,\, u \in \mathbb{R},\, \Pi \in \mathbb{R} \right\}.$$

Its eigenvalues are $\lambda_1 = u - a/\rho$, $\lambda_2 = u$ and $\lambda_3 = u - a/\rho$. All the fields are linearly degenerate.

In the present relaxation model, the Whitham sub-characteristic condition reads: $a > 0$. However, we will see that additional restriction will be imposed to a in order to enforce the required stability conditions.

An uniform mesh, with size Δx, is considered. We set $\mathbf{w}_i^n = (\rho_i^n, (\rho u)_i^n)$, to write the relaxation scheme as follows:

$$\begin{array}{l} \mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \dfrac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+\frac{1}{2}}^n - \mathbf{f}_{i-\frac{1}{2}}^n \right), \\ \mathbf{f}_{i+\frac{1}{2}}^n = \mathbf{F}_{a_{i+\frac{1}{2}}} \left(\mathbf{W}^h(x_{i+\frac{1}{2}}, t^n + \Delta t) \right)|_{[\rho, \rho u]}, \end{array} \tag{37}$$

where $\mathbf{F}_{a_{i+\frac{1}{2}}}$ is the flux function of the system (36) with a relaxation parameter $a_{i+\frac{1}{2}}$. The vector $\mathbf{W}^h(x_{i+\frac{1}{2}}, t^n + \Delta t)$ is the solution of the Riemann problem associated with (36) where we have set $\mu = 0$ and where the left and right states of the initial data are given by the equilibrium states $\mathbf{W}_i^n$ and $\mathbf{W}_{i+1}^n$ with $\mathbf{W}_i^n = (\rho_i^n, (\rho u)_i^n, \Pi_i^n = 0)$ for all $i \in \mathbb{Z}$.

The time step is restricted according to the CFL condition

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(|u_i^n| + \frac{a_{i+\frac{1}{2}}}{\rho_i^n} \right) \leq \frac{1}{2}. \tag{38}$$

When considering the following initial Riemann data:

$$\mathbf{W}_0(x) = \left\{ \begin{array}{ll} \mathbf{W}_L & \text{if } x < 0, \\ \mathbf{W}_R & \text{if } x > 0, \end{array} \right.$$

where $\mathbf{W}_L$ and $\mathbf{W}_R$ are assumed to be equilibrium states, i.e. $\Pi_L = \Pi_R = 0$, let us emphasize that the solution of (36), with $\mu = 0$, is known as soon as the relaxation parameter satisfies:

$$a > \max\left(0,\ \rho_L \frac{u_L - u_R}{2},\ \rho_R \frac{u_L - u_R}{2}\right). \tag{39}$$

Since all the fields of (36) are linearly degenerate, the Riemann problem is easily solved. In addition, this restriction enforces the density ρ to be positive as long as $\rho_L > 0$ and $\rho_R > 0$. The following stability result is thus established:

Theorem 3.3. *Consider the relaxation scheme (37) where the relaxation parameter $a_{i+\frac{1}{2}}$, locally defined on each cell interface $x_{i+\frac{1}{2}}$, satisfies (39). Assume the CFL condition (38). The velocity $(u_i^n)_{i\in\mathbb{Z}}$ satisfies the following properties:*
i) a maximum principle

$$u_i^{n+1} \le \max(u_{i-1}^n,\ u_i^n,\ u_{i+1}^n),$$

ii) a TVD property

$$\mathrm{TV}(\mathbf{u}^n) \le \mathrm{TV}(\mathbf{u}^0),$$

iii) discrete entropy inequalities

$$\frac{S(u_i^{n+1}) - S(u_i^n)}{\Delta t} + \frac{U(u_i^n, u_{i+1}^n) - U(u_{i-1}^n, u_i^n)}{\Delta x} \le 0,$$

for all convex function S, where $U(.,.)$ denotes the numerical entropy flux function.

Regarding the numerical approximation of the vacuum solution, the main difficulty comes from the evaluation of the time increment Δt. Indeed, according to the CFL condition (38), Δt seems to tend to zero when vacuum appears. In fact, a relevant computation of the relaxation parameter a avoids such a difficulty.

Theorem 3.4. *Consider the relaxation scheme (37) and assume the CFL restriction (38). Enforce $a_{i+\frac{1}{2}}$ to satisfy*

$$\begin{aligned} &a_{i+\frac{1}{2}} \ge \frac{u_i^n - u_{i+1}^n}{2} \max(\rho_i^n, \rho_{i+1}^n) \qquad \textit{if} \quad u_i^n \ge u_{i+1}^n, \\ &0 < a_{i+\frac{1}{2}} < \min\left(\epsilon, \frac{1}{2}\min(\rho_i, \rho_{i+1})(C + u_i - u_{i+1})\right) \qquad \textit{if} \quad u_i^n < u_{i+1}^n. \end{aligned}$$

If $\rho_i^n > 0$ for all $i \in \mathbb{Z}$ then $\rho_i^{n+1} > 0$ for all $i \in \mathbb{Z}$. In addition, the following inequality is obtained:

$$\max\left(\left|u_i^n - \frac{a_{i+\frac{1}{2}}}{\rho_i^n}\right|, \left|u_{i+1}^n + \frac{a_{i+\frac{1}{2}}}{\rho_{i+1}^n}\right|\right) \le \left|\frac{u_i^n + u_{i+1}^n}{2}\right| + \frac{1}{2}\mathrm{TV}(\mathbf{u}^0).$$

As a consequence, Δt does not tend to zero as the density decreases to zero.

This numerical method is adopted to perform a 2D simulation on a Cartesian mesh. We consider a square domain with periodic boundary conditions. This numerical experiment comes from large-scale structure simulations [122]. We approximate the solutions of the 2D extension of the system (35):

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x (\rho u) + \partial_y (\rho v) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2) + \partial_y (\rho u v) = 0, \\ \partial_t (\rho v) + \partial_x (\rho u y) + \partial_y (\rho v^2) = 0. \end{array} \right.$$

The initial data must be *well-prepared* where ρ^0 is a Gaussian field (see figure 4) and where the initial velocity vector is the solution of

$$u = -\partial_x \phi, \quad v = -\partial_y \phi, \quad \Delta\phi = 4\Pi G(\rho - \rho_b) \qquad \text{over } \Omega,$$

with $G > 0$ a constant and $\rho_b = \int_\Omega \rho \, dxdy$. After a long time computation, the numerical solution obtained is displayed figure 4 and show the evolution of the large-scale structures.

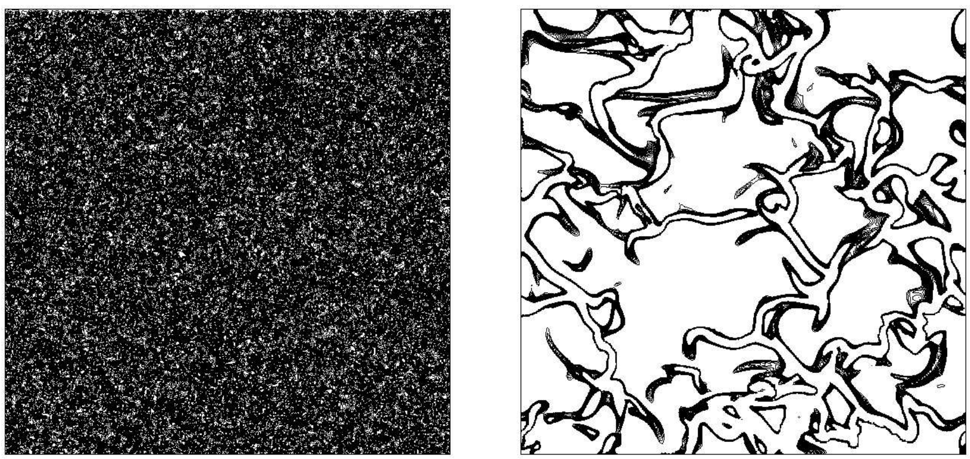

Figure 4: Initial density distribution and large time evolution.

3.3 Petroleum flows [5]

The present work enters the PhD thesis of M. Baudin (grant IFP).

Numerical difficulties arise when considering simulations of petroleum pipeline flows where complex mixtures are considered (gas-oil-water). In the present study, the *Drift-flux model* is considered. Such a model is characterised by a single pressure associated with one momentum conservation law. We set ρ the density of the mixture, ρv the total momentum of the mixture and $Y \in [0,1]$ the mass fraction. The model is governed by the following system of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0, \\ \partial_t \rho v + \partial_x (\rho v^2 + P(\mathbf{w})) = 0, \\ \partial_t \rho Y + \partial_x (\rho Y v - \sigma(\mathbf{w})) = 0, \end{cases} \tag{40}$$

where $\mathbf{w} = (\rho, \rho v, \rho Y)$ is the unknown vector defined over the state space:

$$\Omega = \{\mathbf{w} \in \mathbb{R}^3; \rho > 0, v \in \mathbb{R}, Y \in [0,1]\}.$$

In the present model, one of the main originalities comes from the choice of both pressure $P(\mathbf{w})$ and function $\sigma(\mathbf{w})$. Indeed, the hydrodynamic behavior of the flows can be modeled invoking a relevant choice of these two functions [123]. However, the nonlinearities involved in the system (40) result from P and σ. Moreover, we cannot establish an hyperbolic property satisfied by the system or exhibit any entropy pair (see Benzoni-Gavache [35]) except when assuming restrictive hypothesis.

From these remarks, we immediately deduce that the derivation of a numerical method to approximate solutions of (40) turns out to be difficult. Indeed, after the works of Faille-Heintzé [61] or Flatten-Munkejord [62], the numerical evaluation of the Jacobian of the flux functions is very costly. As a consequence, the usual Riemann solver (see Godlewski-Raviart [67], Toro [116], Gallouët-Masella [65]) must be avoided. We propose to derive a relaxation scheme. Let us stress from now on that the numerical difficulties which arise when considering the usual Riemann solver, will be preserved when establishing the Whitham sub-characteristic condition. The following relaxation model is considered:

$$\begin{cases} \partial_t \rho + \partial_x \rho v = 0, \\ \partial_t \rho v + \partial_x (\rho v^2 + \Pi) = 0, \\ \partial_t \rho \Pi + \partial_x (\rho \Pi v + a^2 v) = \mu \rho (P(\mathbf{w}) - \Pi), \\ \partial_t \rho Y + \partial_x (\rho Y v - \Sigma) = 0, \\ \partial_t \rho \Sigma + \partial_x (\rho \Sigma v - b^2 Y) = \mu \rho (\sigma(\mathbf{w}) - \Sigma), \end{cases} \tag{41}$$

where $a > 0$ and $b > 0$ are the relaxation parameters. With $\mu = 0$, the system is hyperbolic over the state space

$$\mathcal{V} = \{\mathbf{W} = (\rho, \rho v, \rho\Pi, \rho Y, \rho\Sigma) \in \mathbb{R}^5; \rho > 0, v \in \mathbb{R}, , \Pi \in \mathbb{R}, Y \in [0,1], \Sigma \in \mathbb{R}\}.$$

The eigenvalues associated with (40) are v, $v \pm a/\rho$ and $v \pm b/\rho$. All the fields are linearly degenerate which makes the Riemann problem easily solvable.

The main difficulty stays in the establishment of the Whitham condition which must be satisfied by the pair (a, b). In the present study, a formal Chapmann-Enskog expansion is adopted. In general, the stability conditions satisfied by the relaxation parameters can be obtained several ways. For instance, Chen-Levermore-Liu [50] propose considering the

Lax entropies. Unfortunately, the Lax entropy pairs cannot be obtained in the framework of the system (40).

Near an equilibrium state, the following assumption is assumed:

$$\Pi = P(\mathbf{w}) + \mu^{-1}\Pi^1 + O(\mu^{-2}),$$
$$\Sigma = \sigma(\mathbf{w}) + \mu^{-1}\Sigma^1 + O(\mu^{-2}).$$

Setting this relation into (41), after formal computations we obtain:

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = \mu^{-1}\partial_x(\mathcal{D}_{(a,,b)}(\mathbf{w})\partial_x \mathbf{w}), \tag{42}$$

where $\mathbf{f}$ coincides with the flux function of the system (40). The pair (a, b) is chosen in order to enforce the eigenvalues of $\mathcal{D}_{(a,b)}$ to be non-negative. In a sense to be precised, the system (42) must be dissipative. Such a property is obtained as soon as the pair (a, b) satisfies a condition (C). Since its expression is too long to be given here, the condition (C) is not exhibited here.

A relaxation scheme is derived on the basis of (41). We set $\mathbf{w}_i^n = (\rho_i^n, (\rho v)_i^n, (\rho Y)_i^n)$, to write the relaxation scheme in the abstract form (37). Under the CFL condition:

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} \left(|v_i^n \pm \frac{a_{i+\frac{1}{2}}}{\rho_i^n}||v_i^n \pm \frac{b_{i+\frac{1}{2}}}{\rho_i^n}| \right) \leq \frac{1}{2},$$

where $(a_{i+\frac{1}{2}}, b_{i+\frac{1}{2}})$ denotes the pair of the relaxation parameter stated at the cell interface $x_{i+\frac{1}{2}}$, it is possible to prove the following stability properties:
i) if $\rho_i^n > 0$ for all $i \in \mathbb{Z}$ then $\rho_i^{n+1} > 0$ for all $i \in \mathbb{Z}$,
ii) if $Y_i^n \in [0, 1]$ for all $i \in \mathbb{Z}$ then $Y_i^{n+1} \in [0, 1]$ for all $i \in \mathbb{Z}$.

These stability properties are obtained as long as the pair (a, b) satisfies the condition (C) and the following restrictions:

$$a_{i+\frac{1}{2}} \geq \frac{< v_{i+\frac{1}{2}} > + \sqrt{< v_{i+\frac{1}{2}} >^2 + 4 < \Pi_{i+\frac{1}{2}} > \max(1/\rho_i^n, 1/\rho_{i+1}^n)}}{2\min(1/\rho_i^n, 1/\rho_{i+1}^n)},$$
$$b_{i+\frac{1}{2}} \geq \frac{|\rho_i^n \Phi(\mathbf{w}_i^n) + \rho_{i+1}^n \Phi(\mathbf{w}_i^n)|}{2} + \frac{|\rho_i^n \Phi(\mathbf{w}_i^n) - \rho_{i+1}^n \Phi(\mathbf{w}_i^n)|}{2},$$

with $< \varphi_{i+\frac{1}{2}} >= (\varphi_i^n - \varphi_{i+1}^n)/2$. The function $\Phi : \Omega \to \mathbb{R}$ defines the closure hydrodynamics law prescribed by the physics and it satisfies $\sigma(\mathbf{w}) = \rho Y(1 - Y)\Phi(\mathbf{w})$. In addition, let us note that the positiveness of the partial densities is obtained invoking natural extensions of ideas introduced by Larrouturou [82].

The proposed relaxation scheme is robust but also it is less costly that other schemes [61, 62] proposed to approximate the solutions of (40). This remark turns out to be crucial when industrial simulations are considered.

3.4 Multi-fluid flows [22, 24]

The present work enters B. Braconnier's PhD thesis (grant CEA).

In the present analysis, non-conservative systems are adopted to modeled compressible multi-fluid flows. One of the considered models has been derived by Baer-Nunziato [31]. Each component of the flow is characterized by a void fraction $\alpha_k \in (0,1)$, a density $\rho_k > 0$, a velocity $u_k \in \mathbb{R}$ and a pressure $p_k(\rho_k) > 0$. The model is governed by the following system:

$$\left\{ \begin{array}{l} \partial_t \alpha_1 + u_I \partial_x \alpha_1 = 0, \quad \alpha_1 + \alpha_2 = 1, \\ \partial_t \alpha_1 \rho_1 + \partial_x \alpha_1 \rho_1 u_1 = 0, \\ \partial_t \alpha_1 \rho_1 u_1 + \partial_x (\alpha_1 \rho_1 u_1^2 + \alpha_1 p_1) - p_I \partial_x \alpha_1 = 0, \\ \partial_t \alpha_2 \rho_2 + \partial_x \alpha_2 \rho_2 u_2 = 0, \\ \partial_t \alpha_2 \rho_2 u_2 + \partial_x (\alpha_2 \rho_2 u_2^2 + \alpha_2 p_2) + p_I \partial_x \alpha_1 = 0, \end{array} \right. \tag{43}$$

where u_I and p_I respectively are the interface velocity and the interface pressure. After [31], we impose $u_I = u_1$ and $p_I = p_2$.

Two main difficulties arise from this system. Indeed, (43) is not unconditionally hyperbolic. As soon as $\alpha_1 = 0$ or $u_1 = u_2 \pm \sqrt{p_2'(\rho_2)/\rho_2}$, the system is not diagonalizable in $\mathbb{R}$. The second difficulty arises from the non-conservation form of the system. However, it is possible to prove that the non-conservative product does not involve ambiguity (see Bouchut [37]).

To derive a relevant numerical method, a relaxation scheme is adopted. This scheme is based on a relaxation system obtained when substituting the pressures p_k by new unknowns Π_k which are governed by:

$$\partial_t \Pi_k + u_k \partial_x \Pi_k + \frac{a_k^2}{\rho_k} \partial_x u_k = \mu(p_k - \Pi_k).$$

The relaxation parameters a_k must satisfy the Whitham sous-characteristic condition: $a_k^2 > \rho_k p_k'(\rho_k)$.

The relaxation system is hyperbolic and all its fields are linearly degenerate. As a consequence, the associated Riemann problem is easily solvable even if the relaxation system is not in conservation form. After the previous sections, a usual procedure provides a relaxation scheme which satisfies the required robustness. Let us emphasize that the obtained scheme is in non-conservation form.

In the PhD thesis of B. Braconnier, this approach is extended to models involving several physical phenomenon.

4 Other schemes [1, 14, 17]

The present section is devoted to the numerical approximation of the solutions of hyperbolic system obtained involving distinct method than the relaxation scheme; namely the HLLC and the VF-Roe schemes. These methods are applied to model invoking very distinct physics. Indeed, the VF-Roe scheme is considered to approximate the solutions of the turbulence model R_{ij}. By contrast, an HLLC scheme is derived to approximate the solutions of a radiative transfer model.

4.1 A 2D HLLC scheme for the radiative transfer [14, 17]

The communication [17] is an introduction to the article [14].

The radiative transfer is a physical phenomena which enters many applications. Unfortunately, the numerical simulations of radiative transfer turn out to be very costly. A new alternative has been recently introduced by Dubroca-Feugeas [59]; namely the M_1 model. It has been used in a wide literature (for instance, see Turpault [118, 119], Charrier et al. [49], Ripoll [104, 105], Buet-Cordier [44] or Buet-Despres [45, 46]). This model is governed by the following system:

$$\left\{\begin{array}{l} \partial_t E + \nabla \cdot \mathbf{F} = c\sigma\big(aT^4 - E\big), \\ \frac{1}{c}\partial_t \mathbf{F} + c\ \nabla \cdot \mathbf{P} = -\sigma\ \mathbf{F}, \\ \partial_t(\rho\ C_v T) = -c\sigma\big(aT^4 - E\big), \end{array}\right. \tag{44}$$

where $E \geq 0$ denotes the energy, $\mathbf{F} = (F_x, F_y)$ is the radiative flux and T is the temperature. The speed of light c but also ρ, C_v and a are constant. The parameter $\sigma \geq 0$ denotes the opacity.

The radiative pressure is given by

$$\mathbf{P} = \frac{1}{2}\left((1-\chi(f))\mathbf{I} + (3\chi(f)-1)\frac{\mathbf{F}\otimes\mathbf{F}}{||\mathbf{F}||^2}\right)E, \tag{45}$$

$$\chi(f) = \frac{3+4f^2}{5+2\sqrt{4-3f^2}} \quad \text{with} \quad f = \frac{F}{cE}. \tag{46}$$

Let us note that this system is hyperbolic symmetrizable. The eigenvalues of the Jacobian matrix per direction of the flux function are known but too long to be given here.

The solutions of this model satisfy the positiveness of the energy, the following flux limitation:

$$-cE \leq F \leq cE,$$

and the conservation of the total energy:

$$\partial_t(E + \rho C_v T) + \nabla.\mathbf{F} = 0.$$

In addition, this model recovers an asymptotic diffusion regime (see Mihalas-Mihalas [95] or Pomraming [101]). Indeed, as soon as the opacity is large enough, the M_1 model rescales when introducing the Knudsen number ϵ, in order to write:

$$\begin{cases} \epsilon\partial_t E + \nabla \cdot \mathbf{F} = \frac{\sigma c}{\epsilon}(aT^4 - E), \\ \epsilon\partial_t \mathbf{F} + c^2\nabla \cdot \mathbf{P} = -\frac{\sigma c}{\epsilon}\sigma\mathbf{F}, \\ \epsilon\partial_t(\rho C_v T) = -\frac{\sigma c}{\epsilon}(aT^4 - E). \end{cases} \tag{47}$$

Formally, in the limit of ϵ to zero, an equilibrium diffusion equation is obtained:

$$\partial_t(\rho C_v T_0 + aT_0^4) - \nabla \cdot (\frac{c}{3\sigma}\nabla(aT_0^4)) = 0. \tag{48}$$

Recently, several numerical methods have been proposed to approximate the solutions of the M_1 model (for instance, see Buet-Cordier [44] or Buet-Despres [45, 46] or Gosse-Toscani [68]). However, the accuracy of these recent numerical schemes turns out to be insufficient to perform some bi-dimensional simulations.

Here, we propose to derive a predictor-corrector scheme based on an HLLC approximate Riemann solver (see Batten et al. [32], Bouchut [37], Toro et al. [116, 117] for further details). This numerical method will preserve all the properties satisfied by the model and it ensures the required accuracy.

First, the 1D model is considered to derive a numerical scheme. An operator splitting technique is adopted. The first step is devoted to the prediction and the following system is approximated:

$$\begin{cases} \partial_t E + \partial_x F = c\sigma(aT^4 - E), \\ \dfrac{1}{c}\partial_t F + \partial_x(cP) = -\sigma F, \\ \partial_t(\rho C_v T) = -c\sigma(aT^4 - E). \end{cases} \tag{49}$$

After Bouchut [37], the opacity is rewritten as follows: $\sigma(x) = \partial_x\Sigma(x)$. As a consequence, the source term enters the approximate Riemann solver. Since $\partial_t\Sigma = 0$, it appears that the first step of the method is devoted to the numerical approximation of the following system:

$$\begin{cases} \partial_t E + \partial_x F = 0, \\ \dfrac{1}{c}\partial_t F + \partial_x(cP) + F\partial_x\Sigma = 0, \\ \partial_t(\rho C_v T) = 0, \\ \partial_t\Sigma = 0. \end{cases} \tag{50}$$

We develop an approximate Riemann solver to derive an HLLC scheme. Next, into a second step, this solver is corrected to add the temperature relaxation source terms.

Hence, the following system is considered:

$$\begin{cases} \partial_t E = c\sigma(aT^4 - E), \\ \partial_t F = 0, \\ \partial_t(\rho C_v T) = -c\sigma(aT^4 - E). \end{cases}$$

Consider a uniform mesh with size Δx. The resulting scheme reads as follows:

$$\begin{aligned} E_i^{n+1} = E_i^n &- \frac{\Delta t}{\Delta x}\left(\alpha_{i+\frac{1}{2}}\tilde{F}^{HLL}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}\tilde{F}^{HLL}_{i-\frac{1}{2}}\right) \\ &- \frac{\Delta t}{\Delta x}\left((1-\alpha_{i+\frac{1}{2}})b_L^{i+\frac{1}{2}}\left(a\left(T^{\star,HLLC}_{L,i+\frac{1}{2}}\right)^4 - \left(E_i^n - \frac{F_i^n}{b_L^{i+\frac{1}{2}}}\right)\right)\right. \\ &\left. -(1-\alpha_{i-\frac{1}{2}})b_R^{i-\frac{1}{2}}\left(a\left(T^{\star,HLLC}_{R,i-\frac{1}{2}}\right)^4 - \left(E_i^n - \frac{F_i^n}{b_R^{i-\frac{1}{2}}}\right)\right)\right), \end{aligned} \tag{51a}$$

$$\begin{aligned} F_i^{n+1} = F_i^n &- \frac{\Delta t}{\Delta x}c^2\left(\alpha_{i+\frac{1}{2}}\tilde{P}^{HLL}_{i+\frac{1}{2}} - \alpha_{i-\frac{1}{2}}\tilde{P}^{HLL}_{i-\frac{1}{2}} + (\alpha_{i-\frac{1}{2}} - \alpha_{i+\frac{1}{2}})P_i^n\right) + \\ &\frac{\Delta t}{\Delta x}\left((1-\alpha_{i+\frac{1}{2}})b_L^{i+\frac{1}{2}} - (1-\alpha_{i-\frac{1}{2}})b_R^{i-\frac{1}{2}}\right)F_i^n, \end{aligned} \tag{51b}$$

$$\begin{aligned} (\rho C_v T)_i^{n+1} = (\rho C_v T)_i^n &+ \frac{\Delta t}{\Delta x}\left((1-\alpha_{i+\frac{1}{2}})b_L^{i+\frac{1}{2}}\left(a\left(T^{\star,HLLC}_{L,i+\frac{1}{2}}\right)^4 - \left(E_i^n - \frac{F_i^n}{b_L^{i+\frac{1}{2}}}\right)\right)\right. \\ &\left. -(1-\alpha_{i-\frac{1}{2}})b_R^{i-\frac{1}{2}}\left(a\left(T^{\star,HLLC}_{R,i-\frac{1}{2}}\right)^4 - \left(E_i^n - \frac{F_i^n}{b_R^{i-\frac{1}{2}}}\right)\right)\right), \end{aligned} \tag{51c}$$

where

$$\begin{aligned} \alpha_{i+\frac{1}{2}} &= \frac{b_R^{i+\frac{1}{2}} - b_L^{i+\frac{1}{2}}}{b_R^{i+\frac{1}{2}} - b_L^{i+\frac{1}{2}} + c\sigma_{i+\frac{1}{2}}\Delta x} \\ b_L^{i+\frac{1}{2}} &\le \min\left(0, cf_i^n, c\frac{f_i^n - \chi_i^n}{1-f_i^n}, c\frac{f_i^n + \chi_i^n}{1+f_i^n}, \lambda^-(E_i^n, F_i^n)\right), \\ b_R^{i-\frac{1}{2}} &\ge \max\left(0, cf_i^n, c\frac{f_i^n - \chi_i^n}{1-f_i^n}, c\frac{f_i^n + \chi_i^n}{1+f_i^n}, \lambda^+(E_i^n, F_i^n)\right), \end{aligned}$$

with

$$f_i^n = \frac{F_i^n}{cE_i^n} \quad \text{and} \quad \chi_i^n = \chi(f_i^n).$$

The quantities $\lambda^\pm$ denotes the extreme eigenvalues of the Jacobian matrix of the flux function. The numerical flux functions $\tilde{F}^{HLL}_{i+\frac{1}{2}}$ and $\tilde{P}^{HLL}_{i+\frac{1}{2}}$ are defined by

$$\tilde{F}^{HLL}_{i+\frac{1}{2}} = \frac{b_R^{i+\frac{1}{2}} F_i^n - b_L^{i+\frac{1}{2}} F_{i+1}^n - b_L^{i+\frac{1}{2}} b_R^{i+\frac{1}{2}} (E_i^n - E_{i+1}^n)}{b_R^{i+\frac{1}{2}} - b_L^{i+\frac{1}{2}}},$$
$$\tilde{P}^{HLL}_{i+\frac{1}{2}} = \frac{b_R^{i+\frac{1}{2}} P_i^n - b_L^{i+\frac{1}{2}} P_{i+1}^n - b_L^{i+\frac{1}{2}} b_R^{i+\frac{1}{2}} (F_i^n - F_{i+1}^n)}{b_R^{i+\frac{1}{2}} - b_L^{i+\frac{1}{2}}}.$$

Finally, the temperature $T^{\star,HLLC}_{K,i+\frac{1}{2}}$, with $K = L$ or R, is solution of the following equation:

$$\rho C_v T^{\star,HLLC}_{K,i+\frac{1}{2}} + (1 - \alpha_{i+\frac{1}{2}}) a (T^{\star,HLLC}_{K,i+\frac{1}{2}})^4 = \rho C_v T_{K,i+\frac{1}{2}} + (1 - \alpha_{i+\frac{1}{2}}) E_K^0$$

with

$$E_L^0 = E_i^n - \frac{F_i^n}{b_L^{i+\frac{1}{2}}} \quad \text{and} \quad E_R^0 = E_{i+1}^n - \frac{F_{i+1}^n}{b_R^{i+\frac{1}{2}}}. \tag{52}$$

This scheme satisfies all the required properties. Indeed, we have

Theorem 4.1. *Assume $E_i^n \geq 0$, $-cE_i^n \leq F_i^n \leq cE_i^n$ and $T_i^n > 0$ for all $i \in \mathbb{Z}$. Under the CFL condition*

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} (|b_L^{i+\frac{1}{2}}|, |b_R^{i+\frac{1}{2}}|) \leq \frac{1}{2},$$

the following properties are satisfied:
i) the radiative energy is non-negative: $E_i^{n+1} \geq 0$ for all $i \in \mathbb{Z}$,
ii) the radiative flux is limited: $-cE_i^{n+1} \leq F_i^{n+1} \leq cE_i^{n+1}$ for all $i \in \mathbb{Z}$,
iii) the total energy $(E + \rho C_v T)_i^{n+1}$ is conserved.

In addition, the scheme (51) is *asymptotic preserving*. Indeed, let us introduce the

Knudsen number, when rescaling the scheme (51), in order to write:

$$\epsilon\frac{E_i^{n+1}-E_i^n}{\Delta t}+\frac{1}{\Delta x}\left(\alpha_{i+\frac12}\tilde F^{HLL}_{i+\frac12}-\alpha_{i-\frac12}\tilde F^{HLL}_{i-\frac12}\right)=$$
$$-\left(\frac{1-\alpha_{i+\frac12}}{\Delta x}b_L^{i+\frac12}\left(a\left(T^{\star,HLLC}_{L,i+\frac12}\right)^4-\left(E_i^n-\frac{F_i^n}{b_L^{i+\frac12}}\right)\right)\right.$$
$$\left.-\frac{1-\alpha_{i-\frac12}}{\Delta x}b_R^{i-\frac12}\left(a\left(T^{\star,HLLC}_{R,i-\frac12}\right)^4-\left(E_i^n-\frac{F_i^n}{b_R^{i-\frac12}}\right)\right)\right), \tag{53a}$$

$$\epsilon\frac{F_i^{n+1}-F_i^n}{\Delta t}+\frac{c^2}{\Delta x}\left(\alpha_{i+\frac12}\tilde P^{HLL}_{i+\frac12}-\alpha_{i-\frac12}\tilde P^{HLL}_{i-\frac12}+(\alpha_{i-\frac12}-\alpha_{i+\frac12})P_i^n\right)=$$
$$-\left(\frac{1-\alpha_{i+\frac12}}{\Delta x}b_L^{i+\frac12}-\frac{1-\alpha_{i-\frac12}}{\Delta x}b_R^{i-\frac12}\right)F_i^n, \tag{53b}$$

$$\epsilon\frac{(\rho C_vT)_i^{n+1}-(\rho C_vT)_i^n}{\Delta t}=\frac{1-\alpha_{i+\frac12}}{\Delta x}b_L^{i+\frac12}\left(a\left(T^{\star,HLLC}_{L,i+\frac12}\right)^4-\left(E_i^n-\frac{F_i^n}{b_L^{i+\frac12}}\right)\right.$$
$$\left.-\frac{1-\alpha_{i-\frac12}}{\Delta x}b_R^{i-\frac12}\left(a\left(T^{\star,HLLC}_{R,i-\frac12}\right)^4-\left(E_i^n-\frac{F_i^n}{b_R^{i-\frac12}}\right)\right)\right), \tag{53c}$$

where $\alpha_{i+\frac12}$ is now given as follows:

$$\alpha_{i+\frac12}=\frac{\epsilon(b_R^{i+\frac12}-b_L^{i+\frac12})}{\epsilon(b_R^{i+\frac12}-b_L^{i+\frac12})+c\sigma_{i+\frac12}\Delta x}. \tag{54}$$

The following statement establishes that the scheme under consideration is asymptotic preserving and it recovers the expected asymptotic behavior:

Theorem 4.2. *Let the velocities $b_L^{i+\frac12}$ and $b_R^{i+\frac12}$ be given by*

$$\begin{aligned} b_L^{i+\frac12}&=\min\left(0,cf_i^n,c\frac{f_i^n-\chi_i^n}{1-f_i^n},c\frac{f_i^n+\chi_i^n}{1+f_i^n},\lambda^-(E_i^n,F_i^n)\right),\\ b_R^{i-\frac12}&=\max\left(0,cf_i^n,c\frac{f_i^n-\chi_i^n}{1-f_i^n},c\frac{f_i^n+\chi_i^n}{1+f_i^n},\lambda^+(E_i^n,F_i^n)\right).\end{aligned} \tag{55}$$

Assume ϵ small enough. The diffusion limit issued from (53)-(54) is given by

$$\left\{\begin{array}{l} E_i^n=a(T_i^n)^4,\\ F_i^n=0,\\ \dfrac{(E+\rho C_vT)_i^{n+1}-(E+\rho C_vT)_i^n}{\Delta t}=\\ \qquad\qquad\dfrac{c}{3\Delta x^2}\left(\dfrac{1}{\sigma_{i+\frac12}}(E_{i+1}^n-E_i^n)+\dfrac{1}{\sigma_{i-\frac12}}(E_{i-1}^n-E_i^n)\right).\end{array}\right.$$

Next, the 1D scheme (51) is extended to approximate the solutions of the bi-dimensional M_1 model on a Cartesian grid. The full 2D scheme is too long to be written here. However, it is crucial to note that the 2D extended scheme is robust and it preserves all the properties satisfied by the model; namely a non-negative radiative energy, a flux limitation, the conservation of the total energy and an asymptotic regime according to the diffusion limit regime. To conclude, let us note that the accuracy of the method results from the choice of the numerical wave speed $b_L^{i+\frac{1}{2}}$ and $b_R^{i+\frac{1}{2}}$ which are considered independently. Recent work (see Buet-Cordier [44] or Buet-Despres [45, 46]) use numerical wave speed in a symmetrical form $b_R^{i+\frac{1}{2}} = b_L^{i+\frac{1}{2}}$. In general, this symmetrical choice produces lot of numerical diffusion.

To illustrate the method, we perform a numerical experiment where the numerical diffusion plays a crucial role. An opaque region, $\sigma = 2\ 10^5$, is adjacent to a transparent region where $\sigma = 0$ (see figure 5). A radiative temperature $T = 5Kev$ is applied on the

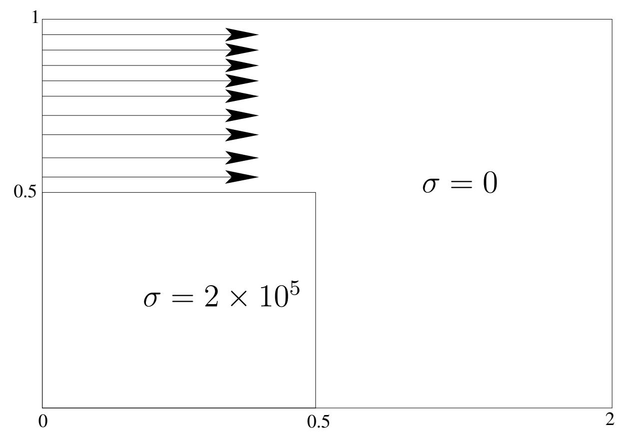

Figure 5: Geometry for the two dimensional case radiative transfer.

left side of the transparent material. The initial temperature is $T = 50000K$ in the dense material and $T = 300K$ elsewhere. The expected solution admits a *simple* structure. Indeed, the dense region produces a shadow cone. Figure 6 displays the approximate solutions obtained when involving two distinct computations of $b_{L,R}^{i+\frac{1}{2}}$. In the left figure (HLLC-var), we exhibit the approximate solution obtained with $b_{L,R}^{i+\frac{1}{2}}$ given by (55). In the right figure (HLLC-c), a variant of the scheme (51) is used. This variant satisfies all the stability properties but for $b_R^{i+\frac{1}{2}} = c = -b_L^{i+\frac{1}{2}}$. It is clear that the choice of the numerical wave speed $b_{L,R}^{i+\frac{1}{2}}$ plays a crucial role. Only scheme (51) produces a relevant approximation of the shadow cone.

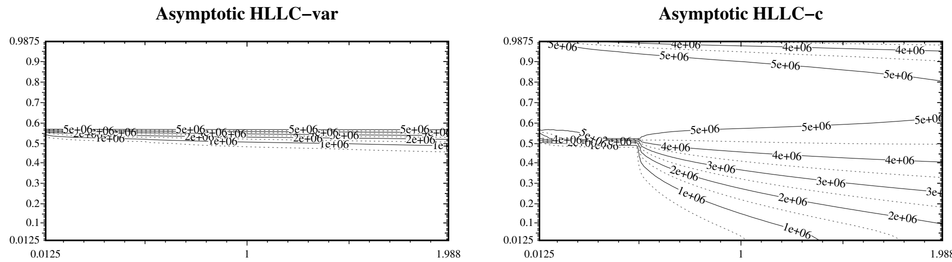

Figure 6: Radiative temperature at time $t = 5 * 10^{-8}$ obtained with the schemes HLLC-c and HLLC-var.

4.2 A VF-Roe scheme for the R_{ij} model [1]

A large number of models are proposed in the literature to simulate turbulent compressible flows (for instance see [74]). The (k, ϵ) model, used in previous sections, is surely one of the most famous. Taking into account the second order closures permits the derivation of more sophisticate models. In the present study, the R_{ij} model is considered for numerical approximation. This model is governed by the following system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t \rho u + \partial_x(\rho u^2 + p + R_{11}) = 0, \\ \partial_t \rho v + \partial_x(\rho u v + R_{12}) = 0, \\ \partial_t E + \partial_x\left((E + p + R_{11}) + v R_{12}\right) = 0, \\ \partial_t R_{11} + \partial_x(R_{11} u) + 2R_{11}\partial_x u = 0, \\ \partial_t R_{22} + \partial_x(R_{22} u) + 2R_{12}\partial_x v = 0, \\ \partial_t R_{12} + \partial_x(R_{12} u) + R_{11}\partial_x v + R_{12}\partial_x u = 0, \\ \partial_t R_{33} + \partial_x(R_{33} u) = 0, \end{cases} \tag{56}$$

where R_{ij} denotes the components of the Reynolds tensor $\mathbf{R}$. The pressure p is defined by the following state law:

$$p = (\gamma - 1)\left(E - \rho\frac{u^2 + v^2}{2} - \frac{R_{11} + R22 + R_{33}}{2}\right).$$

We set $\mathbf{w}$ the unknown vector and we introduce the state space defined by

$$\Omega = \{\mathbf{w} \in \mathbb{R}^8; \rho > 0, p > 0, \mathbf{R} \text{ positive}\}.$$

The system (56) is a non-conservative hyperbolic system over Ω.

For the sake of simplicity, with clear notations, the system (56) is rewritten as follows:

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) + \mathbf{A}(\mathbf{w})\partial_x \mathbf{w} = 0.$$

Here, our purpose is to derive a *simple* numerical scheme despite the non-conservation form of the system under consideration. Indeed, the non-conservation form of the system avoids the usual approximate solvers (see Godlewski-Raviart [67], Toro [116], Roe [107]). After Buffard et al. [47], Gallouët [63, 64], a non-conservative VF-Roe scheme is derived. Because of the non-conservative products, it is clear that such an approach is not relevant. Indeed, these non-conservative products imply considering sophisticated schemes to deal with discontinuous solutions according to the non-conservative hyperbolic system theory (see LeFloch [86], Dal Maso-LeFloch-Murat [58], LeFloch-Liu [89], but also the section 2 devoted to the multi-entropy model). However, the derivation of a VF-Roe scheme allows us to exhibit a numerical scheme able to deal with discontinuous solution of a non-conservative hyperbolic system.

First, to derive the VF-Roe scheme, we introduce the change of variables $\mathbf{W}(\mathbf{w}) = (1/\rho, u, v, p, R_{11}, R_{22}, R_{12}, R_{33})$, to write (56) in the following form:

$$\partial_t \mathbf{W} + \mathbf{C}(\mathbf{W})\partial_x \mathbf{W} = 0.$$

Next, this system is linearized. Let us consider the following initial data:

$$\mathbf{W}^0(x) = \begin{cases} \mathbf{W}(\mathbf{w}_L) & \text{if } x < 0, \\ \mathbf{W}(\mathbf{w}_R) & \text{if } x > 0, \end{cases}$$

to write the following linear problem:

$$\partial_t \mathbf{W} + \mathbf{C}(\bar{\mathbf{W}})\partial_x \mathbf{W} = 0.$$

with $\bar{\mathbf{W}} = (\mathbf{W}(\mathbf{w}_L) + \mathbf{W}(\mathbf{w}_R))/2$. We define $\mathbf{W}^\star(\mathbf{w}_L, \mathbf{w}_R)$ as the exact solution of the above Riemann problem located at $x = 0$. We set $\mathbf{w}^n_{i+\frac{1}{2}} = \mathbf{W}^{-1}(\mathbf{W}^\star(\mathbf{w}^n_i, \mathbf{w}^n_{i+1}))$ to obtain the following scheme:

$$\mathbf{w}^{n+1}_i = \mathbf{w}^n_i - \frac{\Delta t}{\Delta x}(\mathbf{f}(\mathbf{w}^n_{i+\frac{1}{2}}) - \mathbf{f}(\mathbf{w}^n_{i-\frac{1}{2}})) - \frac{\Delta t}{\Delta x}\mathbf{S}^n_i,$$

where $\mathbf{S}^n_i$ is the discrete form of $\mathbf{A}(\mathbf{w})\partial_x \mathbf{w}$:

$$\mathbf{S}^n_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2(R_{11})^n_i(\tilde{u}_{i+\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2}}) \\ 2(R_{12})^n_i(\tilde{v}_{i+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2}}) \\ (R_{11})^n_i(\tilde{v}_{i+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2}}) + (R_{12})^n_i(\tilde{u}_{i+\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2}}) \\ 0 \end{pmatrix},$$

where the centered formula $\tilde{\varphi}_{i+\frac{1}{2}} = (\varphi^n_i + \varphi^n_{i+1})/2$ is adopted.

To conclude, let us note that the positiveness of the Reynolds tensor never fails in the numerical experiments.

5 Robustness of the second order MUSCL schemes [6, 10, 11]

In general, increasing the order of accuracy remains a delicate question when numerical finite volume methods are considered to approximate the solutions of hyperbolic systems. A large literature is devoted to this problem where several approaches are detailed. For instance, the Godunov type methods based on the generalized Riemann problem (see Ben-Artzi-Falcovitz [34])) give useful second-order numerical schemes (Bourgeade et al. [43] or Coquel-LeFloch [56]). However, the characterization of the exact solutions of the generalized Riemann problems remains difficult. As a consequence, these methods are poorly attractive despite stability results which ensure the robustness of the scheme.

Simpler procedures have been introduced to increase the order of accuracy. One of the most popular has been proposed by van Leer [90]; namely the MUSCL scheme. In the present section, our purpose is to analyze this scheme which has been used in many applications (for instance, see Colella [53], Durofsky et al. [60], Sander-Wieser [108]). More recently, distinct numerical procedures have been introduced: for instance, extensions of the kinetic schemes [98], residual distribution schemes [27, 30, 106], ENO and WENO schemes or discontinuous Galerkine methods [69, 111, 112].

These schemes are easier to use since the solutions of the generalized Riemann problem are not useful. But in general robustness or basic stability properties are not ensured. To illustrate our purpose, the CFL type restriction is not rigorously established in the framework of the MUSCL scheme excepted under some restrictive assumptions (for instance, see Khobalatte-Perthame [80], Perthame-Qiu [99], Bouchut [37], or Bouchut et al. [40]). In the present study, such a CFL condition is established in a general framework. However, for the sake of clarity in the presentation, we focus our attention on the Euler equations:

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0, \\ \partial_t \rho E + \partial_x (\rho E + p) u = 0, \end{cases} \tag{57}$$

where the pressure is given by the perfect gas law:

$$p = (\gamma - 1)\left(E - \rho \frac{u^2}{2}\right), \quad \gamma \in (1, 3]. \tag{58}$$

With clear notations, we introduce the following abstract form:

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}) = 0,$$

where $\mathbf{w} : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ denotes the state vector and $\mathbf{f} : \Omega \to \mathbb{R}$ is the flux function. The admissible state space Ω is defined by

$$\Omega = \left\{ \mathbf{w} = (\rho, \rho u, \rho E) \in \mathbb{R}^3;\ \rho > 0,\ u \in \mathbb{R},\ e(\mathbf{W}) = E - \frac{u^2}{2} > 0 \right\}.$$

Because of the shock wave solutions, the system (57) must be supplemented by entropy inequalities (see Lax [83, 84])

$$\partial_t \rho \mathcal{F}(\ln s) + \partial_x \rho \mathcal{F}(\ln s) u \le 0, \quad s := s(\mathbf{w}) = \frac{p}{\rho^\gamma}, \tag{59}$$

where the function $\mathcal{F}$ satisfy

$$\mathcal{F}'(y) < 0 \quad \text{and} \quad \frac{\mathcal{F}''(y)}{\mathcal{F}'(y)} < \frac{1}{\gamma}, \quad \forall y \in \mathbb{R}, \tag{60}$$

so that the function $\mathbf{w} \to \mathcal{F}(\ln s)$ is convex.

In addition, Tadmor [115] establishes a minimum principle satisfied by the specific entropy s:

$$s(x, t+h) \ge \min \{ s(y,t);\ |y-x| \le \| u \|_\infty h \}. \tag{61}$$

A MUSCL scheme is considered to approximate the weak solutions of (57)-(59)-(61). For the sake of simplicity in the notations, we briefly recall the first order conservative scheme. The first order approximation is based on piecewise constant approximation of the solution at time t^n:

$$\mathbf{w}^h(x, t^n) = \mathbf{w}_i^n, \quad x \in (x_{i-\frac{1}{2}} x_{i+\frac{1}{2}}),$$

with $x_{i+\frac{1}{2}} = x_i + (x_{i+1} - x_i)/2$ where $(x_i)_{i\in\mathbb{Z}}$ denote the mesh nodes. To simplify the notations, the mesh is assumed to be uniform with size $\Delta x = x_{i+1} - x_i$. To approximate the solution of (57) at time $t^{n+1} = t^n + \Delta t$, the sequence $(\mathbf{w}_i^{n+1})_{i\in\mathbb{Z}}$ is defined by the following conservative scheme (see Harten-Lax-van Leer [70], Godlewski-Raviart [67], Toro [116], Leveque [85]):

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n) - \mathbf{F}(\mathbf{w}_{i-1}^n, \mathbf{w}_i^n) \Big), \tag{62}$$

where $\mathbf{F} : \Omega \times \Omega \to \mathbb{R}^3$ denotes the numerical flux function assumed to be Lipschitz consistent. The time increment Δt is restricted according to the following CFL condition:

$$\frac{\Delta t}{\Delta x} \max_{i\in\mathbb{Z}} \left(|\lambda^\pm(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n)| \right) \le \frac{1}{2}, \tag{63}$$

where $\lambda^\pm(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n)$ are the numerical velocity of the acoustic waves associated with the numerical flux function $\mathbf{F}(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n)$.

In the sequel, the first-order scheme (62) is assumed to satisfy the numerical invariance of Ω, discrete entropy inequalities and a minimum principle on the specific entropy:

P1 : If $\rho_i^n > 0$ and $e_i^n > 0$ for all $i \in \mathbb{Z}$ then $\rho_i^{n+1} > 0$ and $e_i^{n+1} > 0$ for all $i \in \mathbb{Z}$.

P2 : For all function $\mathcal{F}$ such that the conditions (60) are satisfied, the sequence $(\mathbf{w}_i^{n+1})_{i\in\mathbb{Z}}$ verifies

$$\rho_i^{n+1}\mathcal{F}(\ln s_i^{n+1}) - \rho_i^n\mathcal{F}(\ln s_i^n) + \frac{\Delta t}{\Delta x}\Big(\{\rho\mathcal{F}(\ln s)u\}(\mathbf{w}_i^n, \mathbf{w}_{i+1}^n) - \{\rho\mathcal{F}(\ln s)u\}(\mathbf{w}_{i-1}^n, \mathbf{w}_i^n)\Big) \leq 0,$$

where $\{\rho\mathcal{F}(\ln s)u\}(\mathbf{w}_L, \mathbf{w}_R)$ denotes the numerical flux function.

P3 : The sequence $(\mathbf{w}_i^{n+1})_{i\in\mathbb{Z}}$ satisfies the following minimum principle:

$$s_i^{n+1} \geq \min(s_{i-1}^n, s_i^n, s_{i+1}^n).$$

To illustrate our purpose, the relaxation scheme [57, 76] or the HLLC scheme [32, 116, 117] satisfy all the required properties (see Bouchut [37]).

The MUSCL schemes use a better reconstruction than a piecewise constant functions since piecewise linear functions in the following form:

$$\mathbf{w}^h(x, t^n) = \mathbf{w}_i^n + \sigma_i^n(x - x_i), \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),$$

are considered, where σ_i^n denotes the slope. The gradient choice reconstruction is crucial. A large bibliography [66, 67, 85, 116] is devoted to this problem. In fact, the conservative variables $(\mathbf{w}_i^n)_{i\in\mathbb{Z}}$ are not the most used and several authors [67, 32] consider a piecewise reconstruction based on a relevant change of variables:

$$\kappa(\mathbf{w}^n(x, t^n)) = \kappa(\mathbf{w}_i^n) + \sigma_i^n(x - x_i), \quad x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}),$$

where κ denotes a smooth change of variables. For instance, the primitive variables (ρ, u, p) are usually considered.

In the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, the inner approximations, located at $x_{i\pm\frac{1}{2}}$, are considered:

$$\mathbf{w}_i^{n,\pm} = \mathbf{w}^h(x_{i\pm\frac{1}{2}}, t^n).$$

In the sequel, two kindsof changesof variables will be distinguished. The reconstruction will be *conservative* if

$$\mathbf{w}_i^n = \frac{1}{2}(\mathbf{w}_i^{n,-} + \mathbf{w}_i^{n,+}),$$

while the reconstruction will be *non conservative* if we have:

$$\mathbf{w}_i^n \neq \frac{1}{2}(\mathbf{w}_i^{n,-} + \mathbf{w}_i^{n,+}).$$

In the present analysis, several variants of the MUSCL schemes are proposed involving relevant inner approximations $\mathbf{w}_i^{n,\pm}$.

5.1 Stability of the MUSCL schemes [6]

Arguing the above notations, the space second-order MUSCL scheme reads as follows:

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^{n,+}, \mathbf{w}_{i+1}^{n,-}) - \mathbf{F}(\mathbf{w}_{i-1}^{n,+}, \mathbf{w}_i^{n,-})\Big), \tag{64}$$

where $\mathbf{F}$ is the numerical flux function introduced in (62).

Following an idea introduced by Perthame-Shu [100], the stability of the MUSCL scheme (64) is considered when involving a non-conservative gradient reconstruction. Of course, all the following results can be easily extended in the framework of a conservative gradient reconstruction. The reader is referred to Bouchut [37], Leveque [85], Khobalatte-Perthame [80] where stability results are given when assuming a conservative reconstruction.

The stability of the scheme is based on the existence of a *ghost* state, denoted by $\mathbf{w}_i^{n,\star}$, defined as follows:

$$\alpha_i^- \mathbf{w}_i^{n,-} + \alpha_i^\star \mathbf{w}_i^{n,\star} + \alpha_i^+ \mathbf{w}_i^{n,+} = \mathbf{w}_i^n.$$

where the positive coefficients $\alpha_i^{\pm\star}$ are fixed in order to satisfy

$$\alpha_i^- + \alpha_i^\star + \alpha_i^+ = 1.$$

In fact, a new mesh is thus defined where each cell $I_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ is split into three sub-cells: $I_i^- = (x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}} + \alpha_i^-\Delta x)$, $I_i^\star = (x_{i-\frac{1}{2}} + \alpha_i^-\Delta x, x_{i-\frac{1}{2}} + (\alpha_i^- + \alpha_i^\star)\Delta x)$ and $I_i^+ = (x_{i-\frac{1}{2}} + (\alpha_i^- + \alpha_i^\star)\Delta x, x_{i+\frac{1}{2}})$. The vectors $\mathbf{w}_i^{n,\pm\star}$ are associated with the sub-cell $I_i^{\pm\star}$. The first-order scheme (62) is thus applied on the sub-mesh to obtain:

$$\begin{aligned}
\mathbf{w}_i^{n+1,-} &= \mathbf{w}_i^{n,-} - \frac{\Delta t}{\alpha_i^-\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^{n,-}, \mathbf{w}_i^{n,\star}) - \mathbf{F}(\mathbf{w}_{i-1}^{n,+}, \mathbf{w}_i^{n,-})\Big),\\
\mathbf{w}_i^{n+1,\star} &= \mathbf{w}_i^{n,\star} - \frac{\Delta t}{\alpha_i^\star\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^{n,\star}, \mathbf{w}_i^{n,+}) - \mathbf{F}(\mathbf{w}_i^{n,-}, \mathbf{w}_i^{n,\star})\Big),\\
\mathbf{w}_i^{n+1,+} &= \mathbf{w}_i^{n,+} - \frac{\Delta t}{\alpha_i^+\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^{n,+}, \mathbf{w}_{i+1}^{n,-}) - \mathbf{F}(\mathbf{w}_i^{n,\star}, \mathbf{w}_i^{n,+})\Big).
\end{aligned}$$

As a consequence, the approximate solution obtained by (64) rewrites as follows:

$$\mathbf{w}_i^{n+1} = \alpha_i^- \mathbf{w}_i^{n+1,-} + \alpha_i^\star \mathbf{w}_i^{n+1,\star} + \alpha_i^+ \mathbf{w}_i^{n+1,+}.$$

Since the states $\mathbf{w}_i^{n+1,\pm\star}$ result from a first-order scheme which satisfies the stability properties P1, P2 and P3, the following statement is obtained:

Theorem 5.1. *Assume that the first-order scheme (62) satisfies the properties P1, P2 and P3. Assume that $\mathbf{w}_i^n \in \Omega$ for all $i \in \mathbb{Z}$ and impose $\mathbf{w}_i^{n,\pm\star} \in \Omega$. Consider the following CFL condition:*

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left(|\lambda^\pm(\mathbf{w}_i^{n,-}, \mathbf{w}_i^{n,\star})|, |\lambda^\pm(\mathbf{w}_i^{n,\star}, \mathbf{w}_i^{n,+})|, |\lambda^\pm(\mathbf{w}_i^{n,+}, \mathbf{w}_{i+1}^{n,-})|\right) \le \frac{1}{2}\min_{i\in\mathbb{Z}}(\alpha_i^-, \alpha_i^\star, \alpha_i^+).$$

i) If $\mathbf{w}_i^n \in \Omega$ for all $i \in \mathbb{Z}$ then $\mathbf{w}_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$.
ii) For all function $\mathcal{F}$ which satisfies the conditions (60), the sequence $(\mathbf{w}_i^{n+1})_{i\in\mathbb{Z}}$ satisfies the following discrete entropy inequalities:

$$\begin{aligned}
&\rho_i^{n+1}\mathcal{F}(\ln s_i^{n+1}) - \overline{\rho\mathcal{F}(\ln s)}_i^n + \\
&\frac{\Delta t}{\Delta x}\Big(\{\rho\mathcal{F}(\ln s)u\}(\mathbf{w}_i^{n,+}, \mathbf{w}_{i+1}^{n,-}) - \{\rho\mathcal{F}(\ln s)u\}(\mathbf{w}_{i-1}^{n,+}, \mathbf{w}_i^{n,-}\Big) \le 0,
\end{aligned}$$

with

$$\overline{\rho\mathcal{F}(\ln s)}_i^n = \alpha_i^- \rho_i^{n,-}\mathcal{F}(\ln s_i^{n,-}) + \alpha_i^\star \rho_i^{n,\star}\mathcal{F}(\ln s_i^{n,\star}) + \alpha_i^+ \rho_i^{n,+}\mathcal{F}(\ln s_i^{n,+}),$$

where $s_i^{n,\pm\star} = s(\mathbf{w}_i^{n,\pm\star})$.
iii) The specific entropy satisfies the following minimum principle:

$$s_i^{n+1} = s(\mathbf{w}_i^{n+1}) \ge \min(s_{i-1}^{n,+}, s_i^{n,\pm\star}, s_{i+1}^{n,-}).$$

Let us emphasize that the condition $\mathbf{w}_i^{n,\pm\star} \in \Omega$ turns out to be a new slope limitation. The gradient reconstruction must be modified according to the new conditions. In addition, let us note that the choice of a *conservative* or *non conservative* gradient reconstruction is not specified in the above result. However, concerning the conservative reconstruction, [37, 80] propose a CFL condition which is less restrictive than the CFL condition proposed in the present study.

Now, we exhibit a non-conservative gradient reconstruction based on the primitive variables (ρ, u, p).

In a cell I_i, the inner approximations located at $x_{i\pm\frac{1}{2}}$ write as follows:

$$\begin{cases}
\rho_i^{n,\pm} = \rho_i^n \pm \Delta\rho, \\
u_i^{n,\pm} = u_i^n \pm \Delta u, \\
p_i^{n,\pm} = p_i^n \pm \Delta p,
\end{cases} \tag{65}$$

where $(\Delta\rho, \Delta u, \Delta p)$ denotes the increment vector which must satisfy the new limitations. The increments obtained involving some "usual" limitations (minmod, superbee, MC... [85, 116]) are denoted δX. These standard limiters are thus modified to enforce the condition: $\mathbf{w}_i^{n,\pm\star} \in \Omega$. To illustrate our purpose, let us consider the minmod limiter which writes as follows:

$$\delta X = \frac{1}{2}\mathrm{minmod}(X_i - X_{i-1}, X_{i+1} - X_i).$$

Under the following assumption:

$$\alpha_i^- = \alpha_i^\star = \alpha_i^+ = \frac{1}{3},$$

the choice

$$\Delta\rho = \rho_i^n \max\left(-1, \min(1, \frac{\delta\rho}{\rho_i^n})\right),$$

$$\Delta u = \operatorname{sign}(\delta u)\sqrt{\min\left((\delta u)^2, \frac{p_i^n}{(\gamma-1)\rho_i^n\left(1+2\left(\frac{\Delta\rho}{\rho_i^n}\right)^2\right)}\right)}$$

$$\Delta p = p_i^n \max\left(-1, \min(1, \frac{\delta p}{p_i^n})\right),$$

satisfies all the required conditions.

5.2 L^1-stability of the MUSCL-Hancock scheme [11]

The MUSCL scheme (64) is a space second-order but it remains an accurate time first-order. In [91], van Leer introduces the MUSCL-Hancock scheme which is both space and time second-order accurate. This scheme is written in the following two step form:

$$\begin{cases} \mathbf{w}_i^{n+\frac{1}{2},\pm} = \mathbf{w}_i^{n,\pm} - \dfrac{\Delta t}{2\Delta x}\left(\mathbf{f}(\mathbf{w}_i^{n,+}) - \mathbf{f}(\mathbf{w}_i^{n,-})\right), \\ \mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \dfrac{\Delta t}{\Delta x}\Big(\mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},+}, \mathbf{w}_{i+1}^{n+\frac{1}{2},-}) - \mathbf{F}(\mathbf{w}_{i-1}^{n+\frac{1}{2},+}, \mathbf{w}_i^{n+\frac{1}{2},-})\Big), \end{cases} \tag{66}$$

The aim of the present analysis is to establish a rigorous CFL condition and a relevant limitation procedure on $\mathbf{w}_i^{n,\pm}$ to enforce the robustness of the numerical method; i.e. $\mathbf{w}_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$. When considering the technique of the ghost state (as in the above section), we establish that the states, obtained at the end of each step, belongs to Ω.

To prove that $\mathbf{w}_i^{n+\frac{1}{2},\pm} \in \Omega$, ghost states $\mathbf{w}_i^{\star,\pm}$ are introduced as follows:

$$\frac{1}{4}\mathbf{w}_i^{n,-} + \frac{1}{2}\mathbf{w}_i^{\star,\pm} + \frac{1}{4}\mathbf{w}_i^{n,+} = \mathbf{w}_i^{n,\pm}.$$

Over the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$, we set $\mathbf{w}^{h,\pm}(x,t)$ the weak solution of the Cauchy problem (57) where the initial data is given by

$$\mathbf{w}^{h,\pm}(x,0) = \begin{cases} \mathbf{w}_i^{n,-}, & \text{if } x \in (x_{i-\frac{1}{2}}, x_{i-\frac{1}{4}}), \\ \mathbf{w}_i^{\star,\pm}, & \text{if } x \in (x_{i-\frac{1}{4}}, x_{i+\frac{1}{4}}), \\ \mathbf{w}_i^{n,+}, & \text{if } x \in (x_{i+\frac{1}{4}}, x_{i+\frac{1}{2}}). \end{cases}$$

Under the CFL condition

$$\frac{\Delta t/2}{\Delta x/4}\max_{i\in\mathbb{Z}}\left(|\lambda_e^{\pm}(\mathbf{w}_i^{n,-}, \mathbf{w}_i^{\star,\pm}|), |\lambda_e^{\pm}(\mathbf{w}_i^{\star,\pm}, \mathbf{w}_i^{n,+}|)\right) \leq \frac{1}{2}, \tag{67}$$

where $\lambda_e^{\pm}(\mathbf{w}_L, \mathbf{w}_R)$ denotes the exact velocities of the acoustic waves associated with system (57), the function $\mathbf{w}^{h,\pm}$ turns out to be the juxtaposition of the non-interacting Riemann problems set at the cell interfaces $x_{i\pm\frac{1}{2}}$ and $x_{i\pm\frac{1}{4}}$. After straightforward computations, the following identity is obtained:

$$\mathbf{w}_i^{n+\frac{1}{2},\pm} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{w}^{h,\pm}(x, \frac{\Delta t}{2})\ dx.$$

As a consequence, we have $\mathbf{w}_i^{n+\frac{1}{2},\pm} \in \Omega$ as soon as $\mathbf{w}^{h,\pm}(x, \frac{\Delta t}{2}) \in \Omega$. In other words, we have to impose that the states $\mathbf{w}_i^{n,\pm}$ and $\mathbf{w}_i^{\star,\pm}$ belong to Ω.

Arguing this first property, we must prove that $\mathbf{w}_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$. We invoke the same idea that was introduced in the previous section but for $\alpha_i^{\pm} = 1/4$ and $\alpha_i^{\star} = 1/2$ As a consequence, a state $\mathbf{w}_i^{n+\frac{1}{2},\star}$ is defined as follows:

$$\frac{1}{4}\mathbf{w}_i^{n+\frac{1}{2},-} + \frac{1}{2}\mathbf{w}_i^{n+\frac{1}{2},\star} + \frac{1}{4}\mathbf{w}_i^{n+\frac{1}{2}.+} = \mathbf{w}_i^n.$$

Next, we apply the first-order scheme (62) to the states $\mathbf{w}_i^{n+\frac{1}{2},\pm\star}$. We obtain

$$\begin{aligned}
\mathbf{w}_i^{n+1,-} &= \mathbf{w}_i^{n+\frac{1}{2},-} - \\
&\qquad \frac{\Delta t}{\Delta x/4}\left(\mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},-}, \mathbf{w}_i^{n+\frac{1}{2},\star}) - \mathbf{F}(\mathbf{w}_{i-1}^{n+\frac{1}{2},+}, \mathbf{w}_i^{n+\frac{1}{2},-})\right),\\
\mathbf{w}_i^{n+1,\star} &= \mathbf{w}_i^{n+\frac{1}{2},\star} - \\
&\qquad \frac{\Delta t}{\Delta x/2}\left(\mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},\star}, \mathbf{w}_i^{n+\frac{1}{2},+}) - \mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},-}, \mathbf{w}_i^{n+\frac{1}{2},\star})\right),\\
\mathbf{w}_i^{n+1,+} &= \mathbf{w}_i^{n+\frac{1}{2},+} - \\
&\qquad \frac{\Delta t}{\Delta x/4}\left(\mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},+}, \mathbf{w}_{i+1}^{n+\frac{1}{2},-}) - \mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},\star}, \mathbf{w}_i^{n+\frac{1}{2},+})\right),
\end{aligned}$$

to deduce immediately the following formula satisfied by $\mathbf{w}_i^{n+1}$:

$$\mathbf{w}_i^{n+1} = \frac{1}{4}\mathbf{w}_i^{n+1,-} + \frac{1}{2}\mathbf{w}_i^{n+1,\star} + \frac{1}{2}\mathbf{w}_i^{n+1,+}.$$

As soon as the CFL condition (63) is satisfied, the scheme (62) verifies the P1 property of numerical invariance of Ω. As a consequence, under the CFL condition

$$\frac{\Delta t}{\Delta x/4}\max_{i\in\mathbb{Z}}\left(|\lambda^{\pm}(\mathbf{w}_i^{n+\frac{1}{2},-}, \mathbf{w}_i^{n+\frac{1}{2},\star})|, |\lambda^{\pm}(\mathbf{w}_i^{n+\frac{1}{2},\star}, \mathbf{w}_i^{n+\frac{1}{2},+})|, |\lambda^{\pm}(\mathbf{w}_i^{n+\frac{1}{2},+}, \mathbf{w}_{i+1}^{n+\frac{1}{2},-})|\right) \leq \frac{1}{2}, \quad (68)$$

we have $\mathbf{w}_i^{n+1} \in \Omega$ as soon as the states $\mathbf{w}_i^{n+\frac{1}{2},\pm\star}$ belong to Ω.

To conclude, we have to prove that $\mathbf{w}_i^{n+\frac{1}{2},\star}$ belongs to Ω where

$$\mathbf{w}_i^{n+\frac{1}{2},\star} = \mathbf{w}_i^{\star,\star} - \frac{\Delta t/2}{\Delta x}\left(\mathbf{f}(\mathbf{w}_i^{n,-}) - \mathbf{f}(\mathbf{w}_i^{n,+})\right),$$
$$\mathbf{w}_i^{\star,\star} = 2\mathbf{w}_i^{n} - \frac{1}{2}\left(\mathbf{w}_i^{n,-} + \mathbf{w}_i^{n,+}\right).$$

The establishment of $\mathbf{w}_i^{n+\frac{1}{2},\star} \in \Omega$ turns out to be similar to the proof of $\mathbf{w}_i^{n+\frac{1}{2},\pm} \in \Omega$. Indeed, we easily show that $\mathbf{w}_i^{n+\frac{1}{2},\star}$ belongs to Ω as soon as $\mathbf{w}_i^{n,\pm} \in \Omega$ and $\mathbf{w}_i^{\star,\star} \in \Omega$, with the following CFL condition

$$\frac{\Delta t/2}{\Delta x/4}\max_{i\in\mathbb{Z}}\left(|\lambda_e^{\pm}(\mathbf{w}_i^{n,-},\mathbf{w}_i^{\star,\star})|, |\lambda_e^{\pm}(\mathbf{w}_i^{\star,\star},\mathbf{w}_i^{n,+})|\right) \le \frac{1}{2}, \tag{69}$$

The following statement is thus proved:

Theorem 5.2. *Assume $\mathbf{w}_i^{n+\frac{1}{2},\star} \in \Omega$ for all $i \in \mathbb{Z}$ and consider a (conservative or non-conservative) reconstruction such that $\mathbf{w}_i^{n,\pm} \in \Omega$, $\mathbf{w}_i^{\star,\pm} \in \Omega$ and $\mathbf{w}_i^{\star,\star} \in \Omega$ for all $i \in \mathbb{Z}$. Assume the CFL restrictions (67), (68) and (69). Then the updated state $\mathbf{w}_i^{n+1}$, obtained by the MUSCL-Hancock scheme (66), belongs to Ω for all $i \in \mathbb{Z}$.*

Let us emphasize that the CFL condition (68) is, in fact, *implicit*. Indeed, this restriction needs to evaluate the states $\mathbf{w}_i^{n+\frac{1}{2},\pm\star}$ which depend on the time step Δt. From a numerical point of view, this remark involves several difficulties. However, the MUSCL-Hancock scheme (66) can be modified to make *explicit* the CFL restrictions. The following scheme is thus proposed:

$$\begin{cases} \mathbf{w}_i^{n+\frac{1}{2},\pm} = \mathbf{w}_i^{n,\pm} - \frac{\Delta t_1/2}{\Delta x}\left(\mathbf{f}(\mathbf{w}_i^{n,+}) - \mathbf{f}(\mathbf{w}_i^{n,-})\right), \\ \bar{\mathbf{w}}_i = \frac{1}{2}\left(\mathbf{w}_i^{n+\frac{1}{2},-} + \mathbf{w}_i^{n+\frac{1}{2},+}\right), \\ \bar{\bar{\mathbf{w}}}_i = \mathbf{w}_i^{n} - \frac{\Delta t_2}{\Delta x}\left(\mathbf{F}(\mathbf{w}_i^{n+\frac{1}{2},+},\mathbf{w}_{i+1}^{n+\frac{1}{2},-}) - \mathbf{F}(\mathbf{w}_{i-1}^{n+\frac{1}{2},+},\mathbf{w}_i^{n+\frac{1}{2},-})\right), \\ \mathbf{w}_i^{n+1} = \alpha\bar{\bar{\mathbf{w}}}_i + (1-\alpha)\bar{\mathbf{w}}_i, \end{cases} \tag{70}$$

where α are given by the following formula:

$$\alpha = \begin{cases} \frac{\Delta t_1}{2}\frac{\frac{\Delta t_1}{2} - 2\sqrt{\Delta t_2(\Delta t_1 - \Delta t2)}}{(\frac{\Delta t_1}{2} - \Delta t_2)^2}, & \text{if } \Delta t_2 < \Delta t_1, \\ 1, & \text{if } \Delta t_2 = \Delta t_1. \end{cases}$$

The approximate solution $\mathbf{w}_i^{n+1}$ is evaluated at time $t^n + \Delta t$ with $\Delta t = \alpha\Delta t_2 + (1-\alpha)\frac{\Delta t_1}{2}$. The stability results are preserved as soon as Δt_1 satisfies the CFL conditions (67) and (69) while Δt_2 must satisfy (68).

To conclude this MUSCL-Hancock scheme analysis, let us underline that the new limitation to be considered is more sophisticated than usual. Indeed, we have to impose $\mathbf{w}_i^{n,\pm}$, $\mathbf{w}_i^{\star,\pm}$ and $\mathbf{w}_i^{\star,\star}$ in Ω. However, it is possible to modify the standard limiters to enforce all the required restrictions. Let us set δX the increment issuing from usual gradient reconstructions (minmod, superbee, etc.). With the notations introduced in (65), the primitive variable increments are modified as follows:

$$
\begin{aligned}
\Delta u &= \max\left(-\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n}, \min\left(\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n}, \frac{\Delta x}{2}\sigma_i^u\right)\right), \\
\Delta \rho &= \max\left(-\rho\sqrt{\frac{\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n} - \Delta u^2}{4\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n} - \Delta u^2}},\right. \\
&\qquad \left. \min\left(\rho\sqrt{\frac{\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n} - \Delta u^2}{4\frac{p_i^n}{\frac{3}{2}(\gamma-1)\rho_i^n} - \Delta u^2}}, \frac{\Delta x}{2}\sigma_i^\rho\right)\right), \\
\Delta p &= \max\left(-\frac{1}{2}\left(p_i^n - \frac{3}{2}(\gamma-1)\frac{\Delta u^2}{\rho_i^n + 2\Delta\rho}((\rho_i^n)^2 - \Delta\rho^2)\right),\right. \\
&\left. \min\left(\frac{1}{2}\left(p_i^n - \frac{3}{2}(\gamma-1)\frac{\Delta u^2}{\rho_i^n - 2\Delta\rho}((\rho_i^n)^2 - \Delta\rho^2)\right), \frac{\Delta x}{2}\sigma_i^p\right)\right).
\end{aligned}
\tag{71}
$$

5.3 MUSCL schemes for 2D unstructured meshes [10]

The above stability results concerning the 1D MUSCL schemes are extended when considering 2D unstructured meshes. The reader is referred to Perthame-Qiu [99] where results are given considering a conservative gradient reconstruction. In the present study, we exhibit a CFL restriction and a relevant gradient limitation to enforce the robustness of the MUSCL scheme when considering 2D applications.

Once again, the expected stability results come from the introduction of a relevant sub-mesh associated with relevant ghost states. Let us note from now on that the sub-mesh is an artefact useful in the proof but never informatically computed.

In a cell C_i with $\Lambda(i)$ adjacent cells $j(k)$, $1 \leq k \leq \Lambda(i)$, the edge which separates C_i and $C_{j(k)}$ is denoted $\Gamma_{ij(k)}$ where $\mathbf{n}_{ij(k)}$ is the associated outer unit normal. The state $\mathbf{w}_{ij(k)}$ denotes the second-order approximation of the solution in the cell C_i but located near the edge $\Gamma_{ij(k)}$. With these notations, the 2D MUSCL scheme reads as follows:

$$
\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{|C_i|}\sum_{k=1}^{\Lambda(i)} |\Gamma_{ij(k)}|\phi(\mathbf{n}_{ij(k)}, \mathbf{w}_{ij(k)}, \mathbf{w}_{j(k)i}), \tag{72}
$$

where $\phi(\mathbf{n}, \mathbf{w}_L, \mathbf{w}_R)$ denotes the numerical flux function. For the sake of simplicity, the

2D rotational invariance of the Euler equations alows us to write

$$\begin{aligned} \phi(\mathbf{n}, \mathbf{w}_L, \mathbf{w}_R) &= \phi\left(R.\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{w}_L, \mathbf{w}_R\right), \\ &= \mathcal{R}^{-1}\mathbf{F}(\mathcal{R}.\mathbf{w}_L, \mathcal{R}.\mathbf{w}_R), \end{aligned}$$

where $\mathcal{R}$ denotes an unitary transform and

$$\mathcal{R}.\mathbf{w} = (\rho, \rho\mathcal{R}.\begin{pmatrix} u \\ v \end{pmatrix}, \rho E).$$

Here, $\mathbf{F}$ is the numerical flux function associated with the first-order scheme (62). With some abuses in the notations, let us note that the vector $\mathbf{w}$ denotes the vector state coming from the 2D Euler equations, i.e. with tangential velocity. The tangential velocity is also considered in the numerical flux function.

The cell C_i is thus split into $\Lambda(i) + 1$ sub-cells as displayed in figure 7.

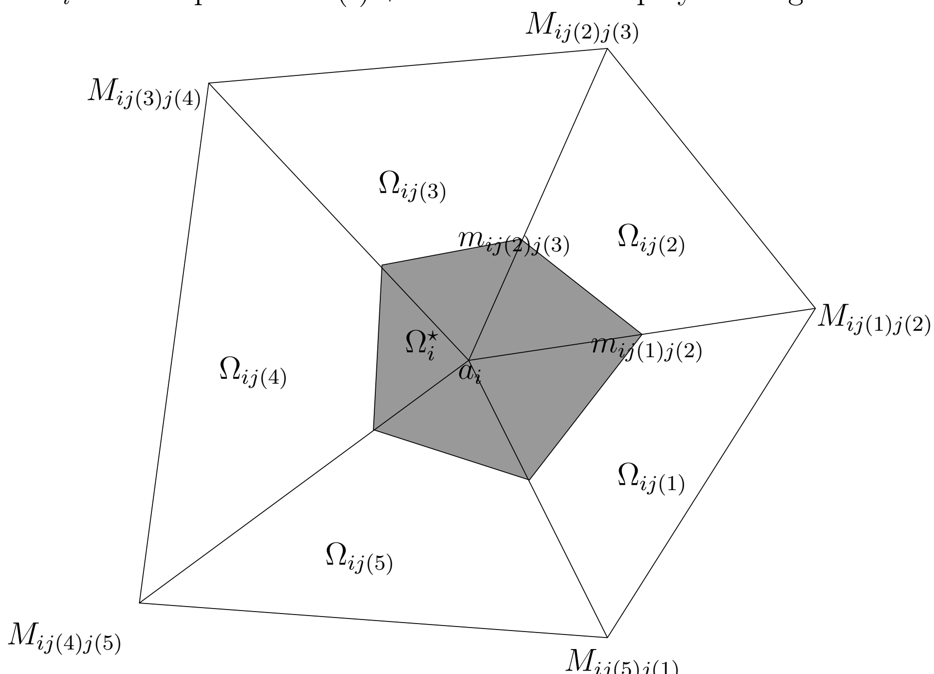

Figure 7: Sub-cell decomposition of the cell C_i.

A ghost state $\mathbf{w}_i^\star$ is defined as follows:

$$\frac{|C_i^\star|}{|C_i|}\mathbf{w}_i^\star + \sum_{k=1}^{\Lambda(i)} \frac{|C_{ij(k)}|}{|C_i|}\mathbf{w}_{ij(k)} = \mathbf{w}_i^n. \tag{73}$$

We apply the same idea used to prove the stability of the 1D MUSCL scheme. To access such an issue, we introduce the following first-order scheme:

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \frac{\Delta t}{|C_i|} \sum_{k=1}^{\Lambda(i)} |\Gamma_{ij(k)}| \phi(\mathbf{n}_{ij(k)}, \mathbf{w}_i, \mathbf{w}_{j(k)}),$$

This first-order scheme is considered to evolve in time the vectors $\mathbf{w}_i^\star$ and $\mathbf{w}_{ij(k)}$ stated on the sub-mesh. The following relation is easily obtained:

$$\mathbf{w}_i^{n+1} = \frac{|C_i^\star|}{|C_i|} \mathbf{w}_i^{n+1,\star} + \sum_{k=1}^{\Lambda(i)} \frac{|C_{ij(k)}|}{|C_i|} \mathbf{w}_{ij(k)}^{n+1}, \tag{74}$$

where $\mathbf{w}_i^{n+1}$ is given by (72). We thus obtain

Theorem 5.3. *Let $(\mathbf{w}_i^n)_{i\in\mathbb{Z}}$ be given by (72). Assume that the states $\mathbf{w}_i^n$, $\mathbf{w}_{ij}$ and $\mathbf{w}_i^\star$ belong to Ω. Assume that the sub-cells (C_{ij} or $C_i^\star$) are split into k elementary triangles T^k. Let us assume the following CFL restrictions:*

$$\Delta t \frac{|\Gamma^k|}{|T^k|} \max |\lambda^\pm(\mathbf{n}_{\Gamma^k}, \mathbf{w}_{\Gamma^k}^+, \mathbf{w}_{\Gamma^k}^-)| \leq 1,$$

where Γ^k is the edge of T^k which separates two sub-cells and where the states $\mathbf{w}_{\Gamma^k}^\pm$ define the value of the vector states in each side of this edge.
i) If $\mathbf{w}_i^n \in \Omega$ for all $i \in \mathbb{Z}$ then $\mathbf{w}_i^{n+1} \in \Omega$ for all $i \in \mathbb{Z}$.
ii) For all function $\mathcal{F}$ which satisfies (60), the following discrete entropy inequalities hold:

$$\rho_i^{n+1} \mathcal{F}(\ln s_i^{n+1}) - \overline{\rho \mathcal{F}(\ln s)}_i^n + \frac{\Delta t}{|C_i|} \sum_{k=1}^{\Lambda(i)} |\Gamma_{ij(k)}| \varphi(\mathbf{n}_{ij(k)}, \mathbf{w}_{ij(k)}, \mathbf{w}_{j(k)i}) \leq 0,$$

where φ denotes the numerical entropy flux function and

$$\overline{\rho \mathcal{F}(\ln s)}_i^n = \frac{|C_i^\star|}{|C_i|} \rho_i^\star \mathcal{F}(\ln s_i^\star) + \sum_{k=1}^{\Lambda(i)} \frac{|C_{ij(k)}|}{|C_i|} \rho_{ij(k)} \mathcal{F}(\ln s_{ij(k)}).$$

iii) The following minimum principle is satisfied by the specific entropy:

$$s_i^{n+1} \geq \min_{1 \leq k \leq \Lambda(i)} (s_i^\star, s_{ij(k)}, s_{j(k)i}).$$

From a numerical point of view, let us emphasize that the scheme is free from the definition of the sub-cells $C_i^\star$ and C_{ij}. For the sake of simplicity, we adopt the following condition:

$$\frac{|C_i^\star|}{|C_i|} = \frac{|C_{ij(k)}|}{|C_i|} = \frac{1}{\Lambda(i)+1}.$$

To illustrate the numerical method, we display figure 8 the results obtained when considering a Mach 3 wind tunnel with a step. The reader is referred to Woodward-Colella [120] for further details concerning this numerical experiment.

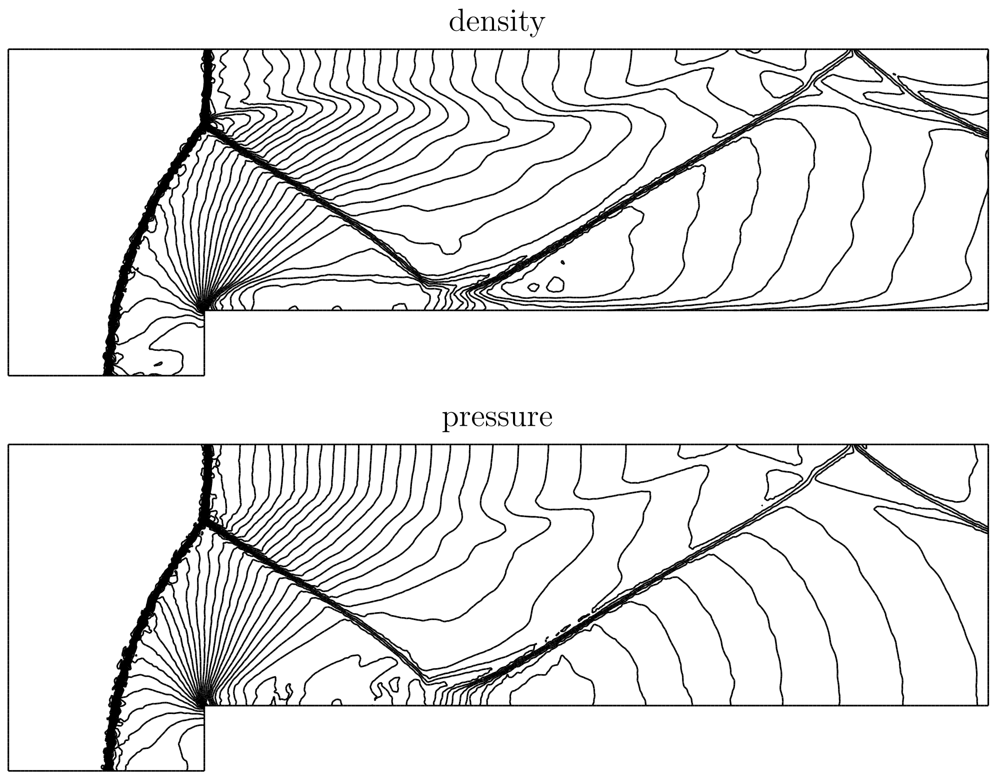

Figure 8: Mach 3 wind tunnel with a step. Density and pressure at time $t = 4$.

6 Conclusion and prospects

The present document considers a whole research tasks which approach several fields whose central topic remains the derivations and analyses of finite volume numerical schemes to approximate the solutions of hyperbolic systems. Particular attention was related to the robustness and the stability when deriving these numerical methods. Two main topics are released.

The first topic concerns the numerical approximation of solutions of systems in non-conservation form with a parabolic regularization. Many compressible flows are modeled involving a suitable extension of the Navier-Stokes equations: the multi-entropy model. Typically, turbulent and/or multi-fluid flows enter such an approach. This model is governed by a non-conservative hyperbolic system supplemented by a parabolic regularization. When viscous shock layers are considered, we can show that the shock structure intrinsically depends on the shape of the parabolic regularization. When considering the numerical approximation, relevant account needs to be taken of the regularization which needs the use of very fine meshes or a suitable scheme. A nonlinear projection scheme is derived to obtain a powerful numerical method which has been applied to perform simulations in several domains of application.

The second topic is devoted to the numerical approximations of the solutions of hyperbolic systems of conservation laws. Relaxation schemes have been derived to propose robust and stable schemes to approximate several hyperbolic systems. The robustness of the relaxation schemes is satisfied even if the considered system is *poorly stable*: weakly hyperbolic or resonant, for instance. In addition, these schemes are accurate which make them very attractive for number of physical applications. Note that some studies use other kinds of numerical schemes distinct from relaxation schemes. An *asymptotic-preserving* HLLC scheme has been derived to simulate radiative transfer. After a rigorous analysis of the scheme, all the required stability properties are proved to be satisfied by the approximate solutions. In addition, the scheme is shown to satisfy a relevant asymptotic behavior. The adopted method has been extended to perform 2D simulation. This extension has preserved all the stability properties. This work concerning the analysis of the conservative first-order scheme is supplemented by a study of the second-order MUSCL schemes. Indeed, the MUSCL schemes have been shown to be a convex combination of states issuing from a first-order scheme. This remark has been obtained after introducing a relevant ghost state. As a consequence, it is easy to establish that the stability properties satisfied by the first-order scheme are preserved by its second-order MUSCL scheme extension. Next, the robust MUSCL schemes have been extended to consider 2D unstructured meshes. Similar stability results are given when considering the second-order time and space MUSCL-Hancock scheme.

All these studies open new prospects in various domains of application. For example, the work devoted to the stability of the MUSCL schemes must be extended to high order accurate schemes. Indeed, when considering some complex applications, the second-order schemes with gradient limiters involve numerical approximations where the accuracy remains poor. Recently, several high order numerical methods have been proposed in the literature (ENO, WENO, residual distribution...) and actually are widely used in many simulations. Few results about the stability of these schemes are proposed. The ideas introduced to prove the robustness of the MUSCL schemes have to be extended. Then we shall establish the robustness of the numerical approximations obtained when involving high order schemes. The consequences of such a result would be numerous and it would raise certain ambiguities concerning the choices of the parameters used in these numerical methods.

Other research orientations must also be approached. For example, the relaxation schemes have shown their ability to approximate the solutions of systems which do not satisfy all the basic stability properties (not unconditionally hyperbolic, weakly hyperbolic or resonant). A natural extension of this work concerns hyperbolic system of conservation laws but for a discontinuous flux function. Recent works devoted to the scalar laws study for the weak entropy solution. The numerical approximation of these solutions constitutes an issue necessary to the realization of many industrial simulations. Indeed, several sophisticated models (to consider complex physics) do not satisfy the smoothness properties usually assumed by the standard theory of hyperbolic system of conservation laws. Similar difficulties arise when considering applications with coupling models. Indeed, for such applications, the flux function turns out not to be a continuous function of the unknowns.

Another extension which must be envisaged relates to the numerical approximation of the non-conservative hyperbolic systems. Indeed, in the present document, we propose several works devoted to a specific model: the multi-entropy model. This system enters many models and the nonlinear projection scheme turns out to be relevant when approximating the solutions of this model. Of course, many models are governed by non-conservative system where the nonlinear projection scheme cannot be applied. Relevant extensions are thus crucial.

7 Publications of the author

Articles

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