THE LINEAR KALMAN-BUCY FILTER WITH RESPECT TO LIOUVILLE FRACTIONAL BROWNIAN MOTION

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ABSTRACT. We suggest a linear filtering method that involves preprocessing of data and postprocessing of estimations.The main advantage of this procedure is that it requires only ordinary linear Kalman-Bucy filtering. The main disadvantage is that it does not give the best estimation of the data given the observation, since it is optimal for a non-classical L^2 criterion.

The problem of linear filtering with fractional Brownian motion noise, in the signal and/or the observations has already been solved on the one hand by Coutin and Decreusefond [3, 4], and on the other hand by Kleptsyna, Kloeden and Anh [6, 7]. However, it does not seem to be very easy to determine explicitly the weight functions and/or to solve the Fredholm/Volterra integral equations. This paper is an attempt to give a more explicit (numerical) scheme (more understandable by a computer engineer). The price to pay is that we do not have the best estimation of the state of the system given the observations. The following archetypal example shall give more insight into the machinery involved.
Assume that the state X of the system and the observations Y can be modelled as the solutions of the linear stochastic differential system

$$\begin{cases} dX_t = aX_t\,dt + dW^H_t \\ dY_t = cX_t\,dt + dB^H_t \end{cases} \tag{1}$$

where W^H, B^H denote two independent Liouville fractional Brownian motions of same Hurst parameter H. We shall use the representation

$$B^H(t) = I^{H+\frac{1}{2}}(\dot{B})(t) = \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\,dB_s\,.$$

$$W^H(t) = I^{H+\frac{1}{2}}(\dot{W})(t) = \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\,dW_s\,.$$

Accordingly, the natural definition of a solution of the system (1) is to consider that $X = I^{H-\frac{1}{2}}\tilde{X}$,$Y = I^{H-\frac{1}{2}}\tilde{Y}$ where $(\tilde{X}, \tilde{Y})$ is the solution of

$$\begin{cases} d\tilde{X}_t = a\tilde{X}_t\,dt + dW_t \\ d\tilde{Y}_t = c\tilde{X}_t\,dt + dB_t \end{cases} \tag{2}$$

where W, B are independent Brownian motions. Indeed, we have then $\tilde{Y} = cI^1(\tilde{X}) + B$, and thus

$$\begin{aligned} I^{H-\frac{1}{2}}(\tilde{Y}) &= I^{H-\frac{1}{2}}(B) + cI^{H+\frac{1}{2}}(\tilde{X}) \\ &= B^H + cI^1(I^{H-\frac{1}{2}}(\tilde{X})) \end{aligned}$$

The filtering process we propose can be decomposed as in the following diagram

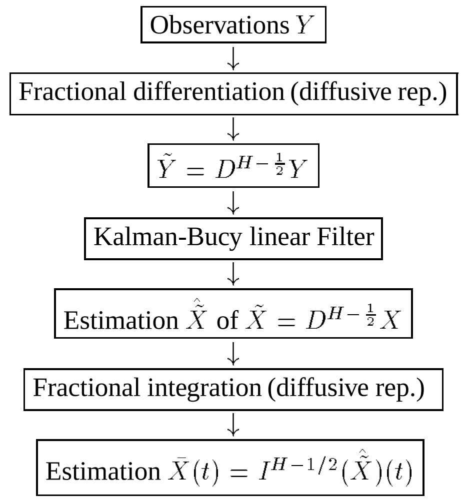

FIGURE 1. The Liouville fractional Brownian motion filter

We need to explain the term *diffusive rep.* (see section 1 and the general references [1, 5, 9, 8, 2]). Fractional integration of order $\alpha \in (0,1)$ has the diffusive representation:

$$I^\alpha f(t) = (const.)\int_0^\infty \xi^{-\alpha} f_\xi(t)\,d\xi\,,$$

where for $\xi > 0$, the function f_ξ is the solution of the linear ordinary differential equation

$$\partial_t f_\xi = -\xi f_\xi + f\,.$$

This leads to the following iterative scheme – also called memoryless or Markovian scheme – the c_ξ denoting suitable coefficients and π a finite partition,

$$I^\alpha f(t) = \sum_{\xi\in\pi} c_\xi f_\xi(t)\,,$$
$$f_\xi(t+\Delta t) - f_\xi(t) = -\xi f_\xi(t) + f(t)\,.$$

The aim of the rest of this paper is to give precise meaning to each of the steps of the proposed filtering scheme. We first recall what is the diffusive representation of the fractional integration of a deterministic function. Then we show how to apply this to Liouville fractional Brownian motion. We then show how to deal with the case of different Hurst indexes in the signal and in the noise.
Eventually, we show that the proposed filter is optimal with respect to a non classical l^2 criterion.

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1. Diffusive representations of fractional integration

Given $\alpha \in (0,1)$ and a locally integrable function f on $\mathbb{R}_+$, the fractional integral of f is

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds \qquad (t>0)\,. \tag{3}$$

Classic properties of fractional integrals may be found in [10].

The diffusive representation of I^α is based on the identity

$$\frac{1}{u} = \int_0^\infty \frac{\xi^{a-1}}{\Gamma(a)} e^{-\xi u}\, d\xi \qquad (u>0, a>0)\,. \tag{4}$$

Injecting this relation into (3) gives

$$\begin{aligned} I^\alpha f(t) &= \int_0^t ds \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty \xi^{-\alpha} e^{-\xi(t-s)} f(s)\, d\xi \\ &= \int_0^\infty \frac{\xi^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} f_\xi(t)\, d\xi\,, \end{aligned}$$

with $f_\xi(t) = \int_0^\infty e^{-\xi(t-s)} f(s)\, ds$ solution of

$$\partial_t f_\xi = -\xi f_\xi(t) + f(t)\,.$$

Remark 1. It is easy to see that Fubini's Theorem applies if either f is locally in L^p for some $p>1$, or f is continuous at 0.

The term diffusive representation has its origin in the equivalent realization

$$I^\alpha f(t) = \int \hat{m}\, \phi\, dx\,, \quad \partial_t \phi = \partial_x^2 \phi + \delta \otimes f\,,$$

where m is the image of the measure $\mu(d\xi) = c_H \xi^{-\alpha}\, d\xi$ under the transformation $\xi \to \zeta$ with $\xi = 4\pi^2\zeta^2$ (we have used the Fourier transform in the real variable ζ).

2. Application to Liouville fractional Brownian motion

From the introduction we can infer that we need a diffusive representation of the operator $\Lambda = I^{H-\frac{1}{2}}$ and a diffusive representation for its inverse Λ^{-1}. This raises the following questions:

1. How can we apply these integrodifferential operators to stochastic processes (and in particular to Brownian motion) ?
2. Depending on wether $H < 1/2$ or not, either Λ, either Λ^{-1} is not a fractional integral operator. What is the diffusive representation of fractional differentiation ?

We are going to answer these two questions simultaneously. The basic ingredient of our proof is the commutation relation

(5)

$$I^\alpha I^{1-\alpha} f(t) = I^1 f(t) = \int_0^t f(s)\, ds \quad (0<\alpha<1)\,,$$

which can be proved by combining Fubini's Theorem with the identity valid for $\alpha \in (0,1)$

(6)

$$\int_0^t r^{\alpha-1}(1-r)^{-\alpha}\, dr = \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin(\alpha\pi)}\,.$$

We shall

1. define the processes $(\Lambda Z(t)\,; t \geq 0)$ and $(\Lambda^{-1} V(t)\,; t\geq 0)$ wher Z is one of the processes $W, B, \tilde{X}, \tilde{Y}$ (defined in the introduction), and V is one of the processes $X, Y, \Lambda W, \Lambda B$.
2. Establish the relation $\Lambda^{-1}\Lambda Z = Z$.
3. Exhibit diffusive representaitons for the processes ΛZ and $\Lambda^{-1} V$.

2.1. **The case $0 < H < \frac{1}{2}$.**

Step 1 : defining ΛZ. Formally $\Lambda Z = I^{H-\frac{1}{2}} Z = I^{H+\frac{1}{2}}(\dot{Z})$. This leads naturally to the definition

$$\Lambda Z(t) \overset{\text{def}}{=} \int_0^t \frac{(t-s)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\, dZ_s\,. \tag{7}$$

We first check that ΛZ is well defined for Z a brownian motion since $s \to (t-s)^{H-\frac{1}{2}}$ is in $L^2(0,t)$. To define $X = \Lambda \tilde{X}$ and Y, we use the explicit expression

$$\tilde{X}_t = \int_0^t e^{a(t-s)}\, dW_s\,,$$

which shows that $\tilde{X}_t$ is a centered Gaussian random variable with variance $\frac{e^{2at}-1}{2a}$. Therefore

$$\mathbb{E}\left[\int_0^t (t-s)^{H-\frac{1}{2}} \left|\tilde{X}_s\right| ds\right] < +\infty$$

Step 2 : the diffusive representation of ΛZ. We now inject into (7) the identity (4). Fubini's Stochastic Theorem yields then

$$\Lambda Z(t) = \int_0^t \frac{\xi^{-(H+\frac{1}{2})}}{\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)} Z_\xi(t)\, d\xi\,,$$

where $Z_\xi(t) = \int_0^t e^{-\xi(t-s)}\, dZ_s$ is the solution of the linear stochastic differential equation

$$dZ_\xi(t) = -\xi Z_\xi(t)\, dt + dZ_\xi(t)\,.$$

It is easy to check that Fubini's Stochastic Theorem applies since

$$\int_0^\infty d\xi\, \xi^{-(H+\frac{1}{2})} \mathbb{E}\left[\left(\int_0^t e^{-\xi(t-s)}\, dW_s\right)^2\right]^{\frac{1}{2}} = \int_0^\infty d\xi\, \xi^{-(H+\frac{1}{2})} \left(\frac{1-e^{-2\xi t}}{2\xi}\right)^{\frac{1}{2}} < +\infty\,.$$

and

$$\int_0^\infty d\xi\, \xi^{-(H+\frac{1}{2})} \mathbb{E}\left[\int_0^t e^{-\xi(t-s)} \left|\tilde{X}_s\right| ds\right] = \int_0^\infty d\xi\, \xi^{-(H+\frac{1}{2})} (const) \int_0^t e^{-\xi(t-s)} \left(\frac{e^{2as}-1}{2a}\right)^{\frac{1}{2}} ds < +\infty\,.$$

Step 3 : defining $\Lambda^{-1} V$. $\Lambda^{-1} V = I^{\frac{1}{2}-H} V$ is just an ordinary pathwise fractional integration:

$$\Lambda^{-1} V(t) = \int_0^t \frac{(t-s)^{-(H+\frac{1}{2})}}{\Gamma(\frac{1}{2}-H)} V(s)\, ds\,.$$

Fubini's Theorem shows that this is well defined for $V = X, Y, \Lambda W, \Lambda B$, and that we have the usual diffusive representation (see section 1).

Step 4 : checking that $\Lambda^{-1}\Lambda Z = Z$. It is another application of Fubini's Stochastic Theorem.

$$\begin{aligned}\Lambda^{-1}\Lambda Z(t) &= \frac{1}{\Gamma(\frac{1}{2}-H)\Gamma(H+\frac{1}{2})} \times \\ &\int_0^t (t-s)^{-(H+\frac{1}{2})}\left(\int_0^s (s-u)^{H-\frac{1}{2}}\,dZ_u\right) ds \\ &= \frac{1}{\Gamma(\frac{1}{2}-H)\Gamma(H+\frac{1}{2})} \times \\ &\int_0^t \left(\int_u^t (t-s)^{-(H+\frac{1}{2})}(s-u)^{H-\frac{1}{2}}\,ds\right) dZ_u \\ &= \int_0^t dZ_u = Z_t\,.\end{aligned}$$

2.2. **The case** $\frac{1}{2} < H < 1$**.** There is a total symmetry with the case $0 < H < \frac{1}{2}$. Now $\Lambda = I^{H-\frac{1}{2}}$ can be defined directly

$$\Lambda Z(t) = \int_0^t \frac{(t-s)^{H-3/2}}{\Gamma(H-\frac{1}{2})} Z(s)\,ds\,.$$

The inverse operator is now

$$\begin{aligned}\Lambda^{-1}V(t) &= I^{\frac{1}{2}-H}(V)(t) = I^{3/2-H}(\dot{V})(t) \\ &= \int_0^t \frac{(t-s)^{\frac{1}{2}-H}}{\Gamma(3/2-H)}\,dV(s)\,.\end{aligned}$$

Fubini's Stochastic Theorem entails that Λ^{-1} has the diffusive representation

$$\Lambda^{-1}V(t) = \int_0^\infty \frac{\xi^{H-3/2}\,d\xi}{\Gamma(3/2-H)\Gamma(H-\frac{1}{2})} V_\xi(t)\,.$$

Eventually we show, using the same commmutation relation and Fubini's Stochastic Theorem, that $\Lambda^{-1}\Lambda Z = Z$.

3. Different Hurst indexes in the signal and the observation

Assume that the state X of the system and the observations Y can be modelled as the solutions of the linear stochastic differential system

$$\begin{cases} dX_t = aX_t\,dt + dW_t^H \\ dY_t = cX_t\,dt + dB_t^{H'} \end{cases} \tag{8}$$

with W^H and $B^{H'}$ Liouville fractional Brownian motions of respective indexes $H \neq H'$. We just need to replace, in our model, the classical Kalman-Bucy linear filter with the linear filter

$$\begin{cases} d\tilde{X}_t = a\tilde{X}_t\,dt + dW_t \\ d\tilde{Y}_t = cA\tilde{X}(t)\,dt + dB_t \end{cases} \tag{9}$$

with $Y = I^{H'-\frac{1}{2}}\tilde{Y}$, $X = I^{H-\frac{1}{2}}\tilde{X}$, and the operator $AZ = I^{H-H'}Z$ defined, as seen in the preceding sections, by

$$I^\alpha Z(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Z(s)\,ds \quad \text{if } 1 > \alpha > 0 \tag{10}$$

$$I^\alpha Z(t) = \int_0^t \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}\,dZ(s) \quad \text{if } 0 > \alpha > -1 \tag{11}$$

4. Comparison with the optimal filter

In the classical optimal filter, we find for every time t a random variable $\hat{X}_t$ which realizes the following infimum

$$\mathbb{E}\left[(\hat{X}_t - X_t)^2\right] = \inf\left\{\mathbb{E}\left[(Z-X)^2\right] : Z \in L^2(\mathcal{Y}_t)\right\},$$

where $\mathcal{Y}_t$ is the sigma-field generated by $(Y_s, s \leq t)$ (the information contained in the observations up to time t.
In our non-classical filter, we find a process $(\bar{X}_t)_{t\geq 0}$ which is optimal in the following sense : for any $t > 0$,

$$\begin{aligned}&\mathbb{E}\left[((\Lambda^{-1}X)(t) - (\Lambda^{-1}\bar{X})(t))^2\right] = \\ &\qquad \inf\left\{\mathbb{E}\left[(Z - (\Lambda^{-1}X)(t))^2\right] : Z \in L^2(\mathcal{Y}_t)\right\},\end{aligned}$$

where Λ denotes the operator $I^{H-1/2}$.

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