A COUNTER EXAMPLE TO THE BUELER’S CONJECTURE.

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Abstract. We give a counter example to a conjecture of E. Bueler stating the equality between the de Rham cohomology of complete Riemannian manifold and a weighted $L^2$ cohomology where the weight is the heat kernel.

1. Introduction

1.1. Weighted $L^2$ cohomology: We first describe weighted $L^2$ cohomology and the Bueler’s conjecture. For more details we refer to E. Bueler’s paper ([2] see also [5]).

Let $(M, g)$ be a complete Riemannian manifold and $h \in C^\infty(M)$ be a smooth function, we introduce the measure $\mu$:

$$d\mu(x) = e^{2h(x)}d\text{vol}_g(x)$$

and the space of $L^2_\mu$ differential forms:

$$L^2_\mu(\Lambda^k T^* M) = \{ \alpha \in L^2_{\text{loc}}(\Lambda^k T^* M), \| \alpha \|^2_\mu := \int_M |\alpha|^2(x) d\mu(x) < \infty \}.$$ 

Let $d^*_\mu = e^{-2h}d^* e^{2h}$ be the formal adjoint of the operator $d : C^\infty_0(\Lambda^k T^* M) \to L^2_\mu(\Lambda^{k+1} T^* M)$. The $k^{\text{th}}$ space of (reduced) $L^2_\mu$ cohomology is defined by:

$$H^k_\mu(M, g) = \{ \alpha \in L^2_\mu(\Lambda^k T^* M), d\alpha = 0 \} / dC^\infty_0(\Lambda^{k-1} T^* M) = dD^{k-1}_\mu(d).$$

where we take the $L^2_\mu$ closure and $D^{k-1}_\mu(d)$ is the domain of $d$, that is the space of forms $\alpha \in L^2_\mu(\Lambda^{k-1} T^* M)$ such that $d\alpha \in L^2_\mu(\Lambda^{k} T^* M)$. Also if $\mathcal{H}^k_\mu(M) = \{ \alpha \in L^2_\mu(\Lambda^k T^* M), d\alpha = 0, d^*_\mu \alpha = 0 \}$ then we also have $\mathcal{H}^k_\mu(M) \simeq H^k_\mu(M)$. Moreover if the manifold is compact (without boundary) then the celebrate Hodge-deRham theorem tells us that these two spaces are isomorphic to $H^k(M, \mathbb{R})$ the real cohomology groups of $M$. Concerning complete Riemannian manifold, E. Bueler had made the following interesting conjecture [2]:

**Conjecture:** Assume that $(M, g)$ is a connected oriented complete Riemannian manifold with Ricci curvature bounded from below. And consider for $t > 0$ and $x_0 \in M$, the heat kernel $\rho_t(x, x_0)$ and the heat kernel measure $d\mu(x) = \rho_t(x, x_0)d\text{vol}_g(x)$, then $0$ is an isolated eigenvalue of the self adjoint operator $dd^*_\mu + d^*_\mu d$ and for any $k$ we have an isomorphism:

$$\mathcal{H}^k_\mu(M) \simeq H^k(M, \mathbb{R}).$$

E. Bueler had verified this conjecture in degree $k = 0$ and according to [3] it also hold in degree $k = \text{dim} M$. About the topological interpretation of some weighted
$L^2$ cohomology, there is results of Z.M. Ahmed and D. Strook and more optimal results of N. Yeganefar ([1],[5]). Here we will show that we can not hope more:

**Theorem 1.1.** In any dimension $n$, there is a connected oriented manifold $M$, such that for any complete Riemannian metric on $M$ and any smooth positive measure $\mu$, the natural map:

$$\mathcal{H}_\mu^k(M) \rightarrow H^k(M, \mathbb{R})$$

is not surjective for $k \neq 0, n$.

Actually the example is simple take a compact surface $S$ of genus $g \geq 2$ and

$$\Gamma \simeq \mathbb{Z} \rightarrow \hat{S} \rightarrow S$$

be a cyclic cover of $S$ and in dimension $n$, do consider $M = \mathbb{T}^{n-2} \times \hat{S}$ the product of a $(n-2)$ torus with $\hat{S}$.

2. **An technical point : the growth of harmonic forms :**

We consider here a complete Riemannian manifold $(M^n, g)$ and a positive smooth measure $du = e^{2h} d\text{vol}_g$ on it.

**Proposition 2.1.** Let $o \in M$ be a fixed point, for $x \in M$, let $r(x) = d(o, x)$ be the geodesic distance between $o$ and $x$, $R(x)$ be the maximum of the absolute value of sectional curvature of planes in $T_x M$ and define $m(R) = \max_{r(x) \leq R} \{ |\nabla dh|(x) + R(x) \}$. There is a constant $C_n$ depending only of the dimension such that if $\alpha \in \mathcal{H}_\mu^k(M)$ then on the ball $r(x) \leq R$:

$$e^{h(x)}|\alpha|(x) \leq C_n \frac{e^{C_n m(2R)R^2}}{\text{vol}(B(o, 2R))} ||\alpha||_\mu.$$

**Proof.** If we let $\theta(x) = e^{h(x)}\alpha(x)$ then $\theta$ satisfies the equation:

$$(dd^* + d^*d)\theta + |dh|^2\theta + 2\nabla dh(\theta) - (\Delta h)\theta = 0.$$
3. Justification of the example and further comments

3.1. Justification. Now, we consider the manifold $M = \mathbb{T}^{n-2} \times \hat{S}$ which is a cyclic cover of $\mathbb{T}^{n-2} \times S$; let $\gamma$ be a generator of the covering group. We assume $M$ is endowed with a complete Riemannian metric and a smooth measure $d\mu = e^{2h}d\text{vol}_g$. For every $k \in \{1,\ldots,n-1\}$ we have a $k$-cycle $c$ such that $\gamma^l(c) \cap c = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$ and a closed $k$-form $\psi$ with compact support such that $\int_c \psi = 1$ and such that $\left(\text{support } \psi \right) \cap \left(\text{support } (\gamma^l)^*\psi \right) = \emptyset$ for any $l \in \mathbb{Z} \setminus \{0\}$. Let $a = (a_p)_{p \in \mathbb{N}}$ be a non zero sequence of real number : then the $k$-form $\psi_a = \sum_{p \in \mathbb{N}} a_p (\gamma^p)^*\psi$ represents a non zero $k$ cohomology class of $M$, indeed $\int_{\gamma^p c} \psi_a = a_p$. We define $R_p = \max\{r(\gamma^l(x)), x \in c, l = 0,\ldots,p\}$, then if the deRham cohomology class of $\psi_a$ contains $\alpha \in H^k_{\mu}(M)$, then according to (2.1), we will have $|a_p| = \left|\int_{\gamma^p c} \alpha \right| \leq M_p \|\alpha\|_{\mu}$ ; where

$$M_p = \text{vol}_g((\gamma^p(c))^C) C_n e^{C_n m(2R_p) R_p^2 \sqrt{\text{vol}(B(o,2R_p))}} \max_{r(x) \leq R_p} e^{-h(x)}.$$

As a consequence, for the sequence defined by $a_p = (M_p + 1)2^p$, $\psi_a$ can not be represented by a element of $H^k_{\mu}(M)$.

3.2. Further comments. Our counter example doesn’t exclude that this conjecture hold for a complete Riemannian metric with bounded curvature, positive injectivity radius on the interior of a compact manifold with boundary.

References


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