L^2 Harmonic Forms on Non-Compact Riemannian Manifolds

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First, I want to present some questions on L^2-harmonic forms on non-compact Riemannian manifolds. Second, I will present an answer to an old question of J. Dodziuk on L^2-harmonic forms on manifolds with flat ends. In fact some of the analytical tools presented here apply in other situations (see [C4]).

1. The Space of Harmonic Forms

Let (M^n, g) be a complete Riemannian manifold. We denote by H^k(M, g) its space of L^2-harmonic k-forms, that is to say the space of L^2 k-forms which are closed and coclosed:

\[ \mathcal{H}^k(M) = \{ \alpha \in L^2(\Lambda^k T^*M), \ d\alpha = \delta \alpha = 0 \}, \]

where

\[ d : C^\infty_0(\Lambda^k T^*M) \to C^\infty_0(\Lambda^{k+1} T^*M) \]

is the exterior differentiation operator and

\[ \delta : C^\infty_0(\Lambda^{k+1} T^*M) \to C^\infty_0(\Lambda^k T^*M) \]

its formal adjoint. The operator d does not depend on g but \( \delta \) does; \( \delta \) is defined with the formula:

\[ \forall \alpha \in C^\infty_0(\Lambda^k T^*M), \ \forall \beta \in C^\infty_0(\Lambda^{k+1} T^*M), \ \int_M \langle d\alpha, \beta \rangle = \int_M \langle \alpha, \delta \beta \rangle. \]

The operator \( d + \delta \) is elliptic hence the elements of \( \mathcal{H}^k(M) \) are smooth and the L^2 condition is only a decay condition at infinity.

2. If the Manifold M is Compact without Boundary

If M is compact without boundary, then these spaces have finite dimension, and we have the theorem of Hodge-DeRham: the spaces \( \mathcal{H}^k(M) \) are isomorphic to the real cohomology groups of M:

\[ \mathcal{H}^k(M) \cong H^k(M, \mathbb{R}). \]

Hence the dimension of \( \mathcal{H}^k(M) \) is a homotopy invariant of M, i.e. it does not depend on g. A corollary of this and of the Chern-Gauss-Bonnet formula is:

\[ \chi(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) = \int_M \Omega^g, \]

where \( \Omega^g \) is the Euler form of \( (M^n, g) \): for instance if \( \dim M = 2 \) then \( \Omega^g = \frac{KdA}{2\pi} \), where K is the Gaussian curvature and dA the area form.
3. What is true on a non-compact manifold

Almost nothing is true in general:

The space \( \mathcal{H}^k(M, g) \) can have infinite dimension and the dimension, if finite, can depend on \( g \). For instance, if \( M \) is connected we have

\[
\mathcal{H}^0(M) = \{ f \in L^2(M, dv_{g}), f = \text{constant} \}.
\]

Hence \( \mathcal{H}^0(M) = \mathbb{R} \) if \( \text{vol } M < \infty \), and \( \mathcal{H}^0(M) = \{0\} \) if \( \text{vol } M = \infty \).

For instance if \( \mathbb{R}^2 \) is equipped with the euclidean metric, we have \( \mathcal{H}^0(\mathbb{R}^2, \text{eucl}) = \{0\} \), and if \( \mathbb{R}^2 \) is equipped with the metric \( g = dr^2 + r^2e^{-2r}d\theta^2 \) in polar coordinates, then \( \mathcal{H}^0(\mathbb{R}^2, g) = \mathbb{R} \). We have also that \( \mathcal{H}^k(\mathbb{R}^n, \text{eucl}) = \{0\} \), for any \( k \leq n \). But if we consider the unit disk in \( \mathbb{R}^2 \) equipped with the hyperbolic metric \( 4|dz|^2/(1 - |z|^2)^2 \) then it is isometric to the metric \( g_1 = dr^2 + \sinh r^2 d\theta^2 \) on \( \mathbb{R}^2 \). And then we have \( \dim \mathcal{H}^1(\mathbb{R}^2, g_1) = \infty \). As a matter of fact if \( P(z) \in \mathbb{C}[z] \) is a polynomial, then \( \alpha = P'(z)dz \) is a \( L^2 \) harmonic form on the unit disk for the hyperbolic metric (this comes from the conformal invariance, see 5.2). So we get an injection \( \mathbb{C}[z]/\mathbb{C} \to \mathcal{H}^1(\mathbb{R}^2, g_1) \). However, the spaces \( \mathcal{H}^k(M, g) \) satisfy the following two properties:

- These spaces have a reduced \( L^2 \) cohomology interpretation:

  Let \( Z^k_2(M) \) be the kernel of unbounded operator \( d \) acting on \( L^2(\Lambda^k T^* M) \), or equivalently

  \[
  Z^k_2(M) = \{ \alpha \in L^2(\Lambda^k T^* M), \; d\alpha = 0 \},
  \]

  where the equation \( d\alpha = 0 \) has to be understood in the distribution sense i.e. \( \alpha \in Z^k_2(M) \) if and only if

  \[
  \forall \beta \in C_0^\infty(\Lambda^{k+1} T^* M), \; \int_M \langle \alpha, \delta \beta \rangle = 0 .
  \]

  That is to say \( Z^k_2(M) = (\delta C_0^\infty(\Lambda^{k+1} T^* M))' \). The space \( L^2(\Lambda^k T^* M) \) has the following Hodge-DeRham-Kodaira orthogonal decomposition

  \[
  L^2(\Lambda^k T^* M) = \mathcal{H}^k(M) \oplus dC_0^\infty(\Lambda^{k-1} T^* M) \oplus \delta C_0^\infty(\Lambda^{k+1} T^* M),
  \]

  where the closure is taken with respect to the \( L^2 \) topology.;We also have

  \[
  Z^k_2(M) = \mathcal{H}^k(M) \oplus dC_0^\infty(\Lambda^{k-1} T^* M),
  \]

  hence we have

  \[
  \mathcal{H}^k(M) \simeq Z^k_2(M)/dC_0^\infty(\Lambda^{k-1} T^* M).
  \]

  A corollary of this identification is the following:

  **Proposition 3.1.** The space \( \mathcal{H}^k(M, g) \) are quasi-isometric invariant of \((M, g)\). That is to say if \( g_1 \) and \( g_2 \) are two complete Riemannian metrics such that for a \( C > 1 \) we have

  \[
  C^{-1} g_1 \leq g_2 \leq C g_1,
  \]

  then \( \mathcal{H}^k(M, g_1) \simeq \mathcal{H}^k(M, g_2) \).

  In fact, the spaces \( \mathcal{H}^k(M, g) \) are biLipschitz-homotopy invariants of \((M, g)\).

- The finiteness of \( \dim \mathcal{H}^k(M, g) \) depends only of the geometry of ends:
**Theorem 3.2.** (J. Lott, [L]) The spaces of $L^2$-harmonic forms of two complete Riemannian manifolds, which are isometric outside some compact set, have simultaneously finite or infinite dimension.

4. A GENERAL PROBLEM

In view of the Hodge-DeRham theorem and of J. Lott’s result, we can ask the following questions:

(1) What geometrical condition on the ends of $M$ insure the finiteness of the dimension of the spaces $\mathcal{H}^k(M)$?

Within a class of Riemannian manifold having the same geometry at infinity;

(2) What are the links of the spaces $\mathcal{H}^k(M)$ with the topology of $M$ and with the geometry "at infinity" of $(M,g)$?

(3) And what kind of Chern-Gauss-Bonnet formula could we hope for the $L^2$-Euler characteristic

$$\chi_{L^2}(M) = \sum_{k=0}^n (-1)^k \dim \mathcal{H}^k(M).$$

There are many articles dealing with these questions. I mention only three of them:

(1) In the pioneering article of Atiyah-Patodi-Singer ([A-P-S]), the authors considered manifolds with cylindrical ends: that is to say there is a compact subset $K$ of $M$ such that $M \setminus K$ is isometric to the Riemannian product $\partial K \times ]0,\infty[$. Then they show that the dimension of the space of $L^2$-harmonic forms is finite; and that these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah-Patodi-Singer in order to obtain a formula for the signature of compact manifolds with boundary.

(2) In [M, M-P], R. Mazzeo and R. Phillips give a cohomological interpretation of the space $\mathcal{H}^k(M)$ for geometrically finite real hyperbolic manifolds.

(3) The solution of the Zucker’s conjecture by Saper and Stern ([S-S]) shows that the spaces of $L^2$ harmonic forms on hermitian locally symmetric space with finite volume are isomorphic to the middle intersection cohomology of the Borel-Serre compactification of the manifold.

5. AN EXAMPLE

I want now to discuss the $L^2$ Gauss-Bonnet formula through one example. The sort of $L^2$ Gauss-Bonnet formula one might expect is a formula of the type

$$\chi_{L^2}(M) = \int_K \Omega^g + \text{terms depending only on } (M - K, g),$$

where $K \subset M$ is a compact subset of $M$; i.e. $\chi_{L^2}(M)$ is the sum of a local term $\int_K \Omega^g$ and of a boundary (at infinity) term. Such a result will imply a relative index formula:

If $(M_1, g_1)$ and $(M_2, g_2)$ are isometric outside compact set $K_i \subset M_i$, $i = 1, 2$, then

$$\chi_{L^2}(M_1) - \chi_{L^2}(M_2) = \int_{K_1} \Omega^{g_1} - \int_{K_2} \Omega^{g_2}.$$
It had been shown by Gromov-Lawson and Donnelly that when zero is not in the essential spectrum of the Gauss-Bonnet operator \( d + \delta \) then this relative index formula is true ([G-L, Do]). For instance, by the work of Borel and Casselman [BC], the Gauss-Bonnet operator is a Fredholm operator if \( M \) is an even dimensional locally symmetric space of finite volume and negative curvature.

In fact such a relative formula is not true in general. The following counterexample is given in [C2]:

\((M_1, g_1)\) is the disjoint union of two copies of the euclidean plane and \((M_2, g_2)\) is two copies of the euclidean plane glued along a disk. As these surface are oriented with infinite volume, we have \( i = 1, 2, \)

\[ H^0(M_i, g_i) = H^2(M_i, g_i) = \{0\}. \]

And we also have \( H^1(M_1, g_1) = \{0\} \).

**Lemma 5.1.** \( H^1(M_2, g_2) = \{0\} \).

This comes from the conformal invariance of this space. Indeed, it is a general fact:

**Proposition 5.2.** If \((M^n, g)\) is a Riemannian manifold of dimension \( n = 2k \), and if \( f \in C^\infty(M) \) then

\[ H^k(M, g) = H^k(M, e^{2f} g). \]

**Proof.** As a matter of fact the two Hilbert spaces \( L^2(\Lambda^k T^* M, g) \) and \( L^2(\Lambda^k T^* M, e^{2f} g) \) are the same: if \( \alpha \in \Lambda^k T^*_x M \), then

\[ \|\alpha\|_{e^{2f} g}^2 = e^{-2k f(x)} \|\alpha\|_{g}^2(x) \]

and \( d\text{vol}_{e^{2f} g} = e^{-2k f} d\text{vol}_g \).

We have

\[ H^k(M, e^{2f} g) = Z^k_2(M, e^{2f} g) \cap dC^\infty(\Lambda^{k-1} T^* M) \]

and \( H^k(M, g) = Z^k_2(M, g) \cap dC^\infty(\Lambda^{k-1} T^* M) \).

As the two Hilbert space \( L^2(\Lambda^k T^* M, g) \) and \( L^2(\Lambda^k T^* M, e^{2f} g) \) are the same, these two spaces are the same. Q.E.D

But \((M_2, g_2)\) is conformally equivalent to the 2-sphere with two points removed. A \( L^2 \) harmonic form on the 2-sphere with two points removed extends smoothly on the sphere. The sphere has no non trivial \( L^2 \) harmonic 1-form, hence Lemma 5.1 follows.

The surfaces \((M_1, g_1)\) and \((M_2, g_2)\) are isometric outside some compact set but

\[ \chi_{L^2}(M_1) - \int_{M_1} \frac{K_{g_1} dA_{g_1}}{2\pi} = \frac{1}{2}\pi = 0 \]

whereas

\[ \chi_{L^2}(M_2) - \int_{M_2} \frac{K_{g_2} dA_{g_2}}{2\pi} = -\frac{1}{2}\pi = -2. \]

Hence the relative index formula is not true in general. A corollary of this argument is the following
Corollary 5.3. If \((S,g)\) is a complete surface with integrable Gaussian curvature, according to a theorem of A. Huber \([H]\), we know that such a surface is conformally equivalent to a compact surface \(\bar{S}\) with a finite number of points removed, then
\[
\dim \mathcal{H}^1(S,g) = b_1(\bar{S}).
\]

6. MANIFOLDS WITH FLAT ENDS

In (1982, \([D]\)), J. Dodziuk asked the following question: according to Vesentini \([V]\) if \(M\) is flat outside a compact set, the spaces \(\mathcal{H}^k(M)\) are finite dimensional. Do they admit a topological interpretation?

My aim is to present an answer to this question. For the detail, the reader may look at \([C4]\):

6.1. Visentini’s finiteness result.

Theorem 6.1. Let \((M,g)\) be a complete Riemannian manifold such that for a compact set \(K_0 \subset M\), the curvature of \((M,g)\) vanishes on \(M - K_0\). Then for every \(p\)
\[
\dim \mathcal{H}^p(M,g) < \infty.
\]

We give here a proof of this result; this proof will furnish some analytical tools to answer J. Dodziuk’s question.

We begin to define a Sobolev space adapted to our situation:

Definition 6.2. Let \(D\) be a bounded open set containing \(K_0\), let \(W^D_0(\Lambda T^*M)\) be the completion of \(C_0^\infty(\Lambda T^*M)\) for the quadratic form
\[
\alpha \mapsto \int_D |\alpha|^2 + \int_M |(d + \delta)\alpha|^2 = N^2_D(\alpha).
\]

Proposition 6.3. The space \(W^D_0\) doesn’t depend on \(D\), that is to say if \(D\) and \(D'\) are two bounded open sets containing \(K_0\), then the two norms \(N_D\) and \(N_{D'}\) are equivalent. We write \(W\) for \(W^D_0\).

Proof. The proof goes by contradiction. We notice that with the Bochner-Weitzenböck formula:
\[
\forall \alpha \in C_0^\infty(\Lambda T^*M), \quad \int_M |(d + \delta)\alpha|^2 = \int_M |\nabla \alpha|^2 + \int_{K_0} |\alpha|^2.
\]

Hence, by standard elliptic estimates, the norm \(N_D\) is equivalent to the norm
\[
Q_D(\alpha) = \sqrt{\int_M |\nabla \alpha|^2 + \int_D |\alpha|^2}.
\]

If \(D\) and \(D'\) are two connected bounded open set containing \(K_0\), such that \(D \subset D'\) then \(Q_D \leq Q_{D'}\). Hence if \(Q_D\) and \(Q_{D'}\) are not equivalent there is a sequence \((\alpha_l)_{l \in \mathbb{N}} \in C_0^\infty(\Lambda T^*M)\), such that \(Q_{D'}(\alpha_l) = 1\) whereas \(\lim_{l \to \infty} Q_D(\alpha_l) = 0\). This implies that the sequence \((\alpha_l)_{l \in \mathbb{N}}\) is bounded in \(W^{1,2}(D')\) and \(\lim_{l \to \infty} \|\nabla \alpha_l\|_{L^2(M)} = 0\). Hence we can extract a subsequence converging weakly in \(W^{1,2}(D')\) and strongly in \(L^2(\Lambda T^*D')\) to a \(\alpha_\infty \in W^{1,2}(D')\). We can suppose this subsequence is \((\alpha_l)\).

We must have \(\nabla \alpha_\infty = 0\) and \(\alpha_\infty = 0\) on \(D\) and \(\|\alpha_\infty\|_{L^2(D')} = 1\). This is impossible. Hence the two norms \(Q_D\) and \(Q_{D'}\) are equivalent. Q.E.D
Corollary 6.4. The inclusion $C_0^\infty \rightarrow W_{loc}^{1,2}$ extends by continuity to a injection $W \rightarrow W_{loc}^{1,2}$.

We remark that the domain of the Gauss-Bonnet operator $D(d + \delta) = \{ \alpha \in L^2, (d + \alpha) \in L^2 \}$ is in $W$. As a matter of fact, because $(M,g)$ is complete $D(d + \delta)$ is the completion of $C_0^\infty(A^*T^*M)$ equipped with the quadratic form

$$\alpha \mapsto \int_M |\alpha|^2 + \int_M |(d + \delta)\alpha|^2.$$ 

This norm is larger that the one used for defined $W$. Hence $D(d + \delta) \subset W$. As a corollary we get that a $L^2$ harmonic form is in $W$. The Visentini’s finiteness result will follow from:

Proposition 6.5. The operator $(d + \delta) : W \rightarrow L^2$ is Fredholm. That is to say its kernel and its cokernel have finite dimension and its image is closed. 

Proof.– Let $A$ be the operator $(d + \delta)^2 + 1_D$, where

$$1_D(\alpha)(x) = \begin{cases} \alpha(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

We have

$$N_D(\alpha)^2 = \langle A\alpha, \alpha \rangle.$$ 

So the operator $A^{-1/2} = \int_0^\infty e^{-tA} \frac{dt}{\sqrt{t}}$ realizes an isometry between $L^2$ and $W$. It is enough to show that the operator $(d + \delta)A^{-1/2} = B$ is Fredholm on $L^2$. But

$$B^*B = A^{-1/2}(d + \delta)^2A^{-1/2} = 1D - A^{-1/2}1_D1DA^{-1/2}.$$ 

The operator $1_D A^{-1/2}$ is the composition of the operator $A^{-1/2} : L^2 \rightarrow W$ then of the natural injection from $W$ to $W_{loc}^{1,2}$ and finally of the map $1_D$ from $W_{loc}^{1,2}$ to $L^2$. $D$ being a bounded set, this operator is a compact one by the Rellich compactness theorem. Hence $1_D A^{-1/2}$ is a compact operator. Hence, $B$ has a closed range and a finite dimensional kernel. So the operator $(d + \delta) : W \rightarrow L^2$ has a closed range and a finite kernel. But the cokernel of this operator is the orthogonal space to $(d + \delta)C_0^\infty(A^*T^*M)$ in $L^2$. Hence the cokernel of this operator is the $L^2$ kernel of the Gauss-Bonnet operator. We notice that this space is included in the space of the $W$ kernel of $(d + \delta)$. Hence it has finite dimension. Q.E.D

We also get the following corollary:

Corollary 6.6. There is a Green operator $G : W \rightarrow L^2$, such that

$$\text{on } L^2 \setminus \{ (d + \delta)G = 1D - P L^2 \}$$

where $P L^2$ is the orthogonal projection on $\bigoplus H^k(M)$.

On $W$, $G(d + \delta) = 1D - P W$ 

where $P W$ is the $W$ orthogonal projection on $\ker W(d + \delta)$. 

Moreover, $\alpha \in Z^k_2(M)$ is $L^2$ cohomologous to zero if and only if there is a $\beta \in W(\Lambda^{k-1}T^*M)$ such that $\alpha = d\beta$. 


6.2. A long exact sequence. In the DeRham cohomology, we have a long exact sequence linking the cohomology with compact support and the absolute cohomology. And this exact sequence is very useful to compute the DeRham cohomology groups. In $L^2$ cohomology, we can also define this sequence but generally it is not an exact sequence.

Let $O \subset M$ be a bounded open subset, we can define the sequence:

\[
\cdots \rightarrow H^k(M \setminus O, \partial O) \xrightarrow{e} H^k(M) \xrightarrow{j^*} H^k(O) \xrightarrow{b} H^{k+1}(M \setminus O, \partial O) \rightarrow \cdots
\]

Here

\[
H^k(M \setminus O, \partial O) = \{ h \in L^2(\Lambda^k T^*(M \setminus O)), \; dh = \delta h = 0 \text{ and } i^* h = 0 \},
\]

where $i : \partial O \rightarrow M \setminus O$ is the inclusion map, and

- $e$ is the extension by zero map: to $h \in H^k(M \setminus O, \partial O)$ it associates the $L^2$ cohomology class of $\bar{h}$, where $\bar{h}$ is 0 on $O$ and $\bar{h} = h$ on $M \setminus O$. It is well defined because of the Stokes formula: if $\beta \in C_0^\infty(\Lambda^k+1 T^* M)$, then

\[
\langle \bar{h}, \delta \beta \rangle_{L^2(O)} - \int_{\partial O} i^* h \wedge i^* \beta = 0
\]

- $j^*$ is associated to the inclusion map $j : O \rightarrow M$: to $h \in H^k(M)$ it associates $[j^* h]$, the cohomology class of $h|_O$ in $H^k(O)$.
- $b$ is the coboundary operator: if $[\alpha] \in H^k(O)$, and if $\tilde{\alpha}$ is a smooth extension of $\alpha$, with compact support, then $b[\alpha]$ is the orthogonal projection of $d\tilde{\alpha}$ on $H^{k+1}(M \setminus O, \partial O)$. The map $b$ is well defined, that is to say, it does not depend on the choice of $\alpha$ nor of its extension.

It is relatively easy to check that

\[
j^* \circ e = 0, \; b \circ j^* = 0 \text{ and } e \circ b = 0 ;
\]

Hence we have the inclusion:

\[
\text{Im } e \subset \text{Ker } j^*, \; \text{Im } j^* \subset \text{Ker } b \text{ and } \text{Im } b \subset \text{Ker } e.
\]

In [C1], we observed that

**Proposition 6.7.** The equality $\text{ker } b = \text{Im } j^*$ always holds.

This comes from the long exact sequence in DeRham cohomology. Moreover, we have the following:

**Proposition 6.8.** On a manifold with flat ends, the equality $\text{Im } b = \text{Ker } e$ always holds.

**Proof.** As a matter of fact, if $h \in \text{Ker } e$ then by (6.6) we get a $\beta \in W$, such that $h = d\beta$ on $M$. Hence $h = b[\beta]|_O$.

Q.E.D

The last fact requires more analysis:

**Theorem 6.9.** If $(M, g)$ is a complete manifold with flat ends and if for every end $E$ of $M$ we have

\[
\lim_{r \to \infty} \frac{\text{vol } E \cap B_x(r)}{r^2} = \infty,
\]

then the long sequence (6.1) is exact.
6.3. **Hodge theorem for manifolds with flat ends.** With the help of the geometric description of flat ends due to Eschenburg and Schroeder ([E-S], see also [G-P-Z]), we can compute the $L^2$-cohomology on flat ends. Then with the long sequence (6.1), we can give an answer to J. Dodziuk’s question; for sake of simplicity, we give here only the result for manifolds with one flat end.

**Theorem 6.10.** Let $(M^n, g)$ be a complete Riemannian manifold with one flat end $E$. Then

1. If $(M^n, g)$ is parabolic, that is to say if the volume growth of geodesic ball is at most quadratic
   \[
   \lim_{r \to \infty} \frac{\text{vol} B_g(r)}{r^2} < \infty,
   \]
   then we have
   \[
   H^k(M, g) \cong \text{Im} \left( H^k_c(M) \to H^k(M) \right).
   \]

2. If $(M^n, g)$ is non-parabolic (i.e. if \( \lim_{r \to \infty} \frac{\text{vol} B_g(r)}{r^2} = \infty \)), then the boundary of $E$ has a finite covering diffeomorphic to the product $S^{n-\nu} \times T$ where $T$ is a flat \((n-\nu)\)-torus. Let $\pi : T \to \partial E$ be the induced immersion, then
   \[
   H^k(M, g) \cong H^k(M \setminus E, \ker \pi^*),
   \]
   where $H^k(M \setminus E, \ker \pi^*)$ is the cohomology associated to the subcomplex of differential forms on $M \setminus E : \ker \pi^* = \{ \alpha \in C^\infty(\Lambda^* T^*(M \setminus E)) : \pi^* \alpha = 0 \}$.

**References**


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