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L^2-COHOMOLOGY OF MANIFOLDS WITH FLAT ENDS

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Abstract

We give a topological interpretation of the spaces of L^2-harmonic forms on manifolds with flat ends. We also prove a Chern–Gauss–Bonnet formula for the L^2-Euler characteristic of some of these manifolds.

Résumé

Nous donnons une interprétation topologique des espaces de formes harmoniques L^2 d'une variété riemannienne complète plate l'infini. Nous obtenons aussi une formule de Chern–Gauss–Bonnet pour la caractéristique d'Euler L^2 de certaines de ces variétés. Ces résultats sont des conséquences de théorèmes généraux sur les variétés riemanniennes complètes dont l'opérateur de Gauss–Bonnet est non-parabolique l'infini.

1 Introduction

Let (M^n, g) be a complete Riemannian manifold, we write $\mathcal{H}^k(M, g)$ or $\mathcal{H}^k(M)$ for its space of L^2-harmonic k-forms. These spaces have a (reduced) L^2-cohomology interpretation. When the manifold is compact, without boundary, the celebrated theorem of Hodge–de Rham identifies these spaces with the real cohomology spaces of M; hence with the Chern–Gauss–Bonnet formula, we know that

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M) = \int_M \Omega^g ,$$

where Ω^g is the Euler form of (M^n, g); for instance if $\dim M = 2$ then $\Omega^g = K dA/2\pi$, where K is the Gaussian curvature and dA the area's form.

If M is non-compact then such results are generally not true; for instance it is no longer true that the spaces $\mathcal{H}^k(M, g)$ have finite dimension; however the following questions are natural:

1. What is the geometry insuring the finiteness of the dimension of the spaces $\mathcal{H}^k(M)$?
2. What are the links of the spaces $\mathcal{H}^k(M)$ with the topology of M and with the geometry "at infinity" of (M, g)?

3. What kind of Chern–Gauss–Bonnet formula could we expect for the L^2- Euler characteristic
$$\chi_{L^2}(M) = \sum_{k=0}^{n} (-1)^k \dim \mathcal{H}^k(M)\,?$$

In 1982, J. Dodziuk ([D]) asked the following question: according to Vesentini ([Ve]), if M is flat outside a compact set, the spaces $\mathcal{H}^k(M)$ are finite dimensional. Do they admit a topological interpretation?

The main result of this paper answers this question. It is known that a complete Riemannian manifold, which is flat outside a compact set, has a finite number of ends. For the sake of simplicity, in the introduction, we only give the result for a manifold with one flat end.

Theorem 1.1. *Let (M^n, g) be a complete Riemannian manifold with one flat end E, then*

1. *If the volume growth of the geodesic ball is at most quadratic,*
$$\lim_{r\to\infty} \frac{\operatorname{vol} B_x(r)}{r^2} < \infty\,,$$
then we have
$$\mathcal{H}^k(M,g) \simeq \operatorname{Im}\big(H_c^k(M) \longrightarrow H^k(M)\big)\,.$$
2. *If $\lim_{r\to\infty} \operatorname{vol} B_x(r)/r^2 = \infty$, then the boundary of E has a finite covering diffeomorphic to the product $\mathbf{S}^{\nu-1} \times \mathbf{T}$ where $\mathbf{T}$ is a flat $(n-\nu)$-torus. Let $\pi : \mathbf{T} \to \partial E$ be the induced immersion, then*
$$\mathcal{H}^k(M,g) \simeq H^k(M \setminus E, \ker \pi^*)\,,$$
where
$$H^k(M\setminus E, \ker\pi^*) = \frac{\big\{\alpha \in C^\infty(\Lambda^k T^*(M\setminus E))\ \ d\alpha = 0\,,\ \pi^*\alpha = 0\big\}}{\{d\alpha, \alpha \in C^\infty(\Lambda^{k-1}T^*(M\setminus E)), \pi^*\alpha = 0\}}\,.$$

In potential theory, the manifold is called parabolic in the first case and non-parabolic in the second case ([A]).

This theorem was already known for an asymptotically Euclidean manifold, i.e. each end is simply connected ([C3], [Me]).

The proof of Theorem 1.1 relies upon the analysis we have developed in [C4,5] and upon the work of Eschenburg and Schroeder describing the ends of such a manifold ([ES], see also [GrPZ]).

Let us explain our results on Dirac operators which are non-parabolic at infinity.

Definition 1.2. *The Gauss–Bonnet operator $d+\delta$ of a complete Riemannian manifold (M,g) is called non-parabolic at infinity when there is a compact set K of M such that for any bounded open subset $U \subset M \setminus K$*

there is a constant $C(U) > 0$ *with the inequality*

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M \setminus K)\big)\,, \quad C(U)\int_U |\alpha|^2 \le \int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2. \tag{1.1}$$

The main property of this operator is the following:

PROPOSITION 1.3. *If the Gauss–Bonnet operator of* (M, g) *is non-parabolic at infinity then*

$$\dim\big\{\alpha \in L^2(\Lambda T^*M)\,,\ d\alpha = \delta\alpha = 0\big\} < \infty\,.$$

Moreover, let D *be a bounded open subset of* M *containing* K*, and let* $W(\Lambda T^*M)$ *be the Sobolev space which is the completion of* $C_0^\infty(\Lambda T^*M)$ *with the quadratic form*

$$\alpha \mapsto \int_D |\alpha|^2 + \int_M |d\alpha|^2 + |\delta\alpha|^2,$$

then this space imbedded continuously in H^1_{loc} *and*

$$d + \delta : W(\Lambda T^*M) \longrightarrow L^2(\Lambda T^*M)$$

is a Fredholm operator.

Furthermore, a differential form which is in the domain of $d+\delta$ is in W, that is to say

$$\{\alpha \in L^2\,,\ d\alpha \in L^2\,,\ \delta\alpha \in L^2\} \subset W\,.$$

Hence any L^2-harmonic form is in W. The first step in proving our Theorem 1.1 is the following result.

PROPOSITION 1.4. *If* (M^n, g) *is a complete Riemannian manifold whose curvature vanishes outside some compact set, then the Gauss–Bonnet operator is non-parabolic at infinity.*

When the Gauss–Bonnet operator is Fredholm on its domain (or equivalently when 0 is not in the essential spectrum of the Gauss–Bonnet operator) then we know that the Gauss–Bonnet operator is invertible at infinity ([Ang]), that is to say there exist a compact K of M and a constant $\Lambda > 0$ such that

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M \setminus K)\big), \quad \Lambda \int_{M\setminus K} |\alpha|^2 \le \int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2.$$

In this case, the Gauss–Bonnet operator is non-parabolic at infinity and the Sobolev space W is the domain of the operator $d+\delta$. For instance, by the work of Borel and Casselman ([BoC]), the Gauss–Bonnet operator is a Fredholm operator if M is an even dimensional locally symmetric space of finite volume and negative curvature.

Another interesting case is when the manifold has a cylindrical end: that is to say there is a compact K of M such that $M \setminus K$ is isometric to the Riemannian product $\partial K \times]0, \infty[$. According to the pioneering article of

Atiyah–Patodi–Singer ([AtPS]), the dimension of the space of L^2-harmonic forms is finite; moreover these spaces are isomorphic to the image of the relative cohomology in the absolute cohomology. These results were used by Atiyah–Patodi–Singer in order to obtain a formula for the signature of compact manifolds with boundary. In fact, on a manifold with a cylindrical end, the Gauss–Bonnet operator is non-parabolic at infinity; harmonic forms in the Sobolev space W are called, by Atiyah–Patodi–Singer, L^2 extended harmonic forms. That is why we have called *extended index* the index of the operator

$$d+\delta : W(\Lambda T^*M)\longrightarrow L^2(\Lambda T^*M)\,.$$

In [C5], we have developed analytical tools in order to compute this index. One of our results was that this index only depends on the geometry of infinity. Recall that we have noted that a harmonic L^2-form is in W; hence we have

$$\mathrm{ind}_e(d+\delta) = \dim \frac{\{\alpha\in W(\Lambda T^*(M)),\, d\alpha+\delta\alpha=0\}}{\{\alpha\in L^2(\Lambda T^*(M)),\, d\alpha+\delta\alpha=0\}}\,.$$

In [C5], we have shown the following:

Theorem 1.5. *If D is a compact set outside of which the estimate* (1.1) *holds, let*

$$h_\infty(M\setminus D) = \dim \frac{\{\alpha\in W(\Lambda T^*(M\setminus D))\cap C^\infty,\, d\alpha+\delta\alpha=0\}}{\{\alpha\in L^2(\Lambda T^*(M\setminus D))\cap C^\infty,\, d\alpha+\delta\alpha=0\}}$$

then we have

$$h_\infty(M\setminus D) = 2\,\mathrm{ind}_e(d+\delta)\,.$$

A consequence of this theorem on the topology of M is the following exact sequence:

Theorem 1.6. *If (M,g) is a complete Riemannian manifold whose Gauss–Bonnet operator is non-parabolic at infinity and assume that* $\mathrm{ind}_e(d+\delta)=0$ *(or equivalently that every harmonic form in W is in L^2) then for every compact D of M, we have the following exact sequence:*

$$\ldots\longrightarrow H^k(D,\partial D)\xrightarrow{\ i\ }\mathcal{H}^k(M)\xrightarrow{\ j^*\ }\mathcal{H}^k_{abs}(M\setminus D)\xrightarrow{\ b\ }H^{k+1}(D,\partial D)\longrightarrow\ldots \tag{1.2}$$

where $\mathcal{H}^k_{abs}(M\setminus D)$ is the space of an L^2 harmonic form on $M\setminus D$ whose normal components vanish along ∂D.

This exact sequence is the L^2-analogue of the exact sequence of the coboundary operator for the de Rham cohomology. It is well known that this exact sequence is true for the non-reduced L^2-cohomology. This implies that if 0 is not in the essential spectrum of the Gauss–Bonnet operator then the sequence (1.2) holds, because the (non-reduced) L^2-cohomology is isomorphic to the space of L^2-harmonic forms. Note that in this case, the hypothesis of Theorem 1.6 is satisfied, because W is the domain of the Gauss–Bonnet operator.

When (M,g) is a manifold with one non-parabolic flat end, then the long exact sequence (1.2) holds, and Theorem 1.1 is proved with this exact sequence. This exact sequence can also be used to obtain a L^2-Chern–Gauss–Bonnet formula.

Theorem 1.7. *If (M^n,g) is a complete oriented Riemannian manifold of even dimension with one flat end E, assume that $\lim_{r\to\infty} \operatorname{vol} B_x(r)/r^2 = \infty$. Then*

$$\chi_{L^2}(M) = \int_M \Omega^g + q(E)\,,$$

where $q(E)$ is computed in terms of $\pi_1(E)$;

1. *When $\pi_1(E)$ has no torsion then $q(E)=0$.*
2. *When $\operatorname{rank}\pi_1(E)=0$ we have $q(E)=\frac{1}{|\pi_1(E)|}-1$.*
3. *In general, $\pi_1(E)$ acts isometrically on the product $\mathbf{S}^{\nu-1}\times\mathbb{R}^{n-\nu}$ and $\pi_1(E)\subset O(\nu)\times[\mathbb{R}^{n-\nu}\rtimes O(n-\nu)]$, let G_E be the image of $\pi_1(E)$ in $O(n-\nu)$, then*
$$q(E) = -\tfrac{1}{|G_E|}\sum_{\gamma\in G_E}\det(\mathrm{Id}-\gamma)\,.$$

This Gauss–Bonnet formula is already known for asymptotically Euclidean manifolds, i.e. each end is simply connected and the curvatures almost vanish (cf. the work of Stern [St], Borisov–Müller–Schrader [BorMS], Brüning [Br] and also [C1]).

This article is organized as follows: In a first section, we recall the main properties of the space of L^2-harmonic forms. In the second section, we present analytical results from ([C4,5]) and we give examples. In the third section, we establish the long exact sequence (1.2), and examples are given where this exact sequence holds. The last section is devoted to the L^2-cohomology of manifolds with flat ends.

In a recent paper [HHM], Tamás Hausel, Eugenie Hunsicker and Rafe Mazzeo obtain a topological interpretation of the L^2-cohomology of a complete Riemannian manifold whose geometry at infinity is a fibred boundary and a fibred cusp (see [MM], [V]). These results have important applications with regard to Sen's conjecture ([Hi], [S1,2]). Moreover, the results of [HHM] can be used to obtain some of our results: according Eschenburg and Schroeder ([ES]), flat ends are classified into three families, and in two of these families, the ends have a finite cover which is a fibred boundary.

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2 The L^2-cohomology

We begin by recalling what reduced L^2-cohomology spaces are.

2.1 Definition. Let (M^n, g) be a complete Riemannian manifold of dimension n: the operator of exterior differentiation is

$$d : C_0^\infty(\Lambda^k T^*M) \longrightarrow C_0^\infty(\Lambda^{k+1}T^*M)$$

and it satisfies $d \circ d = 0$; its formal adjoint is $\delta : C_0^\infty(\Lambda^{k+1}T^*M) \to C_0^\infty(\Lambda^k T^*M)$; we have

$$\forall \alpha \in C_0^\infty(\Lambda^k T^*M)\,,\ \forall \beta \in C_0^\infty(\Lambda^{k+1}T^*M)\,,\ \int_M \langle d\alpha, \beta\rangle = \int_M \langle \alpha, \delta\beta\rangle\,.$$

The spaces $Z_2^k(M)$ and $B_2^k(M)$ are defined as follows:

1. $Z_2^k(M)$ is the kernel of the unbounded operator d acting on $L^2(\Lambda^k T^*M)$. That is to say
$$Z_2^k(M) = \left\{\alpha \in L^2(\Lambda^k T^*M)\,,\ d\alpha = 0\right\},$$
where the equation $d\alpha = 0$ has to be understood in the distribution sense, i.e. $\alpha \in Z_2^k(M)$ if and only if
$$\forall \beta \in C_0^\infty(\Lambda^{k+1}T^*M)\,,\ \int_M \langle \alpha, \delta\beta\rangle = 0\,.$$
Hence we have $Z_2^k(M) = (\delta C_0^\infty(\Lambda^{k+1}T^*M))^\perp$.
2. $B_2^k(M)$ is the closure in $L^2(\Lambda^k T^*M)$ of $d[C_0^\infty(\Lambda^{k-1}T^*M)]$, where $C_0^\infty(\Lambda^{k-1}T^*M)$ is the space of smooth k-differential form with compact support.

Because $d \circ d = 0$, we always have $B_2^k(M) \subset Z_2^k(M)$, the k-space of reduced L^2-cohomology is

$$H_2^k(M) = Z_2^k(M)/B_2^k(M)\,.$$

Thus two weakly closed L^2 k-forms, α and β, are L^2-cohomologous if and only if there is a sequence of smooth $(k-1)$-forms with compact support, $(\gamma_l)_{l=0}^\infty$, such that $\alpha - \beta = L^2 - \lim_{l\to\infty} d\gamma_l$.

The space of (non-reduced) L^2-cohomology is the quotient of $Z_2^k(M)$ by the following space

$$\left\{d\alpha,\ \alpha \in L^2(\Lambda^{k-1}T^*M)\,,\ d\alpha \in L^2\right\}.$$

From now on, we will refer to reduced L^2-cohomology as L^2-cohomology, except where it may cause ambiguity.

2.2 L^2-cohomology and harmonic forms. If $\mathcal{H}^k(M)$ is the space of L^2 harmonic k-forms,

$$\mathcal{H}^k(M) = \left\{\alpha \in L^2(\Lambda^k T^*M)\,,\ d\alpha = \delta\alpha = 0\right\},$$

then the space $L^2(\Lambda^k T^*M)$ has the following of Hodge–de Rham–Kodaira orthogonal decomposition

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*M)} \oplus \overline{\delta C_0^\infty(\Lambda^{k+1}T^*M)}\,,$$

where the closure is taken with respect to the L^2 topology. We also have

$$Z_2^k(M) = \mathcal{H}^k(M) \oplus \overline{dC_0^\infty(\Lambda^{k-1}T^*M)} \quad \text{and} \quad H_2^k(M) \simeq \mathcal{H}^k(M)\,.$$

2.3 Manifolds with compact boundary. If the Riemannian manifold has a compact boundary such that the manifold together with its boundary is metrically complete, then we can also define absolute and relative L^2-cohomology and we again have an identification of the L^2-cohomology space with a space of harmonic forms. When the manifold is compact with boundary, these results are due to G. Duff and D.C. Spencer [DuS], and to P.E. Connor [Co], in fact the arguments of Duff and Spencer generalize easily to this more general setting; these results are well known (see the works of M. Lesch and J. Brüning [BrL], J. Lott [L2], or [C1]).

Let (M^n, g) be a metrically complete Riemannian manifold with compact boundary. We will denote by $C_0^\infty(\Lambda^k T^*M)$ the space of smooth k-differential form with compact support in the interior of M, and by $C_b^\infty(\Lambda^k T^*M)$ the space of smooth k-differential form with bounded support in M, the elements of $C_b^\infty(\Lambda^k T^*M)$ are not zero along the boundary. We define the following four spaces:

1. $Z_{2,abs}^k(M) = \{\alpha \in L^2(\Lambda^k T^*M),\ \forall\beta \in C_0^\infty(\Lambda^{k+1}T^*M),\ \int_M \langle\alpha, \delta\beta\rangle = 0\}$ or equivalently $Z_{2,abs}^k(M) = (\delta C_0^\infty(\Lambda^{k+1}T^*M))^\perp$.
2. $B_{2,abs}^k(M)$ is the closure in $L^2(\Lambda^k T^*M)$ of $d[C_b^\infty(\Lambda^{k-1}T^*M)]$,
3. $Z_{2,rel}^k(M) = \{\alpha \in L^2(\Lambda^k T^*M),\ \forall\beta \in C_b^\infty(\Lambda^{k+1}T^*M),\ \int_M \langle\alpha, \delta\beta\rangle = 0\}$ or equivalently $Z_{2,rel}^k(M) = (\delta C_b^\infty(\Lambda^{k+1}T^*M))^\perp$.
4. $B_{2,rel}^k(M)$ is the closure in $L^2(\Lambda^k T^*M)$ of $d[C_0^\infty(\Lambda^{k-1}T^*M)]$,

Hence a smooth form with bounded support is in $Z_{2,abs}^k(M)$ if and only if it is closed, and a smooth form with bounded support is in $Z_{2,rel}^k(M)$ if and only if it is closed and zero when pulled back to the boundary. We have $B_{2,abs}^k(M) \subset Z_{2,abs}^k(M)$ and $B_{2,rel}^k(M) \subset Z_{2,rel}^k(M)$, the absolute and relative (reduced) L^2-cohomology spaces of M are $H_{2,abs}^k(M) = Z_{2,abs}^k(M)/B_{2,abs}^k(M)$ and $H_{2,rel}^k(M) = Z_{2,rel}^k(M)/B_{2,rel}^k(M)$. We define $\mathcal{H}_{abs}^k(M)$ to be the space

$$H_{2,abs}^k(M) = \left\{h \in L^2(\Lambda^k T^*M)\,,\ dh = \delta h = 0 \text{ and } \mathrm{int}_\nu h = 0\right\},$$

where $\nu : \partial M \to TM$ is the inward unit normal field; and also the space $\mathcal{H}^k_{rel}(M)$ is defined by

$$\mathcal{H}^k_{rel}(M) = \left\{h \in L^2(\Lambda^k T^*M)\,,\ dh = \delta h = 0 \text{ and } \mathrm{i}^* h = 0\right\},$$

where $\mathrm{i} : \partial M \to M$ is the inclusion map. Then the Hilbert space $L^2(\Lambda^k T^*M)$ has two orthogonal decompositions:

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k_{abs}(M) \oplus \overline{dC^\infty_b(\Lambda^{k-1}T^*M)} \oplus \overline{\delta C^\infty_0(\Lambda^{k+1}T^*M)}\,,$$
$$L^2(\Lambda^k T^*M) = \mathcal{H}^k_{rel}(M) \oplus \overline{dC^\infty_0(\Lambda^{k-1}T^*M)} \oplus \overline{\delta C^\infty_b(\Lambda^{k+1}T^*M)}\,.$$

Hence we obtain the identifications

$$H^k_{2,abs}(M) \simeq \mathcal{H}^k_{abs}(M) \text{ and } H^k_{2,rel}(M) \simeq \mathcal{H}^k_{rel}(M)\,,$$

where $\mathrm{i} : \partial M \to M$ is the inclusion map.

If we take the double of M along ∂M to obtain the manifold $X = M\#_{\partial M}M$ with its natural Lipschitz Riemannian metric, X has a natural isometry, the symmetry σ which switches the two copies of M in X; and δ is well defined, so that we can speak of harmonic form on $(X, g\#g)$.

Proposition 2.1. *The absolute L^2-cohomology of (M, g) is naturally isomorphic to the space of a σ-invariant L^2-harmonic form on $(M\#_{\partial M}M, g\#g)$ and the relative L^2-cohomology of (M, g) is naturally isomorphic to the space of a σ-anti-invariant L^2-harmonic form on $(M\#_{\partial M}M, g\#g)$.*

In fact, if M^n is oriented then the Hodge star operator is an isometry between $\mathcal{H}^k_{abs}(M)$ and $\mathcal{H}^{n-k}_{rel}(M)$.

If ∂M has several connected components, then we can put a relative boundary condition on some of the connected components and the absolute boundary condition on the others.

2.4 The Bochner–Weitzenböck formula. When the Riemannian manifold (M, g) is complete, and if $\Delta^k = d\delta + \delta d$ is the Hodge–de Rham Laplacian acting on k-differential forms, we have

$$\mathcal{H}^k(M) = \left\{\alpha \in L^2(\Lambda^k T^*M)\,,\ \Delta^k\alpha = 0\right\},$$

Moreover, in all cases, complete or not we have the Bochner–Weitzenböck decomposition

$$\Delta^k = \bar{\Delta} + \mathcal{R}^k, \tag{2.3}$$

where $\mathcal{R}^k$ is a symmetric linear operator of $\Lambda^k T^*M$ which is defined with the curvature operator of (M^n, g) (cf. [GM]; for instance, $\mathcal{R}^1$ is the Ricci operator), and we always have the bound $|\mathcal{R}^k|(x) \le c(n)|R|(x)$ where R is the curvature tensor of (M, g). A better estimate is obtain with the curvature operator ρ: $|\mathcal{R}^k|(x) \le k(n-k)|\rho|(x)$.

3 When L^2-cohomology Space has Finite Dimension

The purpose of this section is to give conditions which insure that the space of harmonic L^2-forms has finite dimension. These conditions are defined in [C4]. In this paper, we study the Dirac type operators which are non-parabolic at infinity. We give here the definition for the particular case of the Gauss–Bonnet operator.

DEFINITION 3.1. *The Gauss–Bonnet operator $d+\delta$ of a complete Riemannian manifold (M,g) is called non-parabolic at infinity when there is a compact set K of M such that for any bounded open subset $U \subset M \setminus K$ there is a constant $C(U)>0$ with the inequality*

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M \setminus K)\big)\,,\ C(U)\int_U |\alpha|^2 \le \int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2. \tag{3.4}$$

3.1 Properties of these operators.

In fact, we have the following:

PROPOSITION 3.2. *If the Gauss–Bonnet operator of (M,g) is non-parabolic at infinity then*

$$\dim\big\{\alpha \in L^2(\Lambda T^*M)\,,\ d\alpha = \delta\alpha = 0\big\} < \infty\,.$$

This was proved in [C4]; in this paper some characterizing properties of non-parabolicity at infinity are given, moreover these operators satisfy the following:

Theorem 3.3. *If the Gauss–Bonnet operator of (M,g) is non-parabolic at infinity, let D be a bounded open subset of M containing the compact set K outside of which estimate (3.4) holds and define $W(\Lambda T^*M)$ to be the Sobolev space which is the completion of $C_0^\infty(\Lambda T^*M)$ with quadratic form*

$$\alpha \mapsto \int_D |\alpha|^2 + \int_M |d\alpha|^2 + |\delta\alpha|^2.$$

Then this space imbedded continuously in H^1_{loc} and

$$d+\delta : W(\Lambda T^*M) \longrightarrow L^2(\Lambda T^*M)$$

is Fredholm.

A direct consequence of the Fredholmness of this operator is the following Hodge decomposition

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k(M) \oplus dW(\Lambda^{k-1}T^*M) \oplus \delta W(\Lambda^{k+1}T^*M)\,, \tag{3.5}$$

and $Z_2^k(M) = \mathcal{H}^k(M) \oplus dW(\Lambda^{k-1}T^*M)$. Hence, we have

PROPOSITION 3.4. *A L^2 closed k-form α is L^2 cohomologous to zero if and only if there is a $\beta \in W(\Lambda^{k-1}T^*M)$ such that $\alpha = d\beta$ and we can always choose β so that $\delta\beta = 0$.*

Let (M,g) be a complete Riemannian manifold with compact boundary on which the Gauss–Bonnet operator is non-parabolic at infinity, i.e. there is a compact set $K \subset M$ outside of which estimate (3.4) holds. Then the same feature is true on (M,g). The space $W(\Lambda T^*M)$ is the completion of the space $C_b^\infty(\Lambda T^*M)$ with the respect to quadratic form

$$\alpha \mapsto \|\alpha\|^2_{H^{1/2}(\partial M)} + \|\alpha\|^2_{L^2(D)} + \big\|(d+\delta)\alpha\big\|^2_{L^2(M)}\,;$$

where D is a bounded open subset of M containing ∂M and the compact set K; and $W_0(\Lambda T^*M)$ is the closure of $C_0^\infty(\Lambda T^*M)$ in $W(\Lambda T^*M)$. Then we have the orthogonal decompositions,

$$L^2(\Lambda^k T^*M) = \mathcal{H}^k_{abs}(M) \oplus dW(\Lambda^{k-1}T^*M) \oplus \delta W_0(\Lambda^{k+1}T^*M)\,,$$
$$L^2(\Lambda^k T^*M) = \mathcal{H}^k_{rel}(M) \oplus dW_0(\Lambda^{k-1}T^*M) \oplus \delta W(\Lambda^{k+1}T^*M)\,.$$

PROPOSITION 3.5. *$\alpha \in Z^k_{2,abs}(M)$ is L^2 cohomologous to zero if and only if there is an $\eta \in W(\Lambda^{k-1}T^*M)$ such that $\alpha = d\eta$ and we can always choose η so that $\delta\eta = 0$.*

$\alpha \in Z^k_{2,rel}(M)$ is L^2 cohomologous to zero if and only if there is a $\eta \in W_0(\Lambda^{k-1}T^*M)$ such that $\alpha = d\eta$ and we can always choose η so that $\delta\eta = 0$.

We note that if (M,g) is a complete Riemannian manifold whose Gauss–Bonnet operator is non-parabolic at infinity, then for every open set $U \subset M$ with smooth compact boundary, the Gauss–Bonnet operator on (U,g) is non-parabolic at infinity; and moreover we have an exact sequence

$$0 \longrightarrow W_0(\Lambda T^*U) \longrightarrow W(\Lambda T^*M) \longrightarrow W\big(\Lambda T^*(M \setminus U)\big) \longrightarrow 0\,,$$

where the first map is the extension by zero map and the second is the restriction map.

The condition of being non-parabolic at infinity depends only on the geometry in a neighborhood of infinity, so our result of finiteness for the dimension of the space of L^2-harmonic form is in concordance with J. Lott's result [L2], which asserts that the finiteness for the dimension of the space of L^2-harmonic form depends only on the geometry of infinity. That is to say, the spaces of L^2-harmonic form of two complete Riemannian manifolds, which are isometric outside some compact set, have simultaneously finite or infinite dimension.

3.2 Manifolds with flat ends. In [C4], we have shown the following

PROPOSITION 3.6. *If (M^n,g) is a complete Riemannian manifold whose curvatures vanish outside some compact set, then the Gauss–Bonnet operator is non-parabolic at infinity.*

For sake of completeness, we recall the proof.

Proof. Let K be a bounded open subset of M outside of which the curvatures vanish, then by the Bochner–Weitzenböck formula we have

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M \setminus K)\big)\,,\ \int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2 = \int_{M\setminus K} |\nabla\alpha|^2\,;$$

but according to the Kato lemma, we have $|\nabla\alpha| \geq |d|\alpha||$. Hence we obtain

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M \setminus K)\big)\,,\ \int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2 \geq \int_{M\setminus K} \big|d|\alpha|\big|^2.$$

Now if we consider the operator $H = \Delta + \mathbf{1}_K$ acting on functions, H is a positive operator and it satisfies the inequality $\forall u \in C_0^\infty(M)$, $\int_M \langle Hu, u\rangle \geq \int_K |u|^2$, then by a result on Ancona [A], we know that for any bounded open subset U of M we have a positive constant $C(U)$ such that

$$\forall u \in C_0^\infty(M)\,,\quad \int_M \langle Hu, u\rangle \geq C(U) \int_U |u|^2.$$

We take a bounded open subset U of $M \setminus K$, and $\alpha \in C_0^\infty(\Lambda T^*(M \setminus K))$, and we apply this inequality to $u = |\alpha|$. We have

$$\int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2 \geq \int_{M\setminus K} \big|d|\alpha|\big|^2 = \int_M \big|d|\alpha|\big|^2 + \mathbf{1}_K|\alpha|^2 \geq C(U)\int_U |\alpha|^2.$$

That is to say, the Gauss–Bonnet operator is non-parabolic at infinity. □

In fact we have only used the fact that the curvature operator is non-negative outside some compact set.

Remark 3.6. If D is a bounded open set containing K then the proof shows that an equivalent norm on W is given by

$$\alpha \mapsto \sqrt{\int_M |\nabla\alpha|^2 + \int_D |\alpha|^2}.$$

This comes from the Bochner–Weitzenböck formula and of standard elliptic estimates. Moreover if the manifold (M, g) is non-parabolic, then this norm is equivalent to $\alpha \mapsto \sqrt{\int_M |\nabla\alpha|^2}$. As a matter of fact, a Riemannian manifold is called non-parabolic if its Brownian motion is transient or equivalently, if it carries positive Green functions. An analytical characterization has been given by A. Ancona ([A]): (M, g) is non-parabolic if and only if for any bounded open subset $U \subset M$ there is a positive constant $C(U)$ such that

$$\forall f \in C_0^\infty(M)\,,\quad C(U)\int_U f^2 \leq \int_M |df|^2.$$

This result is one of our sources of inspiration to define non-parabolicity at infinity. So if (M, g) is non-parabolic, the result of Ancona and the

Kato inequality show that $W(\Lambda T^*M)$ is the space $H_0^1(\Lambda T^*M)$ obtained by completion of $C_0^\infty(\Lambda T^*M)$ with the norm $\alpha \mapsto \|\nabla\alpha\|_{L^2}$, and we have that the operator $d+\delta : H_0^1(\Lambda T^*M) \to L^2(\Lambda T^*M)$ is a Fredholm operator.

We can improve this result when the negative part of the curvature operator is controlled by the quadratic form $\alpha \mapsto \|\nabla\alpha\|_{L^2}$. Let $\lambda(x)$ be the lowest eigenvalue of the operator $\mathcal{R}(x)$ appearing in the Bochner–Weitzenböck formula (2.3), we note by $\mathcal{R}_-(x)$ the negative part of $\lambda(x)$ that is to say

$$\mathcal{R}_-(x) = \begin{cases} -\lambda(x) & \text{if} \quad \lambda(x) \le 0\,, \\ 0 & \text{if} \quad \lambda(x) > 0\,. \end{cases}$$

Proposition 3.7. *If (M,g) is a complete Riemannian manifold and assume that there is a compact set K of M and a $\varepsilon > 0$ such that*

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M\setminus K)\big)\,, \quad (1+\varepsilon)\int_{M\setminus K} \mathcal{R}_-(x)|\alpha|^2(x)dx \le \int_{M\setminus K} |\nabla\alpha|^2$$

then the Gauss–Bonnet operator is non-parabolic at infinity.

Proof. If $\alpha \in C_0^\infty(\Lambda T^*(M\setminus K))$ we apply the Bochner–Weitzenböck formula

$$\begin{aligned}\int_{M\setminus K} |d\alpha|^2 + |\delta\alpha|^2 &= \int_{M\setminus K} |\nabla\alpha|^2 + \langle \mathcal{R}\alpha,\alpha\rangle \\ &\ge \int_{M\setminus K} |\nabla\alpha|^2 - \mathcal{R}_-(x)|\alpha|^2 \\ &\ge \frac{\varepsilon}{1+\varepsilon}\int_{M\setminus K} |\nabla\alpha|^2.\end{aligned}$$

Then the argument given in the proof of the last proposition applies. □

Remark 3.7. In fact, if we have the estimate

$$\forall \alpha \in C_0^\infty\big(\Lambda T^*(M\setminus K)\big)\,, \quad (1+\varepsilon)\int_{M\setminus K} |\mathcal{R}|(x)|\alpha|^2(x)dx \le \int_{M\setminus K} |\nabla\alpha|^2,$$

Remark 3.6 shows that if (M,g) is non-parabolic then $W = H_0^1$.

3.3 Sobolev–Orlicz inequalities. One application of Proposition 3.7 is based on the Sobolev–Orlicz inequality obtained in [C2]. Let us recall what this inequality is. Let (M,g) be a complete Riemannian manifold and let $(P(t,x,y))_{(t\in\mathbb{R}_+,\,x,y\in M)}$ be its heat kernel, that is to say, this is the minimal solution of the Cauchy problem

$$\begin{cases} \frac{\partial P}{\partial t}(t,x,y) + \Delta_y^g P(t,x,y) = 0\,, \quad (x,y)\in M\,,\ t>0 \\ P(0,x,y) = \delta_x(y)\,. \end{cases}$$

Then let $\varphi : \mathbb{R}^+ \times M \to \mathbb{R}$ defined by

$$\varphi(\lambda,x) = \lambda\left(\int_{1/4\lambda}^\infty \sqrt{\frac{P(s,x,x)}{s}}ds\right)^2,$$

We assume that this integral is finite for some x in M (hence it is finite for all x in M by the Harnack inequality). Now if $u \in C_0^\infty(M)$, we associate

$$N(u) = \sup\left\{ \int_M uv\,,\ v \in C_0^\infty(M) \text{ with } \int_M \varphi\big(|v|(x),x\big)dx \le 1 \right\};$$

then N is a norm, and the completion of $C_0^\infty(M)$ with this norm is a Banach space (called an Orlicz space) which is made of locally integrable functions. Moreover, in [C2], we have shown the following universal Sobolev inequality:

$$\forall u \in C_0^\infty(M)\,, \quad N(u^2) \le C\|du\|_{L^2}^2\,,$$

for an universal constant C. This shows that in this cases the manifold is non-parabolic. As an application of this result we can state the following result.

PROPOSITION 3.8. *There is an universal constant C so that if (M,g) is a complete Riemannian manifold, whose heat kernel $(P(t,x,y))_{(t\in\mathbb{R}_+,\,x,y\in M)}$ and whose curvature operator $\mathcal{R}$ satisfy*

$$\int_M \mathcal{R}_-(x)\bigg(\int_{C/(\mathcal{R}_-(x))}^\infty \sqrt{\frac{P(s,x,x)}{s}}ds\bigg)^2 dx < \infty\,,$$

then the Gauss–Bonnet operator $d+\delta$ is non-parabolic at infinity.

Proof. By definition, we have the Hölder inequality

$$\int_M u^2 v \le N(u^2)\inf\left\{\lambda > 0\,,\ \int_M \varphi\left(\frac{|v(x)|}{\lambda},x\right)dx \le 1\right\}.$$

If $\int_M \varphi(2C\mathcal{R}_-(x),x)dx < \infty$ then there is a compact K of M such that

$$\int_{M\setminus K} \varphi(2C\mathcal{R}_-(x),x)dx \le 1\,.$$

Then if $\alpha \in C_0^\infty(\Lambda T^*(M\setminus K))$, we have

$$\begin{aligned}\big\langle \mathcal{R}_-(x)\alpha,\alpha\big\rangle &\le \frac{1}{2C}N(|\alpha|^2)\\ &\le \frac{1}{2}\int_{M\setminus K}|\nabla\alpha|^2.\end{aligned}$$

Hence the result by Proposition 3.7. □

As we have the bound $|\mathcal{R}^k|(x) \le c(n)|R|(x)$, Remark 3.7 implies that if

$$\int_M |R|(x)\bigg(\int_{Cc(n)/R|(x)}^\infty \sqrt{\frac{P(s,x,x)}{s}}ds\bigg)^2 dx < \infty\,,$$

then the operator $d+\delta : H_0^1(E)\longrightarrow L^2(E)$ is Fredholm.

For instance, if the manifold satisfies the Sobolev inequality

$$\forall u \in C_0^\infty(M)\,, \quad \mu_p(M)\bigg(\int_M |u|^{\frac{2p}{p-2}}(x)dx\bigg)^{1-\frac{2}{p}} \le \int_M |du|^2(x)dx\ .$$

Then, by the result of J. Nash [N], we know that the heat kernel satisfies a uniform bound

$$P(t,x,x) \leq C/t^{p/2}\,, \quad \forall x \in M\,,\ \forall t > 0\,,$$

so that, in this case, we have the uniform bound

$$\varphi(\lambda,x) \leq C\lambda^{p/2}\,, \quad \forall x \in M\,,\ \forall \lambda > 0\,.$$

Thus the hypothesis of the proposition is satisfied if the curvature of (M,g) is in $L^{p/2}$; so Proposition 3.8 generalizes the result of [C3].

3.4 The warped product cases. We recall the results we obtained in [C5] for the Gauss–Bonnet operator on a manifold which is a warped product at infinity.

Proposition 3.9. *If (M,g) is a complete Riemannian manifold and if there is a compact set K of M such that $(M \setminus K, g)$ is isometric to the warped product $(\,]0,\infty[\,\times\partial K, dr^2 + f^2(r)g)$, then the Gauss–Bonnet operator is non-parabolic at infinity in the following two cases:*

1. *$f(r) = ar$ for some $a > 0$,*
2. *$\lim_{r\to\infty} f'(r) = 0$.*

4 Topology and L^2-harmonic Forms

4.1 The exact sequence. The purpose of this section is to give a general condition which insures that the following sequence is exact: if D is a compact domain of the complete Riemannian manifold (M^n,g), we consider

$$\dots\longrightarrow H^k(D,\partial D) \xrightarrow{\ \mathrm{i}\ } \mathcal{H}^k(M) \xrightarrow{\ \mathrm{j}^*\ } H^k_{2,abs}(M\setminus D) \xrightarrow{\ \mathrm{b}\ } H^{k+1}(D,\partial D)\longrightarrow\dots \tag{4.8}$$

This is the long sequence associated to the coboundary operator b. Here the morphisms i, j^* and b are defined as usual:

1. i has the following natural application: To a closed smooth form with compact support in D, i associated its L^2-cohomology class, that is to say
$$\mathrm{i}\left[\alpha \bmod dC_0^\infty(\Lambda^k T^*D)\right] = \alpha \bmod \overline{dC_0^\infty(\Lambda^k T^*M)}^{L^2}.$$
2. j^* is the morphism induced by the restriction map $j : M\setminus D \to M$.
3. The coboundary operator b is defined as usual: for each cohomology class of $M\setminus D$, there is a smooth form α in it which is closed on a neighborhood of ∂D, $\mathrm{b}[\alpha]$ is the cohomology class of $d\bar\alpha$ where $\bar\alpha$ is a smooth extension of α which is closed in a neighborhood of ∂D. b is

well defined, it does not depend on the choice of α and nor on its extension.

By construction, we have
$$\mathrm{j}^* \circ \mathrm{i} = 0\,, \quad \mathrm{b} \circ \mathrm{j}^* = 0 \;\text{ and }\; \mathrm{i} \circ \mathrm{b} = 0\,;$$
so we have the inclusions
$$\mathrm{Im}\ \mathrm{i} \subset \mathrm{Ker}\ \mathrm{j}^*, \quad \mathrm{Im}\ \mathrm{j}^* \subset \mathrm{Ker}\ \mathrm{b} \;\text{ and }\; \mathrm{Im}\ \mathrm{b} \subset \mathrm{Ker}\ \mathrm{i}\,.$$
In [C1], we noted that

PROPOSITION 4.1. *The equality* $\ker \mathrm{b} = \mathrm{Im}\, j^*$ *always hold.*

Proof. This comes from the long exact sequence in the de Rham cohomology. If $[\beta] \in H_2^k(M \setminus D)$ is in the kernel of b, there is a smooth extension of β, say $\bar\beta$, and a smooth $(k-1)$-form γ with compact support in D, so that $d\bar\beta = d\gamma$. So the form $\bar\beta - \gamma$ is closed, square integrable and its restriction to $M \setminus D$ is β; hence $[\beta] = j^*[\bar\beta - \gamma]$. □

In fact, when the Gauss–Bonnet operator is non-parabolic at infinity we have more.

PROPOSITION 4.2. *If the Gauss–Bonnet operator is non-parabolic at infinity, then*
$$\ker j^* = \mathrm{Im}\, \mathrm{i}\,.$$

Proof. Let h be a harmonic L^2 k-form on M such that its restriction to $M \setminus D$ is L^2 cohomologous to zero. By (3.5), we have a smooth form $\eta \in W(\Lambda^{k-1}T^*(M \setminus D))$ such that $h = d\eta$ and $\delta\eta = 0$ on $M \setminus D$. Now if $\overline{\eta} \in W(\Lambda^{k-1}T^*M)$ is a smooth extension of η, then $h - d\overline{\eta}$ is a closed L^2-form which is L^2 cohomologous to h. Then the support of $h - d\overline{\eta}$ is in $\overline{D}$ and $h - d\overline{\eta}$ is zero when pulled back to ∂D. Hence the L^2-cohomology class of h is in the image of the natural map i from $H_c^k(D) \simeq H^k(D, \partial D)$ in $\mathcal{H}^k(M)$. □

For the remaining equality, we conclude with the analysis done in [C5]:

Theorem 4.3. *If the Gauss–Bonnet operator is non-parabolic at infinity and if any harmonic form in* W *is in* L^2 *then the long sequence* (4.8) *is exact.*

Proof. Our hypothesis is
$$h_\infty(M) = \dim \frac{\{\alpha \in W,\, d\alpha + \delta\alpha = 0\}}{\{\alpha \in L^2,\, d\alpha + \delta\alpha = 0\}} = 0\,.$$
In [C5], we computed this dimension, our result is that if we define
$$h_\infty(M \setminus D) = \dim \frac{\{\alpha \in W(\Lambda T^*(M \setminus D)) \cap C^\infty,\, d\alpha + \delta\alpha = 0\}}{\{\alpha \in L^2(\Lambda T^*(M \setminus D)) \cap C^\infty,\, d\alpha + \delta\alpha = 0\}}$$

then we have

$$h_\infty(M) = \tfrac{1}{2} h_\infty(M \setminus D)\,.$$

Our hypothesis implies that a smooth W-harmonic form on $M \setminus D$ is in fact in L^2. We can now show that $\ker \mathrm{i} \subset \mathrm{Im}\,\mathrm{b}$. Let α be a closed k-form with compact support in D which is zero in L^2-cohomology, thus by (3.5), we have a $\beta \in W(\Lambda^{k-1}T^*M)$ with

$$\delta\beta = 0 \quad \text{and} \quad \alpha = d\beta\,.$$

Now β is smooth by elliptic regularity, and $j^*\beta$ is harmonic and in W. By hypothesis, $j^*\beta$ and β must be in L^2. Hence $[\alpha] = \mathrm{b}[\beta|_{M\setminus D}]$. □

Remark 4.9. In fact, in order to show the equality

$$\ker\left\{\mathrm{i} : H^k(D, \partial D) \to \mathcal{H}^k(M)\right\} = \mathrm{Im}\left\{\mathrm{b} : \mathcal{H}^{k-1}_{2,abs}(M \setminus D) \to H^k(D, \partial D)\right\},$$

we only need to verify that on $M \setminus D$, the W harmonic $(k-1)$-form is L^2.

Moreover, we can consider this long exact sequence if (M, g) is a complete Riemannian manifold with compact boundary ∂M, and if we put an appropriate boundary condition on the boundary (the relative condition on some connected component of ∂M and the absolute condition on the other). When $U \subset M$ is an open set with smooth compact boundary with $\partial U \subset \mathrm{int}(M)$. Then we can define the sequence

$$\ldots \longrightarrow H^k_{2,rel}(U) \xrightarrow{\mathrm{i}} H^k_2(M) \xrightarrow{\mathrm{j}^*} H^k_{2,abs}(M \setminus U) \xrightarrow{\mathrm{b}} H^{k+1}_{2,rel}(U) \longrightarrow \ldots$$

On $M \setminus U$, $H^k_{2,abs}(M \setminus U)$ is defined with the absolute boundary condition on ∂U, and on ∂M we put the same boundary condition as for defining $H^k_2(M)$. The map i is well defined because, by definition, an element of $Z^k_{2,rel}(U)$ when extended by zero on M is an element of $Z^k_2(M)$. For the coboundary map b, due to the non-compactness of U, we have to consider only extension with compact support. In fact this coboundary map is the composition of the natural map from $H^k_{2,abs}(M \setminus U)$ to the de Rham cohomology group $H^k_{abs}(M \setminus U)$ then of the usual coboundary map from $H^k_{abs}(M)$ to $H^{k+1}_c(U)$ (the group of cohomology with compact support of U) and finally of the natural map from $H^{k+1}_c(U)$ to $H^{k+1}_{2,rel}(U)$. The same proof leads to the following results:

Theorem 4.4. *If the Gauss–Bonnet operator is non-parabolic at infinity then* $\ker \mathrm{j}^* = \mathrm{Im}\,\mathrm{i}$. *Moreover if any harmonic form in $W(\Lambda T^*M)$ is in $L^2(\Lambda T^*M)$ then the long exact sequence is true. More exactly if $h_\infty(U) = 0$ then* $\ker \mathrm{b} = \mathrm{Im}\, j^*$ *and if $h_\infty(M \setminus U) = 0$ then* $\ker \mathrm{i} = \mathrm{Im}\,\mathrm{b}$.

The fact that $\ker \mathrm{j}^* = \mathrm{Im}\,\mathrm{i}$ implies the following:

COROLLARY 4.5. *If the Gauss–Bonnet operator is non-parabolic at infinity and if U is an open set of M with compact boundary, then* $\mathrm{Im}(H^k_{2,rel}(U) \to H^k_{2,abs}(U))$ *injects in* $H^k_2(M)$.

Proof. If $[\alpha] \in H^k_{2,rel}(U)$ is map to zero in $H^k_2(M)$, then by (3.5) there is a $\beta \in W(\Lambda^{k-1}T^*M)$, such that $\alpha = d\beta$ on M. This identity is also true on U, and $\beta|_U \in W(\Lambda^{k-1}T^*U)$. So $[\alpha]$ is map to zero in $H^k_{2,abs}(U)$ □

If $U = M$, then the result is due to M. Anderson ([An]): $\mathrm{Im}(H^k_{rel}(M) \to H^k_{abs}(M))$ always injects in $H^k_2(M)$.

4.2 Some examples. In [C1], we have shown the following:

Theorem 4.6. *If (M^n, g) is a complete Riemannian manifold, which for a $p > 4$ satisfies the Sobolev inequality*

$$\forall u \in C_0^\infty(M)\,, \quad \mu_p(M)\left(\int_M |u|^{\frac{2p}{p-2}}(x)dx\right)^{1-\frac{2}{p}} \leq \int_M |du|^2(x)dx\,,$$

and whose curvature tensor R satisfies

$$\int_M |R|^{p/2}(x)dx < \infty\,,$$

then the sequence (4.8) *is exact.*

This result can be attained by the analysis done here, because, as we noted earlier, on such a manifold the Gauss–Bonnet operator is non-parabolic at infinity. Moreover, the main analytical tool used in [C1] was that if $\alpha \in H^1_0$ satisfies $\Delta\alpha = (d\delta + \delta d)\alpha \in C_0^\infty(\Lambda T^*M)$ then $\alpha \in L^2$. And this implies that a H^1_0-harmonic form is in L^2. We can generalize this result.

Theorem 4.7. *There is a constant $C(n)$ such that if (M^n, g) is a complete Riemannian manifold whose curvature tensor R and whose heat kernel $(P(t,x,y))_{(t,x,y)\in\mathbb{R}_+^*\times M\times M}$ satisfy*

$$\int_M |R|^2(x)\left(\int_{C(n)/|R|(x)}^\infty \sqrt{P(t,x,x)}dt\right)^2 dx < \infty$$

then the exact sequence (4.8) *holds.*

Proof. First we remark that our assumption implies that we have

$$\int_M |R|(x)\left(\int_{C(n)/|R|(x)}^\infty \sqrt{\frac{P(t,x,x)}{t}}dt\right)^2 dx < \infty\,.$$

Hence there is a choice of the constant $C(n)$ which implies that the Gauss–Bonnet operator is non-parabolic at infinity by Proposition 3.8. Moreover, the Remark 3.7 implies that $W = H^1_0$.

Now we have only to show that if $\alpha \in H_0^1$ is a harmonic form then α is in L^2, because we have in our case $H_0^1 = W$. We will use the Bochner–Weitzenböck formula

$$\Delta\alpha = \bar{\Delta}\alpha + \mathcal{R}\alpha = 0\,.$$

The main point is the following:

If K is a compact of M, large enough and $\Omega = M \setminus K$, then we have that

$$\|\bar{\Delta}_\Omega^{-1}\mathcal{R}\|_{L^2(\Omega)\to L^2(\Omega)} < 1\,,$$
$$\|\bar{\Delta}_\Omega^{-1}\mathcal{R}\|_{H_0^1(\Omega)\to H_0^1(\Omega)} < 1\,,$$

where $\bar{\Delta}_\Omega$ is the minimal self-adjoint extension of $\bar{\Delta} : C_0^\infty(\Lambda T^*\Omega) \to C_0^\infty(\Lambda T^*\Omega)$.

To prove this, we note that we have the following:

$$|\mathcal{R}(x)| \le c(n)|R(x)|\,,$$

where R is the Riemannian curvature tensor; and by a consequence of Kato's lemma, we have

$$|\bar{\Delta}_\Omega^{-1}\alpha|(x) \le \Delta_\Omega^{-1}|\alpha|(x)\,,$$

see for instance [HeSU] or the appendix of [B] written by G. Besson. Hence we need only to show that

$$\|\Delta_\Omega^{-1}|R|\|_{L^2(\Omega)\to L^2(\Omega)} < 1/c(n)$$
$$\|\Delta_\Omega^{-1}|R|\|_{H_0^1(\Omega)\to H_0^1(\Omega)} < 1/c(n)\,,$$

where Δ_Ω is the minimal self-adjoint extension of the symmetric operator $\Delta : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$. Let us show the first bound. We will prove that $\||R|\Delta_\Omega^{-1}\|_{L^2\to L^2} < 1/c(n)$; which is the same result because an operator and its adjoint have the same norm. Recall that, in [C2], we showed the following Sobolev Orlicz inequality: if $u \in C_0^\infty(M)$, we associate to it

$$I(u) = \inf\left\{\int_M uv\,;\ v \in C_0^\infty(M) \text{ s.t. } \int_M \phi\big(|v(x)|, x\big)dx \le 1\right\},$$

where ϕ is the function on $\mathbb{R}_+ \times M$ defined by

$$\phi(\lambda, x) = \lambda\left(\int_{1/\sqrt{\lambda}}^\infty \sqrt{P(t,x,x)}dt\right)^2,$$

where we assume that this integral is finite, otherwise all what we will say will be vacuous. I is a norm and the completion of the space $C_0^\infty(M)$ with respect to I is a Banach space (an Orlicz space) made from a locally integrable function. We have shown that for a universal constant C, we have the Sobolev inequality

$$\forall u \in C_0^\infty(M)\,, \quad I(u^2) \le C\|\Delta u\|\,.$$

By definition, we have the Hölder inequality,

$$\int_\Omega |R|^2 u^2 \le I(u^2)\inf\left\{\lambda, \int_\Omega \phi\big(|R|^2(x)/\lambda, x\big)dx \le 1\right\};$$

so that if we choose Ω in order that

$$\int_\Omega \phi\big(4c(n)^2 C|R|^2(x), x\big)dx \le 1\,,$$

we then obtain that

$$\forall u \in C_0^\infty(\Omega)\,,\quad \big\||R|u\big\|_{L^2} \le \tfrac{1}{2c(n)}\|\Delta u\|_{L^2}\,.$$

This ends the proof of the first bound.

For the second bound, as the operator $\Delta_\Omega^{-1/2}$ is an isometry between $L^2(\Omega)$ and $H_0^1(\Omega)$, we only need to prove that

$$\big\|\Delta_\Omega^{-1/2}|R|\Delta_\Omega^{-1/2}\big\|_{L^2\to L^2} < 1/c(n)\,.$$

We will prove that $\||R|^{1/2}\Delta_\Omega^{-1/2}\|_{L^2\to L^2} < 1/\sqrt{c(n)}$. For some constant C, we have the Sobolev inequality

$$\forall u \in C_0^\infty(M)\,,\quad N(u^2) \le C\|du\|_{L^2}\,.$$

With the Hölder inequality

$$\int_\Omega |R|u^2 \le N(u^2)\inf\left\{\lambda, \int_\Omega \varphi\big(|R|(x)/\lambda, x\big)dx \le 1\right\},$$

and the inequality $\varphi(\lambda,x) \le \phi(\lambda^2,x)$, we obtain

$$\forall u \in C_0^\infty(\Omega)\,,\quad \int_\Omega |R|u^2 \le \frac{1}{2Cc(n)}N(u^2)\,.$$

This ends the proof of our two bounds.

We can now finish the proof. If ρ is a smooth function which is 1 out of compact and with support in Ω, then the form $(\bar\Delta + \mathcal{R})\rho\alpha = \beta$ is a smooth form with compact support in Ω. We remark that

$\bar\Delta_\Omega^{-1}\beta$ *is in* H_0^1 *and in* L^2.

As the matter of fact, the linear form $\phi \mapsto \langle\beta,\phi\rangle_{L^2}$ is continuous on H_0^1, hence there is $\gamma \in H_0^1$ satisfying

$$\langle\gamma, \bar\Delta\phi\rangle_{L^2} = \langle\beta,\phi\rangle_{L^2}\,,\quad \forall\phi \in C_0^\infty(\Lambda T^*\Omega)\,.$$

So that $\gamma = \bar\Delta_\Omega^{-1}\beta \in H_0^1$. To prove that $\bar\Delta_\Omega^{-1}\beta$ is in L^2, we note that if f is a smooth function with compact support which is 1 on the support of β, then

$$\bar\Delta_\Omega^{-1}\beta = \bar\Delta_\Omega^{-1}f\beta\,.$$

But we know that the operator $\phi \mapsto \bar\Delta_\Omega^{-1}f\phi$ is bounded on L^2, so $\bar\Delta_\Omega^{-1}\beta$ is in L^2.

Now we have the equality

$$(\mathrm{Id}_{H_0^1} + \bar\Delta_\Omega^{-1}\mathcal{R})\rho\alpha = \bar\Delta_\Omega^{-1}\beta\,.$$

But the operator $(\mathrm{Id} + \bar{\Delta}_\Omega^{-1}\mathcal{R})$ is invertible on H_0^1 and on L^2 so that

$$\rho\alpha = (\mathrm{Id}_{H_0^1} + \bar{\Delta}_\Omega^{-1}\mathcal{R})^{-1}\bar{\Delta}_\Omega^{-1}\beta\,.$$

As we have

$$(\mathrm{Id}_{H_0^1} + \bar{\Delta}_\Omega^{-1}\mathcal{R})^{-1}\bar{\Delta}_\Omega^{-1}\beta = (\mathrm{Id}_{L^2} + \bar{\Delta}_\Omega^{-1}\mathcal{R})^{-1}\bar{\Delta}_\Omega^{-1}\beta\,,$$

we obtain that $\rho\alpha$ is a squared integrable and $\alpha \in L^2$. □

In [C5], we have given a cohomological interpretation of the dimension of

$$\frac{\{\alpha \in W,\, d\alpha + \delta\alpha = 0\}}{\{\alpha \in L^2,\, d\alpha + \delta\alpha = 0\}}\,.$$

This dimension can be thought of as the dimension of half bound states in quantum mechanics. This dimension is zero when zero is not in the essential spectrum of the Laplacian, or equivalently when the Gauss–Bonnet operator is Fredholm on its domain. But in this case, the spaces of L^2-cohomology and of reduced L^2-cohomology are the same, and the long exact sequence (4.8) always holds for the L^2-cohomology. For instance,

Theorem 4.8. *If M is an even dimensional locally symmetric space of finite volume and negative curvature then the exact sequence* (4.8) *holds.*

This is a consequence of the work of Borel and Casselman [BoC]: the Gauss–Bonnet operator is a Fredholm operator in this case.

With the results of [C5], we can now extend Proposition 3.9.

Theorem 4.9. *If (M,g) is a complete Riemannian manifold of dimension n and if there is a compact set K of M such that $(M \setminus K, g)$ is isometric to the warped product $(\,]0,\infty[\,\times\partial K, dr^2 + f^2(r)g)$, then the Gauss–Bonnet operator is non-parabolic at infinity in the following two cases $f(r) = ar$ for some $a > 0$, or $\lim_{r\to\infty} f'(r) = 0$:*

1. *Moreover let $k = [n/2]$, if $f(r) = ar$, and if the first eigenvalue λ_0^k of the Laplace operator acting on closed k-differential forms of ∂K satisfies $\lambda_0^k > \frac{7+(-1)^n}{8}a$ then the exact sequence* (4.8) *holds. Moreover when the manifold is oriented and of even dimension then*

$$\chi_{L^2}(M) = \chi(M) - \sum_{j=0}^{k-2}(-1)^j \mathrm{b}_j(\partial K)\,.$$

2. *If $\lim_{r\to\infty} f'(r) = 0$ and if we have $\int^\infty f^t < \infty$ for all $t > 1$ then when n is even the exact sequence holds and we have $\chi_{L^2}(M) = \chi(M) + \sum_{j=0}^{k-1}(-1)^j \mathrm{b}_j(\partial K)$; and in odd dimension $\dim M = 2k+1$, the exact sequence holds, provided that $\mathrm{b}_k(\partial K) = 0$.*

In fact, the second Gauss–Bonnet formula was already known by J. Brüning ([Br]).

4.3 Novikov–Shubin invariants and non-parabolicity at infinity. We recall here a result from [C6]. It gives a spectral condition which implies that the Gauss–Bonnet operator is non-parabolic at infinity.

Proposition 4.10. *Assume that (M,g) is a complete Riemannian manifold such that there is a $\alpha > 2$ and a locally bounded function $C(x)$, $x \in M$ with*

$$\left\| e^{-t\Delta}(x,x) \right\| \le C(x) t^{-\alpha/2}\,, \quad \forall t \ge 1\,,\ x \in M\,,$$

then the Gauss–Bonnet operator of (M,g) is non-parabolic at infinity and $h_\infty(M) = 0$.

We make some remarks:

- The non-parabolicity at infinity property and the dimension $h_\infty(M)$ depend only on the geometry at infinity, so the conclusion of the theorem remains valid for any complete Riemannian manifold which is isometric outside some compact set to one satisfying the hypothesis of the theorem.
- By the Karamata theorem, our assumption is equivalent to the following on E, the spectral resolution of Δ,

$$\left\| E([0,\lambda],x,x) \right\| \le \tilde{C}(x) \lambda^{\alpha/2}\,, \quad \forall \lambda \in [0,1]\,,\ \forall x \in M\,.$$

In fact the best possible exponent α is linked with the Novikov–Shubin invariants: if (M,g) is the universal covering of a compact manifold $\overline{M}$ and if $D = d + \delta$ is the Gauss–Bonnet operator acting on differential forms on M, and if $F \subset M$ is a fundamental domain of the covering $M \to \overline{M}$, then

$$\beta = \inf \left\{ \alpha \;\middle|\; \int_F \mathrm{trace}_{\Lambda T_x^* M} e^{-t\Delta}(x,x)dx = O(t^{-\alpha/2}) \right\}$$

is a Novikov–Shubin invariant of $\overline{M}$. It doesn't depend on the metric ([NoS]), nor on the differential structure ([L1]) and it is a homotopy invariant of $\overline{M}$ ([GroS]).

According to the calculus done by M. Rumin ([R]) and by L. Schubert ([S]) we have

Corollary 4.11. *If $n > 1$ then the Gauss–Bonnet operator of the Heisenberg group*

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & ... & x_n & z \\ 0 & 1 & 0 & ... & 0 & y_1 \\ . & .. & . & ... & . & . \\ . & .. & ... & . & . & . \\ . & .. & ... & 0 & 1 & y_n \\ 0 & .. & . & ... & . & 1 \end{pmatrix} ;\quad x_1,..,x_n,y_1,..,y_n,z \in \mathbb{R} \right\}$$

with a left-invariant metric is non-parabolic at infinity. Furthermore, for any complete Riemannian manifold which is isometric outside some compact to such H_{2n+1}*, the exact sequence* (4.8) *holds.*

Proof. It is shown in [R] and [S] that the hypothesis of Proposition 4.10 holds with $\alpha = n+1$ for the Gauss–Bonnet operator of the Heisenberg group H_{2n+1} with a left invariant metric. □

5 Manifolds with Flat Ends

In this section, our results are applied to a complete Riemannian manifold with flat ends. Let (M^n, g) be a complete Riemannian manifold whose curvatures vanish outside some compact set.

5.1 Geometry of flat ends. Then (M^n, g) has a finite number of ends $E_1, E_2, \dots, E_b$, and according to the works of Eschenburg–Schroeder, complete flat ends are classified into three families [ES]:

Theorem 5.1. 1. *E has a finite cover isometric to* $(\mathbb{R}^\nu - B(R)) \times \mathbf{T}^{n-\nu}$*, where* $\nu > 2$ *and* $\mathbf{T}^{n-\nu}$ *is a flat* $(n-\nu)$*-torus.*

2. *E is isometric to the Riemannian product* $]0,\infty[\,\times Y$ *where Y is a compact flat manifold.*

3. *The universal cover of E is isometric to* $Y_{\beta,R} \times \mathbb{R}^{n-2}$*, for* $R > \beta$*, and the* $\pi_1(E)$ *respects this decomposition, where* $Y_{\beta,R}$ *is defined as follows: If* $c_{\beta,R}(s) = (\beta s - R\sin s, R\cos s)$ *is the cycloid and* $n_{\beta,R}$ *its unit normal vector. Then* $Y_{\beta,R}$ *is* $]0,\infty[\,\times\mathbb{R}$ *equipped with the metric* $\Phi^*_{\beta,R}(dx^2+dy^2)$ *where* $\Phi_{\beta,R} :]0,\infty[\,\times\mathbb{R} \to \mathbb{R}^2$ *is the immersion* $\Phi_{\beta,R}(s,t) = c_{\beta,R}(s) + t n_{\beta,R}(s)$*.*

In the third case, a finite cover of the end is isometric to $(Y_{\beta,R}\times\mathbb{R}^{n-2})/\Gamma$ a flat bundle of an $(n-2)$ flat torus on $\bar{Y}_{\beta,R} = Y_{\beta,R}/\{(s,t) \sim (s+k\tau, t),\ k\in\mathbb{Z}\}$ with $\tau = 2\pi$, if $\beta > 0$ and τ is any positive real number if $\beta = 0$.

We also note that in the second and third case the rank of the fundamental group of E is $n-1$, where, as in the first, the rank of the fundamental group of E is less than $n-3$.

5.2 The L^2-cohomology of flat ends. First, we compute the L^2-cohomology of flat ends.

Lemma 5.2. *In case* (2) *of the classification* (*Theorem 5.1*)*, if # is the relative or absolute boundary condition, then* $H^k_{2,\#}(E) = \{0\}$*.*

Proof. According to Proposition 2.1, it is enough to show that the L^2-cohomology of the double manifold $E\#E$ is trivial. But $E\#E$ is the

Riemannian product $\mathbb{R} \times Y$; and it is well known that $H_2^k(\mathbb{R} \times Y) = \{0\}$ (this is, for example, a consequence of the Kunneth formula). □

This lemma can also be obtained from the result of Atiyah–Patodi–Singer ([AtPS]): on a manifold with cylindrical end, the L^2-cohomology is the image of the relative cohomology in the absolute cohomology.

Lemma 5.3. *In case* (3) *of the classification* (*Theorem 5.1*)*, if* $\#$ *is the relative or absolute boundary condition, then* $H^k_{2,\#}(E) = \{0\}$*.*

Proof. Let $\pi : \check{E} \to E$ be the finite cover of E which is a flat bundle of a $(n-2)$ flat torus on $\bar{Y}_{\beta,R}$. We have

$$\alpha \in L^2(\Lambda T^*E) \text{ if and only if } \pi^*\alpha \in L^2(\Lambda T^*\check{E}) .$$

So it is enough to show the result for $\check{E}$. Now, the $(n-2)$ flat torus acts on $\check{E}$ by isometry and this action generates Killing fields of bounded length, so by the argument of N. Hitchin, a $\# - L^2$-harmonic form must be invariant by the torus action (Theorem 3 of [Hi]). Hence there is an identification between $H^k_{2,\#}(\check{E})$ and the space of L^2-harmonic forms on $\bar{Y}_{\beta,R}$ with value in the flat unitary bundle $(Y_{\beta,R} \times \Lambda(\mathbb{R}^{n-2})^*/\Gamma$ satisfying the relative or absolute boundary condition. Now there are no such non-trivial 0 and 2-forms as they must be parallel; and 1-forms being L^2 harmonic is a conformal invariant property. As $\bar{Y}_{\beta,R}$ is conformally equivalent to the cylinder, Lemma 5.2 implies that there are no L^2 harmonic 1-forms with value in this flat unitary bundle. □

Lemma 5.4. *In case* (1) *of the classification* (*Theorem 5.1*)*, if E is isometric to the product* $(\mathbb{R}^\nu - B(R)) \times \mathbf{T}^{n-\nu}$ *then*

$$H^k_{2,rel}(E) = \mathbb{R}\frac{dr}{r^{\nu-1}} \wedge H^{k-1}(\mathbf{T}^{n-\nu}) .$$

Proof. Proposition 4.3 of [C3] implies that $H^k_{2,rel}(\mathbb{R}^\nu - B(R)) = \mathbb{R}\frac{dr}{r^{\nu-1}}$; hence the lemma by the Kunneth formula. □

5.3 The parabolic ends. We then eliminate the parabolic ends. Let E_P be the union of parabolic ends of M, i.e. the union of the ends of type (2) and (3) in the classification (Theorem 5.1).

Proposition 5.5. $H_2^k(M) \simeq \mathrm{Im}\big(H^k_{2,rel}(M \setminus E_P) \longrightarrow H^k_{2,abs}(M \setminus E_P)\big)$.

Proof. First we look at the short sequence

$$H^k_{2,rel}(M \setminus E_P) \xrightarrow{\ \mathrm{i}\ } H_2^k(M) \xrightarrow{\ \mathrm{j}^*\ } H^k_{2,abs}(E_P) .$$

From Theorem 4.4, we know that $\ker \mathrm{j}^* = \mathrm{Im}\,\mathrm{i}$, and from Lemmas 5.2, 5.3, and we know that $H^k_{2,abs}(E_P) = \{0\}$. So in any L^2-cohomology class on M

there is a closed L^2 smooth form with support in $M \setminus E_P$. Second, we look at the short sequence

$$H^k_{2,rel}(E_P) \xrightarrow{\mathrm{i}} H^k_2(M) \xrightarrow{\mathrm{j}^*} H^k_{2,abs}(M \setminus E_P)\,.$$

Again, this sequence is exact and $H^k_{2,rel}(E_P) = \{0\}$.

The injective map between $\mathrm{Im}(H^k_{2,rel}(M \setminus E_P) \to H^k_{2,abs}(M \setminus E_P))$ and $H^k_2(M)$ is surjective hence it is an isomorphism. □

5.4 The non-parabolic ends. We now have to compute the reduced L^2-cohomology of $M \setminus E_P$ with relative or absolute boundary condition on ∂E_P. We will only deal with the case of absolute boundary conditions; the case of relative boundary conditions is identical.

Now on $(M \setminus E_P, g)$ any W-harmonic form must be L^2.

As a matter of fact, by Theorem 1.5, this property only depends on the geometry of ends: if E is an end of $M \setminus E_P$, then E has a finite cover isometric to $\check{E} = (\mathbb{R}^\nu - B(R)) \times \mathbf{T}_E$, where $\nu > 2$ and $\mathbf{T}_E$ is a flat $(n-\nu)$-torus. Now a harmonic form on E is in W (resp. L^2) if and only if it is pulled back to a W (resp. L^2) form on $\check{E}$. And $\check{E}$ is the end of $\mathbb{R}^\nu \times \mathbf{T}_E$ so by Theorem 1.5, it is enough to show the result for $\mathbb{R}^\nu \times \mathbf{T}_E$. But in [C4] it is shown that $\ker_W(d+\delta)_{\mathbb{R}^\nu \times \mathbf{T}_E} = \ker_W(d+\delta)_{\mathbb{R}^\nu} \otimes \ker(d+\delta)_{\mathbf{T}_E}$.

Now, for $\nu > 2$, $\mathbb{R}^\nu$ is non-parabolic, so the topology of W is given by the quadratic forms $\alpha \mapsto \int_{\mathbb{R}^\nu} |\nabla\alpha|^2 = \int_{\mathbb{R}^\nu} |(d+\delta)\alpha|^2$; so a W-harmonic form is parallel. But by the Sobolev inequality, an element of W has to be in $L^{2\nu/(\nu-2)}$. Hence any W-harmonic form on $\mathbb{R}^\nu$ is zero.

So on $M \setminus E_P$, we have $h_\infty(M \setminus E_P) = 0$ and let $E_{NP} = \cup_{i=1}^b E_i$ be the union of the ends of $M \setminus E_P$ and $U = E_P \cup E_{NP}$ be the union of the ends of M. We have the long exact sequence,

$$\ldots \longrightarrow H^k_{2,rel}(E_{NP}) \xrightarrow{\mathrm{i}} H^k_{2,abs}(M \setminus E_P) \xrightarrow{\mathrm{j}^*} H^k_{abs}(M \setminus U) \xrightarrow{\mathrm{b}} H^{k+1}_{2,rel}(E_{NP}) \longrightarrow \ldots$$

Each E_i has a finite cover isometric to $\check{E}_i = (\mathbb{R}^{\nu_i} - B(R)) \times \mathbf{T}_i$ where $\nu_i > 2$ and $\mathbf{T}_i$ is a flat $(n-\nu_i)$-torus. So for a finite subgroup Γ_i of $O(\nu_i) \times \mathrm{Isom}(\mathbf{T}_i)$, $E_i = \check{E}_i/\Gamma_i$. Let $p_i : \check{E}_i \to E_i$ be the covering map. p_i^* induces an isomorphism between $H^k_{2,rel}(E_i)$ and the space of Γ_i-invariant element in $H^k_{2,rel}(\check{E}_i)$:

$$H^k_{2,rel}(E_i) = H^k_{2,rel}(\check{E}_i)^{\Gamma_i}.$$

We have shown that

$$H^k_{2,rel}(\check{E}_i) \simeq \frac{dr}{r^{\nu_i - 1}} \wedge H^{k-1}(\mathbf{T}_i)\,,$$

and the isomorphism between $H^k_{2,rel}(\check{E}_i)$ and $H^{k-1}(\mathbf{T}_i)$ is induced by the map

$$H^{k-1}(\mathbf{T}_i) \to H^{k-1}(\partial \check{E}_i) \xrightarrow{\text{b}} H^k_{2,rel}(\check{E}_i)\,.$$

As Γ_i acts on $\mathbf{T}_i$,we also have $H^k_{2,rel}(E_i) = H^{k-1}(\mathbf{T}_i)^{\Gamma_i}$. Let $\pi : \cup_i \mathbf{T}_i \to \partial E_{NP}$ be the immersion induced by the p_i's; and we define

$$\Omega(M \setminus U, \ker \pi^*) = \left\{\alpha \in C^\infty(\Lambda T^*(M \setminus U)) \text{ such that } \pi^*\alpha = 0\right\}.$$

Then $\Omega(M \setminus U, \ker \pi^*)$ is a subcomplex of the complex of differential forms on $M \setminus U$. Let $H^k(M \setminus U, \ker \pi^*)$ be the associated cohomology spaces. Now we have also the exact sequence

$$... \to \bigoplus_i H^{k-1}(\mathbf{T}_i)^{\Gamma_i} \xrightarrow{\text{b}} H^k(M \setminus U, \ker \pi^*) \xrightarrow{\text{j}^*} H^k_{abs}(M \setminus U) \xrightarrow{\pi^*} \bigoplus_i H^k(\mathbf{T}_i)^{\Gamma_i} \to ...$$

We can build a map $H^k_{2,abs}(M \setminus E_P) \to H^k(M \setminus U, \ker \pi^*)$: if α is a smooth closed L^2-form on $M \setminus E_P$, then from (3.5) and Lemma 5.4 on a end $E_i \subset E_{NP}$, we have

$$\alpha = \frac{dr}{r^{\nu_i - 1}} \wedge \beta_i + d\gamma_i\,,$$

where $\gamma \in W(\Lambda^{k-1}T^*E_i)$ and r is the radial coordinate on E_i; up to some additive constant, on E_i, r is the function distance to ∂E_i. Now let $\overline{\gamma} \in W(\Lambda^{k-1}T^*(M \setminus E_P))$ be an extension of the γ_i's, then $\alpha - d\overline{\gamma}$ is a closed element of $\Omega\,(M \setminus U, \ker \pi^*)$ and its cohomology class in $H^k(M \setminus U, \ker \pi^*)$ depends only on the L^2-cohomology class of α. So we get the desired map. Now, we have a commutative diagram

$$\begin{array}{ccccc}
H^{k-1}_{abs}(M \setminus U) & \xrightarrow{b} & H^k_{2,rel}(E_{NP}) & \xrightarrow{i} & H^k_{2,abs}(M \setminus E_P) \\
\downarrow & & \downarrow & & \downarrow \\
H^{k-1}_{abs}(M \setminus U) & \xrightarrow{\pi^*} & \bigoplus_i H^{k-1}(\mathbf{T}_i)^{\Gamma_i} & \xrightarrow{b} & H^k\,(M \setminus U, \ker \pi^*)
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{j^*} & H^k_{abs}(M \setminus U) & \xrightarrow{b} & H^{k+1}_{2,rel}(E_{NP}) \\
 & \downarrow & & \downarrow \\
\xrightarrow{j^*} & H^k_{abs}(M \setminus U) & \xrightarrow{\pi^*} & \bigoplus_i H^k(\mathbf{T}_i)^{\Gamma_i}
\end{array}$$

The first two and last two vertical arrows are isomorphisms, hence by the fifth arrow lemma, we have an isomorphism between $H^k_{2,abs}(M \setminus E_P)$ and $H^k(M \setminus U, \ker \pi^*)$. We have proved

Proposition 5.6. $H^k_{2,abs}(M \setminus E_P) \simeq H^k(M \setminus U, \ker \pi^*)$.

And we arrive to our main result.

Theorem 5.7. *Let (M^n, g) be a complete Riemannian manifold with flat ends. Let E_P be the union of the parabolic ends and E_{NP} the union of the remaining ends. Each connected component of ∂E_{NP} has a finite cover isometric to $\mathbf{S}^{\nu-1}(R) \times \mathbf{T}^{n-\nu}$ where $\nu > 2$ and $\mathbf{T}^{n-\nu}$ is a flat torus. We get an immersion $\pi : \bigcup_i \mathbf{T}^{n-\nu_i} \to M \setminus (E_P \cup E_{NP})$ and we note $i : \partial E_P \to M \setminus (E_P \cup E_{NP})$ the inclusion map. We define two subcomplexes of $C^\infty(\Lambda T^*(M \setminus (E_P \cup E_{NP})))$:*

$$\Omega\big(M \setminus (E_P \cup E_{NP}), \ker \pi^*\big) = \big\{\alpha \in C^\infty(\Lambda T^*(M \setminus (E_P \cup E_{NP}))) \text{ such that } \pi^*\alpha = 0\big\}\,,$$

and
$$\Omega\big(M \setminus (E_P \cup E_{NP}), \ker \pi^*, \partial E_P\big) = \big\{\alpha \in \Omega(M \setminus (E_P \cup E_{NP}), \ker \pi^*) \text{ such that } i^*\alpha = 0\big\}\,.$$

Let $H^k((M \setminus (E_P \cup E_{NP}), \ker \pi^*)$ and $H^k((M \setminus (E_P \cup E_{NP}), \ker \pi^*, \partial E_P)$ be the associated cohomology spaces. Then we have the isomorphism

$$H_2^k(M) \simeq \operatorname{Im}\big(H^k(M \setminus (E_P \cup E_{NP}), \ker \pi^*, \partial E_P) \to H^k(M \setminus (E_P \cup E_{NP}), \ker \pi^*)\big)\,.$$

5.5 A L^2 Chern–Gauss–Bonnet formula. We now investigate a Chern–Gauss–Bonnet formula for the L^2 Euler characteristic. We only consider manifolds with non-parabolic ends. In this case, the exact sequence (4.8) holds, hence we have

$$\chi_{L^2}(M) = \chi(M) + \chi_{L^2,rel}(E_{NP})\,,$$

where E_{NP} is the union of the ends of M, each of these ends E has a finite cover isometric to $\check{E} = (\mathbb{R}^\nu - B(R)) \times \mathbf{T}_E$, where $\mathbf{T}_E$ is a flat $(n-\nu)$ torus. And for a finite subgroup Γ_E of $O(\nu) \times \operatorname{Isom}(\mathbf{T}_E)$, we have $E = \check{E}/\Gamma_E$. Assume that M is oriented and even dimensional; then we can compute $\chi(M)$ with the Chern formula,

$$\chi(M) = \int_M \Omega^g + \sum_E \int_{\Sigma_E(R)} P(II)\,,$$

where the summation is with respect to all ends E of M, $\Sigma_E(R)$ is the hypersurface $(R\mathbf{S}^{\nu-1} \times \mathbf{T}_E)/\Gamma_E$ and $P(II)$ is a polynomial expression in the curvature and in the second fundamental form of $\Sigma_E(R)$. We can compute $\int_{\Sigma_E(R)} P(II)$. This integral is multiplicative with respect to finite cover hence we obtain

$$\int_{\Sigma_E(R)} P(II) = \frac{1}{\#\Gamma_E} \int_{R\mathbf{S}^{\nu-1} \times \mathbf{T}_E} P(II)\,.$$

Again with the Chern formula, we have

$$\int_{R\mathbf{S}^{\nu-1}\times\mathbf{T}_E} P(II)=\chi(R\mathbf{D}^{\nu}\times\mathbf{T}_E)=\chi(\mathbf{D}^{\nu})\chi(\mathbf{T}_E)\,.$$

Hence, we conclude that

$$\int_{\Sigma_E(R)} P(II)=\begin{cases} 0 & \text{if} \quad \nu<\dim M\,,\\ \frac{1}{\#\Gamma} & \text{if} \quad \nu=\dim M\,.\end{cases}$$

We now compute the L^2-Euler characteristic of an end E of M. We know that

$$H^k_{2,rel}(\check{E})=\frac{dr}{r^{\nu-1}}\wedge H^{k-1}(\mathbf{T}_E)\,.$$

Let G_E be the image of Γ_E in $\mathrm{Isom}(\mathbf{T}_E)$ then we also have the isomorphism

$$H^k_{2,rel}(E)\simeq H^{k-1}(\mathbf{T}_E)^{G_E},$$

and we get $\chi_{L^2}(E)=-\chi(\mathbf{T}_E,G_E)$, where $\chi(\mathbf{T}_E,G_E)$ is the G_E equivariant Euler characteristic of $\mathbf{T}_E$. We can give a more precise formula. We know that the cohomology of the torus $\mathbf{T}_E$ is the exterior algebra of $H^1(\mathbf{T}_E)$ or the exterior algebra of left invariant differential forms on $\mathbf{T}_E$. It is well known that

$$\dim H^k(\mathbf{T}_E)^{G_E}=\tfrac{1}{\#G_E}\sum_{\gamma\in G_E}\mathrm{Tr}_{\mathcal{H}^k(\mathbf{T}_E)}\gamma^*\,.$$

Hence

$$\chi(\mathbf{T}_E,G_E)=\sum_k(-1)^k\dim H^k(\mathbf{T}_E)^{G_E}=\tfrac{1}{\#G_E}\sum_{\gamma\in G_E}\sum_k(-1)^k\mathrm{Tr}_{\mathcal{H}^k(\mathbf{T}_E)}\gamma^*.$$

With the formula

$$\sum_k(-1)^k\mathrm{Tr}_{\mathcal{H}^k(\mathbf{T}_E)}\gamma^*=\det(Id_{\mathcal{H}^1(\mathbf{T}_E)}-\gamma^*)\,,$$

we obtain

$$\chi(\mathbf{T}_E,G_E)=\tfrac{1}{|G_E|}\sum_{\gamma\in G_E}\det(Id_{\mathcal{H}^1(\mathbf{T}_E)}-\gamma^*)\,.$$

This formula can also be obtained from the Lepschetz fixed point formula. And we have proved the following theorem:

Theorem 5.8. *If (M^n,g) is a complete oriented Riemannian manifold of even dimension whose curvature vanishes outside some compact, and if for each end E of M we have*

$$\lim_{r\to\infty}\frac{\mathrm{vol}\,E\cap B_x(r)}{r^2}=\infty\,,$$

then

$$\chi_{L^2}(M)=\int_M\Omega^g+\sum_{E\text{ end of }M}q(E)\,,$$

where $q(E)$ is defined as follows:

- *When* $\pi_1(E)$ *has no torsion then* $q(E) = 0$.
- *When* $\operatorname{rank}\pi_1(E) = 0$ *we have* $q(E) = 1/|\pi_1(E)| - 1$
- *When* $\operatorname{rank}\pi_1(E) > 0$ *then* $\pi_1(E)$ *acts isometrically on* $\mathbf{S}^{\nu-1} \times \mathbb{R}^{n-\nu}$, $n - \nu = \operatorname{rank}\pi_1(E) < n - 1$. *Hence* $\pi_1(E)$ *is a subgroup of* $\operatorname{Isom}(\mathbf{S}^{\nu-1} \times \mathbb{R}^{n-\nu}) = O(\nu) \times [\mathbb{R}^{n-\nu} \rtimes O(n-\nu)]$ *and let* G_E *be the image of* $\pi_1(E)$ *in* $O(n-\nu)$, *then we have*
$$q(E) = -\frac{1}{|G_E|} \sum_{\gamma \in G_E} \det(\operatorname{Id} - \gamma)\,.$$

When the manifold has a parabolic end, the exact sequence does not hold and we cannot give an explicit formula for the L^2-Euler characteristic. As a matter of fact, we cannot expect that the L^2-Euler characteristic is the sum of the integral of the Euler forms and of a contribution of ends. A counterexample is given in [C4]: If $M = \mathbb{R}^2 \# \mathbb{R}^2$ is two copies of $\mathbb{R}^2$ glued along a disk, this surface has two planar ends and no non-trivial L^2-harmonic forms. On $\mathbb{R}^2$, we have $\chi_{L^2}(\mathbb{R}^2) = 0 = \int_{\mathbb{R}^2} KdA/2\pi$. But on M, we have $\chi_{L^2}(M) = 0 \neq -2 = \int_M KdA/2\pi$. The surfaces M and $\mathbb{R}^2 \cup \mathbb{R}^2$ have the same ends, but the differences between the L^2 Euler characteristic and the integral of the Euler forms are not equal.

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