Analysis on singular Space, Oberwolfach, June 2010 :

Finding upper bound for the heat & Green kernel

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Introduction

(M^n, g) will be a complete Riemannian manifold.

Recall that in local coordinate

$$\Delta = -\frac{1}{\Theta} \sum_{i,j} \frac{\partial}{\partial x_i} \Theta g^{i,j} \frac{\partial}{\partial x_j}$$

where

$$\Theta = \sqrt{\det[g_{i,j}]}.$$

Introduction : the heat kernel

The operator $e^{-t\Delta}$ has a smooth Schwartz kernel, the heat kernel $h_t(x,y)$: for $f \in C_0^\infty(M)$, :

$$f_t(x) := \left(e^{-t\Delta} f\right)(x) = \int_M h_t(x,y) f(y) dy$$

solve the evolution equation :

$$\begin{cases} \frac{\partial}{\partial t} f_t + \Delta f_t = 0 & \text{on } (0,+\infty) \times M \\ f_0 = f \end{cases}$$

Introduction : the Green kernel

If moreover, there is some $x, y \in M$ such that

$$\int_1^{+\infty} h_t(x,y)dt < \infty$$

Then for all $x \neq y$, we can define

$$G(x,y) = \int_0^{+\infty} h_t(x,y)dt$$

the Green kernel. And if $f \in C_0^\infty(M)$ then

$$u(x) =: \int_M G(x,y)f(y)dy$$

solve the equation :

$$\Delta u = f$$

with the extra property that $f \geq 0 \Rightarrow u \geq 0$.

Motivations

A good understanding of the Green kernel $G(x, y)$

$$\Rightarrow$$

a good information on solution of the equation :

$$\Delta u = f$$

An example :
On a complete Riemannian $(M^{n>2}, g)$ with

$$\mathrm{Ricci}_g \geq 0$$

and Euclidean growth :

$$\forall r > 0, \forall x \in M \ : \ \mathrm{vol} B(x,r) \geq v\, r^n$$

then Li-Yau 's estimates$_{1986}$ implies that

$$G(x,y) \leq \frac{C(n)}{v\, d(x,y)^{n-2}}.$$

On a complete Riemannian $(M^{n>2}, g)$ with $\mathrm{Ricci}_g \geq 0$ and Euclidean growth, if for some $\alpha > 2$

$$|f(x)| \leq \frac{C}{d(x,o)^\alpha} \ ,$$

then the equation $\Delta u = f$ has a unique solution such that

$$|u(x)| \leq \frac{C'}{d(x,o)^{\alpha-2}} \ .$$

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Li and Yau obtained a upper bound on the gradient of the Green kernel :
If for some $\alpha > 1$

$$|f(x)| \leq \frac{C}{d(x,o)^\alpha}$$

then the equation $\Delta u = f$ has a solution such that

$$|du(x)| \leq \frac{C'}{d(x,o)^{\alpha-1}} \ .$$

These estimates have been used by Mok, Siu and Yau in order to obtain some geometric property of Kaelher manifold with non negative bisectional curvature.

These estimates have been used by Mok, Siu and Yau in order to obtain some geometric property of Kaelher manifold with non negative bisectional curvature.

Another motivation is that a good tool for doing analysis on non compact manifold should be able to give good informations on the heat/Green kernel.

Very precise estimate : harmonic analysis

In the Euclidean space $\mathbb{R}^n$:

$$\widehat{(e^{-t\Delta}f)}(\xi) = e^{-t|\xi|^2}\widehat{f}(\xi)$$

hence

$$h_t(x,y) = \frac{e^{-\frac{d(x,y)^2}{4t}}}{(4\pi t)^{n/2}}$$

and if $n > 2$:

$$G(x,y) = \frac{c(n)}{d(x,y)^{n-2}} \ .$$

Very precise estimate : harmonic analysis

The case of symmetric space $M = G/K$ with $\sec \leq 0$:
Sometimes, there are explicit expression (when the group G is complex):
for instance in the real hyperbolic space $\mathbb{H}^3 = SL_2(\mathbb{C})/SU(2)$:

$$h_t(x,y) = \frac{d(x,y)}{\sinh d(x,y)} \frac{e^{-\frac{d(x,y)^2}{4t}}}{(4\pi t)^{3/2}}$$

$$\text{and } G(x,y) = \frac{c(n)}{\sinh d(x,y)} \ .$$

The case of symmetric space $M = G/K$ with $\sec \leq 0$: The spherical Fourier transform can be used to obtained very precise description (Anker and Ji$_{1999}$ in some case, Anker and Ostellari$_{2004}$ announcement)

$$h_t(o,y) \approx \frac{e^{-\frac{d(o,y)^2}{4t} - \lambda_0 t - \rho(y)}}{t^{n/2}} P(t,y)$$

where we say that $f \approx g$ if both $\frac{f}{g}$ and $\frac{g}{f}$ are bounded.

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where we say that $f \approx g$ if both $\frac{f}{g}$ and $\frac{g}{f}$ are bounded.

- λ_0 is the bottom of the spectrum of the Laplacian Δ.
- $P(t,y)$ is a explicit polynomial in t and y.
- $\rho(y)$ is a function that measure the volume growth of K-orbit :

$$\mathrm{vol} KB(y,1) \approx e^{2\rho(y)}.$$

This method also gives similar estimates on the derivative of the heat/Green kernel and it also give some asymptotic expansion when

$$(t, y) \to \infty.$$

Using perturbation theory (Duhamel formula), on can extend this to space that are asymptotic to symmetric space, a good frame work is the introduction of a good ΨDO calculus.

Very precise estimate :ΨDO calculus

Example 1 : *Asymptotically Hyperbolic manifold* : i.e. the interior of a compact manifold $\overline{M}$ with boundary $\partial\overline{M}$ endowed with a metric

$$g = \frac{\bar{g}}{y^2}$$

where $\bar{g}$ is a smooth Riemannian metric on $\overline{M}$ and $y : \overline{M} \to [0, +\infty)$ a smooth defining function for $\partial\overline{M}$ such that $|dy|_{\bar{g}} = 1$ along $\partial\overline{M}$. For instance the real hyperbolic space

$$\mathbb{H}^n = \left(\mathbb{B}^n, \frac{4\,\mathrm{eucl}}{(1 - \|x\|^2)^2}\right).$$

Very precise estimate :ΨDO calculus, Asymptotically Hyperbolic manifold

In this setting, Mazzeo and Melrose$_{1988}$ have developed a ΨDO calculus (the 0-calculus) and obtained very precise description of the spectrum of the Laplacian and of the behaviour of the resolvent

$$(\Delta - z)^{-1}, \quad z \to \operatorname{Spec} \Delta.$$

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$$(\Delta - z)^{-1}, \quad z \to \operatorname{Spec} \Delta.$$

With $\lambda_0 = \inf \operatorname{Spec} \Delta$, I guess that one can get for any $t \geq 1$:

$$\begin{cases} h_t(x,y) \leq Ce^{-\lambda_0 t} & \text{if } \lambda_0 < (n-1)^2/4 \\ h_t(x,y) \leq C\frac{e^{-\lambda_0 t}}{t^{3/2}} & \text{if } \lambda_0 = (n-1)^2/4 \end{cases}$$

Very precise estimate :ΨDO calculus, Asymptotically conical manifold

Example 2 : *Asymptotically conical manifold* : outside a compact set the metric is asymptotic to a conical metric

$$\left((1,+\infty)\times N,(dr)^2+r^2h\right)$$

N : a compact manifold without boundary.
A ΨDO calculus (the scattering calculus) has been introduced by Melrose_{1994}, a recent extension of this calculus has been recently defined and used by Guillarmou and $\text{Hassell}_{2008-2009}$ in order to obtained a very good description of the resolvent

$$(\Delta+V-z)^{-1}\,,\quad z\to 0.$$

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$$(\Delta + V - z)^{-1}, \quad z \to 0.$$

Some important consequences are new results about the L^p boundness of the Riesz transform :

$$d(\Delta + V)^{-1/2}.$$

Very precise estimate :

The advantage of these tools :

- Very precise estimates are obtained
- The results are not limited to the scalar Laplacian (Laplacian on forms, spinors can be treated also)

Rough estimate

I will now concentrate on Euclidean type upper bound :

$$h_t(x,y) \leq \frac{C}{t^{n/2}}.$$

or

$$G(x,y) \leq \frac{C}{d(x,y)^{n-2}}.$$

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$$h_t(x,y) \le \frac{C}{t^{n/2}}.$$

or

$$G(x,y) \le \frac{C}{d(x,y)^{n-2}}.$$

Although, the same tools will lead to estimate of the type

$$h_t(x,y) \le \frac{C}{\mathrm{vol} B(x,\sqrt{t})}.$$

On manifold with $\mathrm{Ricci} \ge 0$, the above estimates are true according to Li and Yau's estimates.

Rough estimate

We will not speak on lower bound (i.e. we do not search for the optimal upper bound). For instance on the hyperbolic space $\mathbb{H}^3$:

$$h_t(x,x) = \frac{e^{-t}}{(4\pi t)^{3/2}}.$$

Rough estimate

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$$h_t(x,x) = \frac{e^{-t}}{(4\pi t)^{3/2}}.$$

But the Euclidean type upper bound :

$$h_t(x,y) \leq \frac{C}{t^{n/2}},$$

+ a Euclidean's type upper bound on the volume growth implies the lower bound

$$\frac{c}{t^{n/2}} \leq h_t(x,x).$$

Rough estimate : Gaussian upper bound

It is well known that such a Euclidean type upper bound implies Gaussian upper bound, the end point result is due to A. Sikora$_{1996}$:

Theorem

If we have

$$\forall t > 0, \forall x \in M, \quad h_t(x,x) \leq \frac{C}{t^{n/2}}.$$

then

$$\forall t > 0, \forall x, y \in M, \quad h_t(x,y) \leq C \left(1 + \frac{d(x,y)}{\sqrt{t}}\right)^{n-1} \frac{e^{-\frac{d(x,y)^2}{4t}}}{t^{n/2}}.$$

Hence the Euclidean type upper bound on the heat kernel (when $n > 2$) implies an upper bound on the Green kernel :

$$G(x,y) \leq \frac{C}{d(x,y)^{n-2}}.$$

Rough estimate : Gaussian upper bound

A very general result has been obtained by A. Grigor'yan$_{1997}$:

Theorem

Assume that for some $x, y \in M$ we have

$$\forall t > 0, h_t(x,x) \leq \frac{C}{t^{n/2}} \text{ and } h_t(y,y) \leq \frac{C}{t^{n/2}}$$

then for all $D > 4$, there is a constant C' such that

$$\forall t > 0, \quad h_t(x,y) \leq C' \frac{e^{-\frac{d(x,y)^2}{D\,t}}}{t^{n/2}}.$$

Euclidean's type estimate : : the Sobolev inequality

A very useful result has been shown by J. Nash$_{ii)\Rightarrow i),1958}$ and N. Varopoulos$_{i)\Rightarrow ii),1985}$:

Theorem

The following are equivalent
i) $\forall t > 0, \forall x, y \in M \quad h_t(x,y) \leq \frac{C}{t^{n/2}}$.
ii) (M,g) satisfies the Sobolev inequality :

$$\forall f \in C_0^\infty(M),\ \|f\|_{L^{\frac{2n}{n-2}}} \leq A\,\|df\|_{L^2}\,.$$

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This result implies that in order to obtain Euclidean type upper bound on the heat kernel, we only need to understand the geometry up to bilipschitz equivalence : indeed if h, g are two Riemannian metric such that

$$ch \leq g \leq Ch$$

then the Sobolev inequality hold for g if and only if it hold for h.

Euclidean's type estimate : : the Sobolev inequality

Moreover the validity on the Sobolev inequality only depend on the infinity : i.e. if for a compact set $K \subset M$ we have the Sobolev inequality outside K :

$$\forall f \in C_0^\infty(M \setminus K),\ \|f\|_{L^{\frac{2n}{n-2}}} \leq C \|df\|_{L^2},$$

then it hold on M :

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Applications

- If $M^n \subset \mathbb{R}$ is a submanifold with mean curvature $h \in L^n$ then M satisfies the Sobolev inequality (C_{1998}).
- If $\Omega \subset \overline{M}$ is a open set in a compact Riemannian manifold $(\overline{M}, \bar{g})$ endowed with a complete Riemannian metric $g = e^{2f}\bar{g}$ such that

$$\int_\Omega \mathrm{Scal}_+^{n/2} d\mathrm{vol}_g < \infty$$

then (Ω, g) satisfies the Sobolev inequality (C-Herzlich_{2006}).

Euclidean's type estimate :small times behaviour

A control on the local geometry implies a control on the small times behaviour of the heat kernel

Theorem

Assume that the ball $B(x,r) \subset M$ satisfies the Sobolev inequality :

$$\forall f \in C_0^\infty(B(x,r)),\ \|f\|_{L^{\frac{2n}{n-2}}} \leq A\,\|df\|_{L^2}\,.$$

Then

$$\forall t \in (0,\sqrt{r}), \quad h_t(x,x) \leq \frac{C(n)A^n}{t^{n/2}}.$$

That is the estimate is true as soon as not almost all the heat is not outside $B(x,r)$.

Euclidean's type estimate :small times behaviour

For instance if

$$\begin{cases} \mathrm{Ricci} \geq -(n-1)\kappa^2 g \\ v := \inf_{x\in M} \mathrm{vol} B(x,1) > 0 \end{cases}$$

then

$$\forall t \in (0,1), \forall x, y \in M, \quad h_t(x,y) \leq \frac{C(n)}{v t^{n/2}}.$$

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If moreover $\mathrm{Spec}\,\Delta \subset [\lambda_0, +\infty)$ with $\lambda_0 > 0$, then

$$\forall x, y \in M, \quad h_t(x,y) \leq \begin{cases} \frac{C(n)}{v t^{n/2}} & \text{if } t \in (0,1) \\ C e^{-\lambda_0 t} & \text{if } t \geq 1 \end{cases}$$

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so that

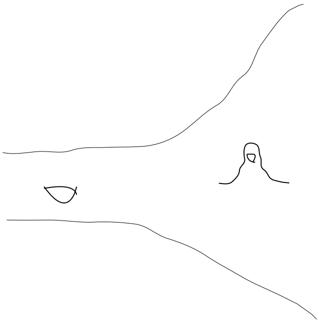
$$\forall t > 0, \forall x,y \in M, \; h_t(x,y) \leq \frac{C}{t^{n/2}}.$$

Euclidean's type estimate : discretisation

Such a manifold can be discretize : choose a discrete set $E = \{x_i\} \subset M$ such that

1. $\forall i \neq j, d(x_i, x_j) \geq 1$
2. $M = \cup_i B(x_i, 1)$

and say that $x_i \sim x_j \Leftrightarrow d(x_i, x_j) \leq 4.$ We obtain a graph.



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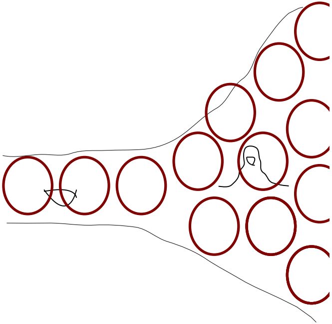

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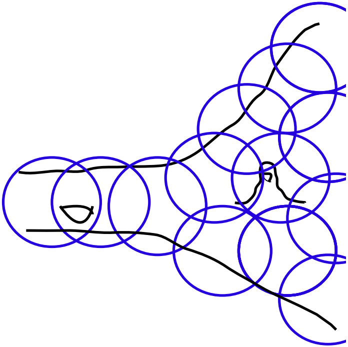

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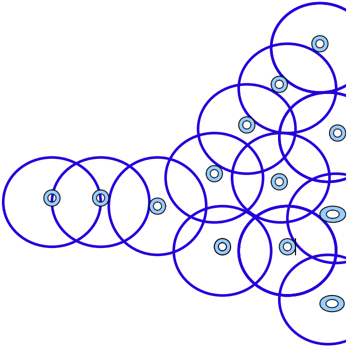

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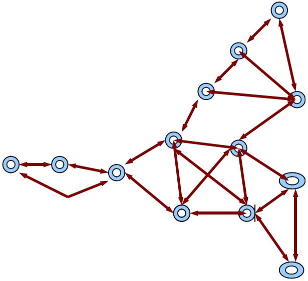

Euclidean's type estimate : discretisation

The result of Kanai_{1986} is that

Theorem

Assume that

$$\begin{cases} \mathrm{Ricci} \geq -(n-1)\kappa^2 g \\ v := \inf_{x\in M} \mathrm{vol} B(x,1) > 0 \end{cases}$$

The Euclidean's type upper bound :

$$\forall t > 0, \forall x, y \in M \ h_t(x,y) \leq \frac{C}{t^{n/2}}$$

is equivalent to the discrete Sobolev inequality : $\forall f \in \mathcal{L}^1_0(E)$ with finite support

$$\left(\sum_i |f(x_i)|^{\frac{2n}{n-2}}\right)^{1-\frac{2}{n}} \leq A \sum_{x_i \sim x_j} |f(x_i) - f(x_j)|^2.$$

Euclidean's type estimate : discretisation

Example (Coulhon and Saloff-Coste$_{1993}$): If $M \to \check{M}$ is a Riemannian covering of a compact Riemannian manifold $(\check{M}, g)$ such that for some $o \in M$ and all $r > 0$:

$$\mathrm{vol} B(o, r) \geq v r^n$$

then

$$\forall t > 0, \forall x, y \in M, \; h_t(x, y) \leq \frac{C}{t^{n/2}}$$

Euclidean's type estimate and the isoperimetric inequality :

The Sobolev inequality

$$\forall f \in C_0^\infty(M),\ \|f\|_{L^{\frac{2n}{n-2}}} \le C\,\|df\|_{L^2}\,.$$

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This L^1-Sobolev inequality is equivalent to the isoperimetric inequality :

$$\forall \Omega \subset M, \quad (\mathrm{vol}\Omega)^{\frac{n-1}{n}} \leq C\,\mathrm{vol}\partial\Omega.$$

Euclidean's type estimate and the isoperimetric inequality :

Theorem

If $\mathrm{Ricci} \geq -(n-1)\kappa^2 g$ and $v := \inf_{x\in M} \mathrm{vol} B(x,1) > 0$ then the Sobolev inequality

$$\forall f \in C_0^\infty(M),\ \|f\|_{L^{\frac{2n}{n-2}}} \leq C\,\|df\|_{L^2}\,.$$

implies the the isoperimetric inequality :

$$\forall \Omega \subset M,\ \text{such that } \mathrm{vol}\Omega \geq 1$$

$$(\mathrm{vol}\Omega)^{\frac{n-2}{n}} \leq C \mathrm{vol}\partial\Omega.$$

Euclidean's type estimate and isoperimetric inequality :

Theorem, (Grigor'yan, C. 1994)

The Euclidean's type upper bound :

$$\forall t > 0, \forall x, y \in M \;\; h_t(x,y) \le \frac{C}{t^{n/2}}$$

is equivalent to the Faber-Krahn isoperimetric inequality :

$$\forall \Omega \subset M, \quad C \,(\mathrm{vol}\Omega)^{-\frac{2}{n}} \le \lambda_0(\Omega).$$

Euclidean's type estimate : Grigor'yan and Saloff-Coste's result

The hypothesis are ($o \in M$ is a fixed point).

- for all $r > 0$: $Ar^n \leq \mathrm{vol} B(o, r) \leq Br^n$.

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- for all $r > 0$: $Ar^n \leq \mathrm{vol} B(o, r) \leq Br^n$.
- (M, g) satisfies the Relatively Connected Annuli conditions : for all $x, y \in M$ such that $d(x, y) = r \geq 1$ can be connected by a C^1 curve γ in $B(o, Cr) \setminus B(o, r/C)$.

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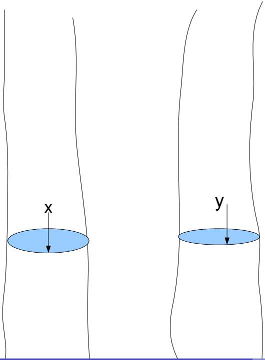

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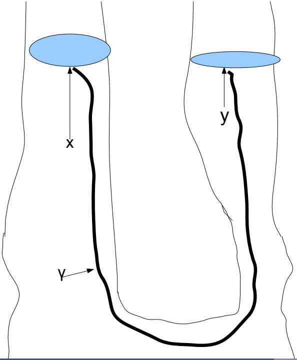

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- (M,g) satisfies the Relatively Connected Annuli conditions : for all $x, y \in M$ such that $d(x,y) = r \geq 1$ can be connected by a C^1 curve γ in $B(o, Cr) \setminus B(o, r/C)$.
- All "remote" balls satisfies the Sobolev and Poincaré inequalities : if $r < d(x,o)/2$ then

$$\forall f \in C_0^\infty(B(x,r)),\ \|f\|_{L^{\frac{2n}{n-2}}} \leq D \|df\|_{L^2}.$$

$$\forall f \in C^\infty(B(x,r)),\ \left\|f - \bar{f}\right\|_{L^2} \leq Er \|df\|_{L^2}.$$

(with $\bar{f} = \fint_{B(x,r)} f = \frac{1}{\mathrm{vol} B(x,r)} \int_{B(x,r)} f$)

Euclidean's type estimate : Grigor'yan and Saloff-Coste's result

Theorem (Grigor'yan and Saloff-Coste$_{2005}$, Minerbe$_{2009}$)

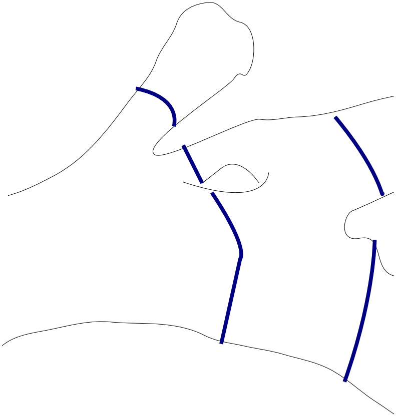
Under these assumptions on (M, g), we have Euclidean's type upper bound for the heat kernel :

$$\forall t > 0, \forall x, y \in M \;\; h_t(x, y) \leq \frac{C}{t^{n/2}}.$$

idea of the proof : used twice the above discretisation.

First step : discretize the annuli $A_k = B(o, 2^{k+a}) \setminus B(o, 2^{k-a})$ by balls on radius 2^k. One get a graph G_k.

- G_k is connected (by the Relatively Connected Annuli conditions condition).



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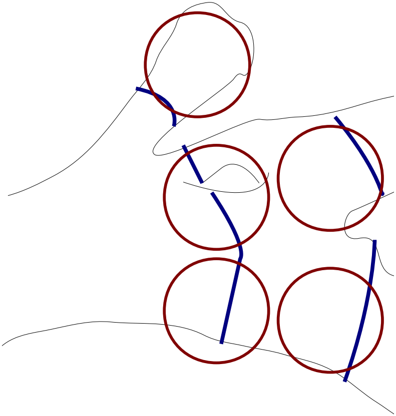

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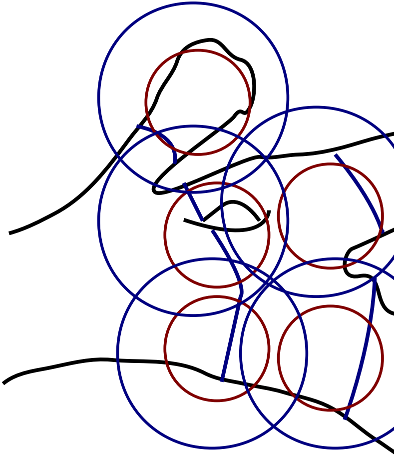

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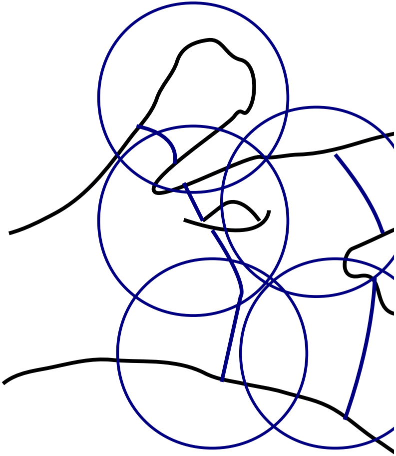

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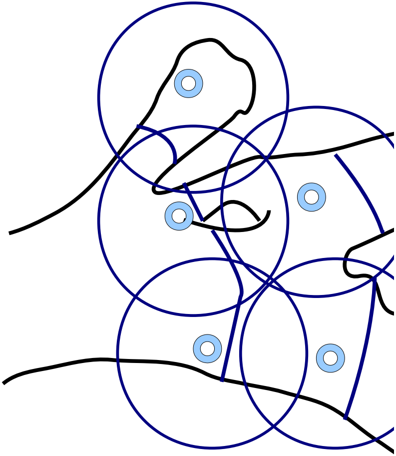

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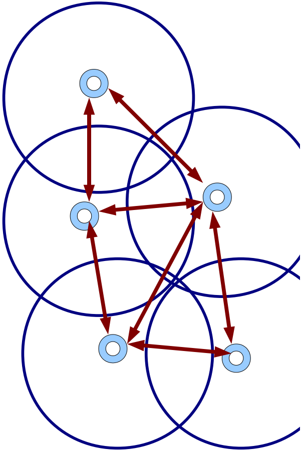

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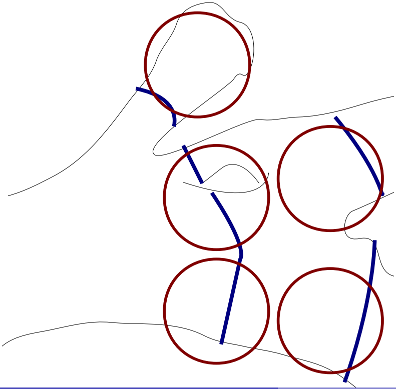
- $\# G_k \leq c_n BD^{n/2}$



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- G_k is connected (by the Relatively Connected Annuli conditions condition).
- $\# G_k \le c_n BD^{n/2}$ There is a uniform Poincaré inequality (with μ independent of k) : $\forall f \, : \, G_k \to \mathbb{R}$ with $m(f) = \frac{1}{\# G_k} \sum_{x \in G_k} f(x)$:

$$\sum_{x \in G_k} |f(x) - m(f)|^2 \le \mu \sum_{x \sim y} |f(x) - f(y)|^2.$$

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Deduce via discretisation that for $\tilde{A}_k = B(o, 2^{k+a+1}) \setminus B(o, 2^{k-a-1})$

$$\forall f \in C_0^\infty(\tilde{A}_k), \; \|f\|_{L^{\frac{2n}{n-2}}(A_k)} \le C \, \|df\|_{L^2(\tilde{A}_k)} .$$

$$\forall f \in C^\infty(\tilde{A}_k), \; \left\|f - \bar{f}\right\|_{L^2(A_k)} \le Cr \, \|df\|_{L^2(\tilde{A}_k)} .$$

idea of the proof, **second step** :
Let $A_0 = B(o, 2^{1+a})$ and $\tilde{A}_0 = B(o, 2^{2+a})$, and $v_k = \mathrm{vol} A_k$ we discretize M by $\mathbb{N}$: to any $f \in C_0^\infty(M)$ we associated the sequence (φ_k) defined by

$$\varphi_k = \fint_{A_k} f, k \geq 0$$

Because $v_k = \mathrm{vol} A_k \approx 2^{kn}$:

$$\sum_k v_k^{1-\frac{2}{n}} \varphi_k^2 \leq C \sum_k v_k^{1-\frac{2}{n}} |\varphi_k - \varphi_{k+1}|^2.$$

Then they used this discrete gap estimate to glue together the different Sobolev inequality and get the Sobolev inequality on M .

idea of the proof, **second step** :
Then

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq C\sum_k \|f\|^2_{L^{\frac{2n}{n-2}}(A_k)} \leq C\sum_k \|f-\varphi_k\|^2_{L^{\frac{2n}{n-2}}(A_k)} + C\sum_k v_k^{1-\frac{2}{n}}|\varphi_k|^2$$

Recall :

$$\sum_k v_k^{1-\frac{2}{n}}|\varphi_k|^2 \leq C\sum_k v_k^{1-\frac{2}{n}}|\varphi_k-\varphi_{k+1}|^2.$$

And moreover using Poincaré and Sobolev inequalities :

$$\|f-\varphi_k\|^2_{L^{\frac{2n}{n-2}}(A_k)} \leq C\,\|df\|^2_{L^2(\tilde{A}_k)}.$$

Hence

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq C\sum_k C\,\|df\|^2_{L^2(\tilde{A}_k)} + C\sum_k v_k^{1-\frac{2}{n}}|\varphi_k-\varphi_{k+1}|^2..$$

$$\leq C\,\|df\|^2_{L^2(M)} + C\sum_k v_k^{1-\frac{2}{n}}|\varphi_k-\varphi_{k+1}|^2.$$

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We have :

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq C\,\|df\|^2_{L^2(M)} + C\sum_k v_k^{1-\frac{2}{n}}|\varphi_k - \varphi_{k+1}|^2.$$

$$v_k^{1-\frac{2}{n}}|\varphi_k - \varphi_{k+1}|^2 = \frac{v_k^{1-\frac{2}{n}}}{(\mathrm{vol}A_k\ \mathrm{vol}A_{k+1})^2}\left|\int_{A_k\times A_{k+1}}(f(x)-f(y))\,dxdy\right|^2$$
$$\leq \frac{v_k^{1-\frac{2}{n}}}{\mathrm{vol}A_k\ \mathrm{vol}A_{k+1}}\int_{A_k\times A_{k+1}}|f(x)-f(y)|^2dxdy$$

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$$\begin{aligned} v_k^{1-\frac{2}{n}} |\varphi_k - \varphi_{k+1}|^2 &= \frac{v_k^{1-\frac{2}{n}}}{(\mathrm{vol} A_k \ \mathrm{vol} A_{k+1})^2} \left| \int_{A_k \times A_{k+1}} (f(x) - f(y))\, dxdy \right|^2 \\ &\leq \frac{v_k^{1-\frac{2}{n}}}{\mathrm{vol} A_k \ \mathrm{vol} A_{k+1}} \int_{A_k \times A_{k+1}} |f(x) - f(y)|^2 dxdy \\ &\leq \frac{v_k^{1-\frac{2}{n}}}{(\mathrm{vol} A_k \cup A_{k+1})^2} \int_{(A_k \cup A_{k+1})^2} (f(x) - f(y))^2\, dxdy \end{aligned}$$

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$$v_k^{1-\frac{2}{n}}|\varphi_k - \varphi_{k+1}|^2 \leq C\int_{\tilde{A}_k} |df(x)|^2 dx$$

$$\|f\|^2_{L^{\frac{2n}{n-2}}} \leq C\,\|df\|^2_{L^2(M)} + C\sum_k \int_{\tilde{A}_k} |df(x)|^2 dx$$

$$\leq C\,\|df\|^2_{L^2(M)}$$

Application to QALE manifold.

Theorem (Joyce$_{2000}$)

A QALE manifold $(M^{n>2}, g)$ satisfies the Sobolev inequality, hence :

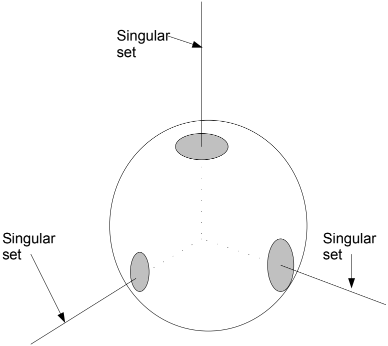
$$G(x,y) \leq \frac{C}{d(x,y)^{n-2}}.$$

What is a QALE manifold ?

When $\Gamma \subset SO(n)$ is a finite group, then

$$(\mathbb{R}^n \setminus \mathbb{B}^n)/\Gamma = U \cup \bigcup_i V_i$$

where U is the smooth part and V_i is "conical" neighbourhood of the singular part.



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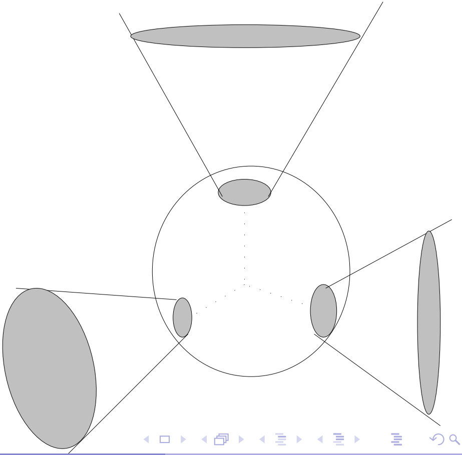


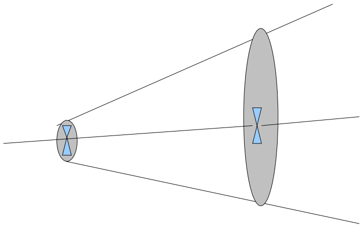
What is a QALE manifold ?

A Quasi Asymptotically Locally Euclidean manifold (M^n, g) asymptotic to $\mathbb{R}^n/\Gamma$ is a manifold such that outside a compact set K

$$M \setminus K = U \cup \bigcup_i \widehat{V}_i$$

$$\text{on } U,\ g \approx \text{eucl}$$

$V_i \subset (\mathbb{R}^{n_i}/A_i) \times \mathbb{R}^{m_i}$ where $A_i \subset SO(n_i)$ is a finite subgroup.



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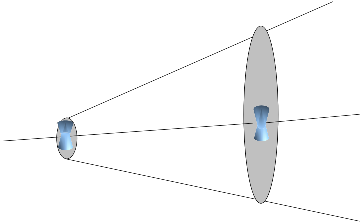
$V_i \subset (\mathbb{R}^{n_i}/A_i) \times \mathbb{R}^{m_i}$ where $A_i \subset SO(n_i)$ is a finite subgroup.
Then $\widehat{V}_i \subset Y_i \times \mathbb{R}^{m_i}$ where Y_i is a QALE manifold asymptotic to $\mathbb{R}^{n_i}/A_i$, and on $\widehat{V}_i$ the metric g is bilipschitz to the product metric.



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$\rho \approx$ distance to a fixed point.
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Theorem (C$_{2010}$)

Let (M^n, g) be a QALE manifold asymptotic to $\mathbb{R}^n/\Gamma$ and assume that the singular set has codimension > 2. If for some $\alpha > 0$

$$|f(x)| \leq \frac{C}{\rho^\alpha \sigma^2}$$

then the equation $\Delta u = f$ has a unique solution such that

$$|u(x)| \leq \frac{C' \log(\rho + 2)}{\rho^\alpha} .$$