An introduction to Fukaya categories

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Introduction.

This notes serve as a support for a Master2 class given in Universit de Nantes in 2014. Their purpose do not go beyond this and I have no ambition of giving any news insight into this subject. They are mostly inspired by the better written and more complete book by P. Seidel [Sei08]. The goal of the book is to give a definition of the Fukaya category of an exact symplectic manifold highlighting the technical details which one might overlook when first work with this concept. Mostly those details concerns the very definition of this $\mathcal{A}_\infty$-category in particular the definition of endomorphism spaces. We present here the approach of Seidel to achieve consistent transverse data which are necessary to define this endomorphism space and their composition.

The $\mathcal{A}_\infty$ equation underlying any $\mathcal{A}_\infty$-category or algebra is

$$\sum_{i=1}^{d}\sum_{j=0}^{d-i}(-1)^{\maltese_j}\mu^{d-i+1}\big(a_d, a_{d-1}, \ldots, a_{j+i+1}, \mu^i(a_{j+i}, \ldots, a_{j+1}), a_j, \ldots, a_1\big) = 0 \tag{1}$$

it reflects the combinatorics of Stasheff's associahedra. One realisation of these associahedra relevant in symplectic geometry is Deligne-Mumford's compactification of pointed Riemann disks with one positive punctures. In order to achieve consistent perturbation of $\overline{\partial}$-equation one needs to decorates this associahedra with perturbation data consistent with this combinatorics. This will occupy Chapters 1 and 3 of the present notes. In Chapter 2 we discuss why those perturbations are necessary to have a well defined category.

In Chapter 4 we discuss how to grade the operation μ^d consistently with the grading of Floer theory and in Chapter 5 we use all those data to define the Fukaya category of an exact symplectic manifolds and prove that its quasi-equivalence class is independent of all those data.

We then turns our focus on applications of this category to symplectic manifolds and explain some generation criterion for some of those category 6. This section could occupy a second full semester class and we will be here rather quick here. We focus on the most effective criterion which is known as Abouzaid's generation criterion and give an explicit implementation of it for surfaces.

Contents

Chapter 1

Deligne-Mumford compactification vs Stasheff associahedra.

Equation (1) is the algebraic counterpart of the combinatorial description of the faces of a particular family of polyhedra called Stasheff's associahedra. One of the most fantastic fact is that the combinatorics of this family coincide with the combinatorics of a particular compactification of Riemann disks with punctures on the boundary called Deligne-Mumford compactification (or stable compactification). A second fact not less fantastic than the previous one is that this particular compactification is relevant in the study of symplectic manifold thanks to the celebrated Gromov's Theorem [Gro85] which state that a family of pseudo-holomorphic disks converge (up to semi-stable breaking) to a pseudo-holomorphic map whose domain appears in the Deligne-Mumford compactification.

In this chapter we present first the combinatorics of Stasheff associahedra in terms of default of associativity of operations (Section 1.1) then we introduce the space of conformal disks (Section 1.2) and describe its stable compactification.

1.1 Stasheff associahedra.

Stasheff associahedra appears in [Sta63] as the polytopes which encodes the non-associativity of multiplication in H-space. It is a family of $d-2$-dimensional polytopes $\mathcal{K}^d$ whose vertices are in bijection with planar rooted tree with $d+1$ leaves whose interior vertices have all valence 3 (such a tree is called a planar rooted binary tree). Two vertices are connected with one edge if by collapsing an interior edge of each of the two trees we get the same rooted tree (see Figure 1.1).

Remark 1.1.1. A leave of a tree is a vertex of valence 1. Our convention is that in a rooted tree, amongst the leave, there is a marked one called the *root*. To go from the definition of rooted tree where any of the vertex can be a root we just need to add an extra edge to the root.

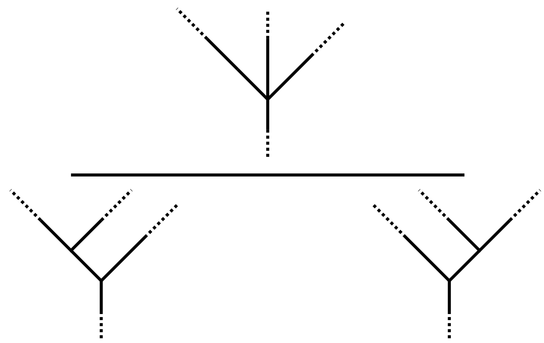

Figure 1.1: An edge of an associahedron.

Definition 1.1.2. *A planar rooted tree is called stable if all its interior vertices have valence at least* 3*. Else it is called semi-stable.*

Note that each edge (i.e. face of codimension $d-1$) of $\mathcal{K}^d$ can be given a label with a stable rooted tree where exactly one of the interior vertices has valence 4 (the common tree obtained by collapsing an edge). Another characterisation of edges is to say that they are in bijection with rooted tree with $d+1$ leaves and $d-2$ interior vertices. Similarly vertices (or faces of codimension $d-2$) correspond to those with $d-1$ interior vertices.

More generally, faces of $\mathcal{K}^d$ of codimension k are in bijection with stable rooted tree with $d+1$ leaves with $k+1$ (or similarly k interior edges) interior vertices they are part of the same faces of codimension $k-1$ if by collapsing one interior edge of each we get the same tree (see Figure 1.2). In particular there is only one face of codimension 0, hence of dimension $d-2$ and it correspond to the tree T_d with on interior vertex and $d+1$ leaves.

Exercise 1.1.1. Let T be the tree corresponding to a codimension 1 face of $\mathcal{K}^d$. Show that as a polytopes it is isomorphic to $\mathcal{K}^{d_1} \times \mathcal{K}^{d_2}$ for some $d_1 + d_2 = d + 1$ (hint: show that all trees corresponding to this face can be cut in two pieces in a natural way).

Remark 1.1.3. We have the following convention for planar rooted tree.

The set of interior vertices is denoted $Ve(T)$, the set of interior edges is denoted $Ed^{ind}(T)$. For a vertex v we denote by $|v|$ its valence.

The root is the 0th leave. All other leaves are numbered counter-clockwise. Each edge going to a leave will have infinite length. The one going to the root will coincide with $(-\infty, -M) \times \{0\}$ and the one correspond to i-th leave for $i > 0$ coincide with $(M, +\infty) \times \{i\}$. Interior edges coincides with this convention with edge of finite length. Each edge (interior or not) as a canonical orientation in by the following convention: the root is at the negative end of the edge it belongs to. At any interior vertices there is only on incoming edge (all the other one being outgoing).

At each interior vertex of T we order the set of edges $e_0, \ldots, e_{|v|}$ counter-clockwise starting at the incoming edge.

There are several description of these polytopes as convex polytopes in Euclidean spaces, we describe here the one by Loday in [Lod04]. It relies on the following order of interior vertices of binary tree. There is a unique total order of the vertices $v \in Ve(T)$ satisfying the following: for all v we denote $e_{v,l}$ the edge going out of v on the left and $e_{v,r}$ the one on the right, we denote by v_g and v_l the other end of this edges (if they are still interior vertices). The order is characterise by $v_l < v < v_g$. We list the interior edges of T in order $v_1 \ldots v_{d-1}$. For $v_i \in Ve(T)$ where T is a vertex of $\mathcal{K}^d$ we define x_i to be the products of its number of right descendants times the number of left descendant. And we define x_T to be the vector $(x_1, \ldots, x_{d-1})$ of $\mathbb{R}^{d-1}$.

Exercise 1.1.2. 1. Show how the coordinates of x_T changes under the move of Figure 1.1.

2. Show that the vector x_T belongs to the hyperplane $H_d = \sum x_i = d(d-1)/2 = \binom{d}{2}$.

Theorem 1.1.4. *[Lod04] The convex envelope of the $\{x_T, T \in Ve(\mathcal{K}^d)\}$ in H_d is $\mathcal{K}^d$.*

1.2 Pointed disks

A *d+1 pointed disk* $\overline{D}$ is a Riemann disk with $d+1$ points marked on the boundary listed in cyclic order $y_0 \cdots y_d$. We denote by D the disk $\overline{D}/\{y_0 \ldots, y_1\}$. The first point y_0 is called a *negative end* of D and all the other are called the *positive ends* of D. A choice of conformal neighbourhoods $\overline{\varepsilon_i}$ of y_i in $\overline{D}$ together with conformal identifications $\overline{\varepsilon_0} \setminus \{y_0\} = \varepsilon_0 \simeq (-\infty, 0) \times [0, 1]$ and $\overline{\varepsilon_i} \setminus \{y_i\} = \varepsilon_i \simeq (0, +\infty) \times [0, 1]$ for $i = 1 \ldots d$ (where infinities correspond to the marked points y_i) such that $\varepsilon_i \cap \varepsilon_j = \emptyset$ is called a *choice of strip like ends* for D.

We denote the moduli space of such disks by $\mathcal{R}^{d+1}$. One way to construct such space is the following: $\mathcal{R}^{d+1} = \{(y_0, y_1 \ldots, y_d\})|y_i \in S^1$ and $y_i \in (y_{i-1}, y_{i+1})\}/G$. With the convention that $y_{-1} = y_d$ and $y_0 = y_{d+1}$. Here G is the group of Möbius transformations $z \to \frac{az+b}{\overline{b}z+\overline{a}}$ with $|a|^2 - |b|^2 = 1$. For $d \geq 3$ in each class there is a unique representative of the form $(1, i, -1, y_3 \ldots, y_d)$ where $y_j = a_j + ib_j$ with $b_j < 0$, we use this to embed $\mathcal{R}^{d+1}$ into $\mathbb{R}^{d-2}$ via the map $\Sigma \to (a_3, \ldots, a_{d+1})$. This gives $\mathcal{R}^{d+1}$ a structure of an open contractible manifold.

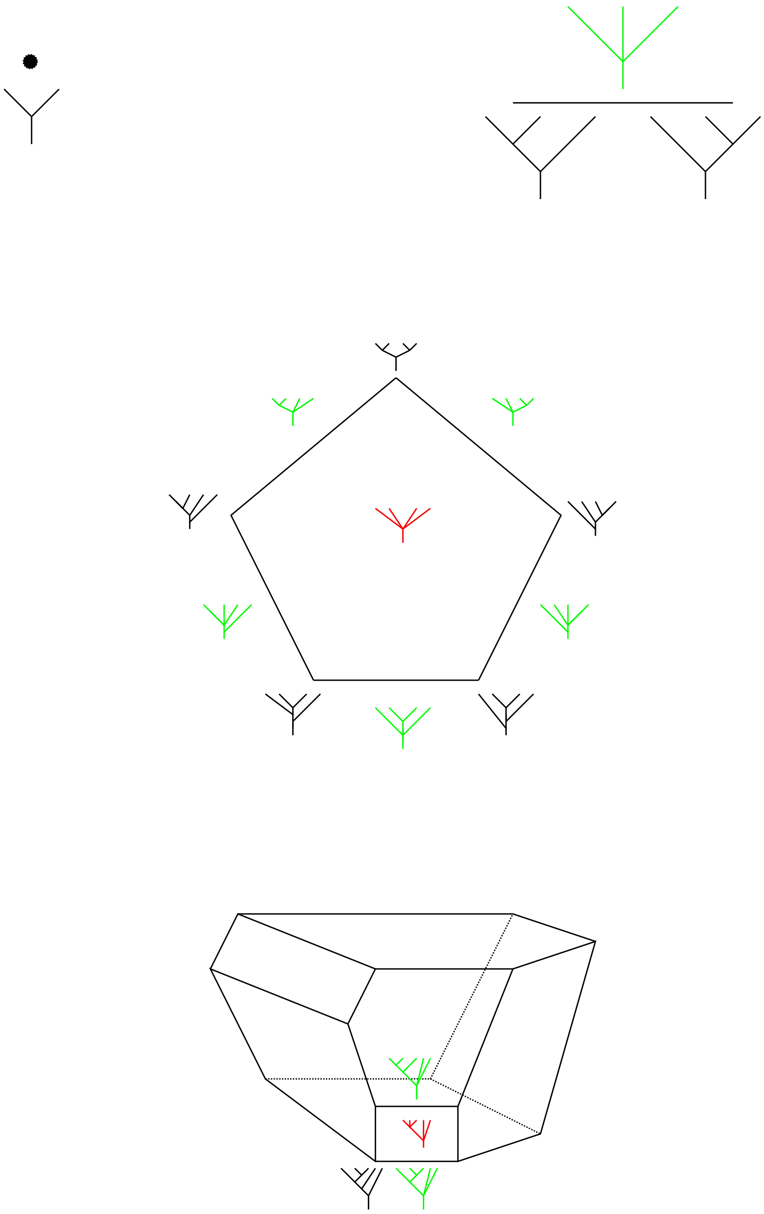

Figure 1.2: The first associahedra

The goal of this section is to show that a natural compactification of $\mathcal{R}^{d+1}$ into a manifold with corners coincides with the Stasheff associahedra $\mathcal{K}^d$.

1.3 Universal Riemann curve

Let $\mathcal{S}^{d+1} = \{(y_0, y_1 \ldots, y_d\})|y_i \in S^1$ and $y_i \in (y_{i-1}, y_{i+1})\} \times D^2/G$ the *universal Riemann curve.* It has a projection $\pi : \mathcal{S}^{d+1} \to \mathcal{R}^{d+1}$ with fibre isomorphic to D^2. For $r \in \mathcal{R}^{d+1}$ we denote by $\overline{\Sigma_r}$ the pre-image $\pi^{-1}(r)$ and by Σ_r the corresponding disk with marked point removed. It is a consequence of uniformisation theorem that any family of $d+1$-pointed Riemann disks is pulled back from $\mathcal{S}^d$.

By construction the projection π_1 comes with $d+1$ sections $s_0 \ldots s_1$ defined by $s_0([(y_0, \ldots, y_d)] = [(y_0, \ldots, y_d), y_i]$. A *universal choice of strip-like ends* for $\mathcal{S}^d$ are neighbourhood $\overline{\varepsilon_i} : \mathcal{R}^d \times (0, +\infty) \times [0, 1] \to \mathcal{S}^d$ of s_i such that for each $r \in \mathcal{R}^{d+1}$ $\varepsilon_i(\{r\} \times (0, +\infty) \times [0, 1])$ is a choice of strip like ends for Σ_r (we let the reader make the appropriate change for the negative end y_0).

Theorem 1.3.1. *Universal choices of strip-like end exists.*

Proof. First we begin the proof with an exercise:

Exercise 1.3.1. Prove that given a curve Σ the choice of strip-like end ε_i is contractible.

Now let η_i be tubular neighbourhood this sections. From the previous exercise the fibration π restricted to $\eta_i \setminus s_i$ is trivial, a trivialisation gives the choice of strip-like ends. ☐

1.4 Deligne-Mumford compactification.

We now describe the Deligne-Mumford compactification of $\mathcal{R}^{d+1}$ and prove that its natural structure of manifold with corners is diffeomorphic to the one of Stasheff associahedra $\mathcal{K}^d$.

Given a stable rooted tree T we denote by $\mathcal{R}^T$ the product $\Pi_{v_i} \mathcal{R}^{|v_i|}$ over its interior edges.

The Deligne-Mumford compactification of $\mathcal{R}^{d+1}$ as a set is simply given by $\overline{\mathcal{R}^{d+1}} \sqcup_T \mathcal{R}^T$ over all stable rooted trees with $d+1$ leaves. Before giving this space a structure of manifold with corner note that the original $\mathcal{R}^d$ correspond to the tree with only one interior vertices which labels the interior face of the Stasheff polyhedra $\mathcal{K}^d$ and codimension 1 faces corresponds to $\mathcal{R}^{|v_1|} \times \mathcal{R}^{|v_2|}$ where $\{v_1, v_2\} = Ve(T)$ ($|v_1| + |v_2| = d + 3$) and thus satisfy the same recursive structure as $\mathcal{K}^d$ as shown in Exercise 1.1.1.

Recall that each faces of $\mathcal{K}^d$ is indexed by a tree T. Let T' be tree which is obtain from T be collapsing a set of interior edges $e_1 \cdots e_k$. In terms of Stasheff's polyhedron the boundary of the face corresponding to T' contains the one corresponding to T. In order to describe the structure of manifold with corner we need to describe partial gluing of curves according to metric data on interior edges of T'.

First we assume that we have chosen a universal choice of strip like end for each of $\mathcal{S}^{|v|}$ for $v \in Ve(T)$. We denote by l_e the length of the edge e. Let $\underline{r} \in \mathcal{R}^{T'}$ i.e. $\underline{r} = \{r_v\}_{v \in Ve(T)}$ and denote by Σ_{r_v} the corresponding punctured curves. An edge e_j connects two of those (say r and r'). We denote by $\widehat{\Sigma_r}$ the Riemann disk $\Sigma_r \setminus \varepsilon_k\big((l_e, +\infty) \times [0,1]\big)$ where ε_k is the strip-like end corresponding to the marked pointed associated to the edge e_j (recall that the set of marked point and the one of edges are given an order). Similarly we denote by $\widehat{\Sigma_{r'}}$ the curves $\Sigma_{r'} \setminus \varepsilon'_0\big((-\infty, -l_e) \times [0,1]\big)$. We obtain a new pointed disks $\widehat{\Sigma_{e_i}} = \widehat{\Sigma_r} \sqcup \widehat{\Sigma_{r'}}/\sim$ where $\varepsilon_k(s,t) = \varepsilon'_0(s - l_e, t)$ for $(s,t) \in [0, l_e] \times [0,1]$. Applying this procedure for all e_j we get a vector of curves $\{\Sigma'_v\}_{v \in Ve(T')}$ taking each equivalence class of its coordinates gives an element of $\mathcal{R}^{T'}$.

The procedure is detailed for two particular trees on Figure 1.3.

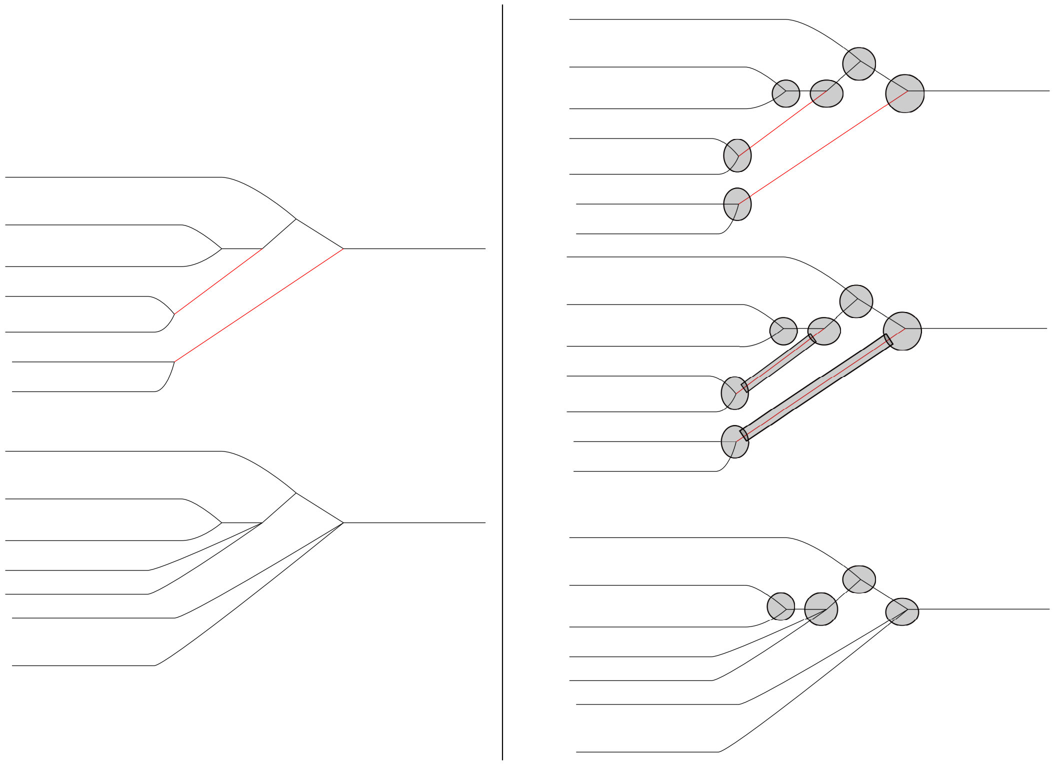

Figure 1.3: An illustration of the map $\gamma^{T,T'}$.

Thus this procedure defines a (smooth) map

$$\gamma^{T,T'} : \mathcal{R}^T \times (-1,0)^k \to \mathcal{R}^{T'} \tag{1.1}$$

where we use $\rho_e = -e^{-\pi l_e}$ as new gluing parameter. This allows us to define a topology on $\overline{\mathcal{R}^{d+1}}$ by setting maps to be continuous when pre-composed with the γ^{T,T_d}.

Theorem 1.4.1. *The Deligne-Mumford space $\overline{R^{d+1}}$ is a manifold with corner where the boundary charts are given by γ^{T,T_d} where T_d is the $d+1$ stable rooted tree with only one interior vertices. We extend γ^{T,T_d} to $\mathcal{R}^T \times (-1,0]^k$ by*

$$\gamma^{T,T_d}(\Sigma, \rho_1, \ldots, 0, \rho_2, 0, \ldots, 0, \rho_i, 0, \ldots, 0, \rho_{i+1} \ldots, 0, \ldots, \rho_k) := \gamma^{T,T'}(\Sigma, \rho_1, \ldots, \rho_k)$$

where T' is obtained from T by collapsing only edges with non zero gluing parameter.

This manifold with corner is diffeomorphic to the polyhedron $\mathcal{K}^d$.

Before giving a proof of this we begin with an exercise.

Exercise 1.4.1. Show that the topology on $\overline{\mathcal{R}^{d+1}}$ does not depend on the choice of strip-like ends.

Proof. The proof decomposes as a sequence of exercises.

Exercise 1.4.2.
1. Show that the maps $\gamma^{T,T'}$ are smooth and prove that when $\underline{\rho} << 1$ the $d(\gamma_{(\underline{r},\underline{\rho})})$ is invertible.
2. Deduce that for small enough parameters $\gamma^{T,T'}$ are open embeddings.
3. Show that with the induced topology $\overline{\mathcal{R}^{d+1}}$ is Hausdorff and compact.
4. * Find an embedding of $\overline{\mathcal{R}^{d+1}}$ into the convex realisation of $\mathcal{K}^d$.

□

1.5 Consistent choice of strip-like ends.

Now given a universal of choice of strip-like ends for each of the $\mathcal{S}^d$ a curve Σ_r for $r \in \mathcal{R}^d$ in the image of a gluing map γ^{T,T_d} admits two sets of strip-like ends: the one given by the universal choice on $\mathcal{S}^d$ and the second given by the universal choice on $\mathcal{S}^{|v|}$ for all vertices v of T and the gluing map (1.1). We say that the universal choices of strip-like ends for each $\mathcal{R}^d$ is *consistent* if for all d there is a neighbourhood $\overline{\mathcal{U}}$ of $\partial\overline{\mathcal{R}^d}$ in $\overline{\mathcal{R}^d}$ on which on those sets of strip-like ends coincides.

Gluing parameters $\underline{\rho}$ for which $\gamma^{T,T_d}(\mathcal{R}^T, \underline{\rho})$ is inside $\overline{\mathcal{U}}$ are said *sufficiently small.*

We now state the following theorem.

Theorem 1.5.1. *Consistent choice of strip-like ends exists.*

Proof. We proceed by induction. Assume that we have universal choices of strip-like ends for each $\mathcal{R}^{d'+1}$ for $d' < d$ such that there exists neighbourhood $\overline{\mathcal{U}^{d'+1}}$ of $\partial\overline{\mathcal{R}^{d'+1}}$ satisfying the consistency condition. We now want to build a universal choice of strip-like ends on $\mathcal{R}^d$ together with a neighbourhood $\overline{\mathcal{U}^{d+1}}$ satisfying the consistency condition.

For $\epsilon > 0$ we denote by $\overline{\mathcal{U}^T_\epsilon}$ the image of $\mathcal{R}^T \times [0, \epsilon)^{|T|}$. We choose ϵ_0 sufficiently small such that

- Each of the $\overline{\mathcal{U}^T_{\epsilon_0}}$ are diffeomorphic images of $\mathcal{R}^T \times [0, \epsilon_0)^{|T|}$
- If T_1 and T_2 have no faces in common (of any codimension) then $\overline{\mathcal{U}^{T_1}_{\epsilon_0}} \cap \overline{\mathcal{U}^{T_2}_{\epsilon_0}} = \emptyset$.
- $\overline{\mathcal{U}^{T_1}_{\epsilon_0}} \cap \overline{\mathcal{U}^{T_2}_{\epsilon_0}} = \overline{\mathcal{U}^{T'}_{\epsilon_0}}$ where T' is the tree which can collapse to both T_1 and T_2.

That such ε_0 exists follows from Exercise 1.4.2.

Let $r \in \mathcal{U}^{T_1}_{\epsilon_0} \cap \mathcal{U}^{T_2}_{\epsilon_0}$. T_1 and T_2 a priori determine two different sets of strip-like ends for Σ_r, we will show that under our induction hypothesis those two sets coincides. First note that since $\mathcal{U}^{T_1}_{\epsilon_0} \cap \mathcal{U}^{T_2}_{\epsilon_0} \neq \emptyset$ this implies that there is a tree T' which is in a face of both ∂T_1 and ∂T_2in other word there are edges $l^1_1 \dots l^1_{k_1}$ and $l^2_1 \dots l^2_{k_2}$ of T' such that the collapse of l^j_i gives T_j.

Let ρ^j_e for e interior edges of T_j and $r^v_1 \in \mathcal{R}^{|v|}$ for v vertex of T_j such that $r = \gamma^{T_j,T_d}(\underline{r^j}, \underline{\rho^j})$.

For gluing parameter $\underline{\rho}$ for γ^{T',T_d} we denote by $\underline{\rho^j}$ the corresponding parameter for γ^{T',T_j} given by the obvious projection. The remaining parameters are denoted $\underline{\rho}^{j,c}$.

From the definition of the gluing maps γ it is clear that $\gamma^{T',T_d}(\underline{r}, \underline{\rho}) = \gamma^{T_j,T_d}(\gamma^{T',T_j}(\underline{r}, \underline{\rho}^j), \underline{\rho}^{j,c})$. And since we restricted the gluing parameters such that each gluing map are open embedding, it follows from the fact that $r = \gamma^{T_1,T_d}(\underline{r^1}, \underline{\rho}^1) = \gamma^{T_2,T_d}(\underline{r^2}, \underline{\rho}^2)$ that there exist $\underline{r}$ and $\underline{\rho}$ such that $r = \gamma^{T',T_d}(\underline{r}, \underline{\rho})$. Since $\gamma^{T',T_j}(\underline{r}, \underline{\rho}^j) = (\underline{r^j})$ our inductive hypothesis implies that all strip-like ends of component of $\underline{\Sigma_{r^j}}$ are induced by the component of $\underline{\Sigma_r}$. This implies that the strip-like ends induced by $\underline{\Sigma_{r^1}}$ on Σ_r are the same as the one induced by $\underline{\Sigma_{r^2}}$ which are the one induced by $\underline{\Sigma_r}$.

All in all, this implies that our hypothesis determines strip-like ends on $\cup\mathcal{U}^T_{\epsilon_0}$. It is now easy to extend this choice of strip-like end on the whole $\mathcal{R}^d$ similarly to the construction of Exercise 1.3.1.

□

From now on we assume that the associahedra $\mathcal{R}^{d+1}$ are equipped with a consistent choice of strip-like ends.

1.6 Lagrangian labels.

A *label* for a stable (or semi-stable) tree by a discrete label set E is simply a continuous function form $\mathbb{R}^2 \setminus T$ to E (i.e. to each connected component of $\mathbb{R}^2 \setminus T$ we assign an element of E).

Exercise 1.6.1. Show that if T belongs to a faces of T' then any label for T extends naturally to a label of T'.

Using the correspondence between $\mathcal{K}^d$ and $\mathcal{R}^{d+1}$ labels has the following meaning. To any interior vertex v of T corresponds a pointed disks D_v. The connected component of ∂D correspond to quadrant of v (a "local" connected component of $\mathbb{R}^2 \setminus T$). Hence a label assign a label to each connected component of D_v. The condition that the labels are the same on global connected component implies that the assignment is consistent with the gluing operation $\gamma^{T,T'}$ (similarly to Exercise 1.6.1).

In the next chapter, the label set will be the set of Lagrangian sub-manifolds of a symplectic manifold and we will use them as boundary condition for holomorphic maps.

Chapter 2

Naive tentative to define Fukaya category.

This chapter serves as a second introduction to the class. We want, using very elementary considerations, to emphasise why consistent choices of perturbation data are necessary to have a well defined Fukaya category. We allows ourselves to be sloppy and won't highlight any of the technical details we will address in the subsequent chapters. The precise statement on transversality and compactness will appears later.

At the end of the chapter there is an exercise which aims to emphasise the differences between higher order operations in the Donaldson category and higher order operations in the Fukaya category.

From now on we assume that $(M, d\theta)$ is the completion of a Liouville manifolds and that Lagrangian labels are taken in $\mathcal{L}ag(M)$ the set of exact Lagrangians which are conical outside a compact set. We also pick a compatible almost complex structure J on M.

2.1 Holomorphic curves.

Let $\underline{L}$ be Lagrangian labels for $\mathcal{R}^{d+1}$ and let $a_i \in L_i \pitchfork L_{i+1}$ (with the convention that L_{d+1} is L_0). Given $r \in \mathcal{R}^{d+1}$, we decompose $\partial(\Sigma_r) = \sqcup_{i=1}^{d}$ according to its connected component ordered counter-clockwise. We denote by $\mathcal{M}^r_{\underline{L}}(a_0; a_1, \dots, a_d))$ the set of maps $u : \Sigma_r \to M$ such that

- $du \circ j_r = J \circ du$.
- $u(\partial_i) \subset L_i$.
- $\lim_{s\to-\infty}(u(\varepsilon_0(s,t)) = a_0$.
- $\lim_{s\to\infty}(u(\varepsilon_i(s,t)) = a_i$.

We denote by $\mathcal{M}_{\underline{L}}(a_0; a_1, \ldots, a_d) := \cup_{r\in\mathcal{R}^{d+1}}\mathcal{M}^r_{\underline{L}}(a_0; a_1, \ldots, a_d)$. By definition the C^k_{loc}-topology on $\mathcal{M}$ is induced by the C^k-topology on $\mathcal{M}^r$ and the requirement that the projection to $\mathcal{R}^{d+1}$. It follows from Gromov's compactness that a 1-parameter family of elements u_t of $\mathcal{M}_{\underline{L}}(a_0; a_1, \ldots, a_d)$ can be compactified into manifolds with boundary where the boundary points are of two types:

- Stable-breaking: the domain r_t of u_t converges to $r_0 \in \partial\overline{R^{d+1}}$ and the maps u_t converges to a pair of holomorphic curves in $\mathcal{M}_{\underline{L}_{T_1}}(a_0; a_1, \ldots, a_{i-1}, a'_i, a_{i+j}, \ldots a_d)\times\mathcal{M}_{\underline{L}_{T_1}}(a'_i; a_{i+1}, \ldots, a_{i+j})$
- Semi-stable breaking: the domain r_t converges to $r_0 \in \mathcal{R}^d$ and the maps u_t converges to a pair of maps in either $\mathcal{M}_{\underline{L}}(a_0; a_1, \ldots, a'_i, \ldots, a_d) \times \mathcal{M}^Z_{L_i,L_{i+1}}(a'_i, a_i)$ or in $\mathcal{M}^Z_{L_d,L_0}(a_0, a'_0) \times \mathcal{M}_{\underline{L}}(a'_0; a_1, \ldots, a_d)$.

2.2 Gromov compactness in action.

Here we explicitly describe an example of the breaking predicted by Gromov's compactness. The Liouville manifold we consider is $\mathbb{C}$ with its standard symplectic structure. As any almost-complex structure uniformise to either $(\mathbb{C}, i)$ or (D^2, i) we will assume that the almost complex structure is i (all consideration in this section will not change whether or not we are actually in $\mathbb{C}$ or D^2). We recall the (piecewise smooth) Riemann mapping Theorem.

Theorem 2.2.1. *(See [Rud87, Theorem 14.19] or [Con95, Theorem 5.6]) Let Ω be simply connected domain of $\mathbb{C}$ such there is a piecewise smooth parametrisation of $\partial\overline{\Omega}$ by S^1. Then there exist a map $u : D^2 \to \overline{\Omega}$ such that $u|_{int(D^2)}$ is a biholomorphism.*

Now consider the four Lagrangians of Figure 2.1. We draw some domain Ω_t (the frontier lies on the L_i's but we thickened it to make its contour clearer). For each t one can applies Riemann mapping theorems and find a map u_t belonging to $\mathcal{M}_{(L_1,L_2,L_3,L_4)}(a_1; a_3, a_4, a_6)$ (note that the actual domain $r \in \mathcal{R}^{d+1}$ doest depend on t). Letting t goes to $\pm\infty$ leads to two breakings. One in $\mathcal{M}_{(L_1,L_3,L_4)}(a_1; a_5, a_6) \times \mathcal{M}_{(L_1,L_2,L_3)}(a_5; a_3, a_4)$ (a stable breaking) and one in $\mathcal{M}_{(L_1,L_2,L_1,L_3,L_4)}(a_1; a_2, a_3, a_5, a_6) \times \mathcal{M}^Z_{(L_1,L_2)}(a_2; a_3)$ (a semi-stable breaking).

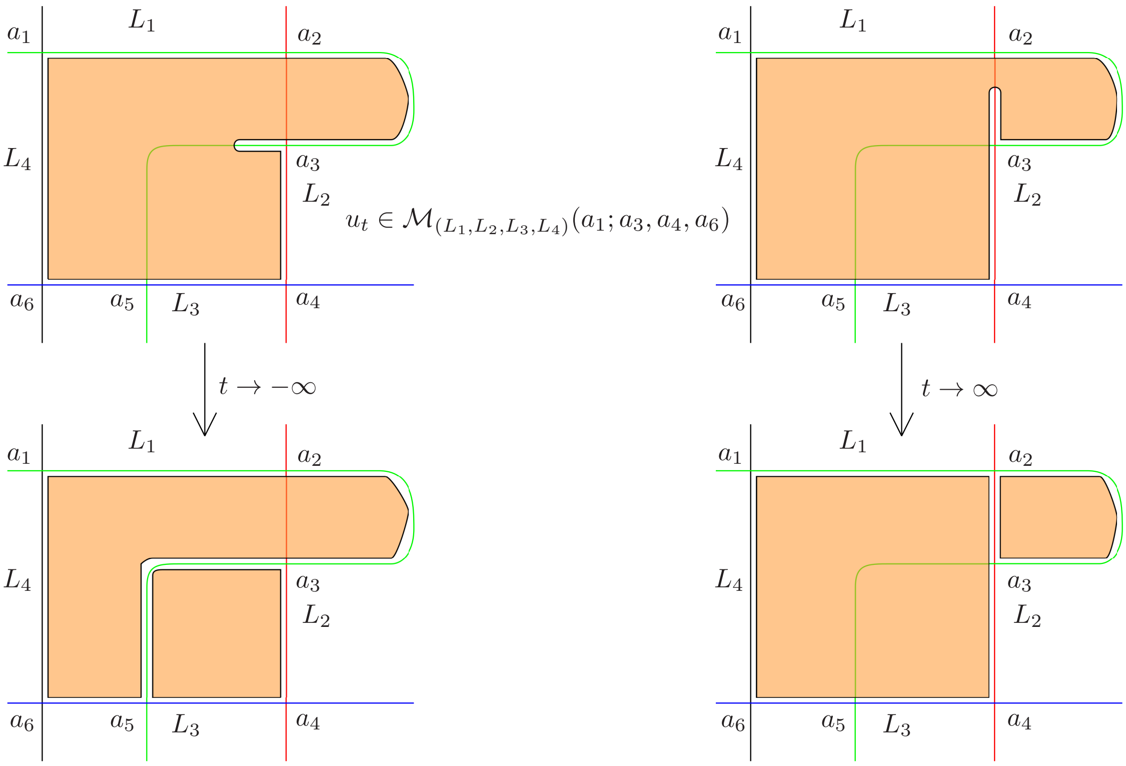

Figure 2.1: One parameter family of holomorphic disks.

2.3 Fukaya categories

Now let's denote by $CF(L_0, L_1)$ the vector space over $\mathbb{Z}_2$ with basis $L_0 \pitchfork L_1$. We denote by $CF(\underline{L})$ the vector space $CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1)$. We define the following operation:

$$\begin{array}{rccc} \mu^d_{\underline{L}}: & CF(\underline{L}) & \to & CF(L_0, L_d) \\ & (a_d, \ldots, a_1) & \to & \sum \mathcal{M}_{\underline{L}}(b; a_d, \ldots, a_1) b. \end{array}$$

where the sum is taken over all b leading to 0-dimensional $\mathcal{M}_{\underline{L}}(b; a_d, \ldots, a_1)$.

Considering 1-dimensional moduli $\mathcal{M}_{\underline{L}}(a_0, a_1, \ldots, a_d)$ the two types of breaking previously discussed lead to the following two types of algebraic terms:

- Stable:
$$\mu^{d_1}_{\underline{L}_{T_1}}(a_d,\ldots,\mu^{d_2}_{\underline{L}_{T_2}}(a_{i+j},\ldots,a_{i+1}),a_{i-1},\ldots,a_1)$$
for $d_1, d_2 > 1$ such that $d_1 + d_2 = d + 1$.

- Semi-stable:
$$(\mu^1 \circ \mu^d)(a_d,\ldots,a_1).$$

As an illustration the two breaking of Figure 2.1 correspond to: $\mu^2(a_6, \mu^2(a_4, a_3))$ and $\mu^3(a_6, a_4, \mu^1(a_3))$ respectively.

We assume that transversality (i.e. $\mathcal{M}_{\underline{L}}$ are manifolds) and gluing (any broken configurations is at the boundary of some 1-dimensional moduli space) holds. Since boundary of compact 1-dimensional manifolds has even cardinality, we get that those terms add up to 0 which implies that the operations μ^d satisfies the $\mathcal{A}_\infty$ Equation (1).

Following this one wish to define an $\mathcal{A}_\infty$-category $\mathcal{F}uk(M)$ in the following way:

- The objects $\mathcal{O}b(\mathcal{F}uk(M))$ is the set of exact Lagrangian sub-manifolds of (M, ω).
- The morphisms spaces are the Lagrangian Floer intersection complex $(CF(L_0, L_1), \mu^1_{L_0,L_1})$.
- Composition of morphisms is given by $\mu^2_{L_0,L_1,L_2} : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$.
- Higher order compositions are given by the operation $\mu_{\overline{L}}$.

On the homological level this category should recover the Donaldson category where morphisms spaces are $HF(L_0, L_1) := H(CF(L_0, L_1), \mu^1_{L_0,L_1})$ and compositions are given by the map induced by $\mu^2_{L_0,L_1,L_2}$. Since Floer homology is invariant under Hamiltonian deformation there are well defined endomorphisms spaces setting $HF(L, L) := HF(L, \phi_H(L))$ for a generic small Hamiltonian. However in the case of Fukaya categories things are different since the actual complex do depend on ϕ_H.

This leads to serious problems as we need to consider perturbations of pair of Lagrangian so that every time this given pair of Lagrangian is involved we actually see this very perturbation. Falling to do so would imply that the applications defined by Equation (1) would not have the appropriate domain or codomain. The same trouble arises when we want to define the higher order operation in the non-transverse case and again perturbation are needed and consistency between those perturbation is required to have a well define theory. In order to illustrate these problems we illustrate this on some very simple examples.

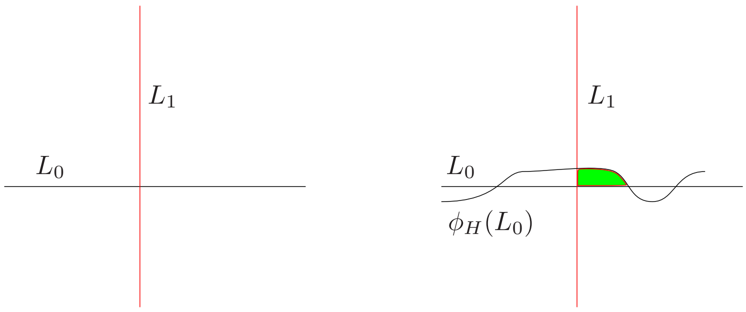

Figure 2.2: Perturbation to compute Floer homology

On Figure 2.2 we introduce a perturbation of L_0 in order to compute $HF(L_0, L_0)$.

This has the effect of introducing a non trivial terms in the $\mu^2_{L_0,L_1,L_0}$ map (in green) as the appropriate domain and codomain for this map is $\mu^2 : CF(L_1, \phi_H(L_0)) \otimes CF(L_0, L_1) \to CF(L_0, \phi_H(L_0))$ instead of $CF(L_1, L_0) \otimes CF(L_0, L_1) \to CF(L_0, L_0)$ as the last term here is not well defined.

Going further, it is true that the situation for the map $\mu^3_{(L_0,L_1,L_0,L_1)} : CF(L_1, L_0) \otimes CF(L_0, L_1) \otimes CF(L_1, L_0) \to CF(L_1, L_0)$ is transverse, and one is tempted to declare it to be 0. However we want to be consistent with the needed perturbation for L_0 and L_1 (as we want the $\mathcal{A}_\infty$ equation to hold this equation have to make sense in the first place). This imply that (in the present example) it is not 0 (see Figure 2.3).

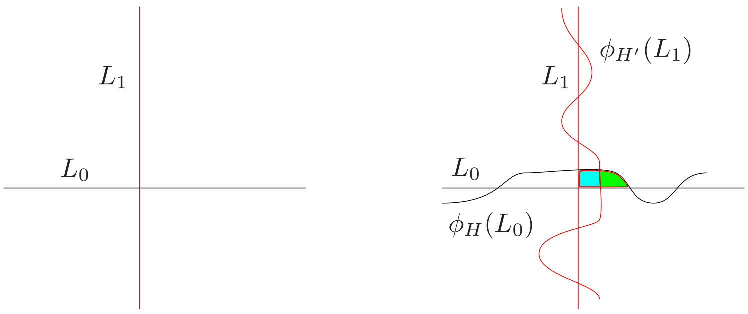

Figure 2.3: Two perturbations create a non zero μ^3.

In order to overcome this difficulties there have been several ways which have more or less satisfactory. First appearing in [KS01] and later in [Abo08] there is the notion of pre-category where the category comes with a class of ordered family of objects called *transverse object*. The operation are then defined on this families only. In this approach there are non endomorphism space and even less (needless to say) identity morphisms. The notion of isomorphism in this category is thus subtle and the price we pay of not developing a more satisfactory theory is that we end up with an object not so easy to manipulate.

The other approach is to package the needed perturbations in a consistent way so that the category we define depends on this family of consistent perturbations. This have been most successfully done in the book of Seidel [Sei08] and we will describe this approach in this course, the next chapter is dedicated to the proof that such consistent choice of data and perturbation exists. Once we have a well define category we need to prove that the quasi-equivalence class of the category does not depends on those. It turns out that this is an easy tasks once we play with lots of algebraic non-sense as we will see in Chapter 5. The price we pay with this approach is that the theory is not so computable any more.

To conciliate the two approaches: having something more computable on one side and an actual category on the other we rely on some method to find a finite set of generator of our category and then on those generators we define the notion of directed category where endomorphism space are trivial (isomorphic to the coefficient ring) and consider that this new category still retains a lot of information of the original category. This approach is again described in the third part of Seidel's book. In the present course we will focus on some methods to find generators of the Fukaya category developed by Abouzaid in [Abo10] and implement it for the case of surface in Chapter 6.

Before we go into the details of the construction we still ask for a little longer the reader to have faith in the well definiteness of this category and give some small intuitive exercise so that he familiarises himself with those operation and see the differences between these and the one which appeared earlier in the Donaldson category.

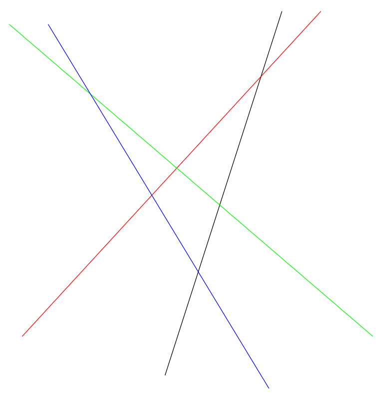

Figure 2.4: Four Lagrangians in $\mathbb{C}$.

Exercise 2.3.1. 1. There is another way to define operation on the complexes $CF(\underline{L})$ using pointed disks leading to the notion of Donaldson-Fukaya category. To define this structure we consider similar moduli space with the difference that we fix the conformal structure on the surface. More precisely for a generic $r \in \mathcal{R}^{d+1}$ we define:

$$\begin{array}{rccc} m^d_{\underline{L}}: & CF(\underline{L}) & \rightarrow & CF(L_0, L_d) \\ & (a_d, \ldots, a_1) & \rightarrow & \sum \mathcal{M}^r_{\underline{L}}(b; a_d, \ldots, a_1)b. \end{array}$$

Convince yourself that this lead to a well defined operation m^d on the complexes which induces application on homology groups.

2. Find an heuristic argument which prove that all maps m^d are characterised in homology by m^2 (which is actually equal to μ^2).

3. Compute μ^1, μ^2, μ^3 and m^3 on the non-compact Lagrangians shown on Figure 2.4.

4. Verify the $\mathcal{A}_\infty$-relation for the operation μ and the Fröbenius type relation found on question 2 for the operation m.

Chapter 3

Perturbed Floer equation and perturbation data.

3.1 Floer data.

To achieve transversality of intersection for any pair of intersection we make the following definition.

Definition 3.1.1. *Let L_0 and L_1 be a pair of Lagrangian sub-manifolds of (M, ω). A Floer datum for (L_0, L_1) is*

- *A time dependent Hamiltonian function $H : M \times [0, 1] \to \mathbb{R}$ such that $\phi_H(L_0) \pitchfork L_1$.*
- *A time dependant almost complex structure J_t $t \in [0, 1]$ compatible with ω.*

Given a pair of Lagrangian sub-manifolds, a Floer datum gives a Floer complex:

$$CF(L_0, L_1; H, J) := \mathbb{Z}_2(\phi_H(L_0) \pitchfork L_1).$$

Note that there is a bijection between $\phi_H(L_0) \pitchfork L_1$ and trajectories $\gamma : [0, 1]$ of X^H such that $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$. For $a \in \phi_H(L_0) \pitchfork L_1$ we denote by γ_a the corresponding trajectory of X^H.

Under transversality assumptions, the Floer complex is equipped with a differential $\mu^1_{L_0,L_1;H,J}(b) = \sum < a, b > a$ where $< a, b >$ is defined as a count of maps $u : Z \to M$ such that:

$$\begin{aligned}
&u(\cdot, 0) \in L_0 \\
&u(\cdot, 1) \in L_1 \\
&lim_{s\to-\infty} u(s, t) = \gamma_a(t) \\
&lim_{s\to\infty} u(s, t) = \gamma_b(t) \\
&\partial_s u + J_t(u)\big(\partial_t u - X_{H_t}(u)\big) = 0.
\end{aligned} \tag{3.1}$$

From now on we assume that any pair of Lagrangian comes with a given Floer datum such that solution to (3.1) are transversely cut-out and we drop both J and H from the notation.

In order to justify the following equation with respect to the consideration of previous chapter let's consider a curve $v : \mathbb{R} \times [0, 1] \to M$ which satisfies the following:

$$\begin{aligned}
&v(\cdot, 0) \in \phi^H(L_0) \\
&v(\cdot, 1) \in L_1 \\
&lim_{s\to-\infty} v(s, t) = a \\
&lim_{s\to\infty} v(s, t) = b \\
&\partial_s v + J(v)\big(\partial_t v - X^{H_t}(v)\big) = 0.
\end{aligned}$$

We define $u(s, t) = (\phi^H_{1-t})^{-1}(v(s, t)$ then

- $u(\cdot, 0) \in (\phi^H)^{-1}(\phi^H(L_0) = L_0$.
- $u(\cdot, 1) \in L_1$.
- $\lim_{s\to-\infty} u(s, t) = (\phi^H_{1-t})^{-1}(a) = \gamma_a(t)$
- $\lim_{s\to\infty} u(s, t) = (\phi^H_{1-t})^{-1}(b) = \gamma_b(t)$.

To check which equation u satisfies we compute:

$$\frac{\partial u}{\partial s} = d\phi^H_{1-t_0}(\frac{\partial v}{\partial s})$$

and

$$\frac{\partial u}{\partial t} = d\phi^H_{1-t_0}(\frac{\partial v}{\partial s} + X^H_{1-t}((\phi^H_{1-t})^{-1}(v)).$$

If we define J_t by $(\phi^H_{1-t})^{-1}J$ then u satisfies

$$\frac{\partial u}{\partial s} + J_t(\frac{\partial u}{\partial t} - X^H_t = 0.$$

Hence the perturbed Floer complex for L_0 and L_1 is the unperturbed on for $\phi^H(L_0)$ and L_1.

3.2 Perturbation data

Let $\Sigma \in \mathcal{R}^d$ together with a Lagrangian label $\underline{L}$ for Σ. In order to interpolate between the different Floer data associated to the marked point of Σ we introduce the notion of perturbation datum.

Definition 3.2.1. *A perturbation datum for $(\Sigma, \underline{L})$ is a pair (K, J) where*

- $K \in \Omega^1(\Sigma, C^\infty_0(M))$ *such that*
  1. *For all ξ $T\partial_i\Sigma_r$ $K(\xi)|_{L_i} = 0$.*
  2. $\varepsilon^*_{y_i}K = H_{y_i}dt$.
- $J \in C^\infty(\Sigma, \mathcal{J})$ *such that* $J|_{\varepsilon_{y_i}(s,t}(s,t) = J_{L_{i-1},L_i}(t)$.

Given such a perturbation datum we have an associated perturbed Cauchy-Riemann equation for maps $u : \Sigma \to M$ such that $u(\underline{\partial}(\Sigma)) \in \underline{L}$ which is

$$\begin{aligned} &du(z)(\zeta) + J(z,u) \circ du(z) \circ j_\Sigma(\zeta) = X^{K(\zeta)} + J(z,u) \circ X^{K(j_\Sigma\zeta)}. \\ &u(\partial_i\Sigma_r) \subset L_i) \\ &\lim_{s\to\pm\infty} u(\varepsilon_i(s,t)) = \gamma_i(t). \end{aligned} \tag{3.2}$$

For short we sometime write the first equation as $du^{0,1} = (X^K)^{0,1}$ or $(du - X^k)^{0,1} = 0$.

Exercise 3.2.1. 1. If the perturbation datum on the Z is given by $J(s,t) = J(t)$ and $K = H(t)dt$ then prove that Equation (3.2) reduces to Equation (3.1).

2. Let H_0 and H_1 be two time-dependent Hamiltonian functions let $K = H^s dt$ be a part of a perturbation datum for Z between the two Floer data associated to H_0 and H_1. Assuming transversality, gluing and compactness show that the counts of solution to Equation (3.2) induces the comparison map from $CF(L_0, L_1, H_0)$ to $CF(L_0, L_1, H_1)$.

3.3 Consistent choices of perturbation data.

Suppose that for all $d' \leq d$ we have chosen perturbation data (K_r, J_r) for all $r \in \mathcal{R}^{d'+1}$

Take a curve r in the image of the gluing map where parameter are sufficiently small in $\mathcal{R}^{d+1}$. As the gluing parameter is sufficiently small, the strip-like end coincide with the one coming from the curves r_v for $v \in Ve(T)$ prior to gluing and thus the data given by (K_v, J_v) are compatible with the strip like ends and thus are perturbation data for r. This implies that r has two set of perturbation data. We need a definition of consistency for these choice of perturbation, which will allows us to say that Gromov breaking of solution of Equation (3.2) converge to broken solution for the boundary perturbation data.

Note that r comes with a thin-thick decomposition where the thin part is made of the image of the strip-like ends of the r_v after the gluing (this is made of the strip-like ends of r plus some interior “rectangle”). Note that from the construction of the gluing map the union of the thick parts of the r_v's is the thick part of r. The appropriate notion of consistency is the following. We say that choices of perturbation data are consistent if

- The two previous perturbation data agrees on the thin part of r.
- If $r_n \to \{r_v\}$ for $n \to \infty$ (i.e. $r_n = \gamma^{T,T_d}(\{\rho_v^n\}, \{r_v^n\})$ with $\rho_v^n \to 0$ and $r_v^n \to r_v$) then the perturbation data on r_n converges to the family of perturbation data $\{K_v, J_v\}$ this convergence make sense if the first condition is satisfied as it ensure that we need only to consider convergence on the thick part.

Note in particular that if the data do agree on the whole of r then the data are consistent, as choices of perturbation data are contractible (because $C_0^\infty(M)$ and $\mathcal{J}(\omega)$ are) we can apply the same proof as of Theorem 1.5.1 to prove

Proposition 3.3.1. *Consistent choices of perturbation data exists.*

Remark 3.3.2. The reason for having a weaker notion of consistent perturbation data will appear in Chapter 5 for transversality consideration, where will modify some consistent choice into generic one to guarantee that solutions to Equation (3.2) form a manifold.

3.4 Linearisation of Equation (3.2)

We now turns our attention to the linearisation of Equation (3.2) and try to describe it as a standard Cauchy-Riemann operator (i.e. the $0,1$-part of a covariant derivative).

We recall the definition of such operators.

Definition 3.4.1. *Let $E \to \Sigma_r$ be a symplectic vector bundle with compatible complex multiplication J. Let F be Lagrangian sub-bundle of E on $\partial\Sigma_r$ (i.e. for $q \in \partial\Sigma_r$ F_q is a Lagrangian subspace of E_q. We denote by $\Gamma(E,F)$ the space of section s of E such that $s_q \in F_q$ for all $q \in \partial\Sigma_r$. A real Cauchy-Riemann operator D on E is a linear map:*

$$D : \Gamma(E,F) \to \Omega^{0,1}_{\Sigma_r}(E)$$

satisfying:

$$\forall f \in C^\infty(\Sigma_r, \mathbb{R})\ D(fs) = fD(s) + \overline{\partial} f \otimes s.$$

Remark 3.4.2. One way to construct such a operator is to take any connexion ∇ and take its $0,1$ part $\nabla^{0,1}$. All real Cauchy-Riemann operator are actually of this forms, for details see [MS04, Appendix C].

The goal of this section is to show that the linearisation of Equation (3.2) is a real Cauchy-Riemann operator. We first describe what is this linearisation process. Let $\mathcal{B}_r$ the space of smooth [1] maps from Σ_r to M with the boundary condition given by the Lagrangian label $\underline{L}$. Its formal tangent space at u is $T_u\mathcal{B}_r = \{V\Gamma(u^*TM) | \forall q \in \partial_i\Sigma_r V_q \in T_{u(q)}L_i\}$ (note that those are sections of a symplectic vector bundle with prescribed Lagrangian condition on the boundary). Let $\mathcal{E}$ be the vector bundle over $\mathcal{B}_r$ whose fibre over u is $\Omega^{0,1}_{\Sigma_r}(u^*TM)$ the space of 1-forms of type $0,1$ over Σ_r with values in TM. Equation (3.2) defines a section of ν of this bundle by $\nu(u)_q(\xi) = du_q(\xi) + J(q)\big(du_q(j_r\xi)\big) - X_q^{K(\xi)} - J(q)\big(X_q^{K(j_r\xi)}\big)$. Let $\pi_u : T_{(u,0)}\mathcal{E} \to \mathcal{E}_u$ be the projection parallel to $T_u\mathcal{B}_r$ where $\mathcal{B}_r$ is the 0-section. Let u_0 be a solution of Equation (3.2), the linearisation of Equation (3.2) at u_0 is

$$D_{u_0} : T_{u_0}\mathcal{B} \to \Omega^{0,1}_{\Sigma_r}(u_0^*TM)$$

defined by $D_{u_0}(V) = \pi_{u_0} d\nu(V)$.

Proposition 3.4.3. *The operator D_{u_0} is a real Cauchy-Riemann operator.*

[1] the actual regularity $W^{1,p}$ necessary to have a Fredholm operator will be detailed in the next section

Proof. It suffices to show that $D_{u_0}(fV) = fD_{u_0}(V) + \overline{\partial}(f)V$ for all $f \in C^\infty(\Sigma_r)$.

∎

Chapter 4

Grading.

4.1 Maslov index.

Let (V, ω_V) be a symplectic vector space of dimension n on which we choose a compatible almost structure J_V. We denote by $Gr(V, \omega_V)$ the Grassmanian of Lagrangian subspace of V. If one choose a reference Lagrangian subspace λ_0, it is identified as the orbit of an $U(n)$ action (the action of $U(n)$ as symplectic transformation is given by J_V). It is thus an homogeneous space $U(n)/O(n)$. This identification lead to a map $\alpha_V := det^2 : Gr(V, \omega_V) \to S^1$ which induce a natural map

$$\mu : H_1(Gr(V, \omega_V) \to \mathbb{Z} = H_1(S^1).$$

In the next section we will need another description of the map α_V. Fix an identification $\eta_V^2 : \Lambda_{\mathbb{C}}^{top} \otimes \Lambda_{\mathbb{C}}^{top} \simeq \mathbb{C}$ and note that for a basis $(v_1, \ldots, v_n)$ of an element $L \in Gr(V, \omega_V)$ we have that $\eta_V^2(v_1 \wedge \ldots \wedge v_n \otimes v_1 \wedge \ldots \wedge v_n) \neq 0$ (since L is Lagrangian we have that $L \oplus JL = V$).

We have the following proposition:

Proposition 4.1.1. *For another basis $(w_1, \ldots, w_n)$ of L we have that $\eta_V^2(w_1 \wedge \ldots \wedge w_n \otimes w_1 \wedge \ldots \wedge w_n) = (detP)^2 \eta_V^2(v_1 \wedge \ldots \wedge v_n \otimes v_1 \wedge \ldots \wedge v_n)$ where P is the change of basis matrix.*

Proof. Simply note that $P \in Gl_n(\mathbb{R})$ extends to a transformation P of V via the inclusion $Gl_n(\mathbb{R}) \subset GL_n(\mathbb{C})$ mapping v_i to w_i. The results follows from the definition of $\Lambda_{\mathbb{C}}^{top}$. □

This implies that the value $\frac{\eta_V^2(v_1 \wedge \ldots \wedge v_n \otimes v_1 \wedge \ldots \wedge v_n)}{||\eta_V^2(v_1 \wedge \ldots \wedge v_n \otimes v_1 \wedge \ldots \wedge v_n)||} \in S^1$ only depends on L. This is related to the function α_V noticing that choosing a referenced subspace λ_0 and a basis for λ_0 gives an identification η_V^2 just mapping $v_1 \wedge \ldots \wedge v_n \otimes v_1 \wedge \ldots \wedge v_n$ to $1 \in \mathbb{C}$. An easy verification shows that the to maps to S^1 corresponds.

The *maslov index* of a loop λ in $Gr(V, \omega_V)$ is $\mu([\lambda]) \in \mathbb{Z}$.

Given a path $\{\lambda(r)\}_{r\in(-\varepsilon,+\varepsilon)}$ in $Gr(V, \omega_V)$ we choose applications ϕ_r such that $\phi_s = Id$ and $\phi_r(\lambda(0)) = \lambda(r)$.

Proposition 4.1.2. *Let $\lambda_1 \in Gr(V, \omega_V)$ the following quadratic forms on $L_{\lambda(0),\Lambda_1} = \lambda(0) \cap \lambda_1$is independent of the choice of the map ϕ_r*

$$q_{\lambda,\lambda_1}(v) := -\frac{d}{dr}|_{r=0}\omega_V(\phi_r(v), v) = -\omega_V(\frac{d}{dr}|_{r=0}\phi_r(v), v) \tag{4.1}$$

Proof. Let ψ_r be a map with the same properties as ϕ_r. Then there are linear maps $f_r : V \to V$ such that $f_0 = Id$ and $\phi_r \circ f_r = \psi_r$ (note that this implies that f_r preserves $\lambda(0)$).

We get that

$$\frac{d}{dr}|_{r=0}\psi_r(v) = \frac{d}{dr}|_{r=0}\phi_r \circ f_r(v)$$
$$= \frac{d}{dr}|_{r=0}\phi_r(f_0(v)) + \phi_0(\frac{d}{dr}|_{r=0}f_r(v)) = \frac{d}{dr}|_{r=0}\phi_r(v) + \frac{d}{dr}|_{r=0}f_r(v)$$

Since f_r preserves λ_0 we get that $f_r(v) \in \lambda_0$ $\forall r$ and thus that $\frac{d}{dr}|_{r=0}f_r(v) \in \lambda_0$. Since λ_0 is Lagrangian we get that $w_V(\frac{d}{dr}|_{r=0}f_r(v), v) = 0$ and the result follows.

□

Definition 4.1.3. *Given a path $\lambda : [0, 1] \to Gr(V, \omega_V)$ we define the crossing formof λ at $s \in (0, 1)$ to be the form on $\lambda(s) \cap \lambda_1$ defined by*

$$q_{\lambda,\lambda_1}(s) := q_{\lambda_s,\lambda_1}$$

where λ_s is defined on some $(-\varepsilon_s, \varepsilon_s)$ by $\lambda_s(r) = \lambda(s + r)$.

For a generic loop, the intersection of $\lambda(s)$ with λ_1 is not $\{0\}$ only for a finite number of time s, therefore the following proposition make sense.

Proposition 4.1.4. *For a generic loop* $\lambda : [0,1] \to Gr(V, \omega_V)$ *such that* $\lambda(0) = \lambda(1) \pitchfork \lambda_1$ *we have*

$$\mu(\lambda) = \sum_{0<s<1} ind(q_{\lambda,\lambda_1}(s)). \tag{4.2}$$

We will need a small variation of this concept for some special paths in $Gr(V, \omega_V)$. We denote by $\mathcal{P}^-(Gr(V, \omega_V))$ the space of paths $\lambda : [0,1] \to Gr(V, \omega_V)$ such that $\lambda(0) \pitchfork \lambda(1)$ and $q_{\lambda,\lambda(1)}(1)$ is negative definite. Similarly to Proposition 4.1.4 we define for a generic path in $\mathcal{P}^-(Gr(V, \omega_V))$:

$$\iota(\lambda) = \sum_{0<s<1} ind(q_{\lambda,\lambda(1)}(s) \tag{4.3}$$

We denote by $Gr^{\#}(V, \omega_V)$ the pull-back of the cover $exp(i\cdot) : \mathbb{R} \to S^1$ by α_V (it is one realisation of the universal cover of $Gr(V, \omega_V)$. And we denote by π the canonical projection to $Gr(V, \omega_V)$. An element $(L, \alpha^{\#})$ in $Gr^{\#}(V, \omega_V)$ is called a *graded Lagrangian* subspace.

Let $(L_1, \alpha_1^{\#})$ and $(L_2, \alpha_2^{\#})$ be two grade Lagrangian subspace such that $L_1 \pitchfork L_2$. There exist up to homotopy a unique path $\lambda^{\#}$ from $(L_1, \alpha_1^{\#})$ to $(L_2, \alpha_2^{\#})$ such that $\pi \circ \lambda^{\#}$ is in $\mathcal{P}^-(Gr(V, \omega_V))$. We define the index of the pair to be

$$\iota((L_1, \alpha_1^{\#}), (L_2, \alpha_2^{\#})) = \iota(\pi \circ \lambda^{\#}) \tag{4.4}$$

We encourage the reader to do the following exercise so that he familiarise himself with the Maslov index of path.

Exercise 4.1.1.

1. Each of the red curves of Figure 4.1 gives a path in $Gr(\mathbb{C}, \omega_0)$ by taking the argument of the curves (pay attention that in $Gr(V)$ we consider **unoriented** Lagrangian). For each of them tells if the given path is in $\mathcal{P}^-$ and if yes compute its index.

2. Let $L_1^{\#} = (\{(t,0)|t \in \mathbb{R}\}, 6\pi)$ and $L_2^{\#} = (\{(t,t)|t \in \mathbb{R}\}, 3\pi/4)$ be two elements of $Gr^{\#}(\mathbb{C}, \omega_0)$. Compute $\iota(L_1^{\#}, L_2^{\#})$ and $\iota(L_2^{\#}, L_1^{\#})$.

3. Show that in general $\iota(L_1^{\#}, L_2^{\#}) = n - \iota(L_2^{\#}, L_1^{\#})$ where $2n$ is the dimension of V.

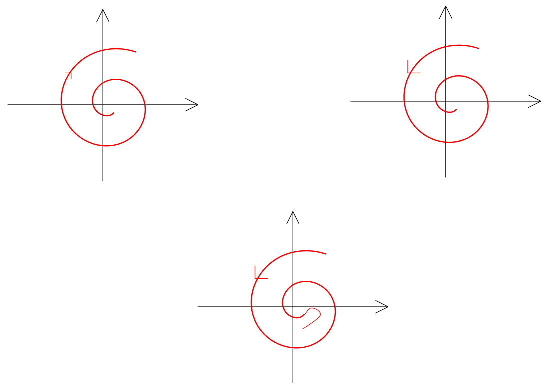

Figure 4.1: Some path in the Grassmanian of Lagrangian line.

4.2 Graded Lagrangians and dimension of moduli spaces.

We now turns our attention to the grading of intersection point between Lagrangian sub-manifolds. Let (M,ω) be a symplectic manifold endowed with a compatible almost complex structure J such that $2c_1(M) = 0$ and let $i : L \hookrightarrow M$ be a Lagrangian sub-manifold. We denote by $Gr(M,\omega) \to M$ the fibre bundle with fibre over $p \in M$ given by $Gr(T_pM, \omega_p)$.

Since $2c_1(M)$ we get that $\Lambda^{top}_{\mathbb{C}}(TM) \otimes \Lambda^{top}_{\mathbb{C}}(TM)$ is trivial and fix a trivialisation $\eta^2_V : \Lambda^{top}_{\mathbb{C}}(TM) \otimes \Lambda^{top}_{\mathbb{C}}(TM) \simeq \underline{C}_M$ where $\underline{C}_M$ is the trivial line bundle over M.

This identification allows us to define a map $\alpha_M : Gr(M,\omega) \to S^1$ and we denote by $Gr^{\#}(M,\omega)$ the pull-back of the universal cover $\mathbb{R} \to S^1$.

The Lagrangian embedding i induce a section α_L of $i^*Gr(M,\omega)$ defined by $\alpha_L(x) = (x, di_x(T_xL))$.

Definition 4.2.1. *The Maslov class of the Lagrangian L is the cohomology class $\mu_L : H^1(L) = Hom(H_1(M), \mathbb{Z} = H_1(S^1)$ defined by*

$$\mu_L = (\alpha_M)_* \circ (\alpha_L)_*.$$

If $\mu_L = 0$ then the section α_L lifts to a section $\alpha^{\#}_L$ of $i^*(Gr^{\#}(M,\omega))$.

Definition 4.2.2. *For a Lagrangian L with vanishing Maslov class a lift of α_L to $\alpha^{\#}_L$ is called a graduation on L. The pair $(L, \alpha^{\#})$ is called a graded Lagrangian sub-manifold.*

We are now able to define the grading of intersection point between grade Lagrangian.

Definition 4.2.3. *Let* $(L_1, \alpha_1^{\#})$ *and* $(L_2, \alpha_2^{\#})$ *be two graded Lagrangian sub-manifold such that* $L_1 \pitchfork L_2$. *For* $p \in L_1 \cap L_2$ *we define the degree of* p *by*

$$\iota(p) = \iota((\alpha_1^{\#}(p), \alpha_2^{\#}(p)) \tag{4.5}$$

Note that the preceding definition make sense because since $p \in L_1 \cap L_2$ then both $\alpha_1^{\#}(p)$ and $\alpha_2^{\#}(p)$ belongs to $Gr^{\#}(T_pM, \omega_p)$.

Let ϕ_H a symplectomorphism. Since symplectomorphisms preserves Lagrangians, $d(\phi_H)_p$ induces a map from $Gr(T_pM, \omega_p)$ to $Gr(T_{\phi_H(p)}M, \omega_{\phi_H(p)})$ which extends to a map from $Gr^{\#}(T_pM, \omega_p)$ to $Gr^{\#}(T_{\phi_H(p)}M, \omega_{\phi_H(p)})$. This implies that a graded Lagrangian $(L, \alpha_L^{\#})$ is naturally mapped to a graded Lagrangian $(\phi_H(L), (\phi_H)_*\alpha_L^{\#})$. If ϕ_H is Hamiltonian then a Hamiltonian chord y from L_1 to L_2 corresponds to an intersection point between $\phi_H(L_1)$ and L_2. If $(L_1, \alpha_1^{\#})$ and $(L_2, \alpha_2^{\#})$ are graded, we define the *degree* of this chord to be $\iota(y) = \iota(p)$ where p is the corresponding intersection point.

We can now state the result which compute the expected dimension of $\mathcal{M}^r(y_0; y_1, \ldots, y_d)$.

Theorem 4.2.4. *Let* $u \in \mathcal{M}^r(y_0; y_1, \ldots, y_d)$ *then*

$$ind(D_u) = \iota(y_0) - \sum_{j=1}^{d} \iota(y_j).$$

The next section is devoted to the proof Theorem 4.2.4.

4.3 Index theorem for Cauchy-Riemann operators on the disk.

We will now explain the index theory for Cauchy-Riemann operators over pointed disks. We start by the compact example.

Let $(E, \omega_E) \to D^2$ be a symplectic vector bundle of dimension n and let $F \subset i^*E$ be a Lagrangian sub-bundle (with $i : S^1 \to D^2$ the boundary inclusion). For $p > 2$ we denote by $W^{1,p}(E, F)$ the space of $W^{1,p}$ section of E with value in F on S^1. Up to homotopy there is a unique symplectic trivialisation of E under which F correspond to a loop $\lambda \in Gr(\mathbb{C}^n, \omega_0)$.

The space $W^{1,p}(E; F)$ is the appropriate domain for real Cauchy-Riemann operators on $\Gamma(E; F)$. So let

$$D : W^{1,p}(E; F) \to L^p(E)$$

be a real Cauchy-Riemann operator (i.e. satisfying $D(f \cdot s) = f \cdot Ds + \overline{\partial} f \cdot s$ for all real valued function).

The following is a relative version of Riemann-Roch Theorem (see [GH78, Chapter 2] for the closed version):

Theorem 4.3.1. *[MS04, Appendix C] The operator D is Fredholm and its index is*

$$ind(D) = n + \mu(\gamma) \tag{4.6}$$

We now turn our attention to the case of disks with marked point on their boundary. Fix $r \in \mathcal{R}^{d+1}$ and let $E \to \Sigma_r$ be a symplectic vector bundle with compatible almost complex structure J. Let F be a Lagrangian sub-bundle of the bundle on the boundary, and we denote by F_i its restriction to $\partial_i \Sigma_r$. We denote again by $W^{1,p}(E;F)$ the space of $W^{1,p}$ sections of E with boundary value in F and consider a real Cauchy-Riemann operator

$$D_r : W^{1,p}(E;F) \to L^p(E).$$

given by the $0,1$-part of a connection ∇.

Definition 4.3.2. *A set of limiting data is given by:*

- *A symplectic vector bundle $(E', \omega_{E'})$ over $[0,1]$.*
- *Some Lagrangian subspace Λ_i of (E'_i, ω_i) for $i = 0, 1$.*
- *A compatible almost complex structure on E'.*
- *A symplectic connection ∇' on E.*

For $i = 0, \dots d$ the operator D_r is compatible with a set of limiting data $(E_i, \omega_i, \Lambda_{i,0}, \Lambda_{i,1}, J_i, \nabla_i)$ if (assuming the end is outgoing)

- There exist an identification $\psi : \varepsilon_i^* E \simeq \pi^* E_i$ (where $\pi : Z \to [0,1]$ is the canonical projection).
- $\lim_{s\to\infty} \psi_{j,t}(F_{j,t}) = \Lambda_{i,j}$ for $j = 0, 1$.
- The difference $\omega_E - \omega_i|_{s=s_0}$, $J - J_i|_{s=s_0}$ and $\nabla - \nabla_i$ converges exponentially to 0 in the C^1-norms.

If D_r has compatible limiting data for $i = 1, \cdots, d$ we say that D_r is admissible if $\forall i$ the parallel transport $\widetilde{\Lambda}_{i,0}$ of $\Lambda_{i,0}$ by ∇_i along $[0,1]$ is transverse to $\Lambda_{i,1}$.

Exercise 4.3.1. Show that the Cauchy-Riemann operator of Section 3.4 is admissible.

Note that since all boundary condition are over open interval, one can lift all of those to $Gr^{\#}(E|_{\partial\Sigma_r})$ and that the convergence induces lifts $\Lambda^{\#}_{i,j}$ which by parallel transport induces lifts $\widetilde{\Lambda}^{\#}_{i,0}$ and $\Lambda^{\#}_{i,1}$

Theorem 4.3.3. *If D_r is admissible then it is Fredholm, its index is*

$$i(\widetilde{\Lambda}^{\#}_{j,0}, \Lambda^{\#}_{0,1}) - \sum_{j=1}^{d} i(\widetilde{\Lambda}^{\#}_{j,0}, \Lambda^{\#}_{j,1}) \tag{4.7}$$

for any lift of the Lagrangian boundary conditions F_i.

Theorem 4.2.4 is then an immediate consequence of Theorem 4.3.3.

We will prove Theorem 4.3.3 in three step. First we proof a gluing formula for CR operator. Then we proof the formula for a particular boundary condition over H and finally we prove the general formula.

4.3.1 Gluing operators

4.3.2 Operators over H

4.3.3 Proof of the index formula

Chapter 5

Fukaya categories.

We are now ready to wrap everything together and define the Fukaya category of an exact symplectic manifold.

5.1 Definition of $\mathcal{F}uk(M,\omega)$.

We are now ready to give the definition of the Fukaya category of an exact symplectic manifolds M. We assume that we have chosen a consistent choice of strip-like ends for all Stasheff polyhedra $\mathcal{R}^{d+1}$, some transverse Floer data for all pairs of Lagrangian together with a compatible choice of perturbation data such that all linearised perturbed Cauchy-Riemann operators are surjective we denote by I those choices. We define the Fukaya category $\mathcal{F}uk(M,\theta,I)$ of the Liouville manifold (M,θ) by

- $\mathcal{O}b(\mathcal{F}uk(M,\theta,I)) = \{L^{\#} | L^{\#}$ graded exact Lagrangian submanifolds of $(M,\theta)\}$.
- $hom(L_0^{\#},L_1^{\#}) = (CF(L_0^{\#},L_1^{\#},\mu^1_{L_0^{\#},L_1^{\#}}) \simeq \mathbb{Z}_2 < \phi^{H_{L_0,L_1}}(L_0)\cap L_1 >$ where

$$\begin{array}{cccc}\mu^1_{L_0^{\#},L_1^{\#}}: & CF(L_0^{\#},L_1^{\#}) & \to & CF(L_0^{\#},L_1^{\#})[1] \\ & y_1 & \to & \sum_{i(y_0)-i(y_1)=1} \#_2\mathcal{M}^*_{L_0,L_1}(y_0;y_1)y_0.\end{array}$$

- Composition and higher order compositions are given by

$$\begin{array}{llll}\mu^d: & CF(L^{\#}_{d-1}, L^{\#}_{d}) \otimes \cdots \otimes, CF(L^{\#}_{0}, L^{\#}_{1}) & \to & CF(L^{\#}_{0}, L^{\#}_{d})[2-d] \\ & y_d, \dots, y_1 & \to & \sum_{i(y_0)-\sum i(y_i)=2-d} \#_2\mathcal{M}_{L_0,\dots,L_d}(y_0; y_1, \dots, y_d) y_0. \end{array}$$

The fact that the operations $\{\mu^d\}_{d=1}^{\infty}$ forms an $\mathcal{A}_\infty$ structures follows from the compactness, transversality and gluing properties of the involved moduli spaces which are guarantied by the fact that all linearised operators are surjective. Indeed the operator D_u on the tangent space to a curve in $\mathcal{M}(y_0; y_1, \dots, y_d)$ is of index $i(y_0) - \sum i(y_j) + d - 2$ therefore if $i(y_0) - \sum i(y_j) = 3 - d$ the dim $\mathcal{M}(y_0; y_1, \dots, y_d) = 1$. This space admits a compactification whose boundary is made of broken curves of three types:

$$\begin{aligned}\partial\mathcal{M}(y_0; y_1, \dots, y_d) = & \bigcup_{y'} \bigcup_{i=0}^{d-1} \bigcup_{j=1}^{d-i} \mathcal{M}(y_0; y_1, \dots, y_{i-1}, y', y_{i+j+1}, \dots, y_d) \times \mathcal{M}(y'; y_i, \dots, y_{i+j}) \\ & \bigcup_{i=1}^{d} \bigcup_{y'} \mathcal{M}(y_0; y_1, \dots, y_{i-1}, y', y_{i+1}, \dots, y_d) \times \mathcal{M}^*(y', y_i) \\ & \bigcup_{y'} \mathcal{M}^*(y_0; y') \times \mathcal{M}(y'; y_1, \dots, y_d) \end{aligned} \tag{5.1}$$

which algebraically turns into the $\mathcal{A}_\infty$ equation (1) as boundary of 1-dimensional compact manifolds has even cardinality.

Remark 5.1.1. Note that we are completely free in our choice of the Hamiltonian perturbation H_{L_0,L_1} as long as the intersection $\phi_{H_{L_0,L_1}}(L_0) \cup L_1$ are transverse. This implies that when L_0 and L_1 are transverse one can choose H to be 0. This implies that this version of the Fukaya category recover the pseudo-category of [Abo08] and [KS01] described in Chapter 2. In this context the problem of endomorphism space is solved from the fact that all our data are chosen consistently.

5.2 Units

Before turning our attention on the dependence of $\mathcal{F}uk(M, \theta, I)$ on I we have to discuss on the fact that this $\mathcal{A}_\infty$-category is c-unital.

In order to justify the discussion we want to interpret the result of invariance of Floer homology under Hamiltonian perturbations. Let (H_{L_0,L_1}, J) and (H'_{L_0,L_1}, J') be two transverse Floer data for the pair L_0, L_1. On $Z \simeq \mathbb{R} \times [0,1]$ with strip-like ends $(\pm T, \infty)$ for $T > 0$ choose some (transverse) perturbation data $(K, \widetilde{J})$ compatible with this given Floer data. Such data allows us to define a map

$$\begin{array}{rcl} \phi_{K,\widetilde{J}}: \; CF(L_0^\#, L_1^\#, H, J) & \to & CF(L_0^\#, L_1^\#, H', J') \\ y_1 & \to & \sum_{i(y_0)-i(y_1)=0} \#_2 \mathcal{M}_{L_0,L_1}(y_0; y_1) y_0 \end{array}$$

(note that we don't have the extra $\mathbb{R}$-symmetry on the moduli space as the Floer data are different). Again investigating boundaries of one dimensional moduli spaces one get that $\phi_{K,\widetilde{J}} \circ d_{H,J} = d_{H',J'} \circ \phi_{K,\widetilde{J}}$. This map is actually the standard comparison map in Floer theory and is thus a quasi-isomorphism and the map in homology does not depends on the choice of $(K, \widetilde{J})$ (as the space of such choices is contactible). In the particular case of $H = H'$ the obvious perturbation data leads to non-trivial 0-dimensional spaces only when $y_0 = y_1$ and implies that ϕ is the identity.

We now describe how to get cohomological unit in $\mathcal{F}uk(M, \theta)$. The only extra piece of data needed is for all exact Lagrangian L_0 a choice of perturbation for the discs with one marked pointed with Lagrangian label L_0 compatible with the Floer data (H_{L_0,L_0}, J). Using this one can define a particular element $c_{K,J} \in CF(L_0^\#, L_0^\#)$ by $c_{K,J} = \sum_{i(y_0)=0} \#_2 \mathcal{M}(y_0) y_0$ where $\mathcal{M}(y_0)$ is the moduli space of discs satisfying the Floer equation perturbed by K with one asymptotic to y_0 and boundary on L.

We claim that such elements $c_{K,J}$ are c-units of $\mathcal{F}uk(M, \theta, I)$. For this note that the map $\mu^2_{L_0,L_0,L_0}(\cdots, c_{K,J})$ applied on a cord y is given by a count of curve shown on the left hand side of Figure.

For gluing parameter sufficiently large, this reduce to count curves shown on the right hand side of Figure which is a comparison map as described before. Since the comparison map does not depend on the perturbation data, it induce the identity in homology and therefore $c_{K,J}$ is a c-unit.

5.3 Well definedness

We prove now that the quasi-equivalence class of $\mathcal{F}uk(M, \theta, I)$ do not depend on I. In order to do so we define the $\mathcal{A}_\infty$-category $\mathcal{F}uk^{\text{tot}}(M, \theta)$ whose objects are pair $(L^\#, I)$ where $L^\#$ is an exact graded Lagrangian and I is a transverse choices of strip-like end, Floer data and perturbation data. The morphism space are define similarly as in $\mathcal{F}uk(M, \theta, I)$ using some compatible choices of perturbation data (bearing in mind that the notion of strip-like might change from one object

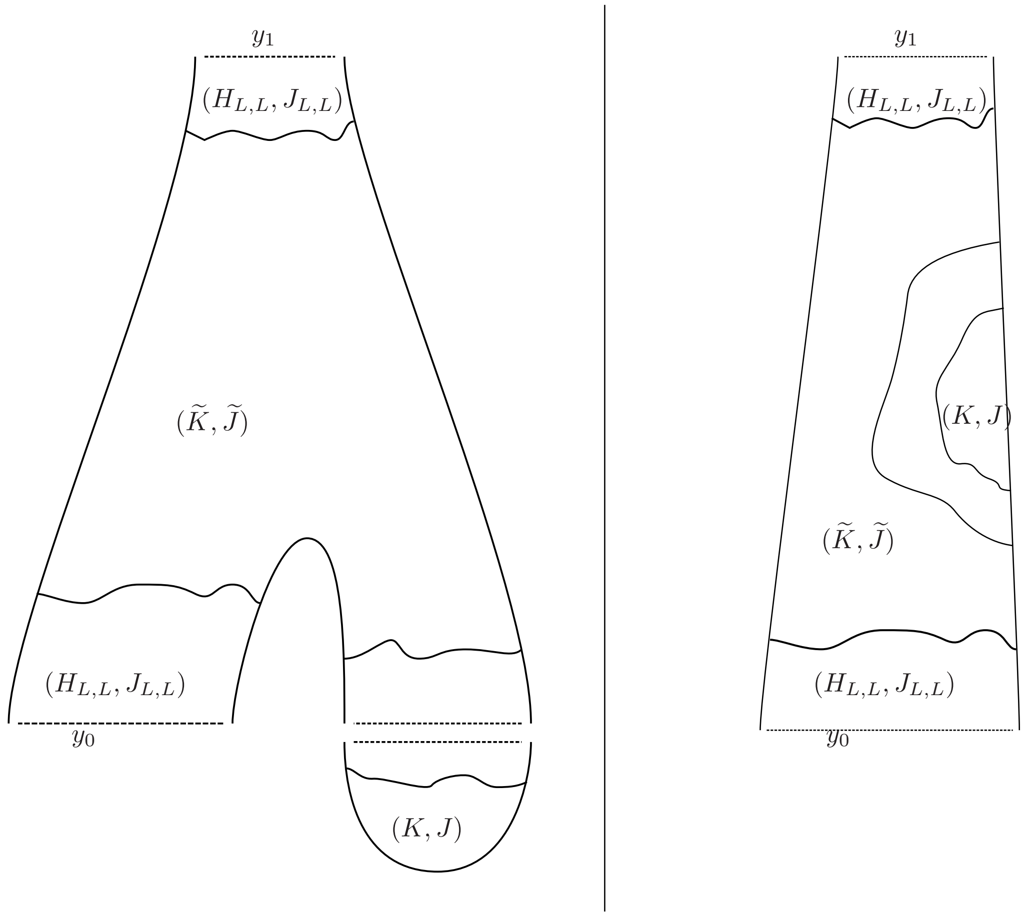

Figure 5.1: Composition with $c_{H,J}$.

to another) with the constrain that when the $d+1$-uple of object $(L_0^{\#}, I_0), \cdots, (L_d^{\#}, I_d)$ satisfies $I_0 = I_1 = \cdots = I_d$ the perturbation data used to define the morphism space is the one given by I_0.

Fix some transverse choices I, the last condition ensure that there is a tautological full and faithful embedding of $\mathcal{F}uk(M, \theta, I)$ into $\mathcal{F}uk^{\text{tot}}(M, \theta)$ for all transverse choices I. We will now argue that this inclusion is actually essentially surjective in cohomology. Let $(L^{\#}, I')$ be an object of $\mathcal{F}uk^{\text{tot}}(M, \theta)$ we will prove that $(L^{\#}, I')$ is quasi-isomorphic to $(L^{\#}, I)$.

Let $c_{H_{L,I,I'}} \in CF(L^{\#}, L^{\#}, H_{L,I,I'})$ defined similarly as in Section 5.2 from the discussion there one get the following:

1. $\mu^1(c_{H_{(L,I),(L,I')}}) = 0$

2. $\mu^2(c_{H_{(L,I),(L,I')}}, c_{H_{(L,I'),(H,I)}}) \in CF(L^\#, L^\#, H_{(L,I),(L,I)})$ is given by the count of curve shown in the left hand side of Figure which after gluing is given by the count of curve on the right hand side of Figure.

It follows from the second point and from Section 5.2 that $\mu^2(c_{H_{(L,I),(L,I')}}, c_{H_{(L,I'),(H,I)}})$ is a c-unit of $CF(L^\#, L^\#, H_{(L,I),(L,I)})$ which means that in cohomology $[c_{H_{(L,I),(L,I')}}] \circ [c_{H_{(L,I'),(H,I)}}]$. Thus $(L^\#, I)$ and $(L^\#, I')$ are quasi-isomorphic. The inclusion functor being cohomologically full faithful and essentially surjective, one get that $\mathcal{F}uk(M, \theta, I)$ is quasi-isomorphic to $\mathcal{F}uk^{\text{tot}}(M, \theta)$ and therefore given to choices I, I' the categories $\mathcal{F}uk(M, \theta, I)$ and $\mathcal{F}uk(M, \theta, I')$ are quasi-equivalent. Note that from the proof one can see that furthermore one can choose the quasi equivalence to send object $(L^\#, I)$ to $(L'^\#, I')$ and thus one can now note those categories $\mathcal{F}uk(M, \theta)$ unambiguously on objects.

5.4 Invariance

The goal of this section is to prove the following:

Theorem 5.4.1. *Let (M, θ) and (M', θ') be two exact symplectic manifolds such that there exists a diffeomorphisms $\phi : M \to M'$ satisfying $\phi^* \theta' = \theta + df$ then $\mathcal{F}uk(M, \theta)$ is quasi-equivalent to $\mathcal{F}uk(M', \theta')$.*

Chapter 6

Some examples and some generation criterion.

The goal of this chapter is to outline the geometric and algebraic idea behind Abouzaid's generation criterion in [Abo10] for Fukaya categories. We will be rather sketchy on the geometrical side and will assume that we are in a context where Lagrangian Floer homology is well defined and the symplectic manifold we consider is symplectically aspherical. It can however be compact (as we will see on our main example). The fact that we get out of the geometrical context is not problematic on the algebraic part of the argument.

6.1 Biran-Cornea's intersection criterion

A linear version of Abouzaid's argument appears in [BC09, Theorem 2.4.1] where the following is proved. Given two Lagrangian submanifolds L and K of (M, ω) (with $(\omega|_{\pi_2(M)} = 0)$ there exists maps $H_*(\mathcal{OC}) : HF(L, L) \to H_*(M)$ and $H_*(\mathcal{CO}) : H_*(M) \to HF(K, K)$ such that if $H_*(\mathcal{CO}) \circ H_*(\mathcal{OC}) \neq 0$ then $K \cap L \neq \emptyset$.

Roughly speaking, the maps $H_*(\mathcal{OC})$ and $H_*(\mathcal{CO})$ are defined using a Morse function on M a counting cardinality of the moduli spaces shown on Figure 6.1. In both case one should think the domain of the curves to be a Riemann disk with one marked point on the boundary and one on the interior and the equation interpolate between the perturbed Floer equation near the marked point on the boundary and the (positive or negative) gradient flow line equation near the marked point on the interior (a neighbourhood of which is identified with $S^1 \times \mathbb{R}_\pm$ the equation being independent of the S^1 parameter).

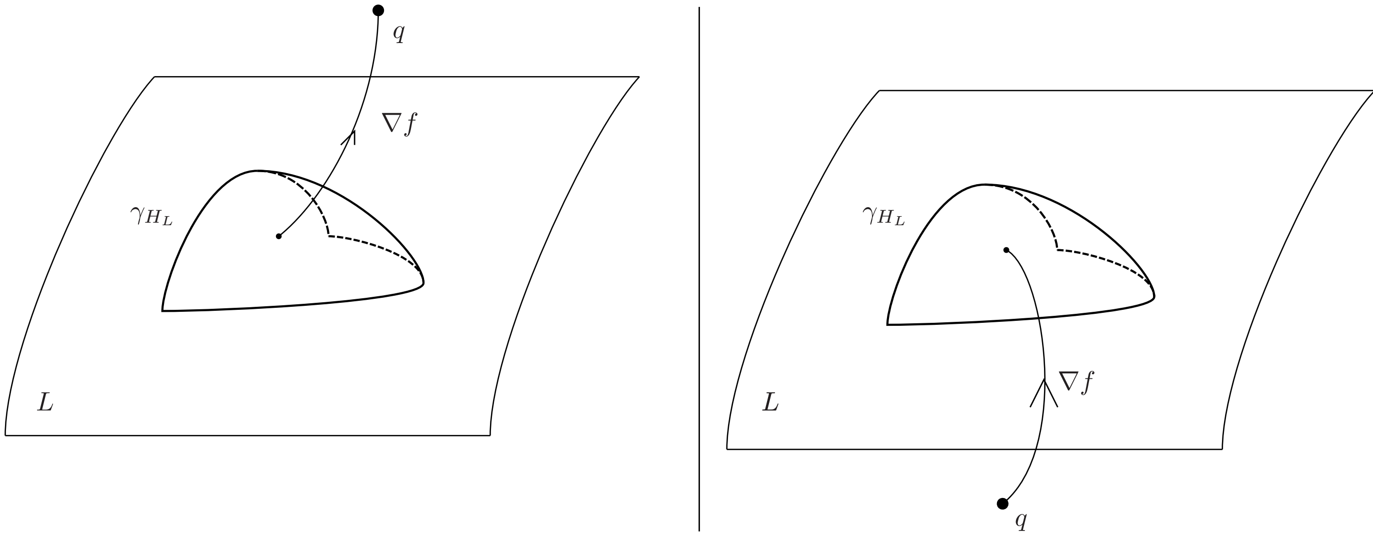

Figure 6.1: Curve contributing to open-closed ($< \mathcal{OC}\gamma_{H_L}, q >$ on the left) and closed-open ($< \mathcal{CO}(q), \gamma_{H_L} >$ on the right) maps (the arrow pointing on positive gradient direction).

We sketch here the argument to prove the theorem. Consider two curve contributing to the term $H_*(\mathcal{CO}) \circ H_*(\mathcal{OC}) \neq 0$ and glue them, the gluing curve belong to a 1-parameter family of holomorphic annuli shown on Figure 6.2. As $H_*(\mathcal{CO}) \circ H_*(\mathcal{OC}) \neq 0$ such family cannot break to another pair of similar curve. The other possible breaking is depicted on Figure 6.2(one cannot have boundary bubbling or interior bubbling since the Lagrangian are exact and M is simplectically aspherical). Note that the marked points all along are 1 and -1 as this is discussed in [Abo10, Appendix C.4] in order to ensure the existence of relevant rigid moduli spaces. From the position of the boundary marked points we deduce from the existence of such a breaking that there are intersection points between L and K.

Note that in the definition of the maps $H_*(\mathcal{CO})$ and $H_*(\mathcal{OC})$ no intersection points between L and K are involved so the existence of the intersection points are really detected from the algebraic maps.

6.2 Toward Abouzaid's criterion

The idea behind Abouzaid's criterion is the following: if the identity maps in $HF(K, K)$ is decomposed as $e = f_n \circ \cdots \circ f_0$ where $f_n \in HF(L_n, K)$, $f_i \in HF(L_i, L_{i+1})$ and $f_0 \in HF(K, L_1)$

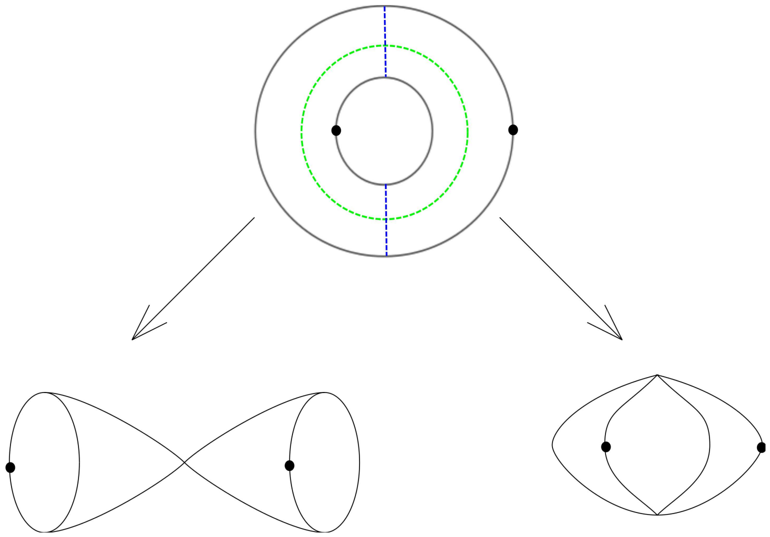

Figure 6.2: One dimensional family of annuli and its boundary. Stretching along the green line leads to the breaking on the left and along the blue on leads to the one on the right.

then K is in the split closed triangulated category generated by $\cup\{L_i\}$. To convince ourselves that it is necessary to take split closure (and not only triangulated) remark that if $id = f \circ g$ where $f : V \to W$ and $g : W \to V$ then $g \circ f$ is an idempotent which project V to a sub-object of V isomorphic to W.

Going back to Biran-Cornea's result the second type of breaking shows to type of map: one map $\Delta : HF_*(L, L) \to HF_*(L, K) \otimes HF_*(K, L) \to HF_*(L, L)$ and second one which is the composition $\mu_2 : HF_*(K, L) \otimes HF_*(L, K) \to HF_*(K, K)$. The degeneration of the moduli space of annuli implies that the following diagram commutes in homology:

$$\begin{array}{ccc} HF_*(L,L) & \xrightarrow{\Delta} & HF_*(K,L) \otimes HF_*(L,K) \\ \downarrow{\scriptstyle H_*(\mathcal{OC})} & & \downarrow{\scriptstyle \mu_2} \\ H_*(M) & \xrightarrow{H_*(\mathcal{CO})} & HF(K,K) \end{array} \tag{6.1}$$

The proof of the intersection criterion reads now as if $H_*(\mathcal{CO}) \circ H_*(\mathcal{OC}) \neq 0$ then $\mu_2 \circ \Delta \neq 0$ and thus $HF_*(K, l) \neq 0$ which implies the existence of intersection between K and L. Going further toward algebraic non-sense we can state the following proposition which is the linear version of the criterion we will state in the next section, it appears in [Smi14, Section 4.6.2].

Proposition 6.2.1. *If the unit $e \in HF(K,K)$ is in the image of $H_*(\mathcal{CO}) \circ H_*(\mathcal{OC})$ then K is in the split-closure of the sub-category generated by L.*

Proof. The hypothesis of the theorem and the commutativity of diagram (6.1) implies that $e = \sum \mu_2(p_i, q_i)$ for some $p_i \in HF(K,L)$ and $q_i \in HF(L,K)$. It suffices now to note that the maps $\alpha : \oplus_i HF(L,L) \to HF(L,K)$ given by $\mu^2(p_i, \cdot)$ and $\beta : HF(L,K) \to \oplus_i HF(L,L)$ given by $\beta = (\mu_2(q_1, \cdot), \dots, \mu_2(q_k, \cdot))$ satisfy $\alpha \circ \beta = Id$ and thus $\beta \circ \alpha$ is a projector of $\oplus HF(L,L)$ with image isomorphic to $HF(L,K)$. In terms of Yoneda's embedding this implies that $\mathcal{Y}(K)$ is quasi-isomorphic to a sub-object of $\oplus_{i=1}^{k-1} \otimes \mathcal{Y}(L)$, by definition of being in the split closure, this implies that K is in the split closure of the subcategory generated by L. □

6.3 Geometric criterion

The idea of the geometric criterion is to find a geometric argument which guarantees that the unit is in the image of the composition. First note that both $H*(M)$ and $HF(K,K)$ are ring and that the map $H_*(\mathcal{CO})$ is a unital ring morphism it therefore maps the unit to the unit so that it suffices to check that the unit of $H_*(M)$ is hit by the map $H_*(\mathcal{OC})$.

This is however very unlikely to be the case as this would implies that the Fukaya category is generated by a single object L which is seldom the case. However one can consider a family of object $\{L_1, \dots, L_k\}$ and add more marked point to define a map (on the chain level) $\mathcal{OC} : HC(\mathcal{B}, \mathcal{B}) \to C_*(M, f)$ where $\mathcal{B}$ is the subcategory of $\mathcal{F}uk(M)$ generated by the objects $\{L_i\}_{i=1}^k$ and $HC(\mathcal{B}, \mathcal{B})$ is the Hochschild complex of $\mathcal{B}$ with coefficient in $\mathcal{B}$ (seen as a bimodule over itself). That the domain is the Hochschild complex follows from the definition of the Hochschild differential and the degeneracies of one parameter family of disks with one marked point on the boundary.

Abouzaid's criterion read now as the following

Theorem 6.3.1. *(Abouzaid [Abo10] for the Wrapped Floer homology case, Abouzaid-Fukaya-Oh-Ohta-Ono for the closed and bulk Floer homology case) If the unit $1 \in H_*(M)$ is in the image of $H_*(\mathcal{OC}) : HH_*(\mathcal{B}, \mathcal{B}) \to H_*(M)$ then $\mathcal{B}$ split-generate the category $\mathcal{F}uk(M)$.*

In order to prove this theorem one must extend the previous commutative diagram to the non linear case of Hochschild homology. The map Δ extend to a map from $HC(\mathcal{B}, \mathcal{B})$ to $HC(\mathcal{B}, \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r)$. Algebraically this is just a change of coefficient in Hochschild homology and geometrically this amount (roughly) to a count of of curves with two outgoing ends. Note that the Hochschild complex of $\mathcal{B}$ with coefficient in the bimodule $\mathcal{Y}_K^l \otimes \mathcal{Y}_K^r$ is isomorphic to the complex given by

$C_*(\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K)$ (where $\otimes_{\mathcal{B}}$ refer to the derived tensor product described in the algebraic part of the book) where the isomorphism is given by the map τ sending $(a \otimes b) \otimes a_d \otimes \cdots \otimes a_1$ where $(a \otimes b) \in \mathcal{Y}^l_K \otimes \mathcal{Y}^r_K(L_0, L_d) = hom(K, L_0) \otimes hom(L_d, K)$ and $a_i \in hom(L_{i-1}, L_i)$ to $b \otimes a_d \otimes \cdots \otimes a_1 \otimes a$ in $hom(L_d, K) \otimes hom(L_{d-1}, L_d) \otimes \cdots \otimes hom(K, L_0)$ which is a summand of $\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K$.

In order to familiarised the reader with definition of Yoneda's module and Hochschild differential of the algebraic part of the book we encourage him to solve the following easy exercise.

Exercise 6.3.1. Show that τ is a chain map.

The maps μ_2 of the previous section extend naturally to the map $\mu : C_*(\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K) \to CF(K, K)$ given by the compositions $\{\mu^d\}_d$.

The extension of the previous commutative diagram reads now as the following

Theorem 6.3.2. *[Abo10, Proposition 1.3] The following diagram commutes in homology:*

$$\begin{array}{ccc} HC_*(\mathcal{B}, \mathcal{B}) & \xrightarrow{\Delta} & C_*(\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K) \\ \downarrow \mathcal{OC} & & \downarrow \mu \\ C_*(M, f) & \xrightarrow{\mathcal{CO}} & CF_*(K, K) \end{array} \tag{6.2}$$

The hypothesis of Theorem 6.3.1 (and because $H_*(\mathcal{CO})$ maps the unit to the unit) therefore implies that the unit of $HF_*(K, K)$ is in the image of $H_*(\mu)$ for all object K of $\mathcal{F}uk(M)$. The proof of the Theorem 6.3.1 is completed once one prove the following:

Proposition 6.3.3. *If $e \in HF(K, K)$ is in the image of $H_*(\mu)$ then the Yoneda module $\mathcal{Y}^r_K$ of K is an idempotent image in a twisted complex over $\mathcal{B}$.*

Proof. Recall that the complex $\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K$ is $\bigoplus_k \bigoplus_{X_0, \ldots X_k \in \mathcal{B}} hom(X_k, K) \otimes hom(X_{k-1}, X_k) \otimes \cdots \otimes hom(K, X_0)$ with the obvious differential given by partial composition of the terms. Since e is in the image of $H_*(\mu)$ there exist an N such that e in the image of the maps induced in homology by the composition $\bigoplus_{k \leq N} \bigoplus_{X_0, \ldots X_k \in \mathcal{B}} hom(X_k, K) \otimes hom(X_{k-1}, X_k) \otimes \cdots \otimes hom(K, X_0) \subset C_*(\mathcal{Y}^r_K \otimes_{\mathcal{B}} \mathcal{Y}^l_K) \to CF(K, K)$.

Consider $\mathcal{U}^N_K$ the twisted complex over $\mathcal{B}$ given by $\mathcal{U}^N_K = \bigoplus_{k \leq N} \bigoplus_{X_0, \ldots X_k \in \mathcal{B}} hom(X_k, K) \otimes hom(X_{k-1}, X_k) \otimes \cdots \otimes \mathcal{Y}^r_{X_0}$ with the differential given by all possible partial multiplication of the terms. There is a full composition map f which maps $\mathcal{U}^N_K$ to $\mathcal{Y}^r_K$.

Note that the $\mathcal{A}_\infty$ version of Yoneda's lemma implies that $hom_{mod}(\mathcal{U}_k^N, \mathcal{Y}_K^r) \simeq \mathcal{U}_N^r(K) = (\mathcal{Y}_K^r \otimes_\mathcal{B} \mathcal{Y}_K^l)^{\leq N} := \bigoplus_{k\leq N} \bigoplus_{X_0,\ldots X_k\in\mathcal{B}} hom(X_k, K)\otimes hom(X_{k-1}, X_k)\otimes\cdots\otimes hom(K, X_0)$ and contemplate the following (tautologically) commutative diagram

$$\begin{array}{ccc} hom_{mod}(\mathcal{U}_k^N, \mathcal{Y}_K^r) & \xrightarrow{f\circ} & hom_{mod}(\mathcal{Y}_K^r, \mathcal{Y}_K^r) \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathcal{Y}_K^r \otimes_\mathcal{B} \mathcal{Y}_K^l)^{\leq N} & \xrightarrow{\mu} & hom(K,K) \end{array} \tag{6.3}$$

We get that $Id \in hom_{mod}(\mathcal{Y}_K^r, \mathcal{Y}_K^r)$ is in the image of the composition maps $\circ : hom_{mod}(\mathcal{U}_k^N, \mathcal{Y}_K^r)\otimes hom_{mod}(\mathcal{Y}_K^r, \mathcal{U}_K^N)$ since e is in the image of μ.

The same argument as in the proof 6.2.1 implies that $\mathcal{Y}_K^r$ is a split summand of some $\bigoplus_i \mathcal{U}_K^N$ which by definition implies that K is in the split-closure of $\mathcal{B}$.

□

6.4 Example

In order to give a (non rigorous) illustration of this criterion let's try to understand how can we determine if the unit is in the image of $H_*(\mathcal{OC})$. The unit of $H_*(M)$ being represented in the Morse complex by the minimum m out which no rigid gradient flow line exits we get that the number $< a_d \otimes a_0, m >$ is given by a count of rigid holomorphic disks shown in Figure 6.3.

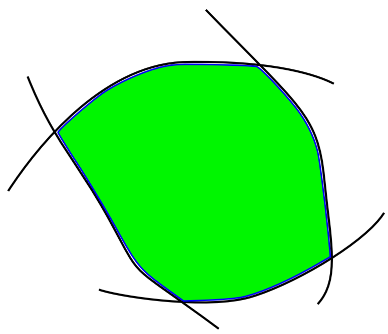

Figure 6.3: A rigid curve contributing to

Hence if one can find a family of object $\underline{L}$ such that for any generic point of m there is a "unique" holomorphic disks passing through this point with boundary on $\underline{L}$ then we get that $H_*(\mathcal{OC})(a_d, \otimes, a_0) = \overline{m}$ where $a_0, \ldots, a_d$ are the corner of the unique disk passing through m.

A basic example is the following. One can easily adapt the definition of the Fukaya category to the case of closed surface with genus greater than 2 (to keep the coefficient simple) by example by considering Lagrangian which are exact in the complement of a given point $z_0 \in \Sigma$ (or by considering Lagrangian which are Hamiltonian isotopic to a closed geodesic for a metric with constant scalar curvature, or to those lifting to Legendrian submanifolds of one of its prequantisation...). In this situation take some representative $\gamma_0, \ldots \gamma_{2g}$ so that cutting those gives a disks. Then by construction there exist a unique holomorphic disks with boundary on $\gamma_0 \cup \cdots \gamma_{2g}$ passing through any generic point. Therefore those objects generate the Fukaya category of Σ (see Figure 6.4).

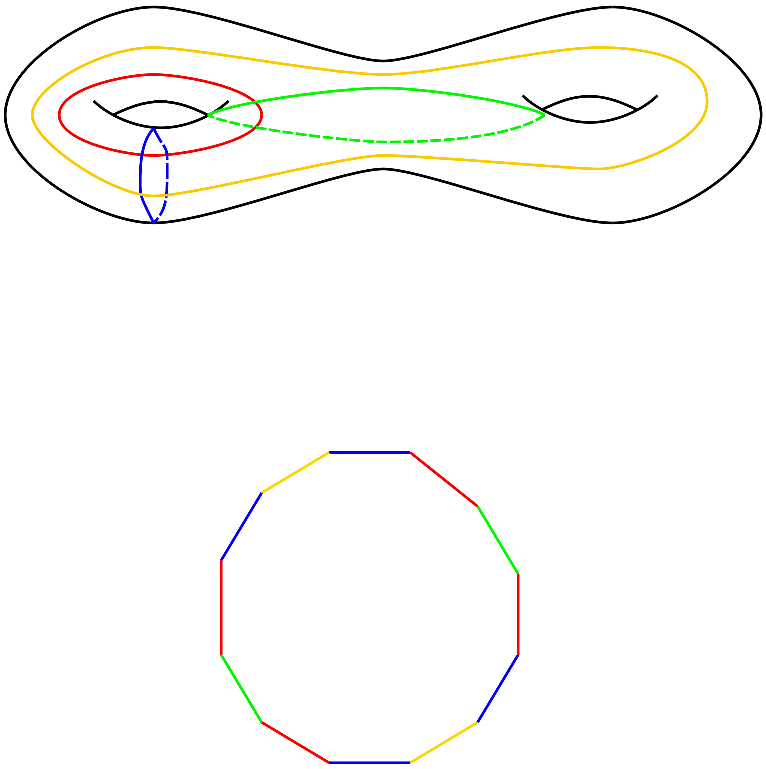

Figure 6.4: Split generators of $\mathcal{F}uk(\Sigma_2)$

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