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Augmentations des sous-variétés legendriennes: invariants associés et applications.

Mémoire présenté par

Baptiste Chantraine

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Rapporteurs:

- **Octav Cornea**, Professeur, Université de Montréal
- **Alexandru Oancea**, Professeur, Sorbonne Université, Paris
- **Claude Viterbo**, Professeur, Ecole Normale Supérieure, Paris

Jury:

- **Jean-François Barraud**, Professeur, Université de Toulouse
- **Vincent Colin**, Professeur, Université de Nantes
- **Tobias Ekholm**, Professeur, Uppsala Universitet
- **Lisa Traynor**, Professeur, Bryn-Mawr College
- **Claude Viterbo**, Professeur, Ecole Normale Supérieure, Paris

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Contents

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Introduction

At the International Congress of Mathematics 1998 in Berlin Y. Eliashberg reported on new invariants of contact manifolds and their Legendrian submanifolds (see [Eli98]). He defined the contact homology algebra and its relative version: Legendrian contact homology. The latter has been developed independently by Y. Chekanov in [Che02] for Legendrian knots in the standard three dimensional contact space. In this dimension the theory has the benefit of being combinatorial (thanks both to Reidemeister theorem and the uniformisation theorem). These invariants became part of what is known as Symplectic Field Theory (developed in [EGH00]) which has been feeding mathematicians ever since. First it is a very rich algebraic invariant which admits many variations allowing to distinguish many objects. And second its definition in full generality is yet to come, leaving room for a lot of foundational work to be done. I am one of those mathematicians who has been able to put food on his plate thanks to this important foundational work. This memoir will try to expose my contribution to the field.

Legendrian sub-manifolds. When we imagine a car moving along a prescribed trajectory or think on how to orient the skate on the ice to follow someone in front of us we are doing Legendrian lifts of front projections. When on a piece of paper we try to draw the contour of the mountains that we glance at or draw on a blackboard a torus in $\mathbb{R}^3$, with its centre hole that we know well not to draw as an ellipse, we draw front projections of Legendrian sub-manifolds. Looking at a wave propagating after we throw a rock in the water or turn on the light in space (OK we usually do not have so much time to enjoy that one) we watch Legendrian isotopies. All is to say that my biased feeling is: our eyes and brain experiment Legendrian sub-manifolds and their isotopies all the time.

The previous examples described Legendrian sub-manifolds in the so-called space of contact elements. This is the collection of all possible tangent hyper-planes of a given configuration space. In this situation Legendrian sub-manifold arises as the collection of hyper-planes which are all tangent to a given sub-manifold. Those are the building blocks of contact geometry and Legendrian sub-manifolds. As usual it was the question of classification of such objects (up to deformations) which led to the definitions of the rich invariants that were mentioned in the opening of this introduction.

Augmentations. My contribution to the realm of Symplectic Field Theory stays modest I have mostly focused on the juice that can be extracted for this very first relative invariant that Y. Eliashberg talked about in 1998: the Legendrian Contact Homology. It is an invariant defined applying ideas from [Flo88] and [Gro85] (for closed Lagrangian sub-manifolds) to the Lagrangian cylinder over a Legendrian sub-manifold. However in this situation the new phenomenon that one encounters is the bubbling off of holomorphic half-planes toward the negative end (which is concave). This forces the complex to be a freely generated differential graded algebra (DGA). The full Legendrian contact homology is thus the homology of an algebra, called the Chekanov-Eliashberg algebra, being both freely generated and non commutative. Such objects are quite complicated and without any other tools it is very hard to manipulate and distinguish two of them. Fortunately such tools exist: in [Che02], Y. Chekanov introduced the idea of representing the Chekanov-Eliashberg algebra into simpler algebrasssss to extract some finite dimensional complexes out of this DGA. This lead to the definition of the so-called linearised Legendrian contact homology. Such a representation is called an augmentation of the Legendrian sub-manifold. Augmentations have then been used in many places as ways to weights possible degeneration that might be tricky to handle otherwise: for instance they serve as bounding co-chain for Lagrangian Floer homology in [Fuk+09]. In my career I have been trying to understand those objects as much as I could and use them for what there are good for: define linear homology theories.

Linearised contact cohomology and augmentation categories. Augmentations arise geometrically as Lagrangian fillings of a given Legendrian. Intersecting Lagrangian sub-manifolds leads to Floer theory. Floer complexes can all be organised into an $\mathcal{A}_\infty$-category called the Fukaya category (see [Fuk93] and [Sei08]). Trying to do the same with augmentations as abstract algebraic objects led to the collaboration [BC14b] of which the main result is:

Theorem 0.0.1. *There is an $\mathcal{A}_\infty$-category $\mathcal{A}ug(\Lambda)$ whose objects are augmentations of Λ and morphisms spaces are linearised Legendrian contact homology complexes. The quasi-equivalence type of this $\mathcal{A}_\infty$-category is invariant under Legendrian isotopy.*

Being similar to other $\mathcal{A}_\infty$ or dg categories associated to a given Legendrian sub-manifold this category is the ground of many conjectures relating different invariants (generating families, fillings, augmentations and sheaves with micro-support on Λ).

Its structure, and specifically its morphism spaces, has been shown to be a more refined invariant than linearised contact homology. For instance in [Cha15b] it allowed me to show that the relation of Lagrangian concordance (some Lagrangian cobordism with the simplest topology) is not symmetric. This a good point to start talking about those Lagrangian cobordisms.

Lagrangian cobordisms. Fronts of deformation of Legendrian sub-manifolds look like front of Legendrian sub-manifolds one dimension higher. As this front comes from an isotopy, the topology of the manifold cannot change along such a deformation. If we allow the topology to change we are led to the definition of Legendrian cobordisms from [Arn80a] and [Arn80b]. This relation seems to lack rigidity when talking about holomorphic type invariants but this might suggest that we are just not there yet (see [Lim18] for recent developments of this relation). The Lagrangian counterpart of this relation has been successfully studied in [BC13]. But if we think of this cobordism to be the lift of a Lagrangian cobordism then we can ask for this cobordisms to be embedded in the so-called symplectisation of the contact manifold. This leads to the definition of Lagrangian cobordism between Legendrian sub-manifolds which are the geometrical objects I have been staring at since my PhD. They are the underlying objects that leads to functorial properties of relative SFT. In the first few years of my career I've been focusing on the structure of such cobordisms and how to build them (see [Cha10; Cha12; Cha15b] and [Cha15a]).

In the early age of symplectic topology the realisation that intersection properties of Lagrangian submanifolds seemed driven by the Morse theory of the underlying manifold led to one of the famous Arnol'd conjectures, a proof of which [Flo88] was devoted. In simple manifolds (such as cotangent bundles) exact Lagrangian submanifolds seem to have simple topology (the one of the 0-section). This is the content of the so-called nearby Lagrangian conjecture:

Conjecture 0.0.2 (Nearby Lagrangian conjecture.). Any exact Lagrangian in T^*Q is Hamiltonian isotopic to the 0-section.

Both aspects of this conjecture (the diffeomorphism type of the Lagrangian first and its symplectic unknotedness on top of that) are still open but progress have been made in seeing the homotopy type of the Lagrangian and how holomorphic curves cannot distinguish it symplectically from the 0-section (a little more on that later). The main tool to study both the symplectic and topological properties of Lagrangian is the Floer complex that we mentioned many times.

In [Cha+15b] we develop a Floer theory for Lagrangian cobordisms which make use of augmentations of the negative end. The main result we obtain is:

Theorem 0.0.3. *Given two Lagrangian cobordisms Σ_0 and Σ_1 in the symplectisation of Liouville manifold with negative ends Λ_i^- and positive ends Λ_i^+ ($i = 0, 1$) and two augmentations ε_0^- and ε_1^- of Λ_0^- and Λ_1^- respectively. There is a filtered chain complex $\mathrm{Cth}(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1)$ whose quasi-isomorphism type is invariant under Hamiltonian deformations of Σ_0 and Σ_1 that are cylindrical at infinities. The first page of the spectral sequence associated to the filtration is*

$$\begin{aligned} &LCH^\bullet(\Lambda_0^+, \Lambda_1^+; \varepsilon_0^+, \varepsilon_1^+) \oplus HF_+^\bullet(\Sigma_0, \Sigma_1; \varepsilon_0^-, \varepsilon_1^-) \\ &\oplus LCH^\bullet(\Lambda_0^-, \Lambda_1^-; \varepsilon_0^-, \varepsilon_1^-) \oplus HF_-^\bullet(\Sigma_0, \Sigma_1; \varepsilon_0^-, \varepsilon_1^-). \end{aligned}$$

The motivation comes from the fact that some augmentations find geometric incarnations in Lagrangian fillings of the negative ends. Our Floer theory should recover the Floer Homology of the cobordism capped with that filling. This suggested us to add to the Floer complex of hidden intersection points (which are the generator of the Legendrian contact homology of the negative end).

The terms $HF_\pm$ have in general not real meanings do not have relevant invariant properties. They are generated by intersection points which (under the previous analogy) lie over (reps. under) negative Reeb chords. As those might slide from one to the other (and to chords) along an Hamiltonian isotopy those do not satisfy any interesting invariance properties.

From the study of this so-called Cthulhu complex we were able to deduce many results both on the symplectic aspect of Lagrangian cobordisms as well as on the topology of Lagrangian cobordisms.

Topology Lagrangian cobordisms. One of the early remarks when studying a Lagrangian cobordism is that in dimension 2 (I speak about the dimension of the cobordism here) its Euler characteristic (and hence its topology if oriented) is characterised by classical Legendrian invariant of its extremities.

This is of course not so simple in higher dimension (as the topology is not characterised just by the Euler characteristic). But if the negative end of a cobordisms Σ has an augmentation then similar results are true. Indeed in [Cha+15b] we prove

Theorem 0.0.4. *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. There is a long exact sequence*

$$\begin{array}{l}\cdots \to LCH^{k-1}_{\mathrm{rel}}(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+) \\ \qquad\qquad\downarrow \\ \qquad H^{k-1}(\overline{\Sigma},\partial_+\overline{\Sigma};R) \longrightarrow LCH^k_{\mathrm{rel}}(\Lambda^-;\varepsilon_0^-,\varepsilon_1^-) \to LCH^k_{\mathrm{rel}}(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+) \to ,\end{array} \tag{1}$$

where the map from $LCH^\bullet_{\mathrm{rel}}(\Lambda^-;\varepsilon_0^-,\varepsilon_1^-)$ to $LCH^\bullet_{\mathrm{rel}}(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+)$ is the adjoint of the linearised DGA morphism Φ_Σ induced by Σ.

Our work is motivated by the so-called Seidel isomorphism (see [Ekh12]) which motivated the definition of the augmentation category. It shows that when the Legendrian sub-manifold Λ^- is Lagrangian fillable by a Lagrangian L then $LCH_{\mathrm{rel}}(\Lambda^-)$ is the singular cohomology of L relative to Λ^- (see Theorem 2.1.19). In this context this exact sequence becomes the exact sequence of the triple $\Lambda^+ \subset \overline{\Sigma} \subset \overline{L} \cup \overline{\Sigma}$. A similar exact sequence exists for generating family homology when cobordisms are induced by some generating functions (see [ST13]) or for wrapped Floer homology when the cobordisms can be filled (see [CO18]). Using a different perturbation we also find an analogue of Mayer-Vietoris exact sequence for the decomposition $\overline{L} \cup \overline{\Sigma}$. I want to step back a little toward the second paragraph of the intro on the example of the torus: the contour of the torus on the blackboard is enough for us to understand that we are looking at the torus. The content of all the statement relating the Legendrian invariants to the topology of fillings or cobordisms seems to suggest that a Legendrian remember a lots of the topology of object it is the boundary of in a similar fashion of the fact that the 1-dimensional front on the blackboard represent the 2-dimensional object well enough.

A third type of perturbation leads to a long exact sequence relating the relative Legendrian contact homology to Legendrian contact cohomology.

Theorem 0.0.5. *Let Σ be an exact graded Lagrangian cobordism from Λ^- to Λ^+ and let ε_0^- and ε_1^- be two augmentations of $\mathcal{A}(\Lambda^-)$ inducing augmentations ε_0^+, ε_1^+ of $\mathcal{A}(\Lambda^+)$. Then there is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & LCH^{k-1}(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+) \arrow[d] & & & \\ & H^{k-1}(\overline{\Sigma};R) \arrow[r] & LCH^k_{\mathrm{rel}}(\Lambda^-;\varepsilon_0^-,\varepsilon_1^-) \arrow[r] & LCH^k(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+) \arrow[r] & {} \end{tikzcd} \tag{2}$$

If $\Sigma = \mathbb{R}\times\Lambda$, then $H^\bullet(\Sigma) = H^\bullet(\Lambda)$ and thus when Λ^- is fillable this is the exact sequence of the pair (L,Λ^-). Furthermore if Λ^- is horizontally displaceable then $LCH^k_{\mathrm{rel}}(\Lambda^-,\varepsilon_0^-,\varepsilon_1^-) \simeq LCH_{n-1-k}(\Lambda^-,\varepsilon_0^-,\varepsilon_1^+)$. For $\Sigma = \mathbb{R}\times\Lambda$ this is exactly the duality exact sequence of [Sab06] for Legendrian knots and later generalised to arbitrary Legendrian sub-manifolds in [EES09]. This duality long exact sequence was generalised in [BC14b] when two augmentations are used. We use the exact sequence (2) to prove that the fundamental class in LCH defined in [Sab06] and in [EES09] is functorial with respect to the maps induced by exact Lagrangian cobordisms.

These exact sequences work in great generality regarding coefficient and allow to study them in $R[\pi_1(\Sigma,*)]$. This has allowed us to prove the following

Theorem 0.0.6. *Let Σ be an n-dimensional Legendrian homotopy sphere and assume that $\mathcal{A}(\Lambda;\mathbb{Z})$ admits an augmentation. Then any exact Lagrangian cobordism Σ from Λ to itself is an h-cobordism. In particular:*

1. *If $n \neq 3,4$, then Σ is diffeomorphic to a cylinder;*
2. *If $n = 3$, then Σ is homeomorphic to a cylinder; and*
3. *If $n = 4$ and Λ is diffeomorphic to S^4, then Σ is diffeomorphic to a cylinder.*

When $n = 1$, a stronger result is known. Namely, in [Cha+15a, Section 4] we proved that any exact Lagrangian cobordism Σ from the standard Legendrian unknot Λ_0 to itself is compactly supported Hamiltonian isotopic to the trace of a Legendrian isotopy of Λ_0 which is induced by the complexification of a rotation by $k\pi$, $k \in \mathbb{Z}$. This classification makes use of the uniqueness of the exact Lagrangian filling of Λ_0 up to compactly supported Hamiltonian isotopy, which was proved in [EP96]. In contrast, the methods we developed in [Cha+15b] give restrictions only on the smooth type of the cobordisms and little information is known about their symplectic knottedness in higher dimension. Of course this is not as spectacular as any variation of the nearby Lagrangian conjecture, but the flavour is that simple topology of the end of a Lagrangian cobordisms forces simple topology of the cobordisms itself. To what extend the extremities dictate the cobordism is unclear: we know some counter-examples when we allow more topology on the extremities (or don't have the existence of the augmentation) but we don't know where the border is.

Obstruction to Lagrangian cobordisms. On the other side, one can use exact sequences (1) and (2) to obstruct certain types of cobordisms. A first immediate corollary of Theorem 0.0.4 is that linearised contact homology is invariant under Lagrangian concordance.

Corollary 0.0.7. *Let Σ be an exact Lagrangian concordance from Λ^- to Λ^+. If, for $i = 0, 1$, ε_i^- is an augmentation of $\mathcal{A}(\Lambda^-; R)$ and ε_i^+ is the pull-back of ε_i^- under the DGA morphism induced by Σ, then the map*

$$\Phi_\Sigma^{\varepsilon_0^-,\varepsilon_1^-} : LCH^\bullet_{\varepsilon_0^-,\varepsilon_1^-}(\Lambda^-) \to LCH^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda^+)$$

is an isomorphism.

This criterion gives obstructions for two sub-manifolds to be Lagrangian concordant. In particular this allows to recover the result from [Cha15b] showing that there is no Lagrangian concordance from a Legendrian representative of the $m(9_{46})$ knot (see Figure A.3) to the trivial Legendrian knot even though a concordance in the other direction exists. Using the same obstruction we were able to generalise this counter-example to higher dimension (previous known examples in higher dimensions appealed to flexible techniques with loose Legendrian ends, see [EM13]).

The Cthulhu complex also has a remarkable structure when applied to cobordisms coming from positive isotopies. A positive Legendrian isotopy is an isotopy that always moves positively transverse to its front; it is in that sense pretty close from the propagating wave in the preamble. In this situation the term HF_+ vanishes, thus we can build a long exact sequence allowing us to obstruct existence of such isotopy in some particular situation. We exploited this stream of ideas in [CCD19] which allowed us to give new example of non-orderable contact manifolds (in the sense of [EP00]).

Generation of the wrapped Fukaya category. The paper [FSS09] showed that homotopical methods such as $\mathcal{A}_\infty$ algebras allow to restrict the topology of Lagrangian sub-manifolds. In [AS10] the Fukaya category was extended allowing non-compact objects with Legendrian ends leaving more possible geometric representations of the 0-object. This makes detection of quasi-isomorphism more geometric (as cone of such are 0-objects). In [Abo10] and then [Abo11] M. Abouzaid showed that these larger categories could have small sets of generators under some geometric hypothesis. Such hypothesis is satisfied for a single cotangent fibre in T^*Q. Those three foundational papers began to exhibit a series of topological constraints on the topology of Lagrangian sub-manifolds culminating in [AK18] showing that exact and compact Lagrangians in a cotangent bundle has the simple homotopy type of the 0-section. The fact that the wrapped Fukaya category is generated by one object reduces the study of Lagrangian to module over this object.

Going back to more modest results we proved in [Cha+17] that all Weinstein manifolds and sectors (a version allowing boundary) have the same type of generators.

Theorem 0.0.8. *The wrapped Fukaya category of the Weinstein sector* $(S, \theta, \mathfrak{f})$ *is generated by the Lagrangian co-core planes of its completion* $(W, \theta_W, \mathfrak{f}_W)$ *and by the spreading of the Lagrangian co-core planes of its belt* $(F, \theta_F, \mathfrak{f}_F)$.

Note that a cotangent bundle is a Weinstein manifold with a single co-core being the cotangent fibre. A first consequence is to conclude the proof that Symplectic homology is Hochschild homology of the wrapped category (as conjectured in [Sei09]). Other consequences of this result start to appear in the literature: it allows to prove that some triangulated categories (of sheaves, Lagrangian or generating family) are isomorphic only checking on those generators (assuming similar generation results have been proved on the other sides). See [Laz18], [GPS18] or [Kar18] for recent applications.

What is in there. I have tried in this memoir to give an account of the objects and ideas used in the proof of the theorems that I have stated in the introduction. Chapter 1 is a preliminary chapter where I introduce the basic definitions building the field of contact and symplectic geometry. It also describes the notion of Lagrangian cobordisms which allow me to speak a little about some early papers of mine about the theory ([Cha15b], [Cha12] and [Cha15a]). Then we will rush through the definition of the various moduli spaces of holomorphic curves we will consider in the rest of the document. This chapter is neither comprehensive nor introductory to the field (even if the first three pages might give that impression); I hope though that it fixes notations and defines objects well enough to make it less painful for the reader to follow the rest of the text. In Chapter 2 the reader will find the definition of the central object of text: augmentations of Legendrian sub-manifolds. I tried to give a definition general enough in terms of coefficients taking into account the homotopy classes of the boundary of holomorphic discs to cover the many uses we will make of it. In the whole text I restrict to graded theory and assume thus that Lagrangians and Legendrians are all graded. The reader familiar with the theory will know immediately how to modify some of the statements in the ungraded case (or graded modulo some Maslov class). We then proceed to define the Floer complex associated to Lagrangian cobordisms from [Cha+15b] . This chapter is therefore and account of the first half (the foundational part) of that paper. Applications will be discussed in the last chapter. Chapter 3 is a summary of the construction of the augmentation category from [BC14b] with the modification from [Cha+16] to take into account coefficients in group rings. We also discuss the main functorial property of this category. Finally Chapter 4 presents applications of the theory of the two precedent chapters to the topology of Lagrangian cobordisms, the structure of this relation and the generation of the wrapped Fukaya category. It gives account of the application part of [Cha+15b], presents the results from [CCD19] and summarises some of the arguments from [Cha+17].

Every once in while I use a “Perspective” environment where I want to highlight that some ideas could be explored further for future development of the subjects. Some of those perspectives emerge from my work and suggest where I want my research to possibly go. Some others present open questions of work of other authors about which I believe some of the work exposed here can be relevant. I also sometime use a "Verification" environment where the reader will find explicit formulas on some concrete Legendrian sub-manifolds and Lagrangian cobordisms. Every time I proved something and failed to do this type of verification, the result turned out to be wrong on a scale evolving from very wrong to some of the grading being incorrect (one can be surprised on the impact of the fact that Lagrangian cobordisms have dimension $n+1$). I should ask my subconscious why I still decide not to do those verifications on some occasions. In an appendix I have made a list of some of those explicit objects that I find useful to have in order to do those verifications.

What is not. The first thing that is not here are proofs. I have decided not to reproduce proofs of results which are written in the paper I talk about. Instead I used this memoir to organise and present my work in a unified way and tried to convey some of the intuitions behind constructions I described and motivations for why my work followed certain directions. After most of the statements I present some arguments which highlight the scheme of the proof or the geometric idea that lead to the construction of the actual proof. All the proofs are in the paper and are sometime different from this first geometric ideas (they are usually quite long and try to use the analytic technology available and not speculated one). For that reason this memoir will not have the pretension to provide any original results.

The second thing that is not here is some account of the work in [CM19]. It studies Lagrangian intersection in conformal symplectic manifolds (manifolds with a conformal symplectic atlas). It proves notably some intersection properties in conformal cotangent bundle in terms of Morse-Novikov Betti number of some closed 1-forms. Though it relates to Legendrian sub-manifolds (as exact Lagrangians in this context lifts to Legendrian sub-manifolds), the whole article is too disconnected from the rest of the results exposed here and would not fit in any of the sections. We however feel that this subject could potentially be of importance to study some rigidity properties of contact manifolds through special conformal symplectic fillings.

The last notable thing that is note here are signs. I decided to focused on the details of the results for fields of characteristic 2 as coefficients (or algebras over such) and not having signs in the formulas. This is first because I do not think I have a true contribution to the sign treatment of the subject (I have been using the existing literature) and second because all those results are interesting even with mod 2 coefficients. Theorem 0.0.6 is an exception where $\mathbb{C}$ coefficients are important but I will explain how signs are needed in due time.

Chapter 1

Preliminaries.

In this chapter we collect various definitions and constructions in the field of contact and symplectic geometry. Our goal is to introduce the notion of Lagrangian cobordism between Legendrian sub-manifolds and we try to highlight some questions relevant to the study of such objects. Most of the results precede any of the author's work but some subtle points of the definition of Lagrangian cobordisms and some of its properties were raised and studied in [Cha10; Cha15a] and [Cha12]. In Section 1.1 we introduce the notion of symplectic and contact manifolds together with the notions of Lagrangian and Legendrian sub-manifolds. We also introduce the notions of Weinstein manifolds and sectors. Then in Section 1.3 we define Liouville and Lagrangian cobordisms. Section 1.4 rushes through the definitions essential for the study of holomorphic curves in symplectic cobordisms. It is not intended to be complete but we hope we give enough of the theory so that the reader can follow the definition of moduli spaces we will consider in the rest of the document, see what type of action of asymptotics we consider and find a dimension formula for these moduli spaces.

1.1 Contact and symplectic geometry.

1.1.1 Contact manifolds.

We begin with the contact manifold in which Legendrian sub-manifolds arises the most naturally. The definition of contact manifold will rely on this example.

Example 1.1.1. Let Q be a smooth manifold. We denote by $\mathbb{P}(T^*Q)$ the set $\{(q,H)|q \in Q\ H$ hyperplane of $T_qQ\}$. It is called the *space of contact elements* of Q. It has the structure of a smooth manifold obtained via the identification $\mathbb{P}(T^*Q) \simeq (T^*Q \setminus Q_0)/\mathbb{R}^*$ (where Q_0 is the zero section and $\mathbb{R}^*$ acts on fibres). We denote the obvious projection to Q by $\pi : \mathbb{P}(T^*Q) \to Q$. There is a canonical hyperplane distribution ξ^{can} on $\mathbb{P}(T^*Q)$ determined by $X \in \xi^{can}_{(q,H)} \Leftrightarrow d\pi(X) \in H$.

Definition 1.1.2. A *contact manifold* is given by a couple (Y, ξ) where Y is a smooth manifold of dimension $2n-1$ and ξ is a hyperplane distribution such that for all $x \in Y$ there exists a neighbourhood $\mathcal{U}_x$ of x and an embedding $\phi : \mathcal{U}_x \to \mathbb{P}(T^*\mathbb{R}^n)$ such that $\phi_*\xi = \xi^{can}$.

Definition 1.1.3. A *contactomorphism* between two contact manifolds (Y, ξ) and (Y', ξ') is a diffeomorphism $\phi : Y \to Y'$ such that $\phi_*\xi = \xi'$. The group of contactomorphisms from Y to itself is denoted $\mathrm{Cont}(Y, \xi)$.

The distribution ξ from Definition 1.1.2 is called a *contact structure*. A more manageable definition follows from the equivalence given by the so-called Darboux Theorem.

Theorem 1.1.4 (Darboux Theorem). *Let (Y, ξ) a smooth manifold with a hyperplane distribution. Then ξ is a contact structure iff for any local equation α of ξ (i.e. locally $\xi = \ker \alpha$) we have $\alpha \wedge d\alpha^{n-1}$ is non degenerate.*

When ξ is co-orientable, there exists a globally defined equation α, such a 1-form is called a *contact form*.

Example 1.1.5. The contact structure ξ^{can} on $\mathbb{P}(T^*Q)$ is not co-orientable, i.e. it does not admit a globally defined equation. Indeed if H_t is a path of hyperplane in T_qQ which makes a rotation of angle π along an axis (of dimension $n-2$) in H_0 then the co-orientation of ξ^{can} along (q, H_t) is reversed. The co-orientation cover of ξ^{can} on $\mathbb{P}(T^*Q)$ is identified with

$$S(T^*Q) = \{(q,p)|p \in T^*_qQ\ ||p|| = 1\}$$

for a choice of a bundle metric on T^*Q. A contact form is then given by the pullback α of λ. Here λ is the tautological form on T^*Q defined by $\alpha^*\lambda = \alpha$ for any section $\alpha : Q \to T^*Q$. Indeed the hyperplane at (q,p) is given by $\pi^{-1}\ker p$ which is by definition $\ker \alpha$.

Another natural example is given by the space of 1-jets of functions.

Example 1.1.6. The space of 1-jet of functions on a manifold Q, $\mathcal{J}^1(Q)$, is identified with $T^*Q \times \mathbb{R}$. Given a 1-jet $[f]$ at q we denote its 0-jet $\Pi_q[f] = f(q) \in \mathbb{R}$, this defines a projection $\Pi : \mathcal{J}^1(Q) \to Q \times \mathbb{R}$. We denote by b the projection to the base Q. At a jet $(q, [f])$ we define a natural hyperplane $\xi_{(q,[f])}$ by

$$X \in \xi_{(q,[f])} \Leftrightarrow df_q(db(X)) = d\Pi_q(X).$$

This contact structure actually admits a natural equation: under the identification of $\mathcal{J}^1(Q)$ with $T^*Q\times\mathbb{R}$ it is given by $\xi = \ker(dz - pdq)$. It naturally embeds in the space of contact elements of $Q \times \mathbb{R}$: to a jet $[f]$ at q we associates the (well defined!) contact element $(q, f(q), \ker(dz - df_q))$. One checks easily that this embedding maps ξ to ξ^{can} (in terms of equations this is implied by the fact that any non-vertical hyperplane has an equation of the form $z = \sum_i a_i q_i$).

A sub-manifold $N \subset Q$ gives a natural sub-manifold

$$\mathrm{Nil}(N) = \{(q, p) | q \in N\ , p(T_qN) = 0\} \subset \mathbb{P}(T^*\mathbb{Q})$$

which is tangent to ξ^{can} called the *co-normal* of N. Whatever the dimension of N is, $\mathrm{Nil}(N)$ is always of dimensions $\dim(Q) - 1$.

In general let Λ be a sub-manifold tangent to ξ^{can} and let $(q, H) \in \Lambda$. We denote by $W = d\pi_{(q,H)}(T_{(q,H)}\Lambda)$ and $V = T_{(q,H)}\Lambda \cap T_{(q,H)}\pi^{-1}(q)$. Recall that in a vector space E of dimension n, k-dimensional sub-spaces of $T_H\mathbb{P}(E^*)$ corresponds to $n - 1 - k$ dimensional sub-spaces of H. So to V corresponds a $n - 1 - \dim V$ subspace V' of H. From the fact that Λ is tangent to ξ^{can} a computation shows that $W \subset V'$. This implies that $\dim \Lambda = \dim W + \dim V \leq n - 1 - \dim V + \dim V = n - 1$.

Thus a co-normal always has the maximal dimension that a sub-manifold tangent to ξ^{can} can have. This lead to the following definition:

Definition 1.1.7. A sub-manifold Λ of a contact manifold (M^{2n+1}, ξ) is *Legendrian* if $dim\Lambda = n$ and $T\Lambda \subset \xi$.

Example 1.1.8. The jet of a function $f : Q \to \mathbb{R}$, $j^1(f) = \{(q, f(q), df_q)|q \in Q\}$ is a Legendrian sub-manifold of $\mathcal{J}^1(Q)$. (Through the contact embedding $\mathcal{J}^1(Q) \to \mathbb{P}(T^*(Q \times \mathbb{R})$ of Example 1.1.6 the jet of function becomes the co-normal of the graph of f).

Definition 1.1.9. A *Legendrian isotopy* is an isotopy $\{\Lambda_t\}$ such that for all t Λ_t is a Legendrian sub-manifold. We say in that situation that Λ_0 and Λ_1 are Legendrian isotopic.

Perspective 1. Of course isotopic sub-manifolds of Q leads to Legendrian isotopic sub-manifolds of $\mathbb{P}(T^*Q)$. To which extend the converse is true or not is an exciting question admit a few partial answers (see for instance [She16] and [ENS18] for co-normals of knot in $\mathbb{R}^3$). Exhibiting similar phenomena in higher dimension is an exciting project that would test the precision of the known Legendrian sub-manifold invariants.

Infinitesimal symmetries of contact manifolds $(M, \xi = \ker \alpha)$ are encoded by functions on M. Indeed let X be a vector field whose local flow preserves ξ; we claim that X is characterised by the function $H : M \to \mathbb{R}$ (called the *contact Hamiltonian*) given by $H(q) = \alpha_q(X_q)$. Indeed this function characterises X on the quotient TQ/ξ. On the other end Cartan's formula gives $\mathcal{L}_X \alpha = d(\alpha(X)) + \iota_X d\alpha$, the kernel of α is preserved along the flow of X and $d\alpha$ is non degenerate; thus if X is in ξ then it vanishes (as $\alpha(X) = 0$). This implies that two such contact vector fields with the same contact Hamiltonian coincide.

Remark 1.1.10. *We could (and should) have not chosen a contact form α and then have seen more elegantly H as a section of TQ/ξ. But considering holomorphic curves forces us to choose some contact form and thus we will always talk about contact Hamiltonians as being functions.*

The contact vector field R_α associated to $H \equiv 1$ is characterised by $\iota_{R_\alpha} d\alpha = 0$ and $\alpha(R_\alpha) = 1$. It is called the *Reeb* vector field of the contact form α. In the example of $\mathcal{J}^1(Q)$ it is given by the vertical vector field $\frac{\partial}{\partial z}$.

We conclude this section recalling that any Legendrian sub-manifold has a standard neighbourhood: if $\Lambda \subset (M, \xi = \ker \alpha)$ then there exist a neighbourhood $\mathcal{U}$ of Λ and an embedding $\phi : \mathcal{U} \to \mathcal{J}^1(\Lambda)$ such that

- $\phi(\Lambda) = \{(q, 0, 0) | q \in \Lambda\}$.
- $\phi^*(dz - \lambda) = \alpha$ (in particular ϕ maps the contact distribution to the canonical distribution).

We refer to such a neighbourhood as a *Weinstein* neighbourhood of Λ.

1.1.2 Symplectic manifolds.

As this document relates Legendrian invariants to symplectic invariants, we proceed now to give briefly the basic definitions from symplectic geometry necessary in order to follow the exposition. This will be more succinct than the (already succinct) exposition on contact geometry but we hope that the examples we give are sufficient for the understandability of the text.

We begin by deprojectivising the space of contact elements. A point (q, p) of the cotangent bundle T^*Q such that $p \neq 0$ determines a contact element (q, H) via $H = \ker p$. This gives a smooth projection $\Pi : T^*Q \backslash Q_0 \twoheadrightarrow \mathbb{P}(T^*Q)$. As it replaces the space of all hyper-planes in Q by the space of all equations of hyper-planes, the space $T^*Q \setminus Q_0$ carries a natural one form λ defined by $\lambda_{p,q}(X) = p(d\pi(X))$ where π is the natural projection to Q (this is the tautological form from Example 1.1.6). This form (defined similarly on the whole T^*Q) is called the *Liouville form* of T^*Q. Its kernel is the pre-image of ξ^{can} via the projection Π.

The cotangent bundle with its canonical form is the prototype of a Liouville manifold (an exact symplectic manifold). A *symplectic manifold* is a manifold M together with an atlas $\{\mathcal{U}_i \to \mathbb{R}^{2n}, i \in I\}$ such that all transition functions ϕ_{ij} maps pdq to $pdq + dG$ for some function G. This is the same as asking for the existence of an atlas such that transitions functions preserves the 2-forms $\omega_0 = dq \wedge dp$ (one might have to take a refinement of the latter so that it satisfies the exactness condition). Thus a symplectic manifold comes with a natural 2-forms ω called the symplectic form, which is closed and non-degenerate.

Remark 1.1.11. *The symplectic analogue of Darboux Theorem state that such a non-degenerate closed two form determines an equivalence of symplectic atlases.*

Being non degenerate ω determines an identification $T^*M \simeq TM$, the vector associated to a 1-form α being denoted X_α. As there might be some sign ambiguity with this identification we fix it once for all saying that X_α is characterised by $\omega(X_\alpha, Y) = \alpha(Y)$.

When ω is exact, (M, ω) is called an exact symplectic manifold. A primitive λ of ω gives a particular vector field X_λ called the *Liouville vector field*. Note that $\mathcal{L}_{X_\lambda}\omega = d(\iota_{X_\lambda}\omega) = d\lambda = \omega$. Thus the flow of X_λ expands the symplectic form. We call (M, λ) a *Liouville manifold* if:

1. X_λ is complete.

2. Outside a compact set $\overline{M}$ with smooth boundary Y X_λ has no critical points.

3. X_λ points outside $\overline{M}$ along Y.

The cotangent bundle T^*Q with the form $-pdq$ is an example of a Liouville manifold, Y can be chosen to be the sphere bundle $S(T^*Q)$ for a metric on T^*Q. It is not a coincidence that $S(T^*Q)$ carries a contact forms: indeed for any Liouville manifolds (M, λ) the manifold Y carries a natural contact structure given by $\ker i^*\lambda$ where i is the inclusion of Y in M. Since X_λ has no zeros outside of $\overline{M}$ we have an identification $M \setminus \overline{M}$ with $Y \times (0, \infty)$. Under this identification X_λ is mapped to $\frac{\partial}{\partial t}$ and λ becomes $e^t\alpha$.

If ϕ is a contactomorphism of $\mathbb{P}(T^*Q)$ then it induces an exact symplectomorphism of T^*Q: let $(q, [p]) \in \mathbb{P}(T^*Q)$ and denote $\phi(q, [p]) = (q', [p'])$. Since ϕ is a contactomorphism we have $\Pi^*p(X) = 0 \Leftrightarrow \phi^*\Pi^*p'(X) = 0$, therefore $\phi^*\Pi^*p' = \mu_{q,p,p'}\Pi^*p$ for some non-zero $\mu_{q,p,p'}$. The numbers $\mu_{q,p,p'}$ satisfy $\mu_{q,ap,bp'} = a^{-1}b\mu_{q,p,p'}$, and this allows us to defines $\tilde{\phi}(q, p) = (q', \mu_{q,p,p'}^{-1}p')$. One computes that $\tilde{\phi}^*\lambda = \lambda$ and that $\widetilde{\phi}$ commutes with the projection to $\mathbb{P}(T^*Q)$. Since any contact manifold is modelled on $\mathbb{P}(T^*Q)$, this implies that any contact manifold (Y, ξ) "deprojectivises" to an exact symplectic manifold $(S(Y, \xi), \lambda)$. One can describes $S(Y, \xi)$ by $S(Y, \xi) = \{(q, p) | \ker p = \xi_q\} \subset T^*Y$ with the Liouville form given by the restriction of the canonical form. The choice of a contact form α identifies $(S(Y, \xi), \lambda)$ with $(\mathbb{R}^* \times Y, s\alpha)$, the connected component corresponding to positive reals being usually identified with $(\mathbb{R} \times Y, e^t\alpha)$ and called the *symplectisation* of (Y, ξ).

Remark 1.1.12. *The exact symplectic type of the symplectisation only depends here on the co-orientation of ξ given by α. One could (and probably should) call the whole $S(Y, \xi)$ the symplectisation of (Y, ξ) but we will not make that choice as it would be less convenient when talking about holomorphic curves type invariants.*

A natural notion for symmetries of symplectic manifold is the one of *symplectomorphism*, namely diffeomorphism which preserves ω. A stronger type of symmetries is given by Hamiltonian diffeomorphisms. Given a function $H : M \times [0, 1] \to \mathbb{R}$, one defines a time-dependent vector field $X_H(t)$ by $X_H(t) = X_{dH_t}$. The induced isotopy preserves ω, and ϕ_H^1 is called a *Hamiltonian diffeomorphism*. The set of Hamiltonian diffeomorphisms form a normal subgroup of the group of symplectomorphisms and these will be the main symmetries we will consider in the rest of the document.

Verification 1. At that point we had better verify that with all our conventions the Hamiltonian vector for the kinetic energy $\frac{1}{2}p^2$ on $T^*\mathbb{R}^n$ has the correct sign. Indeed $dH = pdp$ and $\omega = dq \wedge dp$ thus $\omega(X_H, \cdot) = pdp$ gives $X_H = p\frac{\partial}{\partial q}$, that looks fine.

Pre-image of Legendrian sub-manifolds under the projection $\Pi : S(Y, \xi) \to (Y, \xi)$ are sub-manifolds on which the symplectic form vanishes. Considerations similar to the one in previous section shows that those have maximal dimension amongst sub-manifolds having this properties. This lead to the following definitions:

Definition 1.1.13.

- An immersion $i : L \looparrowright M$ into a symplectic manifold (M^{2n}, ω) is called *Lagrangian*, if $\dim L = n$ and $i^*\omega = 0$. If furthermore i is an embedding then $i(L)$ is called a *Lagrangian* sub-manifold of (M, ω)

- An immersion $i : L \looparrowright M$ into a Liouville manifold (M^{2n}, λ) is called *exact* if $i^*\lambda$ is an exact form. If in addition i is an embedding then $i(L)$ is called an *exact Lagrangian* sub-manifold of (M, λ)

As it was the case for Legendrian sub-manifolds a Lagrangian sub-manifold has a standard neighbourhood. If $L \subset M$ is Lagrangian, then there exist a neighbourhood $\mathcal{U}$ of L in M and an embedding $\phi : \mathcal{U} \to T^*L$ such that:

1. $\phi(L) = \{(q, 0) | q \in L\}$.

2. $\phi^*(dq \wedge dp) = \omega$.

We finish this section by showing another interaction between contact and symplectic manifolds. Associated to a Liouville manifold (P, λ) there is a contact manifold $(P \times \mathbb{R}, \ker(dz + \lambda))$ called its *contactisation*. Any exact Lagrangian immersion lifts to Legendrian immersion via $\tilde{i}(q) = (i(q), -f(q))$ which is well defined up to a choice of primitive f of $i^*\lambda$. If i is generic then $\tilde{i}$ is an embedding.

Note that the symplectisation of such a contactisation is identified with $(P \times T^*(0, \infty), \lambda + tds)$ via the map

$$(t, q, z) \to (\phi^t_\lambda(q), t, z). \tag{1.1}$$

(Where ϕ^t_λ is the the flow of X_λ).

1.2 Grading.

From now on we assume that all symplectic manifolds (including symplectisations!) we consider will have $2c_1(M) = 0$. We also assume that all Lagrangian sub-manifolds considered have vanishing Maslov class. This has the following consequence: denote $Gr(M, \omega) \to M$ the fibration where the fibre over a point q consists of all n-plane in T_qM on which ω vanishes (called the *Lagrangian Grassmanian fibration*). The Gauss map of a Lagrangian immersion $i : L \to M$ gives a map $\tilde{i} : L \to Gr(M, \omega)$. We denote by $Gr^\sharp(M, \omega)$ the fibration whose fibre over each point q is given by the universal cover of the fibre of $Gr(M, \omega)$. For a Lagrangian with vanishing Maslov class $\tilde{i}$ lifts to $Gr^\sharp(M, \omega)$. The pair formed by the Lagrangian and the choice of such a lift gives the notion of a *graded Lagrangian* immersion. We might forget to state that Lagrangians are graded and apologise in advance if that causes confusion but **all Lagrangians are graded in this document**.

Similarly for a contact manifold of dimension $2n+1$ we denote by $Gr(M, \xi)$ the fibration whose fibre is given by n-planes in ξ_q one which $d\alpha$ vanishes (note that this does not depend on the local equation α) and we have the similar notion of *graded* Legendrian sub-manifold.

We will not recall the construction here but given two graded Lagrangian immersions L_0 and L_1, a transverse intersection point q between L_0 and L_1 is equipped with an index $i_{L_0,L_1}(q) \in \mathbb{Z}$. This is standard and we refer to [Sei08, Section 11] for a clear treatment.

Given a contact form α on a contact manifold (M, ξ) and a Legendrian sub-manifold Λ, a trajectory γ (of length T) of the Reeb vector field of α that starts and ends on Λ is called a *Reeb chords* of Λ. It is called non degenerate of $\phi_\alpha^T(T_q\Lambda)$ is transverse to $T_{q'}\Lambda$ where q and q' are the starting and endings point of γ. A non-degenerate Reeb chord is endowed with an index $i_\Lambda(\gamma) \in \mathbb{Z}$. Again we do not give details and refer to [EES05] for details. We give a few values on some examples via the following

Verification 2. The trivial Legendrian unknot Λ_0 (the Whitney sphere) of Appendix A in $\mathcal{J}^1(\mathbb{R})$ has a unique Reeb chord γ. We have $i_{\Lambda_0}(\gamma) = 1$. In general the Legendrian Whitney sphere in $\mathcal{J}^1(\mathbb{R}^n)$ has a unique Reeb chord of grading n. Thus $i_\Lambda(\gamma) = n$.

If L is the exact Lagrangian in T^*Q given by (q, df_q) for a Morse function f on Q, then an intersection between the zero section Q_0 and L corresponds to critical point q of f. Both Q_0 and L come with a natural grading (since L is a the image of Q_0 via the Hamiltonian flow given by $f \circ \pi$), for this choice we have $i_{Q_0,L}(q) = i_f(q)$ (i.e. the number of negative eigenvalues of the Hessian of f at q). Lifting L and Q_0 to Legendrians in $\mathcal{J}^1(Q)$ (using the function f as potential) each critical point q give a Reeb chord γ_q of the Legendrian link and we have

$$i(\gamma_q) = \begin{cases} i_f(q) - 1 \text{ if } f > 0 \\ n - i_f(q) \text{ if } f < 0. \end{cases}$$

Figure A.2 shows the projection of a Legendrian knot Λ in $\mathcal{J}^1(\mathbb{R})$ to $T^*\mathbb{R}$ with 5 Reeb chords. We have $i(a_i) = 1$ and $i(b_i) = 0$.

If Λ is the lift of an exact Lagrangian immersion L in a Liouville manifold P as in the end of Section 1.1.2 then a Reeb chord γ of Λ corresponds to a self-intersection point of L. A grading of L immediately induces one of Λ, for that choice we have $i_L(q) = i_\Lambda(\gamma) + 1$.

The index of chords or intersection points will be related to the degree of the generators of the chain complexes that will be considered in Chapters 2 and 3. For an intersection point we will denotes its degree by $gr(q) := i_{L_0,L_1}(q)$ and for a chord it will be $gr(\gamma) := i_\Lambda(\gamma) + 1$. The reason for this convention is that the index allows us to write dimension formulas more conveniently (see Equation (1.3) and subsequent ones) whereas the degree is more in adequation with the expected degree for operations in $\mathcal{A}_\infty$-algebras.

1.3 Cobordisms and fillings

We now introduce the various notions of cobordisms that we will consider in the rest of the document.

1.3.1 Liouville cobordisms

Let (Y^-, ξ^-) and (Y^+, ξ^+) be two contact manifolds. A *Liouville cobordism* from Y^- to Y^+ is an exact symplectic manifold $(X; \lambda)$ so that:

1. The Liouville vector field X_λ is complete.

2. There is a compact set $\overline{X}$ with oriented boundary $Y^+ \sqcup -Y^-$ outside of which X_λ has no critical points.

3. X_λ points inward $\overline{X}$ at Y^- and outward $\overline{X}$ at Y^+.

4. The pullback of λ on Y^- is a contact form α^- for ξ^-.

5. The pullback of λ on Y^+ is a contact form α^+ for ξ^+.

The complement of $\overline{X}$ is thus made of two connected component that we identify with $(-\infty, 0) \times Y^-$ with Liouville form $e^t\alpha^-$ (called the *negative end* of X) and $(0, \infty) \times Y^+$ with Liouville form $e^t\alpha^+$ (called the *positive end* of X).

The symplectisation is a cobordism from a contact manifold to itself. A Liouville manifold is a Liouville cobordism from the empty set to a contact manifold (Y, ξ) (we refer to such object as a *Liouville filling* of (Y, ξ)). The fact that the symplectisation does not depend on the contact form allows us to concatenate Liouville cobordisms (note that one of the Liouville form usually has to be re-scaled). Therefore being Liouville cobordant forms a reflexive transitive relation on the set of contact manifolds.

In order to simplify the notation we will sometime denote by $\overline{X}_T$ the compact manifold $[-T, 0] \times Y^- \cup \overline{X} \cup [0, T] \times Y^+$. Given a cobordism X_0 from (Y^-, α^-) to (Y, α) and X_1 from (Y, α) to (Y^+, α^+) (note that we fixed the contact forms) we denote by $X_0 \odot_T X_1$ the Liouville cobordism from (Y^-, α^-) to $(Y^+, e^T\alpha^+)$ obtained by concatenation of X_0 and X_1 with $(0, T) \times Y$ as overlapping piece in the middle.

1.3.2 Lagrangian cobordisms.

We define now the notion of Lagrangian cobordism.

Definition 1.3.1. Let Λ^- and Λ^+ be two closed Legendrian sub-manifolds in some contact manifold (Y^+, ξ^+) and (Y^-, ξ^-) respectively. An *exact Lagrangian cobordism from* Λ^- *to* Λ^+ in a (X, λ) is a properly embedded submanifold $\Sigma \subset \mathbb{R} \times Y$ without boundary satisfying the following conditions:

1. for some $T \gg 0$,

   (a) $\Sigma \cap ((-\infty, -T) \times Y^-) = (-\infty, -T) \times \Lambda^-$,

   (b) $\Sigma \cap ((T, +\infty) \times Y^+) = (T, +\infty) \times \Lambda^+$, and

   (c) $\Sigma \cap \overline{X}_T$ is compact;

2. There exists a smooth function $f_\Sigma : \Sigma \to \mathbb{R}$ for which

   (a) $\lambda|_{T\Sigma} = df_\Sigma$,

(b) $f_\Sigma|_{(-\infty,-T)\times\Lambda^-}$ is constant.

We will call $(T,+\infty)\times\Lambda_+\subset\Sigma$ and $(-\infty,-T)\times\Lambda_-\subset\Sigma$ the *positive end* and the *negative end* of Σ, respectively. We will call a cobordism in a symplectisation from a sub-manifold to itself an *endocobordism*.

Remark 1.3.2. *Condition 2(b) is empty when the negative end of the cobordisms is connected. The similar condition as 2(b) for the positive end is vacuous as there is enough room in the positive end of the symplectisation to adjust the values of the primitive on the connected component as we want: just make a tiny push of the trivial part of the cobordism at time T and wait for the exponential to be big enough so that pushing it back to its original position make the desired adjustment.*

As mentioned in Section 1.1.2 the cylinder over a Legendrian sub-manifolds Λ is a Lagrangian endocobordism of Λ. More generally Legendrian isotopic sub-manifolds are Lagrangian endocobordant. This was shown in [EG98, Lemma 4.2]. Not being aware of this result we gave an alternative description of such an endocobordism in [Cha10, Theorem 1.2]. The former construction rely on a real result on decomposition of Legendrian isotopies into positive and negative pieces whereas the latter is a trivial consequence of the fact that contactomorphisms are generated by Hamiltonian functions (and hence the name theorem and lemma should be probably swapped). We tend to favour the first construction because it allows us to relate Lagrangian cobordisms to positive Legendrian isotopies (see Section 4.4). The second construction has been used to extract information on the length of cobordisms between two fixed Legendrian sub-manifolds (see [ST17] for instance).

Condition (2b) will later be used to rule out certain bad breaking of pseudoholomorphic curves. That this condition is of importance for the study of functorial properties of Legendrian contact homology and rigidity of cobordisms has been shown in a small note [Cha15a] where some relevant explicit examples of cobordisms are given to exhibiting unwanted behaviour of cobordisms when this condition is removed.

In the case when there exists an exact Lagrangian cobordism from Λ^- to Λ^+ we say that Λ^- *is exact Lagrangian cobordant to* Λ^+. If Σ is an exact Lagrangian cobordism from the empty set to Λ, we call Σ an *exact Lagrangian filling* of Λ. We then say that Λ is *exactly fillable.*

The group $\mathbb{R}$ acts on $\mathbb{R}\times Y$ by translations in the first factor. For any

$s \in \mathbb{R}$ we define

$$\tau_s \colon \mathbb{R} \times Y \to \mathbb{R} \times Y,$$
$$\tau_s(t,p) = (t+s,p).$$

It is easy to check that the translate of an exact Lagrangian cobordism still is an exact Lagrangian cobordism.

Definition 1.3.3. Given exact Lagrangian cobordisms Σ_a from Λ^- to Λ in (X_0, λ_0) (a Liouville cobordism going from Y^- to Y) and Σ_b from Λ to Λ^+ in (X_1, λ_1) (going from Y to Y^+), their *concatenation* $\Sigma_a \odot \Sigma_b$ is defined as $\Sigma_a \cup \Sigma_b$ in $X_0 \odot_T X_1$ (note that by construction they match on the overlapping piece).

Condition (2b) of Definition 1.3.1 together with Remark 1.3.2 imply that (if one perturbs slightly the positive end of Σ_a) $\Sigma_a \odot \Sigma_b$ is an exact Lagrangian cobordism from Λ^- to Λ^+. When both the cobordisms are in the symplectisation of a contact manifold $X_0 = X_1 = \mathbb{R} \times Y$ then $X_0 \odot X_1$ is identified with $\mathbb{R} \times Y$ and thus $\Sigma_a \odot \Sigma_b$ is a cobordism in the symplectisation as well.

1.3.3 Construction of cobordisms.

Front of cobordisms. The most useful trick to construct Lagrangian cobordisms is to use the symplectomorphisms (1.1) when the contact manifolds is the Jet space of $\mathbb{R}^n$ with coordinate (q,p,z). Indeed in this situation the symplectisation has coordinates (t,q,p,z) with $t > 0$ and Liouville form $e^t(dz - pdq)$. Its differential is $d(e^t) \wedge dz + dq \wedge d(e^t p)$ and therefore it is symplectomorphic (as stated in Equation (1.1)) to $T^*\mathbb{R}^n \times \mathbb{R}^*_+$ by the map $(t,q,p,z) \to (q, e^t, e^t p, z)$. The pull-back of $pdq + zds$ is then $e^t pdq + zd(e^t) = -e^t(dz - pdq) + d(e^t z)$. Therefore a Lagrangian that is exact for the Liouville form on the symplectisation with potential f stays exact for the standard form on the cotangent bundle with potential $-e^t f + e^t z$ (where z should be understood as the restriction of z to the Lagrangian. This allows us to lift exact Lagrangian cobordisms to Legendrians $\mathcal{J}^1(M \times \mathbb{R}^*_+)$ and look at their front projection to draw such cobordisms. Note that through this identification the ends of the cobordism (near 0 and ∞) have an exponential growth. Since it comes from an embedded Lagrangian, this front has no parallel tangent planes over any fixed value $(q,t) \in \mathbb{R}^n \otimes \mathbb{R}^*_+$. Looking at such a front as a $\mathbb{R}^*_+$ family of fronts, this imply that the length of a Reeb chords along this family always varies (i.e. the length function has non-zero derivative). Combining this fact with the restriction near the end (lengths there are always increasing), that means that if you draw a bifurcation of fronts you can safely think of it as an exact Lagrangian cobordisms as long as Reeb chords have increasing length. That means you can easily create Reeb chords but it makes it harder for you to destroy them (going toward ∞). With that in mind this recover the list of possible bifurcations along a cobordisms in dimension 1 given by Figure 1.1.

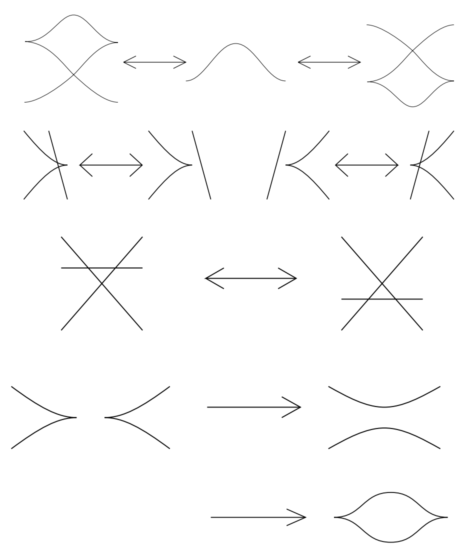

Figure 1.1: Bifurcations of the front projection of a Legendrian knot along a Lagrangian cobordisms.

The first three moves corresponds to Legendrian Reidemeister moves of Legendrian isotopies. The last one is a minimum cobordism which makes a Whitney sphere appearing and the 4th is a saddle surgery. In the last two moves a chord is created and hence we can only go toward ∞ with this local moves.

Remark 1.3.4. *This discussion also shows that if one reverses the direction on the front coming from a Lagrangian cobordism, rectifying it to have the correct exponential growth creates as many immersion points of the Lagrangian projection as we have chords at both ends of the cobordisms. When* *ends have no chords we can perform this reversion of time and also we can make the end linear leading to cobordisms considered in [BC14a] and [BC13]*

Lagrangian surgery. The fourth move of Figure 1.1 generalises to all dimensions, which gives a Legendrian description of the local modification of a Lagrangian near its intersection called Lagrangian surgery (see [LS91] and [Pol91]). The local Legendrian picture of the front of such a surgery is given by Figure 1.2. In this picture the surgery corresponds to removing the Reeb chord going from right to left. As we can create chord along a Lagrangian cobordisms, the two local picture from left to right can be linked by an exact Lagrangian cobordism.

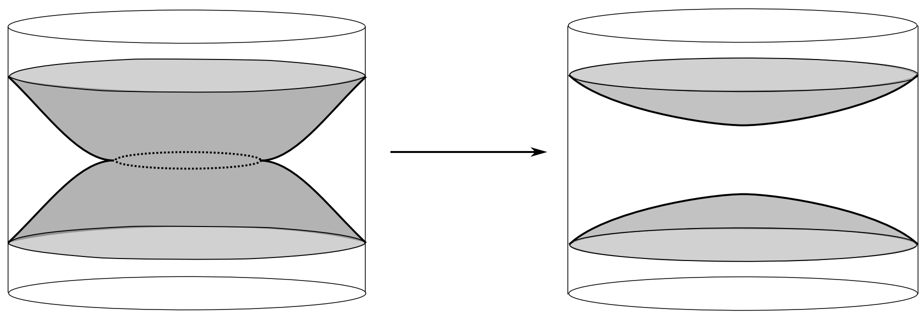

Figure 1.2: The (critical) surgery cobordism.

Remark 1.3.5. *Usually Lagrangian surgery is described on the level of Lagrangian intersection points and does not consider Legendrian lifts (explicitely). The description in terms of Legendrian sub-manifolds has the advantage to guarantee that the results of the surgery projects to an exact Lagrangian. The general Lagrangian surgery procedure does not usually preserve exactness (indeed we can do it on any transverse intersection point and not only on one which has a neighbourhood lifting as in Figure 1.2). We can however describes this general surgery procedure using the Legendrian picture in the following way: we take Legendrian lifts locally in a standrad Darboux chart (which amount to lifting two Lagrangian disks), we perform the modification to this lift and then project back to the symplectic space. As we can choose which disk we put on top of the other we get the description of the two standard Lagrangian surgeries this way.*

In order to perform such a surgery to several point at the same time we introduce the following terminology.

Definition 1.3.6. A set of Reeb chords $\{a_1, \ldots, a_k\}$ on Λ is called *contractible* if,

- The chords a_i have index 1.
- Each chord a_i have a neighbourhood as the one on the right side Figure 1.2. All those neighbourhoods can be choosen to be disjoints (see [Cha+17, Definition 8.5] for the precise quantitative description of these neighbourhoods).

Remark 1.3.7. *This is a restrictive assumption because, in general, the lengths of the chords $a_1, \ldots, a_k$ cannot be modified independently. An example when this is possible is when $\mathbb{L}^+$ is a link with $k+1$ components, all a_i are mixed chords, and each component contains either the starting point or the end point of at least one of the a_i. In this situation we can indeed modify the Legendrian link by Legendrian isotopies of each his components so that its Lagrangian projection is unchanged and all the previous conditions on the neighbourhoods are satisfied. (Warning: this might not be an isotopy of the Legendrian* link.*)*

It follows from the previous consideration that if $\{a_1, \ldots, a_k\}$ is a contractible set of Reeb chords then there is a cobordisms from $\Lambda(a_1, \cdots, a_k)$ to Λ where $\Lambda(a_1, \cdots, a_k)$ is obtained from Λ by performing surgeries on each of the a_i.

Other cobordisms. There are cobordisms which cannot be described using this bifurcations, namely Lagrangian caps from [Lin16] and [EM13]. All those have negative ends which are flexible (loose). We do not recall here the definition of such an object since they are on the opposite side of those that we will consider as they do not admit augmentations.

In the literature we can find various conjectures regarding the structure of cobordisms when those are excluded. For instance it often happens that article focused on dimensions 3 restrict their attention to so-called decomposable cobordisms (i.e. those arising from bifurcation from Figure 1.1).

Perspective 2. Study decomposability of fillings of Legendrian knots in S^3. Same question for cobordisms with negative ends admitting augmentations.

Perspective 3. Are all Lagrangian fillings regular? i.e. can we deform the Weinstein structure such that the Liouville vector field is tangent to a given filling? Find classes of fillings/cobordisms for which it is true.

A related question is when can we cut a Lagrangian sub-manifold by a transverse contact hypersurface and expect to see a Legendrian. In [Cha12] we provided some example of such slice that cannot be deformed to be Legendrian. Regarding this question [Cha10] still contains something interesting: in Section 5 there we make the slice of a surface Legendrian modifying the Liouville vector field through transverse Liouville field so that it becomes tangent to the surface near the slice. To do so we modify the vector field by subtracting the Hamiltonian of a function H and the function H must satisfy two things: its differential should be a primitive of the original Liouville form restricted to the slice (this exists assuming the Lagrangian is exact) and its variation along the Reeb direction should be smaller than 1. Thus obstructing sliceness implies existence of Reeb chords whose length is smaller than the amplitude of a primitive of θ on the slice.

Perspective 4. Can the quantity $\frac{\text{minimal length of Reeb chords}}{\text{amplitude of primitive of } \theta}$ be studied? Is it interesting? For instance can we consider some capacity like quantities for this type of slices, or some metric properties?

For instance the previous consideration means that slices of compact Lagrangians tend to have Reeb chords, which is dynamically interesting. Either they are small and obstruct collarability or the slice can be made Legendrian and the Lagrangian is a filling of this Legendrian. Thus Theorem 2.1.19 provides existence of chords from the topology of the Lagrangian. Of course for the latter the contact form is different.

Perspective 5. Do all Lagrangian slices of a compact Lagrangian have Reeb chords?

At the end of reading this memoir, the reader will be convinced that we can formulate an equivalent question replacing compact by negative Legendrian ends with augmentation.

1.3.4 Weinstein manifolds

We recall now the main definitions of Weinstein manifolds and sectors. We do not discuss much of the theory and refer to [CE12] for further details on the former concept and [GPS17] for the latter.

Definition 1.3.8. A *Weinstein manifold* $(W, \lambda, \mathfrak{f})$ consists of:

(i) an even dimensional smooth manifold W without boundary,

(ii) a one-form λ on W such that (W, λ) is a Liouville manifold, and

(iii) a proper Morse function $\mathfrak{f}\colon W \to \mathbb{R}$ bounded from below such that the Liouville vector field X_λ is a pseudo-gradient of $\mathfrak{f}$ in the sense of [CE12, Equation (9.9)]: i.e.

$$d\mathfrak{f}(X_\lambda) \geq \delta(\|X_\lambda\|^2 + \|d\mathfrak{f}\|^2),$$

where $\delta > 0$ and the norms are computed with respect to some Riemannian metric on W.

The function $\mathfrak{f}$ is called a *Lyapunov function* (for X_λ). It is easy to check that stable manifolds of X_λ are isotropic, indices of critical points of $\mathfrak{f}$ are all less than n

If $\mathfrak{f}$ has finitely many critical points, then $(W, \theta, \mathfrak{f})$ is a Weinstein manifold of finite type. From now on, Weinstein manifold will always mean Weinstein manifold of finite type.

Given a regular value M of $\mathfrak{f}$ the compact manifold $\{\mathfrak{f} \leq M\}$ is called a *Weinstein domain*.

For each critical point p of $\mathfrak{f}$ of index n, there is a stable manifold Δ_p and an unstable manifold D_p which are both exact Lagrangian sub-manifolds. We will call the unstable manifolds D_p of the critical points of index n the *Lagrangian co-core planes.*

Definition 1.3.9. Let $W_0 \subset W$ be a Weinstein domain containing all critical points of $\mathfrak{f}$. The *Lagrangian skeleton* of $(W, \theta, \mathfrak{f})$ is the attractor of the negative flow of the Liouville vector field on the compact part of W, i.e.

$$W^{sk} := \bigcap_{t>0} \phi^{-t}(W_0),$$

where ϕ denotes the flow of the Liouville vector field X_λ. Alternatively, W^{sk} can be defined as the union of unstable manifolds of all critical points of $\mathfrak{f}$.

The stable manifolds of the index n critical points form the top dimensional stratum of the Lagrangian skeleton.

By the combination of [CE12, Lemma 12.18] and [CE12, Corollary 12.21] we can assume that X_λ is Morse-Smale. This implies that we can assume that handles of higher index are attached after handles of lower index. The deformation making X_λ Morse-Smale can be performed without changing the symplectic form $d\theta$ and so that unstable manifolds corresponding to the critical points of index n before and after such a deformation are Hamiltonian isotopic.

We now turn to the definition of sectors, which are Weinstein manifolds with boundary. But first let us discuss an example. The prototypical example of a Weinstein manifold is the cotangent bundle of a manifold Q, with Liouville form λ and exhausting function p^2 (this is of course not a Morse-Smale situation, one usually uses a Morse function on Q to change both the Liouville form and the function). Similarly the prototypical example of a Weinstein sector will be the cotangent bundle of a manifold with boundary. Choosing a collar neighbourhood of the boundary $\partial Q \times [0, \epsilon)$ the cotangent bundle is $T^*\partial Q \times T^*[0, \epsilon)$ and thus the boundary is $T^*\partial Q \times \mathbb{R} \simeq \mathcal{J}^1\partial Q$. This is not just a smooth identification: the symplectic form restricts to a two forms whose kernel is exactly the Reeb direction.

Definition 1.3.10. A *Weinstein sector* $(S, \theta, I, \mathfrak{f})$ consists of:

1. an even dimensional smooth manifold with boundary S;

2. a one-form θ on S such that $d\theta$ is a symplectic form and the associated Liouville vector field X_λ is complete and everywhere tangent to ∂S;

3. a smooth function $I \colon \partial S \to \mathbb{R}$ which satisfies

   (a) $dI(X_\lambda) = \alpha I$ for some function $\alpha \colon \partial S \to \mathbb{R}_+$ which is constant outside of a compact set and

   (b) $dI(C) > 0$, where C is a tangent vector field on ∂S such that $\iota_C d\theta|_{\partial S} = 0$ and $d\theta(C, N) > 0$ for an outward pointing normal vector field N;

4. a proper Morse function $\mathfrak{f} \colon S \to \mathbb{R}$ bounded from below having finitely many critical points, such that X_λ is a pseudo-gradient of $\mathfrak{f}$ and satisfying moreover

   (a) $d\mathfrak{f}(C) > 0$ on $\{I > 0\}$ and $d\mathfrak{f}(C) < 0$ on $\{I < 0\}$,

   (b) the Hessian of a critical point of $\mathfrak{f}$ on ∂S evaluates negatively on the normal direction N, and

(c) there is a constant $c \in \mathbb{R}$ whose sub-level set satisfies $\{\mathfrak{f} \leq c\} \subset S \setminus \partial S$ and contains all interior critical points of $\mathfrak{f}$.

For simplicity we will often drop part of the data from the notation. We will always assume that S is a Weinstein sector of *finite type*, i.e. that $\mathfrak{f}$ has only finitely many critical points. A Weinstein sector is a particular case of an exact Liouville sector in the sense of [GPS17].

To a Weinstein sector $(S, \theta, I, \mathfrak{f})$ we can associate two Weinstein manifolds in a canonical way up to deformation: the *completion* and the *belt*. The completion of S is the Weinstein manifold $(W, \theta_W, \mathfrak{f}_W)$ obtained by completing the Weinstein domain $W_0 = \{\mathfrak{f} \leq c\}$, which contains all interior critical points of $\mathfrak{f}$. The belt of S is the Weinstein manifold $(F, \theta_F, \mathfrak{f}_F)$ where $F = I^{-1}(0)$, $\theta_F = \theta|_F$ and $\mathfrak{f}_F = \mathfrak{f}|_F$. To show that the belt is actually a Weinstein manifold it is enough to observe that $d\theta_F$ is a symplectic form because F is transverse to the vector field C, and that the Liouville vector field X_λ is tangent to F because $dI(X_\lambda) = \alpha I$, and therefore the Liouville vector field of θ_F is $X_{\theta_F} = X_\lambda|_F$. A neighbourhood of the boundary of a sector can be identified with the stabilisation of the belt $F \times T^*(-2\epsilon, 0]$.

Let $\kappa \in \mathbb{R}$ be a number such that all critical points of $\mathfrak{f}$ are contained in $\{\mathfrak{f} \leq \kappa\}$. We denote $S_0 = \{\mathfrak{f} \leq \kappa\}$ and $F_0 = F \cap S_0 = \{\mathfrak{f}_F \leq \kappa\}$. By Condition (4a) of Definition 1.3.10, the boundary ∂S_0 is a contact manifold with convex boundary with dividing set ∂F_0. Moreover $S \setminus S_0$ can be identified to a half symplectisation.

Definition 1.3.11. Let ϕ be the flow of X_λ. The *skeleton* $S^{sk} \subset S$ of a Weinstein sector $(S, \theta, \mathfrak{f})$ is given by

$$S^{sk} = \bigcap_{t>0} \phi^{-t}(S_0).$$

Remark 1.3.12. *Let W and F be the completion and the belt, respectively, of the Weinstein sector S. To understand the skeleton S^{sk} it is useful to note the following:*

1. *critical points of $\mathfrak{f}$ on ∂S are also critical points of $\mathfrak{f}|_{\partial S}$ and vice versa,*

2. *any critical point $p \in \partial S$ of $\mathfrak{f}$ lies inside $\{I = 0\} = F$ and is also a critical point of $\mathfrak{f}_F$,*

3. *the Morse indices of the two functions satisfy the relation*

$$\mathrm{ind}_{\mathfrak{f}}(p) = \mathrm{ind}_{\mathfrak{f}_F}(p) + 1,$$

4. *the skeleton satisfies* $S^{sk} \cap \partial S = F^{sk}$.

The top stratum of the skeleton of $(S, \theta, \mathfrak{f})$ is given by the union of the stable manifolds of the critical points of $\mathfrak{f}$ of index n, where $2n$ is the dimension of S. Those are of two types: the stable manifolds Δ_p where p is an interior critical point of $\mathfrak{f}$, which are also stable manifolds for $\mathfrak{f}_W$ in the completion, and the stable manifolds Θ_p where p is a boundary critical point of $\mathfrak{f}$, for which $\Delta'_p = \Theta_p \cap \partial S$ is the stable manifold of p for $\mathfrak{f}_F$ in F.

Definition 1.3.13. Let S be a Weinstein sector and let L be a Lagrangian sub-manifold of its belt F. The *spreading* of L is

$$\mathrm{spr}(L) = L \times T^*_{-\varepsilon}(-2\varepsilon, 0] \subset F \times T^*(-2\varepsilon, 0) \subset S.$$

Remark 1.3.14. *The spreading of* L *depends on the choice of symplectic standard neighbourhood of the collar. However, given two different choices, the corresponding spreadings are Lagrangian isotopic. Furthermore, if* L *is exact in* F*, then* $\mathrm{spr}(L)$ *is exact in* S*, and thus two different spreadings are Hamiltonian isotopic.*

Example 1.3.15. When the Weinstein sector is the cotangent bundle of a manifold with boundary, the spreading of a cotangent fibre of $T^*\partial Q$ is simply a cotangent fibre of T^*Q.

We will not detail here how the dynamics of Hamiltonian vector field is controlled near the boundary but the reader can have in mind the following picture: in order for Reeb trajectories on the pre-image of $\mathfrak{f}$ in the cotangent bundle to stay in the interior of T^*Q (those trajectory corresponds to geodesics) one should use a complete metric on the interior of Q.

1.4 Holomorphic curves

1.4.1 Almost complex structures.

Let (X, λ) be a Liouville cobordism with negative (resp. positive) end (Y^+, α^+) (resp. (Y^-, α^-)). We denote by $\mathcal{J}(X)$ the set of almost complex structures J on X compatible with $d\lambda$ such that outside of $\overline{X}$:

- J is invariant under the Liouville flow.
- $J\frac{\partial}{\partial t} = R_{\alpha^\pm}$.

- $J\xi^{\pm} = \xi^{\pm}$.

For $J \in \mathcal{J}(X)$ we denote by $J^{\pm}$ the $\mathbb{R}$-invariant complex structure induced on the symplectisation of $(Y^{\pm}, \xi^{\pm})$.

Remark 1.4.1. *In the situation where the extremities of X are contactisations of Liouville manifolds $P^{\pm}$ we also usually assume that outside a compact set $J^{\pm}$ are cylindrical lifts of almost complex structures on $P^{\pm}$.*

1.4.2 Moduli spaces of holomorphic curves.

In this document we will consider holomorphic curves with boundary on Lagrangian cobordisms with asymptotics toward intersection points or Reeb chords. We recall first the general definition and then restricts to some explicit combination of asymptotics to give formulas for the dimension of the moduli spaces and for the energy of such curves.

General definitions. We assume that we have chosen some consistent choices of strip-like ends for Riemann disks with $d+1$ marked point on their boundary. We will not go into the details of such strip like ends and refer to [Sei08, Section 9] for details. We just note that this means that whenever we consider a disk D with $(d+1)$-marked point on its boundary $(a_0, \cdots, a_d)$ then for any i there is a preferred choice of holomorphic parametrisation of a (punctured) neighbourhood of a_i by (s,t) for $t \in [0,1]$ and $s \in (0,\infty)$ if $i=0$ and $s \in (-\infty, 0)$ otherwise. We refer to $D \setminus \{a_0, \cdots, a_d\}$ as a punctured disk. A decoration of each connected component of ∂S by Lagrangian cobordisms is called a Lagrangian label for S. We list counter clockwise those connected component $(\partial_0 S, \partial_1 S, \cdots, \partial_d S)$ starting from a_0.

Let S be a $d+1$-punctured disc ($d \geq 0$) and $\underline{L}$ a Lagrangian label for S with values in a pair of Lagrangian cobordisms (Σ_0, Σ_1). Suppose we have a (possibly domain dependent) almost complex structure J on X. We say that a map $u : S \to X$ is *J-holomorphic with boundary conditions in $\underline{L}$* if $\forall z \in S$

$$d_z u \circ j = J(z) \circ d_z u, \tag{1.2}$$

where j denotes the standard complex structure on D^2, and $u(\partial_i S) \subset \underline{L}(i)$.

Asymptotics Given an intersection point $p \in \Sigma_0 \cap \Sigma_1$, we say that u is *asymptotic* to p at a_i if

- the Lagrangian label change when we cross the puncture, and
- $\lim_{z \to a_i} u(z) = p$.

Let γ be a Reeb chord of $\Lambda_0^{\pm} \sqcup \Lambda_1^{\pm}$ of length T. The map $u = (a, v)$ into $\mathbb{R} \times Y$ has a *positive asymptotic* to γ at a_i if:

- $\lim_{s \to +\infty} v(\varepsilon_i(s,t)) = \gamma(Tt)$ and $\lim_{s \to +\infty} a(\varepsilon_i(s,t)) = +\infty$, given that $a_i = a_0$ is the incoming puncture, or
- $\lim_{s \to -\infty} v(\varepsilon_i(s,t)) = \gamma(T(1-t))$ and $\lim_{s \to -\infty} a(\varepsilon_i(s,t)) = +\infty$, given that $i \neq 0$.

Let γ be a Reeb chord of $\Lambda_0^{\pm} \sqcup \Lambda_1^{\pm}$ of length T. The map $u = (a, v)$ has a *negative asymptotic* to γ at a_i, $i \neq 0$, given that

- $\lim_{s \to -\infty} a(\varepsilon_i(s,t)) = -\infty$ and $\lim_{s \to -\infty} v(\varepsilon_i(s,t)) = \gamma(Tt)$.

We will never consider holomorphic curves which have a negative asymptotic at the incoming end x_0.

Associated to a pair of cobordisms there are three types of possible targets for Lagrangian labels that will be considered here: (Σ_0, Σ_1), $(\mathbb{R} \times \Lambda_0^{\pm}, \mathbb{R} \times \Lambda_1^{\pm})$ and $\mathbb{R} \times \Lambda_i^{\pm}$ for $i = 0, 1$; note that the asymptotics of the latter two are subsets of those of the first pair. We will use $\underline{L}$ to denote any of those labels. Given $x_0, \ldots, x_d$ a set of asymptotic (chords or intersection points) and $r \in \mathcal{R}^{d+1}$, we denote by

$$\mathcal{M}^r_{\underline{L}}(x_0; x_1, \ldots, x_d; J)$$

the space of J-holomorphic maps from S_r to X with asymptotics to x_i at a_i modulo reparametrisations of S_r. In other words, x_0 is the asymptotic of the incoming puncture. We denote by

$$\mathcal{M}_{\underline{L}}(x_0; x_1, \ldots, x_d; J)$$

the union of the $\mathcal{M}^r_{\underline{L}}(x_0; x_1, \ldots, x_d; J)$ over all $d+1$-punctured discs. Note that once the asymptotics for the moduli space is fixed, the actual Lagrangian label is uniquely determined and, hence, we do not need to specify it.

In the case when both $\underline{L}$ and J are invariant under translations of the symplectisation coordinate, there is an induced $\mathbb{R}$-action on $\mathcal{M}_{\underline{L}}(x_0; x_1, \ldots, x_d; J)$. We use

$$\widetilde{\mathcal{M}}_{\underline{L}}(x_0; x_1, \ldots, x_d; J)$$

to denote the quotient of the moduli space by this action.

Strips and half-planes As already seen, the pseudoholomorphic discs considered here will have a number of different types of asymptotics. However, it will be useful to make a distinction between the following two types of discs considered.

- A pseudoholomorphic disc where the Lagrangian label does not change will be called a *(punctured) half-plane*; while

- A pseudoholomorphic disc for which the Lagrangian label changes exactly twice (once at the positive puncture) will be called a *(punctured) strip*. The puncture corresponding to the unique incoming end will be called the *output* while the puncture corresponding to the unique outgoing end at which a jump occurs will be called the *input*.

Remark 1.4.2. *The fact that outgoing ends are inputs and incoming ends are outputs might seem confusing. The incoming/outgoing dichotomy comes from the notion of incoming and outgoing edges in a rooted tree and refers to particular coordinates of the domain (the strip like ends). This follows the convention of [Sei08]. The dichotomy input/output refers to what will belong to the domain/co-domain of the differential as defined in Section 2.2.*

1.4.3 Structure of the moduli spaces

We recall that the cobordisms Σ_i have dimension $n+1$. The proof of the following proposition is a patchwork of results from the literature, see [Dra04, Section 2.2], [CEL10, Theorem A.1], [EES09, Lemma 2.5].

Proposition 1.4.3. *For generic almost complex structures $J^{\pm}$ and $J_{\bullet}$, the moduli spaces described in the previous section are transversely cut out and* *therefore are smooth manifolds. Their dimensions are*

$$\dim \mathcal{M}(\gamma^+; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-) = i_{\Lambda_0^+,\Lambda_1^+}(\gamma^+) - i_{\Lambda_0^-,\Lambda_1^-}(\gamma^-) - i_{\Lambda_1^-}(\boldsymbol{\delta}) - i_{\Lambda_0^-}(\boldsymbol{\zeta}), \tag{1.3}$$

$$\dim \mathcal{M}(\gamma^+; \boldsymbol{\delta}^-, q, \boldsymbol{\zeta}^-) = i_{\Lambda_0^+,\Lambda_1^+}(\gamma^+) - i_{\Sigma_0,\Sigma_1}(q) - i_{\Lambda_1^-}(\boldsymbol{\delta}) - i_{\Lambda_0^-}(\boldsymbol{\zeta}) + 1, \tag{1.4}$$

$$\dim \mathcal{M}(p; \boldsymbol{\delta}^-, q, \boldsymbol{\zeta}^-) = i_{\Sigma_0,\Sigma_1}(p) - i_{\Sigma_0,\Sigma_1}(q) - i_{\Lambda_1^-}(\boldsymbol{\delta}) - i_{\Lambda_0^-}(\boldsymbol{\zeta}) - 1, \tag{1.5}$$

$$\dim \mathcal{M}(p; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-) = i_{\Sigma_0,\Sigma_1}(p) - i_{\Lambda_0^-,\Lambda_1^-}(\gamma^-) - i_{\Lambda_1^-}(\boldsymbol{\delta}) - i_{\Lambda_0^-}(\boldsymbol{\zeta}) - 2, \tag{1.6}$$

$$\begin{aligned}\dim \mathcal{M}(\gamma_{1,0}; \boldsymbol{\delta}^-, \gamma_{0,1}, \boldsymbol{\zeta}^-) = {} & i_{\Lambda_1^-,\Lambda_0^-}(\gamma_{1,0}) + i_{\Lambda_0^-,\Lambda_1^-}(\gamma_{0,1}) - i_{\Lambda_1^-}(\boldsymbol{\delta}) \\ & - i_{\Lambda_0^-}(\boldsymbol{\zeta}) - n + 2.\end{aligned} \tag{1.7}$$

When the natural $\mathbb{R}$-action is well defined and free, the moduli spaces after taking the quotient are still transversely cut out and their dimension is one less. Moreover the zero-dimensional moduli spaces (after quotient, when possible) are compact, and the one-dimensional moduli spaces can be compactified by adding two-levels pseudoholomorphic buildings where each level belong to a zero-dimensional moduli space.

1.4.4 Energy

In this section, we recall the notion of energy for pseudoholomorphic curves in the symplectisation of a contact manifold as introduced in [Hof93] and [Bou+03]. See also [Abb04] for the relative case.

Finite volume symplectic forms.

Let $i : \Sigma \to X$ be an exact Lagrangian cobordisms that is standard on the cylindrical ends $(-\infty, -T) \times Y^-$ and $(T, \infty) \times Y^+$. Given any function $\rho : \mathbb{R} \to \mathbb{R}$ such that

1. $\rho'(t) > 0$.
2. $\rho(t) = t$ if $|t| < T$

we denote by λ_ρ the form which is equal to λ on the compact part $\overline{X}$ and where $e^t\alpha_\pm$ is replaced by $e^{\rho(t)}\alpha_\pm$. The exterior derivative of λ_ρ is still symplectic and we have $i^*\lambda_\rho = i^*\lambda$. Of particular interest we have the case when $\lim_{t\to\pm\infty}\rho = \pm T \in \mathbb{R}$: in this situation the volume of X becomes finite.

Hofer energy

Assume that Σ_0 and Σ_1 are two exact Lagrangian cobordisms in a Liouville cobordisms $(X, d\lambda)$. Let $f_i \colon \Sigma_i \to \mathbb{R}$ be primitives of $e^t\alpha|_{\Sigma_i}$ which are constant at the cylindrical ends. Without loss of generality we will assume that both constants are 0 on the negative ends, while the constants on the positive end of Σ_i will be denoted by $\mathfrak{c}_i$, $i = 0, 1$.

Fix a function ρ as in the previous section such that $\lim_{t\to\pm\infty}\rho = \pm T$ and let λ_ρ be the corresponding Liouville form whose exterior derivative has finite volume.

With this choices a primitive of $\lambda|_{\Sigma_i}$ (which exists by exactness) is hence also a primitive of $\lambda_\rho|_{\Sigma_i}$.

Definition 1.4.4. Let S be a punctured disc and let $u \colon S \to X$ be a smooth map. The *$d\lambda_\rho$-energy* of u is given by

$$E_{d\lambda_\rho}(u) = \int_S u^*(d\lambda_\rho).$$

Non-constant holomorphic curves have positive total energy, as stated in the following lemma.

Lemma 1.4.5. *If u is non-constant punctured pseudoholomorphic disc with boundary on a pair of exact Lagrangian cobordisms, and if the almost complex structure is cylindrical outside of $[-T+\epsilon, T-\epsilon] \times Y$, then $E_{d\lambda_\rho}(u) > 0$.*

Action and energy

Consider a pair of exact Lagrangian cobordisms Σ_i from Λ_i^- to Λ_i^+ in a Liouville cobordism (X, λ) from (Y^-, ξ^-) to (Y^+, ξ^+). Denote the associated potential functions $f_i \colon \Sigma_i \to \mathbb{R}$, $i = 0, 1$ and let $\alpha^\pm$ be the contact form for $\xi^\pm$ induced by the cobordism.

Remark 1.4.6. *Note that in the case that X is a symplectisation we still have $\alpha^+ \neq \alpha^-$ as those are defined with respect to where the Lagrangian cobordisms is trivial. In $e^t\alpha$ notations we would have $\alpha^- = e^{-T}\alpha$ and $\alpha^+ = e^T\alpha$.*

For a Reeb chord c of $\Lambda_0^\pm \cup \Lambda_1^\pm$ we define

$$\ell(c) := \int_c \alpha^\pm.$$

Recall that the definition of the $E_{d\lambda_\rho}$-energy depends on the choice of a constant $T \geq 0$, where equality is possible only when both cobordisms are trivial cylinders. The *action* of a Reeb chord γ of $\Lambda_1^\pm \cup \Lambda_0^\pm$ is defined by

$$\begin{aligned} \mathfrak{a}(\gamma) &:= \ell(\gamma) + (\mathfrak{c}_i - \mathfrak{c}_j) && \text{if } \gamma \text{ is a chord of } \Lambda_0^+ \cup \Lambda_1^+, \text{ and} \\ \mathfrak{a}(\gamma) &:= \ell(\gamma) && \text{if } \gamma \text{ is a chord of } \Lambda_0^- \cup \Lambda_1^-. \end{aligned}$$

In particular, the action of a pure Reeb chord γ of $\Lambda_i^\pm$ is always $\mathfrak{a}(\gamma) = \ell(\gamma)$. Given a word $\boldsymbol{\gamma} = \gamma_1 \dots \gamma_d$, we denote $\mathfrak{a}(\boldsymbol{\gamma}) = \sum\limits_{i=1}^{d} \mathfrak{a}(\gamma_i)$.

The *action* of an intersection point $p \in \Sigma_0 \cap \Sigma_1$ is defined by

$$\mathfrak{a}(p) := f_1(p) - f_0(p).$$

For simplicity when $\boldsymbol{x} = (x_1, \cdots, x_2)$ is a collection of asymptotic we will denote by $\mathfrak{a}(\boldsymbol{x})$ the sum $\sum \mathfrak{a}(x_i)$.

Stokes's theorem gives the following proposition (see [Dim16a] for details), whose proof heavily relies on the fact that each cobordism Σ_i, $i = 0, 1$, is exact.

Proposition 1.4.7. *Let $\gamma^\pm \in \mathcal{R}(\Lambda_1^\pm, \Lambda_0^\pm)$ be mixed Reeb chords, $\boldsymbol{\delta}^- = \delta_1^- \dots \delta_{i-1}^-$ and $\boldsymbol{\zeta}^- = \zeta_{i+1} \dots \zeta_d$ words of pure Reeb chords on Λ_1^- and Λ_0^-, respectively, and $p, q \in \Sigma_0 \cap \Sigma_1$ intersection points.*

- *If $u \in \mathcal{M}(\gamma^+; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-)$, then*
$$E_{d\lambda_\rho}(u) = \mathfrak{a}(\gamma^+) - \mathfrak{a}(\gamma^-) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{1.8}$$
- *If $u \in \mathcal{M}(\gamma^+; \boldsymbol{\delta}^-, p, \boldsymbol{\zeta}^-)$, then*
$$E_{d\lambda_\rho}(u) = \mathfrak{a}(\gamma^+) - \mathfrak{a}(p) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{1.9}$$
- *If $u \in \mathcal{M}(p; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-)$, then*
$$E_{d\lambda_\rho}(u) = \mathfrak{a}(p) - \mathfrak{a}(\gamma^-) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{1.10}$$

- *If* $u \in \mathcal{M}(p; \boldsymbol{\delta}^-, q, \boldsymbol{\zeta}^-)$*, then*

$$E_{d\lambda_\rho}(u) = \mathfrak{a}(p) - \mathfrak{a}(q) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{1.11}$$

- *If* $u \in \mathcal{M}(\gamma_{1,0}; \boldsymbol{\delta}^-, \gamma_{0,1}, \boldsymbol{\zeta}^-)$*, then*

$$E_{d\lambda_\rho}(u) = \big(\mathfrak{a}(\gamma_{1,0}) + \mathfrak{a}(\gamma_{0,1})\big) - \big(\mathfrak{a}(\boldsymbol{\delta}^-) + \mathfrak{a}(\boldsymbol{\zeta}^-)\big). \tag{1.12}$$

Chapter 2

Augmentations of Legendrian sub-manifolds and Floer complex for Lagrangian cobordisms.

In this chapter we describe the construction of the Cthulhu complex from [Cha+15b]. The goal is to associate to a pair of cobordisms Σ_0 and Σ_1 a Floer complex (partially) generated by intersection points between Σ_0 and Σ_1. The crucial ingredient that allows us to deal with holomorphic curves bubbling off in the concave end of the cobordisms is the introduction of augmentations of the negative ends of the cobordisms. We recall the definition of this concept in Section 2.1 and we described how those allow to define linearised Legendrian contact cohomology from [Che02] and [EES05]. We recall then the so-called Seidel isomorphism in Theorem 2.1.19 stating that when the augmentation is induced by a filling then the Linearised contact cohomology is the cohomology of the filling. This is the fundamental motivation for our understanding of augmentations: they should reflect hidden fillings of our Legendrian sub-manifold.

We use augmentations to assign weights to extra legs of holomorphic curves; this allows us to ignore bubbling off half planes. One parameter families of holomorphic strips can however bubble off other curves. The first situation is when an actual holomorphic plane asymptotics to a closed Reeb orbit of (Y^-, ξ^-) bubble off. This situation do not appear in the manifolds considered in [Cha+15b] as it studies cobordisms in symplectisations of contactisation of Liouville manifolds. The second one is when a strip bubble off a holomorphic band with two positive asymptotics: one toward a Reeb chords from Λ_0^- to Λ_1^- and one from Λ_1^- to Λ_0^- (see Figure 2.1).

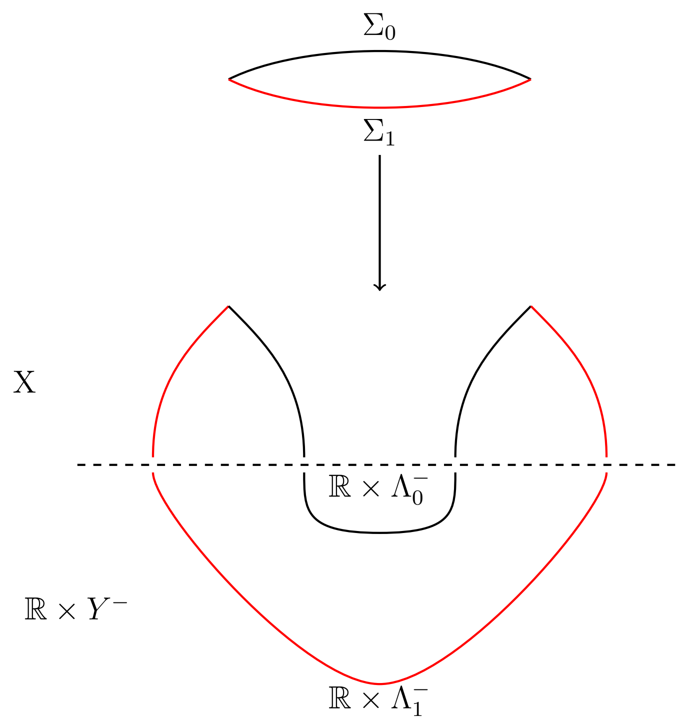

Figure 2.1: Bubbling off a band in the negative end.

There are no reasonable ways (and indeed we should have no reasons to want to do so) to discard these breakings. Instead this suggests that this three components broken holomorphic curves should be part of a d^2 operator and thus that chords between Legendrian ends of the cobordisms are part of the generators of our complex. This leads us to define in Section 2.2.1, fixing two augmentations ε_0^- and ε_1^- of Λ_0^- and Λ_1^- respectively, a chain complex $CF_{-\infty}(\Sigma_0, \Sigma_1)$ generated by intersection points between Σ_0 and Σ_1 and Reeb chords from Λ_0^- to Λ_1^-. This complex is invariant under Hamiltonian isotopies supported away from the positive end of the cobordism (Theorem 2.2.2).

Following the idea that augmentations reflect hidden Lagrangian fillings of the Legendrians at the negative ends, the complex $CF_{-\infty}(\Sigma_0, \Sigma_1)$ is thought as the Lagrangian Floer cohomology complex of the fillings of Λ_0^+ and Λ_1^+

obtain by “capping” the negative ends of Σ_0 and Σ_1 with ε_0^- and ε_1^-.

To prove Theorem 2.1.19 in [Ekh12], T. Ekholm uses a Legendrian contact homology presentation of the wrapped Floer cohomology complex whose generator includes intersection points between Lagrangian fillings and Reeb chords between the positive ends of these fillings. This complex is acyclic when the ambient Liouville domain has vanishing symplectic homology, and this acyclicity shows the isomorphisms of Theorem 2.1.19. In Section 2.2.2, we mimic this strategy following our idea that augmentations accounts of hidden fillings of the negative ends, we enlarge the complex $CF_{-\infty}(\Sigma_0, \Sigma_1)$ to the so called *Cthulhu* complex incorporating Reeb chords from Λ_0^+ to Λ_1^+. After describing the holomorphic curves involved in the differential we state that this is a complex (Theorem 2.2.4). Under cylindrical Hamiltonian isotopies, this complex changes under quasi-isomorphisms. In the symplectisation of the contactisation the cylindrical lift of the Reeb flows allows to remove all generators of this complex, and thus the Chulhu complex is acyclic (Corollary 2.2.5). This is our analogue of Theorem 2.1.19 in the context of Lagrangian cobordisms.

In this chapter we focus on the foundational aspect of the theory, the applications are discussed in Chapter 4. We begin the chapter (section 2.1.1) with a discussion on how to associate homotopy classes of loops on Legendrian and Lagrangian cobordisms to holomorphic maps, which allows us to consider coefficients in algebras over the group ring of the fundamental group. We try to give a unified description of all possible coefficients that will be useful later on for applications. But the price we pay is sometime confusing notation (due partially to non-commutativity of the coefficients), on first approximation we suggest the reader to simply have in mind $\mathbb{F}_2$ coefficients and all complexes being vector spaces.

2.1 Augmentations and linearised contact cohomology.

2.1.1 Homotopy class of the boundary component of holomorphic curves.

Let Λ be a Legendrian sub-manifold, and denote by $\Lambda_1 \cdots, \Lambda_m$ its connected components. Fix a base point $\star_i$ for each of those connected components.

Let R be an algebra over $\mathbb{F}_2$ We denote by $R[\Lambda]$ the group $R[\pi_1(\Lambda_1, \star_1) * \pi_1(\Lambda_2, \star_2) \cdots * \pi_1(\Lambda_m, \star_m)]$. (Here $\pi_1(\Lambda_1, \star_1) * \pi_1(\Lambda_2, \star_2) \cdots * \pi_1(\Lambda_m, \star_m)$ is thought as the fundamental group of all connected components of Λ wedged at their base points.) In order to consider coefficient in $R[\Lambda]$ we need to be able to associate to holomorphic curves various homotopy classes of loops in Λ.

We assume that Λ is chord generic and for each Reeb chords γ of Λ we fix two paths, s_γ and e_γ, on Λ connecting the starting and ending points of γ to the base point of the appropriate connected component of Λ. Let p be the projection from $\mathbb{R} \times Y$ to Y.

Let $u \in \mathcal{M}_{\mathbb{R}\times Y}(\gamma^+; \gamma_1, \cdots, \gamma_k)$.

- For $1 \leq i < k$ we denote by $c_{\gamma_i,\gamma_{i+1}}(u)$ the element $l_{\gamma_i,\gamma_{i+1}} \in R[\Lambda]$ where $l_{\gamma_i,\gamma_{i+1}}$ is the loop (in the wedged space) given by $e_{\gamma_{i+1}} * p \circ u(\partial_i) * s_{\gamma_i}^{-1}$
- We denote by $c_{\gamma,\gamma_1}(u)$ the element $l_{\gamma_i,\gamma_{i+1}} \in R[\Lambda]$ where $l_{\gamma_i,\gamma_{i+1}}$ is the loop given by $e_{\gamma_1} * p \circ u(\partial_0) * e_\gamma^{-1}$.
- We denote by $c_{\gamma_k,\gamma}(u)$ the element $l_{\gamma_k,\gamma} \in R[\Lambda]$ where $l\gamma_k, \gamma$ is the loop given by $s_\gamma * p \circ u(\partial_0) * s_{\gamma_k}^{-1}$.

As we will consider how Legendrian invariants behave under the relation of cobordism we will consider curves with boundary also on Lagrangian cobordisms, and this imposes some choices. Whenever we have a family of cobordisms $\{\Sigma_i\}$ with Legendrian ends $\{\Lambda_i^+\}$ and $\{\Lambda_i^-\}$ we assume that each connected component of each Σ_i comes with a base points in the negative cylindrical part. We also choose a path connecting all the base points of the inclusion of $\Lambda_i^\pm$ to the appropriate base point, and we denote $R[\mathbf{\Sigma}]$ the group $R[\pi_1(\Sigma_1) * \cdots \pi_1(\Sigma_k)]$. For a holomorphic curve with boundary on the family Σ_i we define the element $c_{\gamma,\gamma'} \in R[\mathbf{\Sigma}]$ similarly as before.

Remark 2.1.1. *When the inclusion of Λ_i^- into Σ_i is π_0-injective we assume that we choose the base point of Σ_i to be the one coming from the inclusion of Λ_i^- into $\Sigma_i \cap \{-T\} \times Y$. If furthermore one of the Σ_i is a trivial cylinder then we choose the path connecting the base point of the positive end to the negative one to be the vertical line.*

We will always assume that all those choice have been made whenever we consider such geometric situations and we will never include those choices in all our notations as the overall results will not really depend on them.

Remark 2.1.2. *A last convention we will use is that when some choices have been made for a given Legendrian or Lagrangian, any isotopy of these induces some canonical choices for the other end of the isotopy. We will always assume that those has been picked in that situation.*

2.1.2 Definition of augmentations.

Let Λ be a Legendrian sub-manifold of a contact manifold (Y, ξ). Let R_α be a Reeb vector field of (Y, ξ).

Remark 2.1.3. *There are various conditions one can ask on the Reeb vector field when Y in non-compact. In most of our application Y will be the contactisation $P \times \mathbb{R}$ of a Liouville manifold (P, λ), in that situation we ask for the Reeb vector field to coincide with the one coming from the form $dz - \lambda$.*

Let A be a ring (the example we have in mind is $A = R[\Sigma]$ for a cobordism Σ).

We denote by $\mathcal{R}(\Lambda)$ the set of Reeb chords of Λ.

Definition 2.1.4. An *augmentation* of Λ over A is a pair given by

- A ring homomorphism $\iota : R[\Lambda] \to A$.
- A map $\varepsilon : \mathcal{R}(\Lambda) \to A$ such that:

$$\forall \gamma \in \mathcal{R}(\Lambda) \sum_{u \in \widetilde{\mathcal{M}}(\gamma)} \iota(c_{\gamma,\gamma}(u)) = \\ \sum_{\gamma_1, \dots, \gamma_k \in \mathcal{R}(\Lambda)} \sum_{u \in \widetilde{\mathcal{M}}(\gamma; \gamma_1, \dots, \gamma_k)} \iota(c_{\gamma,\gamma_1}(u))) \cdot \varepsilon(\gamma_1) \cdot \iota(c_{\gamma_1,\gamma_2}(u)) \cdots \varepsilon(\gamma_k) \cdot \iota(c_{\gamma_k,\gamma}(u)). \tag{2.1}$$

Remark 2.1.5. *The indices in all sums in the previous formula are, as usual, to be understood to be taken only when the involved moduli spaces have dimension 0.*

Remark 2.1.6. *There are several ways to understand Equation* (2.1)*; one we favour amongst others is summarised using the terminology from Section 1.4.2, in the following sentence:*

Equation (2.1) *says that the number of (punctured) half-planes asymptotics to a given chords γ^+ weighted by the augmentation ε is* 0.

Remark 2.1.7. *One instance of this equation arises when Λ is filled by an exact Lagrangian sub-manifold L: then the curve on the left of the Equation* (2.1) *all belong to one dimensional moduli spaces of curves with boundary on L. Each of those one dimensional moduli spaces partially compactify in two ways:*

- *As a copy of $\mathbb{R}$ meaning that the one dimensional space goes through the cobordism to end on both sides on the cylindrical part. Both ends contribute to cancelling terms in the sum on the left of Equation* (2.1)*.*
- *As a copy of $[0, \infty)$ meaning that the family of curves converges to a broken building with two levels: one level on the trivial part which is a curve contributing to a summand on the right of Equation* (2.1)*, and the other level is made of disks capping all the negative ends of this curves.*

The value of the augmentation therefore account to those hidden disks. This motivates our interpretation of augmentations as encoding the same type of information as an exact Lagrangian fillings does that we develop a little further in Section 3.2.

Remark 2.1.8. *As they allow to rigidify the theory of Legendrian sub-manifolds, it is obvious that augmentations should not exist for all Legendrian sub-manifolds. Indeed if we construct a Legendrian sub-manifold by some flexible construction then we create many chords admitting only one curve (no matter how many negative ends we allow) positively asymptotic to it. Therefore Equation* (2.1) *can never be satisfied at these chords. On the other end of the spectrum, Legendrians arising as boundary of exact Lagrangian sub-manifolds naturally carry augmentations as discussed in Remark 2.1.7.*

Remark 2.1.9. *Definition 2.1.4 is our attemp to find a unified way to speak about the various notions of augmentation that arised in the litterature since the seminal work [Che02]. There an augmentation is a differential graded algebra map from the Chekanov-Eliashberg algebra to the coefficient ring R. It coincides with the case $A = R$ and $\iota : R[\Lambda] \to R$ the standard augmentation map. Then homological and homotopical coefficients for the Chekanov-Eliashberg dga started to appear (see for instance [Ekh+13], [EHK16], [EL17] or [Cha+15b, Section 8]). In this situation two possibilites occur: first (this is the situation in the various augmentations categories from [BC14b], [Ng+15],* *[Cha+16], [CNS19]) augmentations still take values in R (or a matrix algebra with coefficient in R) and in this case the map ι varies from one augmentation to another (it accounts of the value of the augmentation on the t variable in the 1-dimensional case). Second (this is the situation in the part of [Cha+15b] where we study fundamental groups of a cobordism Σ) the augmentations take values in $R[\pi_1(\Sigma)]$ with the map ι being the one given by the inclusion $\Lambda \subset \Sigma$. Therefore in most situations the maps ι will be implicit and when we consider Lagrangian cobordisms then we have natural maps $R[\Lambda^+] \to R[\Sigma]$ and $R[\Lambda^-] \to R[\Sigma]$ (using all the choices we have made in Section 2.1.1). We assume that all the maps $\iota_i^{\pm} : R[\Lambda_i^+] \to A$ are induced by a single maps $\iota : R[\Sigma] \to A$.*

Remark 2.1.10. *If $f : A \to A'$ is a ring morphism and (ε, ι) is an augmentation over A then $(f \circ \varepsilon, f \circ \iota)$ is an augmentation over A'. In particular any augmentation over $R[\pi_1(\Lambda)]$ will induce an augmentation over R.*

We now introduce a notation which will allow us to simplify all formulas we are going to give, it reflects that given some augmentation an holomorphic curves can be thought as some curve with less ends.

Definition 2.1.11. Let $\Sigma_0, \cdots, \Sigma_k$ be some Lagrangian cobordisms in a symplectic cobordism X. Let ε_i be some augmentation of Λ_i^-, the negative end of Σ_i. Let $\delta_i^1, \cdots \delta_i^{k_i}$ be some Reeb chords of Λ_i^- and $x_0, \cdots x_k$ be some other asymptotics (intersection points or Reeb chords). For an holomorphic curves $u \in \mathcal{M}_{\Sigma_0, \cdots, \Sigma_k}(\boldsymbol{\delta}_0, x_0, \boldsymbol{\delta}_1, x_1, \cdots, x_{k-1}, \boldsymbol{\delta}_k)$ its *i-th weighted boundary component* is $c_i(u) =$

$$\iota(c_{x_{i-1}, \delta_i^1}(u)) \cdot \varepsilon_i(\delta_i^1) \cdot \iota(c_{\delta_i^1, \delta_i^2}(u)) \cdot \varepsilon_i(\delta_i^2) \cdots \iota(c_{\delta_i^{k_i - 1}, \delta_i^{k_i}}(u)) \cdot \varepsilon_i(\delta_i^{k_i}) \cdot \iota(c_{\delta_i^{k_i}, x_i}(u)).$$

Remark 2.1.12. *Two noticeable case will be when u is a punctured half plane or a punctured strip (as defined in Section 1.4.2) where then u has respectively one or two weighted boundary components.*

Consider now Σ an exact cobordism from Λ^- to Λ^+ and let $\varepsilon^- : \mathcal{R}(\Lambda^-) \to A$ be an augmentation of Λ^-. Then for $\gamma \in \mathcal{R}(\Lambda^+)$ we define:

$$\varepsilon^+(\gamma) = \sum_{\gamma_1, \cdots \gamma_k} \sum_{v \in \mathcal{M}_\Sigma(\gamma_1, \cdots, \gamma_k)} c_1(v). \tag{2.2}$$

In the preceding notation all v's are thought of as punctured half-planes. Note that the dependency in ε^- is hidden in the notation $c_1(v)$.

In other terms $\varepsilon^+(\gamma)$ counts all weighted (punctured) half-planes asymptotic to γ with boundary on Σ. (Note that when Λ^- is empty this is exactly the definition of the augmentation induced by a filling, this expands the idea of thinking of augmentations as counting disks on an hidden filling as in Remark 2.1.7.)

The following result follows from standard breaking analysis that we will see a lot in the rest of this document. However we prefer here to refer to [EHK16] for a proof:

Theorem 2.1.13. *The map $\varepsilon^+ : \mathcal{R}(\Lambda^+) \to A$ is an augmentation of Λ^+.*

We call ε^+ the *pull-back augmentation* induced by Σ, if we want to keep track of the ε^- and the cobordism we will denote it $\Sigma^*\varepsilon^-$.

2.1.3 Augmentations as a correction to $d^2 = 0$ and weights.

We now turn to the definition of linearised contact cohomology. This will use augmentations of Legendrian sub-manifolds following the same idea as in Remark 2.1.7 that they allow to weight moduli spaces of holomorphic curves in order to compensate unexpected breaking.

Let Λ_0 and Λ_1 be two graded Legendrian sub-manifolds such that $\Lambda_0 \cup \Lambda_1$ is chord generic. Fix an algebra R over $\mathbb{F}_2$ and let A_0 and A_1 be two rings as before. Let $\mathcal{R}(\Lambda_0, \Lambda_1)$ be the set of Reeb chords starting on Λ_0 and ending on Λ_1. We denote by $LCC(\Lambda_0, \Lambda_1)$ the free $A_0 \otimes A_1^{\mathrm{op}}$-module with basis given by $\mathcal{R}(\Lambda_0, \Lambda_1)$. We assume that it is a graded vector space where each generator γ is graded by $gr_\Lambda(\gamma)$.

We want to define a differential μ^1 on $LCC(\Lambda_0, \Lambda_1)$ counting holomorphic strips between Reeb chords. However one parameter family of such strips may not only break at broken strips (which would prove that $\mu_1^2 = 0$) but can break as shown on Figure 2.2.

On this figure we see two type of breaking. On the right side the one parameter family of curves in $\widetilde{\mathcal{M}}(\gamma^+; \gamma^-)$ breaks in two curves $(u_1, u_2) \in \widetilde{\mathcal{M}}(\gamma^+; \delta) \times \widetilde{\mathcal{M}}(\delta; \gamma^-)$ which would contribute to $\mu_1^2(\gamma^-)$. The breaking on the left corresponds to a building of height 2:

- the top level is a curve in $\widetilde{\mathcal{M}}(\gamma^+; \delta_1, \gamma^-)$.
- The bottom one is a disconnected curve. One component is in $\widetilde{\mathcal{M}}(\delta_1)$ the second being in $\widetilde{\mathcal{M}}(\gamma^-; \gamma^-)$.

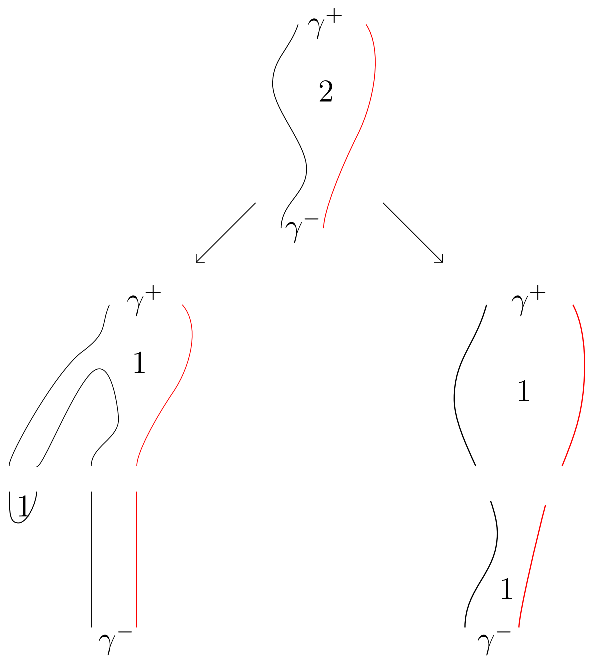

Figure 2.2: Possible degenerations of 1-parameter family of holomorphic strips.

This second type of breaking prevents possibly μ_1^2 to be 0. In order to compensate this we will use two augmentations ε_0 of Λ_0 and ε_1 of Λ_1 which will allow curves like the one on the top level of this building to contribute to the differential.

More precisely, let $\mu^1_{\varepsilon_0,\varepsilon_1}$ be the $\mathbb{F}_2$-linear map defined by:

$$\mu^1_{\varepsilon_0,\varepsilon_1}(a_0 \otimes a_1 \cdot \gamma^-) = \sum_{\gamma^+} \sum_{\boldsymbol{\delta} \in \mathcal{R}(\Lambda_1), \boldsymbol{\zeta} \in \mathcal{R}(\Lambda_0)} \sum_{u \in \mathcal{M}(\gamma^+; \boldsymbol{\delta}, \gamma^-, \boldsymbol{\zeta})} c_0(u) \cdot a_0 \otimes a_1 \cdot c_1(u) \cdot \gamma^+. \tag{2.3}$$

Remark 2.1.14. *The definition of the operator $\mu^1_{\varepsilon_0,\varepsilon_1}$ is the extension of the definition of the linearised Legendrian contact cohomology differential in [Che02] to non-commutative coefficients proposed in [Cha+16]. As a map it does not preserve any form of multiplication by elements of A_0 or A_1. It is a natural dual of the linearised contact homology differential which is a bimodule map(see [Cha+16, Section 4.1]). We choose not to talk about* *the homological side here as the cohomology differential fits more in the $\mathcal{A}_\infty$ picture.*

That it is indeed a differential is the content of the following theorem:

Theorem 2.1.15 ([Che02],[EES07])**.** *Given two graded Legendrian sub-manifolds Λ_0 and Λ_1 with augmentations ε_0 and ε_1 we have:*

- $(\mu^1_{\varepsilon_0,\varepsilon_1})^2 = 0$.
- *If Λ_1' is Legendrian isotopic to Λ_1 in the complement of Λ_0 then there is an augmentation ε_1' of Λ_1' such that $(LCC(\Lambda_0,\Lambda_1),\mu^1_{\varepsilon_0,\varepsilon_1})$ is quasi-isomorphic to $(LCC(\Lambda_0,\Lambda_1'),\mu^1_{\varepsilon_0,\varepsilon_1'})$.*
- *Symmetrically if Λ_0' is Legendrian isotopic to Λ_0 in the complement of Λ_1 then there is an augmentation ε_0' of Λ_0' such that $(LCC(\Lambda_0,\Lambda_1),\mu^1_{\varepsilon_0,\varepsilon_1})$ is quasi-isomorphic to $(LCC(\Lambda_0',\Lambda_1),\mu^1_{\varepsilon_0',\varepsilon_1})$.*

The proof of this theorem goes usually into proving that the algebra freely generated by Reeb chords is a differential graded algebra with a differential counting curves such as in the top building of the left degeneration in Figure 2.2. Then augmentation are used to twist this differential in order that the length 1 term of this differential is a differential on the $A_0 \otimes A_1$-module. Our differential is then the dual of this one. Here we will sketch a proof of this theorem not using (explicitly) the DGA. The geometric idea behind the fact that augmentations compensates bad breakings being the main intuitive idea which led us to the definition of the Cthulhu complex from Section 2.2.

Idea of proof. Consider a 1-parameter family of (punctured) strips as in Figure 2.2. This can degenerate in two types of holomorphic buildings:

- **($\mu^1 \circ \mu^1$ breaking).** One building of height 2 where each level is a (punctured) strips.
- **(∂-breaking).** One building of height 2 where the bottom level is made of one (punctured) half-plane union some trivial strips and the top is a punctured strip with one extra negative asymptotics.

We claim that this second type of breaking cancels out thanks to the augmentation Equation (2.5). Indeed this equation exactly says that the number of (punctured) half-planes you can attached to a fixed negative asymptotics counted with weights is 0. As gluing holomorphic curves allows us to see that all broken configurations are possible, this implies that all ∂-breakings cancel each other.

This leads to $(\mu^1_{\varepsilon_0,\varepsilon_1})^2 = 0$ when the map ι is the trivial map to R. To consider coefficients in $R[\Lambda]$ with $\iota = \mathrm{id}$ (from which the case with general A and ι follows) note the following: let u and v be strips with a matching asymptotics γ (negative for u and positive for v). Let δ denotes the asymptotics before γ and ζ the one after (see Figure 2.3).

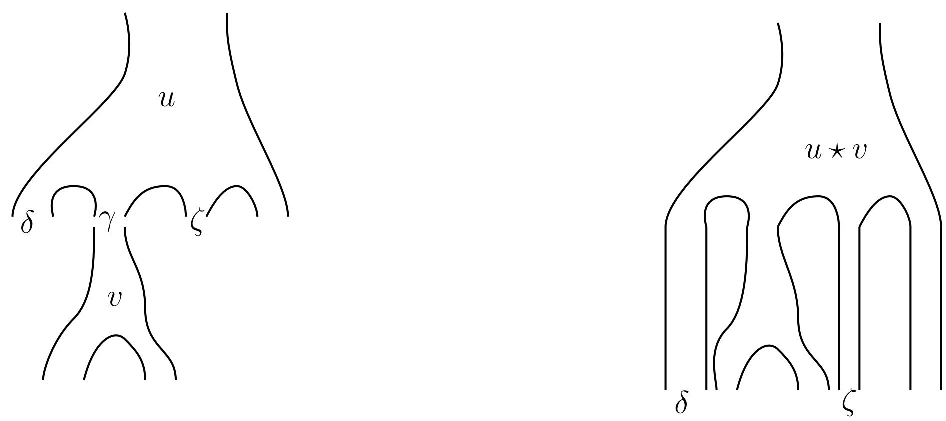

Figure 2.3: A gluing configuration.

Assume that δ is the i-th asymptotics of u (and thus one of $u \star v$), then $c_i(u \star v) = c_{\delta,\gamma}(u) \star c_1(v) \star c_{\gamma,\zeta}$ where v is thought as a (punctured) half planes. This implies $(\mu^1_{\varepsilon_0,\varepsilon_1})^2 = 0$ for general coefficients.

To show invariance let C be the Lagrangian cylinder associated to the isotopy from Λ_1 to Λ_1'. As the isotopy is disjoint from Λ_0 we can assume that this cylinder is disjoint from the trivial cylinder over Λ_0. Let ε_1' be the augmentation obtain by pull-back of ε_1 along C (by Equation (2.2)). We define now the map from $F_C : LCC(\Lambda_0, \Lambda_1) \to LCC(\Lambda_0, \Lambda_1')$ on the generators by:

$$F(a \cdot \gamma^-) = \sum_{\gamma^+} \sum_{\boldsymbol{\delta} \in \mathcal{R}(\Lambda_1), \boldsymbol{\zeta} \in \mathcal{R}(\Lambda_0)} \sum_{u \in \mathcal{M}_{\mathbb{R} \times \Lambda_0, C}(\gamma^+; \boldsymbol{\delta}, \gamma^-, \boldsymbol{\zeta})} c_0(u) \cdot a \cdot c_1(u) \cdot \gamma^+. \quad (2.4)$$

Degeneration of 1-parameter family of weighted (punctured) strips with boundary on the pair $\mathbb{R} \times \Lambda_0, C$ are of three types:

- Building with the top level being a (punctured) strip on the pair of trivial cylinders over Λ_0 and Λ'_1. The bottom level being disconnected consisting of one (punctured) strips on the pair $\mathbb{R} \times \Lambda_0, C$ and some (punctured) half-planes with boundary on C.

- Building with the top level being a (punctured) strip with boundary on the pair $\mathbb{R} \times \Lambda_0, C$, the bottom one being a a (punctured) strip on the pair of trivial cylinders over Λ_0 and Λ_1.

- Some ∂-breakings.

Similarly as before the ∂-breaking cancels out. The second type contribute to terms in $F \circ \mu^1_{\varepsilon_0,\varepsilon_1}$. The first type contributes to $\mu^1_{\varepsilon_0,\varepsilon'_1} \circ F$ as half planes with boundary on C are exactly what contributes to the definition of ε'_1. Thus F is a chain map. To conclude that it is a quasi-isomorphism let C' be the cylinder associated to the reversed isotopy. Then $C \odot C'$ is a cylinder from Λ_1 to itself inducing a chain. Standard stretching the neck argument show that it is homotopic to $F_{C'} \circ F_C$. Now notice that the construction of $C \odot C'$ can be done parametrically, showing that it is Hamiltonian isotopic to the trivial cylinder. A standard (but tedious because of the presence of negative asymptotics) analysis allows us to conclude that in homology $F_{C \odot C'} = I$. The case of a Legendrian isotopy of Λ_0 is absolutely identical. □

Remark 2.1.16. *We hope that it is clear from the argument in the previous theorem why we introduced the terminology of (punctured) strips and half planes in 1.4.2. From now on we will remove the punctured parenthesis when talking about those. Similarly in every picture we will draw from now it should be understood as having any numbers of extra negative punctures each weighted by the appropriate augmentations. All ∂-breaking can be disregarded as they all cancel out from equation* (2.5)*.*

The homology of the complex $(LCC^\bullet(\Lambda_0, \Lambda_1), \mu^1_{\varepsilon_0,\varepsilon_1})$ is called the *linearised Legendrian contact cohomology complex* from Λ_0 to Λ_1.

Remark 2.1.17. *Note that it is clear that from the constuction we could have take as complex $LCC(\Lambda_0, \Lambda_1; \varepsilon_0, \varepsilon_1)$ to be $\oplus_{\gamma \in \mathcal{R}_\alpha} M[\gamma]$ for any $A_0 \otimes A_1^{op}$ module M and the definition would have worked similarely. For instance when $A_0 = A_1 = A$ we could take A, this is what is done in Section 3.2. We do not use another notation for the LCC complex in that situation though, we hope that this will not create confusion.*

Considering only one Legendrian Λ, one can make take a two-copy link by take Λ' to be the graph of the jet a C^2 small function f such that $f > 0$. The Reeb chords of Λ' are in bijection with the one of Λ and any augmentation ε of Λ induces one ε' of Λ'.

Given two augmentations ε_0 and ε_1, over A_0 and A_1 respectively, of Λ. We define the *linearised Legendrian contact cohomology* of Λ to be

$$LCH^{\bullet}(\Lambda;\varepsilon_0,\varepsilon_1) := H_*(LCC(\Lambda,\Lambda'),\mu^1_{\varepsilon_0,\varepsilon_1'}).$$

The *relative linearised contact cohomology* is defined to be

$$LCH^{\bullet}_{\mathrm{rel}}(\Lambda;\varepsilon_1,\varepsilon_0) := H_*(LCC(\Lambda',\Lambda),\mu^1_{\varepsilon_1',\varepsilon}).$$

It follows from the second point in Theorem 2.1.15 that it does not depend on the choice of f. Also notice that if f is chosen sufficiently small then Reeb chords from Λ to Λ' are in bijection with critical points of f and Reeb chords of Λ whereas chords from Λ' to Λ are in bijection with those of Λ.

Remark 2.1.18. *At this point reader familiar with the literature might be concerned by a list of things:*

- *This definition of linearised Legendrian contact cohomology differs from other definitions from the literature. The author opinion is that it is a matter of choice (mostly on the ambiguity between homology and cohomology), this choice guarantees that this cohomology groups can be equipped with a product turning it into a unital ring. This coincides with what is suggested in [Ng+15, Remark 5.9].*

- *The relative cohomology is what in the literature is called the Linearised contact cohomology, we added the relative adjective here (knowing very well that there is few chances that the terminology will gain popularity) as it is more related to relative cohomology of a filling as in Theorem 2.1.19.*

As our grading convention are different from the usual one the literature in LCH we give some example via the following:

Verification 3. We consider $R = A = \mathbb{F}_2$ and ι the canonical augmentation maps $R[\pi] \to R$.

- For the Whitney sphere from Section A Λ_0 in $\mathcal{J}^1(\mathbb{R}^n)$ (which admit a unique augmentation), we have $LCH^\bullet_{\mathrm{rel}}(\Lambda_0, \varepsilon_0) = \mathbb{Z}_2[n+1]$ generated by the unique Reeb chord. On the other end $LCH^\bullet(\Lambda_0, \varepsilon_0) = \mathbb{Z}_2[0]$ and is generated by the minimum of the Morse perturbation (the Reeb chord γ is killed by the maximum).

- For the zero section Q_0 in $\mathcal{J}^1(Q)$ (which again have a unique augmentation) we have $LCH^\bullet_{\mathrm{rel}}(Q_0, \varepsilon_0) = 0$ (as Q_0 has no Reeb chords) whereas $LCH^\bullet(Q_0; \varepsilon_0) = H^\bullet(Q)$.

- Finally for the knot shown on Figure A.2, if we choose the augmentation that maps b_1 to 1 and all other generator to 0 we have that

  $$LCH^\bullet_{\mathrm{rel}}(\Lambda) = \mathbb{Z}_2\langle[a_1]\rangle \oplus \mathbb{Z}_2\langle[b_1 + b_3], [b_2]\rangle = \mathbb{Z}_2[2] \oplus \mathbb{Z}_2^2[1]$$

  and

  $$LCH^\bullet(\Lambda) = \mathbb{Z}_2^2[1] \oplus \mathbb{Z}_2[0].$$

  In the latter case the degree 0 generator is again given by the minimum whereas the maximum kills the degree 2 class.

Let L_0 and L_1 be two exact Lagrangians in a Liouville manifold P such that the primitive of the Liouville form is bounded on L_0 and L_1. We can lift L_0 to a Legendrian sub-manifold $\tilde{L}_0$ and L_1 to $\tilde{L}_1$ in $P \times \mathbb{R}$ such that $\max_{\tilde{L}_0} z < \min_{\tilde{L}_1} z$. As they have no Reeb chords those Legendrian admits a unit augmentation. For J a cylindrical lift of an almost complex structure of P the complex $LCC(\tilde{L}_0, \tilde{L}_1)$ is independent of the choice of the lift and is identify with the well-known *Floer complex* $CF(L_0, L_1)$. Its homology is the *Lagrangian Floer cohomology* of L_0 and L_1 from [Flo88].

Verification 4. Let's see on a simple example what all our convention gives. For $f : Q \to \mathbb{R}$ we choose L_0 to be the 0-section in T^*Q and L_1 to be the graph of df (recall that the Liouville form is $-pdq$). Choosing the 0-section of $\mathcal{J}^1(Q)$ as lift of L_0, the lift $\tilde{L}_1$ of L_1 is $j^1(f)$ provided $f > 0$. In this situation $LCC(\tilde{L}_0, \tilde{L}_1)$ is generated by critical points of f and the differential is increasing the value of f. In the ends it computes the Morse cohomology of f and thus $HF^\bullet(L_0, L_1) = H^\bullet(Q)$.

Seidel isomorphism. When the augmentation arises geometrically as in Remark 2.1.7 then this groups are characterised topologically, this is what is known as the Seidel isomorphism proved in [Ekh12] and [Dim16b]:

Theorem 2.1.19. *Let L_0 and L_1 be exact Lagrangian fillings of a Legendrian Λ in the contactisation $P \times \mathbb{R}$ of a Liouville manifold. Let A_i $i = 0, 1$ be some rings as before. We assume that the maps $\iota_i : R[\Lambda] \to A_i$ factors through maps $R[L_i] \to A_i$. We think of A_i as local coefficient systems on L_i. Let ε_i^L be the augmentation over A_i induced by L_i as in Remark 2.1.7 then*

$$LCH^\bullet(\Lambda; \varepsilon_0^L, \varepsilon_1^L) = HF^\bullet(L_0, L_1'; A_0, A_1),$$

and

$$LCH^\bullet_{\mathrm{rel}}(\Lambda; \varepsilon_0^L, \varepsilon_1^L) = HF^\bullet(L_0', L_1; A_0, A_1).$$

(Where L_i' denote a small Hamiltonian deformation of L_i which induce a small translation of the cylindrical end in the Reeb direction.)

When $L_0 = L_1 = L$ the Floer homology groups are topological which gives:

$$LCH^\bullet(\Lambda; \varepsilon_0^L, \varepsilon_1^L) = H^*(\overline{L}; A_0 \otimes A_1),$$

and

$$LCH^\bullet_{\mathrm{rel}}(\Lambda; \varepsilon_0^L, \varepsilon_1^L) = H^*(\overline{L}, \Lambda; A_0 \otimes A_1).$$

A few particular cases can be highlighted. When $A_0 = A_1 = \mathbb{F}_2$ then this recover the cohomology of L with $\mathbb{F}_2$-coefficient. Indeed in verification 3: the Whitney sphere is fillable by an $n+1$ dimensional disc, whereas the augmentation of the trefoil is realised by a punctured torus. If $A_0 = \mathbb{F}_2[\pi_1]$ and $A_1 = \mathbb{F}_2$ then this recover the cohomology of the universal cover of L.

2.2 Cthulhu complex.

In this Section we present the part of the work from [Cha+15b] where we construct a Floer complex associated to a pair of Lagrangian cobordisms. To a pair of cobordisms we associate a complex $\mathrm{Cth}(\Sigma_0, \Sigma_1)$ whose homology is invariant under Hamiltonian deformations of Σ_1. The construction stems from the construction of the Floer complex of to transversely intersecting Lagrangians. Always having in mind that augmentations of the negative ends represents hidden fillings of those ends, we proceed in two steps: first we construct what mimic the Floer complex of the fillings (obtained capping the cobordisms with the hidden fillings) then we construct the complex $\mathrm{Cth}(\Sigma_0, \Sigma_1)$ which mimic the wrapped homology and will be used to prove the theorem corresponding to Seidel's isomorphism.

2.2.1 The Floer complex for a pair of cobordisms

Consider two cobordisms Σ_0 and Σ_1 such that fo $i = 0,1$ Σ_i goes from Λ_i^- to Λ_i^+. Let ε_i^- $i = 0,1$ be augmentations of Λ_i^- and let ε_i^+ be the induced augmentations of Λ_i^+ induced by pullback along the cobordisms Σ_i^+. We assume that the cobordisms intersects transversely (note that this implies that $\Lambda_0^\pm$ and $\Lambda_1^\pm$ are disjoint) and that the links $\Lambda_0^\pm \cup \Lambda_1^\pm$ are generic. Finally for $i = 0,1$ we denote by $\mathfrak{f}_i$ the primitive of the pullback of λ on Σ_i which vanishes on the negative end.

As in Section 2.1.2 we want to describe the complex with coefficients in rings which are module over some group rings. To do so fix a $\mathbb{F}_2$-algebra R and let $j_i : R[\pi_1(\Sigma_i)] \to A_i$ $i = 0,1$ be some ring maps (note that this induce ring maps from $R[\pi_1(\Lambda_i^\pm)]$ into A_i for $i = 0,1$) that we will assume to be the underlying ring map of all our augmentations in this section. The module underlying our Floer complex is $CF_{-\infty}(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1) = CF_+(\Sigma_0, \Sigma_1) \oplus LCC(\Lambda_0, \Lambda_1) \oplus CF_-(\Sigma_0, \Sigma_1)$ where $CF_\pm(\Sigma_0, \Sigma_1)$ is the free $A_0 - A_1^{\mathrm{op}}$-module with basis given by the intersection points between Σ_0 and Σ_1 of positive, reps. negative, action. The differential with respect to this decomposition is given by a 3×3 matrix:

$$d_{-\infty}^{\varepsilon_0^-, \varepsilon_1^-} = \begin{pmatrix} d_{0^+0^+} & d_{0^+-} & d_{0^+0^-} \\ 0 & d_{--} & d_{-0^-} \\ 0 & 0 & d_{0^-0^-} \end{pmatrix}.$$

The terms d_{--} is the linearised contact cohomology differential $\mu^1_{\varepsilon_0^-, \varepsilon_1^-}$ of Λ^- defined in Section 2.1.3. The other terms are R-modules map given by various counts of holomorphic curves, we proceed to describe them now:

The Floer "differential".

The maps $d_{0^+0^+}$ $d_{0^+0^-}$ and $d_{0^-0^-}$ are all modifications of the differential in Lagrangian Floer homology. For an intersection point q, it is defined as

$$d_{0^\pm 0^\pm}(a_0 \otimes a_1 \cdot q) := \sum_p \sum_{\boldsymbol{\delta}^-, \boldsymbol{\zeta}^-} \sum_{u \in \mathcal{M}_{\Sigma_0, \Sigma_1}(p; \boldsymbol{\delta}^-, q, \boldsymbol{\zeta}^-)} c_0(u) \cdot a_0 \otimes a_1 \cdot c_1(u) \cdot p. \tag{2.5}$$

From Equation (1.5) we deduce that this map is of degree 1.

The Cultist map.

The map d_{0+-} is defined by

$$d_{0+-}(a_0 \otimes a_1 \cdot \gamma^-) := \sum_p \sum_{\boldsymbol{\delta}^-, \boldsymbol{\zeta}^-} \sum_{u \in \mathcal{M}_{\Sigma_0, \Sigma_1}(p; \boldsymbol{\delta}^-, \gamma^-, \boldsymbol{\zeta}^-)} c_0(u) \cdot a_0 \otimes a_1 \cdot c_1(u) \cdot p, \quad (2.6)$$

Equations (1.4) implies that d_{0+-} has degree 1.

The Nessie map.

The map d_{-0^-} is defined using broken curves:

$$d_{-0^-}(a_0 \otimes a_1 \cdot q) = \sum_{\gamma_{10} \in \mathcal{R}(\Lambda_1^-, \Lambda_0^-)} \sum_{\gamma^-} \sum_{\substack{u \in \widetilde{\mathcal{M}}_{\mathbb{R} \times \Lambda_0^-, \mathbb{R} \times \Lambda_1^-}(\gamma_{10}; \boldsymbol{\delta}, \gamma^-, \boldsymbol{\zeta}; J^-) \\ u' \in \mathcal{M}_{\Sigma_0, \Sigma_1}(p; \boldsymbol{\delta}^-, \gamma_{10}, \boldsymbol{\zeta}^-)}} c_0(u') \cdot c_0(u) \cdot a_0 \otimes a_1 \cdot c_1(u) \cdot c_1(u') \cdot \gamma^-. \quad (2.7)$$

Curves contributing to this term are depicted on Figure 2.4. Observe that, compared to the pseudoholomorphic Cthulhus used in the definition of the map d_{0--}, the neck of a Nessie has reversed boundary conditions.

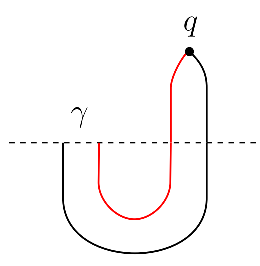

Figure 2.4: A building contributing to $\langle d_{-0} q, \gamma \rangle$.

It follows from Equation (1.7) and (1.4) that d_{-0^-} has degree 1.

The proof of $(d_{-\infty}^{\varepsilon_0^-, \varepsilon_1^-})^2 = 0$

We proceed now to the statement that this complex is indeed a chain complex:

Theorem 2.2.1. *([Cha+15b, Theorems 4.1 and 8.3]) If X is the symplectisation of a contactisation of a Liouville manifold then for generic $J \in \mathcal{J}(X)$ we have that*

$$(d_{-\infty}^{\varepsilon_0^-,\varepsilon_1^-})^2 = 0.$$

Sketch of proof. The proof uses standard analysis of breaking of 1-parameter families of holomorphic curves with various boundary conditions. Note that any curve in those families is considered as a (punctured)-strip and all ∂-breaking cancel each others (as we are using augmentations as weights). Hence we need only to consider breaking into strip. The square of the matrix $d_{-\infty}^{\varepsilon_0^-,\varepsilon_1^-}$ have 6 terms. This reduces the study of $d_{-\infty}^{\varepsilon_0^-,\varepsilon_1^-} \circ d_{-\infty}^{\varepsilon_0^-,\varepsilon_1^-} = 0$ to 6 equations:

1. $d_{0^+0^+}^2 = 0.$
2. $d_{0^+0^+}d_{0^+-} + d_{0^+-}d_{--} = 0.$
3. $d_{0^+0^+}d_{0^+0^-} + d_{0^+-}d_{-0^-} + d_{0^+0^-}d_{0^-0^-} = 0.$
4. $d_{--}^2 = 0.$
5. $d_{--}d_{-0^-} + d_{-0^-}d_{0^-0^-} = 0.$
6. $d_{0^-0^-}^2 = 0.$

The fourth one is the corresponding result for linearised Legendrian contact cohomology. For the others we note first that any holomorphic curve must increase action. Therefore the first and last equations follow from consideration similar to the usual proof of $d^2 = 0$ in Lagrangian Floer theory. The remaining equations follow from studying 1-parameter family of curves with fixed appropriate asymptotics. For instance the broken curve from Figure 2.1 is contributing to the middle term in the left side of

$$d_{0^+0^+}d_{0^+0^-} + d_{0^+-}d_{-0^-} + d_{0^+0^-}d_{0^-0^-} = 0.$$

It arises studying a family of curves in $\mathcal{M}(q^+; \boldsymbol{\delta}, q^-, \boldsymbol{\zeta})$ for q^+ an intersection point of positive action and q^- an intersection of negative action. So in addition to the standard degeneration toward a broken strip, as we go from negative to positive action, such a family can "bubble" a banana toward the negative end. This leads to the three breakings on the figure corresponding respectively to broken curves in:

1. $\mathcal{M}(q^+; q_0^+) \times \mathcal{M}(q_0^+; q^-)$ or $\mathcal{M}(q^+; q_0^-) \times \mathcal{M}(q_0^-; q^-)$ depending on the action of q_0.

2. $\mathcal{M}(q^+, \gamma_{01}) \times \mathcal{M}(\gamma_{01}; \gamma_{10}) \times \mathcal{M}(\gamma_{10}; q^-)$.

(For simplicity we remove the pure chords asymptotics from the notation but of course they are in all of those and have to match appropriately). The first type gives the terms of the equation involving only intersection points and the second corresponds to the term $d_{0^+-}d_{-0^-}$. □

Note that since $d_{-\infty}$ is triangular each of the summand is a complex with the differential given by the corresponding diagonal term we denote the homology groups of the complex made of positive and negative generator by $HF_+(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1)$ and $HF_-(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1)$. The homology of the whole complex is denoted $HF_{-\infty}(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1)$.

We now state the invariance properties for this complex.

Proposition 2.2.2. *Let (Σ_0^s, Σ_1), $s \in [0, 1]$, be a compactly supported one-parameter family of pairs of exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i = 0, 1$. Also, consider a one-parameter family $\{J_s\}_{s\in[0,1]}$ of admissible almost complex structures which agree outside of a compact set. There is an induced homotopy equivalence*

$$\Psi_{\{(\Sigma_0^s, J_s)\}} : (CF_{-\infty}(\Sigma_0^0, \Sigma_1), d_{-\infty}^{\varepsilon_0^-\varepsilon_1^-}(J_0)) \to (CF_{-\infty}(\Sigma_0^1, \Sigma_1, d_{-\infty}^{\varepsilon_0^-\varepsilon_1^-}(J_1)).$$

Remark 2.2.3. *The proof given in [Cha+15b] does not rely on abstract perturbations developed in [Ekh08]. In order to do so we trade negative Reeb chords with intersection points by gluing to the negative end of Σ_1 a "tail" which is a Lagrangian cylinder obtained as the graph of the Reeb flow applied to Λ_1^-. In [Cha+15b, Section 5] we describe the effect of this gluing on the complex which allows us to deduce that this does no change its quasi-isomorphism type. This allows to use standard bifurcation analysis to prove the results. An alternative proof, more closely adapted to the SFT formalism, would be simply to generalise [Ekh12, Section 4.2.1] to the current setting. The latter approach depends on the abstract perturbation scheme outlined in [Ekh08, Appendix B]. We do not reproduce here the construction of the maps allowing this comparison and therefore will not say more on the proof of the previous proposition.*

Note that the complex $CF_{-\infty}$ is a double cone, but if one of the maps d_{0^+-} or d_{-0^-} vanishes then it become a cone complex. This will be exploited in Section 4.1 when we will construct long exact sequences relating the Legendrian contact cohomology of the end of a cobordism to its singular cohomology.

2.2.2 The Cthulhu complex.

We now turn to the construction of the Cthulhu complex. One can motivates its definition following the idea that $CF_{-\infty}(\Sigma_0, \Sigma_1)$ is the standard Floer complex of a capped version of the cobordisms: the Cthulhu complex should mimic the wrapped Floer homology complex of [AS10] (though our description uses more the complex from [Ekh12]) if Σ_0 and Σ_1 had no concave end. This means that there are some generator at infinity coming from intersection points between Σ_1 and a version of Σ_0 wrapped along the Reeb flow. Hence any Reeb chords from Λ_0 to Λ_1 gives such an intersection. Thus the underlying vector space of the Cthulhu complex is

$$\mathrm{Cth}^\bullet(\Sigma_0, \Sigma_1) := LCC^\bullet_{\varepsilon_0^+,\varepsilon_1^+}(\Lambda_0^+, \Lambda_1^+)[1] \oplus CF^\bullet_{-\infty}(\Sigma_0, \Sigma_1).$$

Using the splitting $CF_{-\infty}(\Sigma_0, \Sigma_1; \varepsilon_0, \varepsilon_1) = CF_+(\Sigma_0, \Sigma_1) \oplus LCC(\Lambda_0, \Lambda_1) \oplus CF_-(\Sigma_0, \Sigma_1)$ the differential is given by the 4×4 matrix

$$\mathfrak{d}_{\varepsilon_0,\varepsilon_1} = \begin{pmatrix} d_{++} & d_{+0^+} & d_{+-} & d_{+0^-} \\ 0 & d_{0^+0^+} & d_{0^+-} & d_{0^+0^-} \\ 0 & 0 & d_{--} & d_{-0^-} \\ 0 & 0 & 0 & d_{--} \end{pmatrix}.$$

The entries $d_{++} = d_{\varepsilon_0^+,\varepsilon_1^+}$ is again the linearised Legendrian contact cohomology differential described in Section 2.1.3. The other three terms that we have not encountered yet, d_{+0^+},d_{+-} and d_{+0^-}, are defined similarly as before. In Figure 2.5 we provide a schematic picture of all the strips involved in the matrix $\mathfrak{d}_{\varepsilon_0,\varepsilon_1}$. We denote each time which end is considered as an input and which is an output and specify the boundary condition on each of the curves. Recall that, typically, the strips also have additional negative asymptotics to Reeb chords of Λ_i^-, $i = 0, 1$, and that all counts are 'weighted' by the values of the chosen augmentations on these chords (as in Formula (2.3)).

The main theorem of [Cha+15b] state that this is a complex invariant under cylindrical Hamiltonian deformation:

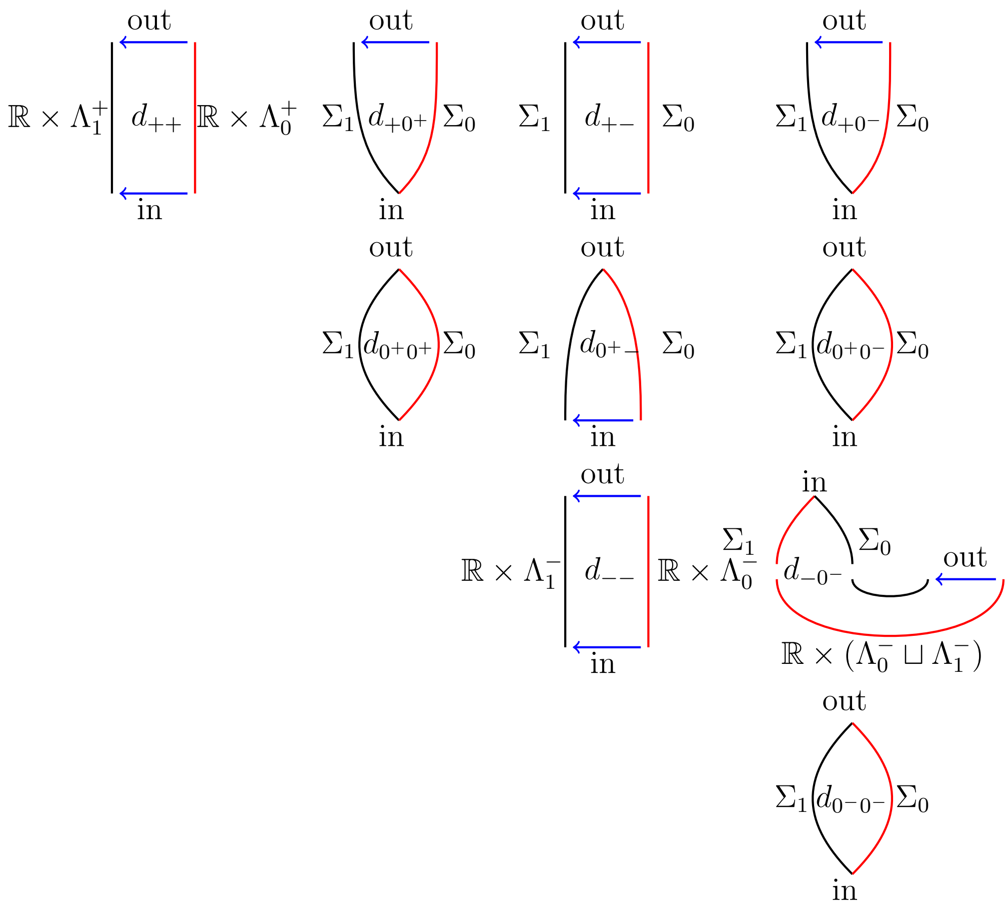

Figure 2.5: Curves contributing to the Cthulhu differential; *in* and *out* denote the input and output of the respective component of the differential.

Theorem 2.2.4. *([Cha+15b, Theorem 4.1, 6.6 and 8.3]) For generic J the differential $\mathfrak{d}_{\varepsilon_0,\varepsilon_1}$ satisfies*

$$\mathfrak{d}^2_{\varepsilon_0,\varepsilon_1} = 0.$$

Furthermore if $\{\Sigma_0^t\}$ is a cylindrical an isotopy of exact Lagrangian cobordism then the complexes $(\mathrm{Cth}_*(\Sigma_0, \Sigma_1), \mathfrak{d}_{\varepsilon_0,\varepsilon_1})$ and $(\mathrm{Cth}_*(\Sigma_0^1, \Sigma_1), \mathfrak{d}_{\varepsilon_0^1,\varepsilon_1})$ are quasi-isomorphic. (Here ε_0^1 denotes the augmentation of the negative ends of Σ_0^1 induced by the Legendrian isotopy given by the negative ends of $\{\Sigma_0^t\}$).

Since X is the symplectisation of a contactisation $P \times \mathbb{R}$ there is a Hamiltonian flow (namely $s \to (q, z + s, t)$ which pushes Σ_0 far enough so that for

s big enough the complex $\mathrm{Cth}(\Sigma_0^s, \Sigma_1)$ has no generators thus we have

Corollary 2.2.5. *The complex* $\mathrm{Cth}(\Sigma_0, \Sigma_1), \mathfrak{d}_{\varepsilon_0^1, \varepsilon_1})$ *is acyclic.*

The proof of Theorem 2.2.4 is similar to the one of Theorem 2.2.1 and Proposition 2.2.2. The entries of the matrix d^2 involve no moduli spaces which are more complicated than those discussed in the case of $CF_{-\infty}(\Sigma_0, \Sigma_1)$. For the invariance part we glue a tail to the top and bottom of the cobordism to trade all chords with intersection point and then we are left with a pair of cobordism that we can displace using a compactly supported Hamiltonian allowing us to apply Proposition 2.2.2.

Perspective 6. It is naturally expected that with an appropriate analytic framework under which all the counts of holomorphic curves are possible Theorem 2.2.4 holds for any pair of cobordisms in a Liouville cobordism X. Note that in this situation then the involved holomorphic curves must have asymptotics toward closed Reeb orbits of the negative end and therefore the Cthulhu complex is a module over the Legendrian contact homology algebra (we can use an augmentation of this one to simplify coefficients). In general though, as cobordisms are not displaceable, we should not expect the acyclicity of this complex. An interesting complex to speculate on begin the core and co-core disks in a Weinstein critical handle attachment. We let the reader's imagination wanders to see the full picture, in a forthcoming work the author of [Cha+15b] plan to define the Cthulhu complex in a more general set up and provide various applications.

The differential $\mathfrak{d}_{\varepsilon_0, \varepsilon_1}$ is the cone of the map $\begin{pmatrix} d_{+0^+} & d_{+-} & d_{+0^-} \end{pmatrix}$ from $CF_{-\infty}(\Sigma_0, \Sigma_1)$ to $LCC(\Lambda_0^+, \Lambda_1^+)$. It follows from Corollary 2.2.5 that it is a quasi-isomorphic, it is in this sense that we see Theorem 2.2.4 as a relative version of Seidel's isomorphism.

Chapter 3

The augmentation category.

In the previous chapter we used the idea that augmentations gave information of hidden Lagrangian fillings of Λ. Theorem 2.1.19 indeed states that when the two augmentations are geometric $LCC_{\mathrm{rel}}(\Lambda, \varepsilon_0, \varepsilon_1)$ computes some Floer homology groups. Since the work of Fukaya [Fuk93] It is well known that several Lagrangian (in transverse position) defines operations on those Floer complexes which all together satisfies the so-called $\mathcal{A}_\infty$-equation. Thus one should naturally expect that a $d+1$-uple of augmentations leads to some operation on the groups $LCC(\Lambda; \varepsilon_i, \varepsilon_{i+1})$ satisfying similar relation. This is the content of our paper [BC14b] where we define the so called *augmentation category* associated to a Legendrian sub-manifolds.

It is an $\mathcal{A}_\infty$-category whose objects are the augmentations of a given Legendrian sub-manifolds. The morphisms spaces are (as chain complexes) the linearised Legendrian contact cohomology complexes and higher order compositions are defined using families of augmentations to augment various ends of holomorphic planes. On Figure 3.1 we see an example of a contribution to the operation $\mu^2_{\varepsilon_0,\varepsilon_1,\varepsilon_2}$.

We show that this $\mathcal{A}_\infty$-category is invariant (in an appropriate sense) under Legendrian isotopies of Λ. Since it first appeared this category have been upgraded in various ways: in [Ng+15] a variation of it including Morse generators is given in dimension 3, in [Cha+16] we extend the original definition to non commutative coefficient ring (note that this still an $\mathcal{A}_\infty$-category over a commutative ring R though as the coefficient ring is an algebra over R) and in [EL17] a partial description as the dg-category of modules over the Chekanov algebra is given. We try to give an overview of these developments in the present chapter.

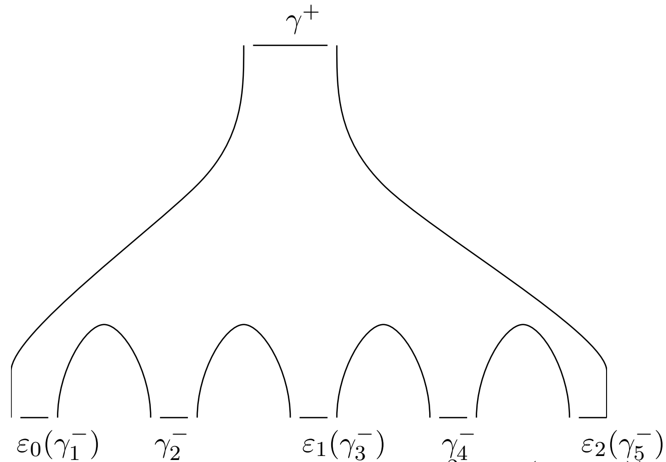

Figure 3.1: Curve contributing to $\mu^2_{\varepsilon_0,\varepsilon_1,\varepsilon_2}(\gamma_2^-, \gamma_4^-))$.

In Section 3.1 we recall some background aspects of the theory of $\mathcal{A}_\infty$-categories which does not intend to be complete at all but we hope is enough to follow the constructions that we will give (this is also relevant for Section 4.5). Then in Section 3.2 we proceed to give the definition of the augmentation category $\mathcal{A}ug(\Lambda)$ with a general coefficient ring. We adjunct a strict unit to the general description which allows to speak about quasi-equivalence without having to go through the Yoneda embedding (as it was originally done in [BC14b]). We describe the invariance and functorial property of $\mathcal{A}ug(\Lambda)$. In Section 3.3 we describe the variation of this definition from [Ng+15] but giving a general definition in all dimension using a localisation procedure we learnt from [Sei13] and [GPS17]. In this section give a general discussion on how those category should correspond to various category associated to Liouville manifold and Legendrian sub-manifolds, this part of the field is extremely active and the general picture is constantly evolving so it is most likely that between the writing of those line and the moment a reader see them things have evolved. As an illustration we describe the outcome of some explicit computation from [CNS19] that support some of those conjectures. In Section 3.4 we describes some work of [Leg18] that constructs and $\mathcal{A}_\infty$-product on the complex $CF_{-\infty}(\Sigma_0, \Sigma_1)$ from the previous chapter and compare with the one on $\mathcal{A}ug(\Lambda^+)$.

3.1 $\mathcal{A}_\infty$-categories.

3.1.1 Definitions.

We briefly recall here some of the basic definitions of the theory of $\mathcal{A}_\infty$-categories. We refer the reader to the first part of [Sei08] where the theory is developed in depth. We begin by the definition of $\mathcal{A}_\infty$-category.

Definition 3.1.1. An $\mathcal{A}_\infty$-category $\mathcal{A}$ over $\mathbb{F}_2$ is given by:

- A class of objects $\mathcal{O}b(\mathcal{A})$.
- For any pairs of objects X_0, X_1 a graded chain complex of vectors spaces $(\hom(X_0, X_1), \mu^1_{X_0,X_1})$ where $\mu^1_{X_0,X_1} : \hom(X_0, X_1) \to \hom(X_0, X_1)[1]$.
- For any $d+1$-uple $X_0, \dots X_d$ some composition of order d: $\mu^d_{X_0,\dots,X_d}$: $\hom(X_{d-1}, X_d) \otimes \cdots \otimes \hom(X_0, X_1) \to \hom(X_0, X_d)[2-d]$.
- The operation μ^d satisfy: for each d and $a_1, \cdots, a_d$:

$$\sum_{i=1}^{d} \sum_{j=0}^{d-i} \mu^{d-i+1}(a_d, a_{d-1}, \dots, \mu^i(a_{i+j}, \cdots, a_i), \dots, a_1) = 0. \tag{3.1}$$

  (For the sake of clarity we dropped in this equation all subscripts and indices).

The Etiquette requires cases $d = 1, 2$ and 3 to be detailed. The case $d = 1$ states that

$$\mu^1(\mu^1(a_1))$$

which implies that hom spaces are chain complexes, as required in the second point of the definition. The case $d = 2$ gives

$$\mu^2(\mu^1(a_2), a_1) + \mu^2(a_2, \mu^1(a_1)) + \mu^1(\mu^2(a_2, a_1)) = 0,$$

which means that μ_2 descends to a (degree 0) map $H_*(\hom(X_1, X_2) \otimes \hom(X_0, X_1)) \to H_*(\hom(X_0, X_2))$. Finally the case $d = 3$ gives

$$\begin{gathered}\mu^3(\mu^1(a_3), a_2, a_1) + \mu^3(a_3, \mu^1(a_2), a_1) + \mu^3(a_3, a_2, \mu^1(a_1)) \\ + \mu^1(\mu^3(a_3, a_2, a_1)) + \mu^2(\mu^2(a_3, a_2), a_1) + \mu^2(a_3, \mu^2(a_2, a_1)) = 0.\end{gathered}$$

The last two terms computes the default of associativity of the operation μ^2, applied on cycles the first four is exact, that implies that μ^2 is associative in homology.

This does not make quite the homology of hom-spaces a category yet: we have to address the question of unit. In the present document we will only see explicitly strictly unital $\mathcal{A}_\infty$ category. Though it is a convenient way to speak about unit it is not the most versatile one as strict unitality is not preserved under quasi-equivalence. But our goal is not to give the full scope of the realm so we just stick to the following definition:

Definition 3.1.2. An $\mathcal{A}_\infty$ category is strictly unital if for each object X there an element e_X such that:

- $\mu^1_X(e_X) = 0$.
- $\mu^2(e_{X_1}, a) = a$ and $\mu^2(a, e_{X_0}) = a$ for all $a \in \hom(X_0, X_1)$.
- For any $d > 2$ $\mu^d(a_d, \dots, a_0) = 0$ as soon as one of the entry a_i is one of the e_X.

Now we can claim that the homology of hom-spaces of a strict unital category $\mathcal{A}$ is indeed a category, called the *homological category* that we denotes $H^\bullet(\mathcal{A})$.

In order to compare $\mathcal{A}_\infty$-categories we need the notion of $\mathcal{A}_\infty$-functors, we recall the definition here:

Definition 3.1.3. An $\mathcal{A}_\infty$-functor $\mathcal{F}$ between two $\mathcal{A}_\infty$-categories $(\mathcal{A}, \{\mu^d_{\mathcal{A}}\})$ and $(\mathcal{B}, \{\mu^d_{\mathcal{B}}\})$ consists of the following:

- A map $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$
- For each $d \geq 1$ and $(X_0, \dots, X_d) \in Ob(\mathcal{A})^{d+1}$, a map $F^d_{X_0\dots,X_0} : \hom(X_d, X_{d-1}) \otimes \dots \otimes \hom(X_0, X_1) \to \hom(F(X_0), F(X_d))[1-d]$ satisfying:

$$\sum_{r=1}^{d} \sum_{s_1+\ldots+s_r=d} \mu^r_{\mathcal{B}}\big(F^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, F^{s_1}(a_{s_1}, \dots, a_1)\big) \tag{3.2}$$
$$= \sum_{i=1}^{d} \sum_{j=0}^{d-i} F^{d-i+1}\big(a_d, \dots, \mu^i_{\mathcal{A}}(a_{j+i}, \dots, a_{j+1}), \dots, a_1\big).$$

(Again we dropped some subscripts from the equation for readability.)

When $\mathcal{A}$ and $\mathcal{B}$ are strictly unital, if in addition we have that $F^1(e_a) = e_{F(A)}$ and F^d ($d > 1$) vanishes whenever one of the entries is a unit, we say that $\mathcal{F}$ is *strictly unital*.

We do not forget the Etiquette and develop the terms $d = 1, 2$ of equation (3.2). The case $d = 1$ reads as

$$F^1(\mu^1_{\mathcal{A}}(a)) = \mu^1_{\mathcal{B}} F^1(a)$$

i.e. F^1 is a chain map and thus descends to a map in $H^\bullet(\mathcal{A})$. The case $d = 2$ reads as

$$\mu^2(F^1(a), F^1(b)) + \mu^1(F^2(a, b)) = F^2(a, \mu^1(b)) + F^2(\mu^1(a), b) + F^1(\mu^2(a, b)).$$

Applied on cycle we see that this implies that $H(F^1)$ preserves the composition in $H^\bullet(\mathcal{A})$. Together with strict unitality this implies that a strictly unital $\mathcal{A}_\infty$ functor induces a functor in the homological category. In the case when $\mathcal{F}$ is not strictly unital, if the induced application in $H^\bullet(\mathcal{A})$ is unital (and thus a functor), we will say that $\mathcal{F}$ is c-unital.

If $\mathcal{F}$ is a c-unital functor, we say it is a quasi-equivalence if $H(\mathcal{F})$ is an equivalence.

Remark 3.1.4. *There is one question which have not been addressed: how can we compose two functors? The answer is of course in the book [Sei08]. We also refer the reader to [Lef03], here it is explained how $\mathcal{A}_\infty$-operation corresponds so some co-differential on the free co-algebra. Morphisms between two such co-algebras corresponds to $\mathcal{A}_\infty$-functors. Composition becomes trivial then.*

3.1.2 Twisted complexes.

Given an unital A_∞-category $\mathcal{A}$, we describe the category $Tw\mathcal{A}$ of twisted complexes over $\mathcal{A}$ and recall its basic properties. We introduce the following notation: given a number d of matrices A_i with coefficients in the morphism spaces of an A_∞-algebra, we denote by $\mu^d_{\mathcal{A}}(X_d, \dots, X_1)$ the matrix whose entries are obtained by applying $\mu^d_{\mathcal{A}}$ to the entries of the formal product of the X_i's.

Definition 3.1.5. A *twisted complex* over $\mathcal{A}$ is given by the following data:

- a finite collection of objects $X_0, \dots, X_k$ of $\mathcal{A}$ for some k,
- integers κ_i for $i = 0, \dots, k$, and
- a matrix $X = (x_{ij})_{0 \le i,j \le k}$ such that $x_{ij} \in \hom_{\mathcal{A}}(X_i, X_j)$ and $x_{ij} = 0$ if $i \ge j$, which satisfies the *Maurer-Cartan equation*

$$\sum_{d=1}^{k} \mu^d_{\mathcal{A}}(\underbrace{X, \dots, X}_{d \text{ times}}) = 0.$$

The integers κ_i are degree shifts.

We represent twisted complexes by writing the objects X_i in a row going from X_0 to X_k. The map x_{ij} by hypothesis only go in the right direction. The Maurer-Cartan equation is equivalent to the following: fix any two objects X_i and X_j, the sum of possible compositions going from X_i to X_j is 0. We will detail a few cases now in order to digest the definition (this is however not imposed by the Etiquette):

Example 3.1.6. If two objects are involved then the only non trivial possibility is to have a map $c \in \hom(X_0, X_1)$, the twisted complex is $X_0 \xrightarrow{c} X_1$, the Maurer-Cartan simply states that $\mu^1(c) = 0$ therefore c is a closed map.

Example 3.1.7. When three objects are involved there are two non trivial cases:

1. The twisted complex is $X_0 \xrightarrow{x_{01}} X_1 \xrightarrow{x_{12}} X_2$. The Maurer-Cartan give three relation: $\mu^1(x_{01}) = \mu^1(x_{12}) = 0$ and $\mu^2(x_{12}, x_{01}) = 0$ (as there is only one composition going from X_0 to X_2). Therefore the twisted complex consists of two closed morphisms which compose to 0.

2. The other case is when the twisted complex is $\begin{tikzcd} X_0 \arrow[r, "x_{01}"] \arrow[rr, bend left, "x_{02}"] & X_1 \arrow[r, "x_{12}"] & X_2 \end{tikzcd}$. Here there is two compositions going from X_0 to X_2 so that the third relation becomes $\mu^2(x_{12}, x_{01}) + \mu^1(x_{02}) = 0$. Thus the two maps do not compose to 0 but to an exact term (which is thus 0 in homology).

Given two twisted complexes $\mathfrak{L} = (\{L_i\}, \{\kappa_i\}, X)$ and $\mathfrak{L}' = (\{L'_i\}, \{\kappa'_i\}, X')$ we define $\hom_{Tw\mathcal{A}}(\mathfrak{L}, \mathfrak{L}') := \bigoplus_{i,j} \hom_{\mathcal{A}}(L_i, L_j)[\kappa_i - \kappa'_j]$ and, given $d+1$ twisted complexes $\mathcal{L}_0, \dots, \mathcal{L}_d$, we define A_∞ operations

$$\mu^d_{Tw\mathcal{A}} \colon \hom_{Tw\mathcal{A}}(\mathfrak{L}_{d-1}, \mathfrak{L}_d) \otimes \dots \otimes \hom_{Tw\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_1) \to \hom_{Tw\mathcal{A}}(\mathfrak{L}_0, \mathfrak{L}_d)$$

by

$$\mu^d_{Tw\mathcal{A}}(q_d,\ldots,q_1) = \sum_{k_1,\ldots,k_d\geq 0} \mu_{\mathcal{A}}^{k_1+\ldots+k_d+d}(\underbrace{X_d,\ldots,X_d}_{k_d}, q_d, X_{d-1},\ldots,X_1,q_1,\underbrace{X_0\ldots,X_0}_{k_0}). \tag{3.3}$$

We will not go into too much details regarding this formula but note if the twisted complex is of the type of those given in Example 3.1.6 then a map from A to $X_0 \xrightarrow{c} X_1$ is given by a pair of map (a,b) from A to X_0 and to X_1 respectively. The differential of such a pair is $\mu^1((a,b)) = (\mu^1(a), \mu^1(b) + \mu^2(c,a))$. This is the differential of the cone of the chain map given by $\mu^2(c,\cdot)$ (this is a chain map since c is closed). We will refer to the twisted complex $X_0 \xrightarrow{c} X_1$ as the *cone* of the closed map c.

Another useful remark is that a map $A = (a_{ij})$ between two twisted complexes

$$\begin{tikzcd}
X_0 \arrow[r, "x_{01}"] \arrow[rr, bend left, "x_{0i}"] \arrow[rrrr, bend right, "x_{0k}"'] & X_1\cdots & X_i\cdots \arrow[r, bend left, "x_{ij}"] & X_j\cdots & X_k
\end{tikzcd}$$

and

$$\begin{tikzcd}
X'_0 \arrow[r, "x'_{01}"] \arrow[rr, bend left, "x'_{0i}"] \arrow[rrrr, bend right, "x'_{0k}"'] & X'_1\cdots & X'_i\cdots \arrow[r, bend left, "x_{ij}"] & X'_j\cdots & X'_{k'}
\end{tikzcd}$$

is closed if an only

$$\begin{tikzcd}
X_0 \arrow[r, "a_{01}"] \arrow[rr, bend left, "a_{0k}"] \arrow[rrr, bend left=40, "a_{00}"] \arrow[rrrr, bend right, "a_{0k'}"'] & X_1\cdots & X_k \arrow[r, "a_{k0}"] \arrow[rr, bend left, "a_{kk'}"] & X'_0\cdots & X'_{k'}
\end{tikzcd}$$

is a twisted complex.

It is shown in [Sei08, Section 3.k] that the set of twisted complexes with operations $\mu^d_{Tw\mathcal{A}}$ constitutes an A_∞-category $Tw\mathcal{A}$ which contains $\mathcal{A}$ as a full subcategory. Furthermore it is shown in [Sei08, Lemma 3.32 and Lemma 3.33] that $Tw\mathcal{A}$ is the pre-triangulated envelope of $\mathcal{A}$ and thus $H^0Tw(\mathcal{A})$ is the derived category of $\mathcal{A}$.

Definition 3.1.8. We say that a collection of objects $X_1, \dots, X_k$ of $\mathcal{A}$ generates $\mathcal{A}$ if and only if any object A of $\mathcal{A}$ is quasi-isomorphic in $Tw\mathcal{A}$ to a twisted complex built from the objects X_i's.

Remark 3.1.9. *Generation criterion allows to reduce the study of an object in the Fukaya category just by studying its morphism toward the generating sets. In other word the Yoneda embedding into the module over the generating objects is cohomologically full and faithful.*

The following Lemma is useful to check that a collection of objects generates:

Lemma 3.1.10. *If there is a twisted complex $\mathfrak{L}$ built from $X_0, \dots, X_k$ (here we assume that X_0 only appear on the left of the twisted complex) such that, for every object T of $\mathcal{A}$ we have $H\hom_{Tw\mathcal{A}}(T, \mathfrak{L}) = 0$, then X_0 is quasi-isomorphic in $Tw\mathcal{A}$ to a twisted complex built from $X_1, \dots, X_k$.*

Proof. This follows from the iterated cone description of twisted complexes from [Sei08, Lemma 3.32]. More precisely, from the definition of twisted complexes, for any object T we have that $\hom_{\mathcal{A}}(T, X_0)$ is a quotient complex of $hom_{Tw\mathcal{A}}(T, \mathfrak{L})$ by the twisted complex $\mathfrak{L}'$ built from $\mathfrak{L}$ starting at X_1 (i.e. "chopping" out X_0 from the twisted complex $\mathfrak{L}$), and thus those three objects fit in an exact triangle. The vanishing of $H\hom_{Tw\mathcal{A}}(T, \mathfrak{L})$ implies then that

$$H\hom_{\mathcal{A}}(T, X_0) \cong H\hom_{Tw\mathcal{A}}(T, \mathfrak{L}').$$

The result follows now because the map from X_0 to $\mathfrak{L}'$, which is given by the maps (x_{0j}), is a map of twisted complexes. □

3.1.3 $\mathcal{A}_\infty$-quotient and localisation.

The last notion we need to continue our exposition with is the one of $\mathcal{A}_\infty$-quotient. It derives from the notion of *dg*-quotient defined in [Dri04] and an explicit formula is given in [LO06]. The goal is to set some objects to be quasi-isomorphic to 0 (thus creating lots of new quasi-isomorphisms) but keeping interesting $\mathcal{A}_\infty$-operations.

Definition 3.1.11. Let $\mathcal{A}$ be a strictly unital category and $\mathcal{B}$ a full subcategory. The *quotient category* $\mathcal{A}/\mathcal{B}$ is the $\mathcal{A}_\infty$-category defined by:

1. $\mathcal{O}b(\mathcal{A}/\mathcal{B}) = \mathcal{O}b(\mathcal{A})$.

2. Given two objects X_0, X_1,

$$\hom_{\mathcal{A}/\mathcal{B}}(X_0, X_1) =$$
$$\hom(X_0, X_1) \bigoplus_j \bigoplus_{B_1, \cdots B_j \in \mathcal{O}b(\mathcal{B}} \hom(B_1, X_1) \otimes \hom(B_2, B_3) \cdots \otimes \hom(X_0, B_j)$$

(this is huge!).

3. The differential and compositions are given by the formula

$$\begin{aligned}
&\mu^d_{\mathcal{A}/\mathcal{B}}((a_d^{k_d} \otimes a_d^{k_d-1} \otimes \cdots \otimes a_d^2 \otimes a_d^1), (a_{d-1}^{k_{d-1}} \otimes \cdots a_{d-1}^1 \otimes a_{d-1}^1), \cdots \\
&\cdots (a_1^{k_1} \otimes \cdots a_1^1 \otimes a_1^1)) \\
&= \sum_{0 \leq i \leq k_1 \leq} \sum_{0 \leq j \leq k_d} a_d^{k_d} \otimes a_d^{k_d-1} \otimes \mu_{\mathcal{A}}^{d+i+j+\sum_{l=1}^{d-1} k_l}(a_d^j, \cdots, a_1^{k_1-i}) \otimes \cdots \\
&\cdots \otimes a_1^2 \otimes a_1^1
\end{aligned} \tag{3.4}$$

We apologise the reader who got use to us developing the formula but we will not detail much more. We just want to remark that for every object B in $\mathcal{B}$ then there is a particular element $\hom_{\mathcal{A}/\mathcal{B}}(B, B)$ given by $\epsilon = e_b \otimes e_b$ such that $\mu^1_{\mathcal{A}/\mathcal{B}}(\epsilon) = \mu^2(e_B, e_B) = e_B$. Therefore the identity of B is exact and thus in the homological category of $\mathcal{A}/\mathcal{B}$ B becomes the 0 object. This has the following consequence: let c be a closed morphism such that the cone of c, $X_0 \xrightarrow{c} X_1$ (a priori an object of $Tw\mathcal{A}$) is (quasi-isomorphic to) an object of $\mathcal{B}$. Then the map given by $\mu^2(\mu^2(x, \epsilon), y)$ where $y : X_1 \to (\ X_0 \xrightarrow{c} X_1\ )$ is the inclusion given by e_{X_1} and $x : (\ X_0 \xrightarrow{c} X_1\ ) \to X_0$ is the projection given by e_{X_0} is quasi-inverse to c. (This is a fun computation playing with many of the definition given here, remember the behaviour of strict unit with higher order composition! The details are given in the dg-case in [CNS19, Remark 10].)

Now given a set C of closed morphisms in $\mathcal{A}$ consider the subcategory $\mathcal{C}$ in $Tw(\mathcal{A})$ generated by the set $\mathrm{Cone}(C)$ of all cones of elements in C. We denote $\mathcal{D}$ the quotient $Tw(\mathcal{A})/\mathcal{C}$. We define the *localisation* of $\mathcal{A}$ at C, $\mathcal{A}[C^{-1}]$, to be the image of $\mathcal{A}$ in $\mathcal{D}$ via the natural map. It follows from the preceding paragraph that all maps from C are now quasi-isomorphisms in $\mathcal{A}[C^{-1}]$. We highlight two useful properties of the localisation

1. $\mathcal{A}$ and $\mathcal{A}[C^{-1}]$ have the same objects,
2. The localisation functor $\mathcal{A} \to \mathcal{A}[C^{-1}]$ is the identity on objects and morphisms. It has trivial higher order terms (i.e. it matches A_∞ operations on the nose).

3.2 The augmentation category.

We proceed now to the description of an $\mathcal{A}_\infty$-category whose objects are augmentations of a given Legendrian link Λ. Its construction was given first [BC14b] where augmentation was taken in a field but works equally well in a commutative ring. In [Cha+16] we extend the construction to any ring. We propose there two different definitions in the case where the coefficient ring admits an involution (for instance transposition in matrix algebra or inversion in group rings). In [CNS19] we verified on the Legendrian $(2,m)$-torus knots that one of the definition was isomorphic to the category of sheaves with micro-support on those. We proceed now to describe it slightly modifying the presentation from [BC14b] in order to incorporate units in the definitions simplifying the invariance statement of these categories.

In this section, following Remark 2.1.17, we consider the Legendrian contact homology complex $LCC(\Lambda_0, \Lambda_1, \varepsilon_0, \varepsilon_1)$ to be $\oplus_\gamma A\langle\gamma\rangle$. Indeed since all augmentations take value in a fixed A we can see A as an $A_0 \otimes A_1^{\mathrm{op}}$ module.

We denote by $\mathcal{A}ug_-(\Lambda, A)$ the $\mathcal{A}_\infty$-category constructed as follows:

1. Objects of $\mathcal{A}ug_-(\Lambda, A)$ are augmentations of Λ with value in A.
2. The morphisms space between two augmentations ε_0 and ε_1 is given by
$$\hom(\varepsilon_0, \varepsilon_1) = A\langle e\rangle \oplus LCC(\Lambda, \Lambda'; \varepsilon_0, \varepsilon_1')$$
where e as grading 0. The differential is given by the linearised contact cohomology differential on the second summand and $\mu^1 e = \sum_\gamma (\varepsilon_0(\gamma) - \varepsilon_1(\gamma)) \cdot \gamma$ on the first.
3. Compositions are given by
$$\begin{aligned}&\mu^k(a_1\gamma_1, a_2\gamma_2, \cdot, a_k\gamma_k)\\ &= \sum_{\gamma^+}\sum_{\boldsymbol{\delta}_i}\sum_{u\in\mathcal{M}(\gamma^+;\boldsymbol{\delta_1},\gamma_1,\boldsymbol{\delta_2},\gamma_2,\cdots,\boldsymbol{\delta_{k-1}},\gamma_k,\boldsymbol{\delta_k})} c_0(u)a_1c_1(u)a_2\cdots c_k(u)\cdot\gamma^+.\end{aligned} \tag{3.5}$$

The other composition are characterised by the fact that e acts as a strict unit.

Remark 3.2.1. *As mentioned earlier this is the augmentation category from [BC14b] and [Cha+16] to which we add a strict unit e. The formula for the differential of e when thought as a morphism between different objects is justified by the following: an augmentation is a map from the algebra generated by Reeb chords to A, thus turning A into a module of $\mathcal{A}(\Lambda)$ (the Chekanov-Eliashberg algebra). As pointed out in [EL17, Section 1.2] the hom-spaces between two augmentations ε_0 and ε_1 in $\mathcal{A}ug_-(\Lambda)$ should compute the derived hom of the two associated $\mathcal{A}(\Lambda)$ modules. Under this correspondence, the element e is the identity $A \to A$ and the formula for the differential is simply the differential on the set of maps whose closed element are exactly modules map.*

The main theorem from [BC14b] (generalised to non-commutative coefficients in [Cha+16]) is:

Theorem 3.2.2. *The category $\mathcal{A}ug(\Lambda, A)$ is a strictly unital $\mathcal{A}_\infty$-category over $\mathbb{F}_2$. A Legendrian isotopy from Λ_0 to Λ_1 induces a quasi-equivalence from $\mathcal{A}ug(\Lambda_0, A)$ to $\mathcal{A}ug(\Lambda_1, A)$ (note that the isotopy induces an isomorphism from $R[\pi_1(\Lambda_0)]$ to $R[\pi_1(\Lambda_1)]$).*

The proof follows algebraically from the invariance properties of Chekanov-Eliashberg algebra from [Che02] and [EES05]. One can also give one using Lagrangian cobordisms. Indeed we have the following:

Theorem 3.2.3. *Let Σ be a graded exact Lagrangian cobordisms from Λ^- to Λ^+. Let A be a ring together with a map $\iota : \mathbb{F}_2[\pi_1(\Sigma)] \to A$. Then the maps $\mathcal{F}^d : \hom_{\mathcal{A}ug(\Lambda^-,A)}(\varepsilon_{d-1}^-, \varepsilon_d^-) \otimes \cdots \otimes \hom_{\mathcal{A}ug(\Lambda^-,A)}(\varepsilon_0^-, \varepsilon_1^-) \to \hom_{\mathcal{A}ug(\Lambda^+,A)}(\varepsilon_0^+, \varepsilon_d^+)$ defined by*

$$\begin{aligned}&\mathcal{F}^d(a_d\gamma_d^-, \cdots, a_1\gamma_1^-) = \\ &\sum_{\gamma^+} \sum_{\boldsymbol{\delta}_0, \cdots, \boldsymbol{\delta}_d} \sum_{u \in \mathcal{M}_\Sigma(\gamma^+, \boldsymbol{\delta}_d, \gamma_d^-, \cdots, \gamma_1^-)} = c_0(u) a_1 c_1(u) a_2 \cdots c_k(u) \cdot \gamma^+ \end{aligned} \tag{3.6}$$

gives an strictly unital $\mathcal{A}_\infty$-functor from $\mathcal{A}ug(\Lambda^-, A)$ to $\mathcal{A}ug(\Lambda^+, A)$.

Sketch of proof. This follow from similar analysis of degeneration of holomorphic curves as we sketched several times: aside from ∂-breakings such a family can degenerate in height two buildings giving contribution the left hand side or right hand side of Equation (3.2) depending if the top level of the building has boundary on $\mathbb{R} \times \Lambda^+$ or Σ . □

Perspective 7. This functor is well-behaved under concatenation of cobordisms. If $\Sigma_0 \odot \Sigma_1$ is obtained as concatenation of Σ_0 and Σ_1 then the functor induced by $\Sigma_0 \odot \Sigma_1$ is homotopic to the composition of the one induced by Σ_0 with the one induced by Σ_1. This is hardly a functor because this homotopy is far from canonical (if we want to stay at the chain level in order to retain interesting homotopical information). Actually the category of cobordism is not in any reasonable way a category, it is more canonically a quasi-category (∞-category). It would be interesting to write this construction has a morphism between this quasi-category and the homotopy nerve of $\mathcal{A}ug(\Lambda, A)$.

Theorem 3.2.2 follows from Theorem 3.2.3 together with this concatenation property applied to the cylinder induced by a Legendrian isotopy.

Note that for any pairs of object ε_0 and ε_1 the complex $LCC_{\varepsilon_0,\varepsilon_1}(\Lambda; A)$ is a subcomplex of $\hom(\varepsilon_0, \varepsilon_1)$, this leads to a long exact sequence:

$$\begin{tikzcd} LCH^{\bullet}_{\mathrm{rel}}(\Lambda; \varepsilon_0, \varepsilon_1) \arrow[r, "i"] & H^{\bullet}(\hom(\varepsilon_0, \varepsilon_1), \mu^1_{\varepsilon_0,\varepsilon_1}) \arrow[dl, "\pi"] \\ A\langle 0\rangle \arrow[u, "f"', "{[1]}"] & \end{tikzcd} \tag{3.7}$$

For a single object ε it follows from the definition of the differential of e that the map f in this sequence is 0. Therefore in $\mathcal{A}ug(\Lambda, A)$ the homology of the endomorphism space of ε is $A \oplus LCH_{\mathrm{rel}}(\Lambda; \varepsilon)$.

The first factor is somewhat confusing. It reflect that we added a unit to our category algebraically. However recall that Seidel–Ekholm–Dimitroglou-Rizell isomorphism in its simplest case states when $A = \mathbb{F}_2$ and ε is induced by an exact Lagrangian filling L then $LCH_{\mathrm{rel}}(\Lambda; \varepsilon) \simeq H^{\bullet}(\overline{L}, \Lambda; \mathbb{F}_2)$ which is not a unital algebra, the homology of the endomorphism space should be understood through this isomorphism as the cohomology of L where we attached the cone of its boundary (creating this a unit in cohomology). When the involved Legendrian is a sphere then this is simply the cohomology of the closed Lagrangian obtained by capping L with the Lagrangian core in the handle attachment along Λ (a viewpoint adopted in [EL17]).

For two different objects the behaviour of f is related to homotopies of augmentations.

Definition 3.2.4. We say that two augmentations ε_0 and ε_1 are *homotopic* if the map f in (3.7) is 0.

Remark 3.2.5. *Note that by definition $f(e) = \sum_\gamma(\varepsilon_0(\gamma)-\varepsilon_1(\gamma))\cdot\gamma$ thus f is trivial in homology if and only if there is chain $\sum a_i\gamma_i$ for which $\mu^1_{\varepsilon_0,\varepsilon_1}(\sum a_i\gamma_i) = \sum_\gamma(\varepsilon_0(\gamma)-\varepsilon_1(\gamma))\cdot\gamma$. This implies that for any γ^+*

$$\varepsilon_0(\gamma^+) - \varepsilon_1(\gamma^+) = \sum_\gamma \sum_{\boldsymbol{\delta},\boldsymbol{\zeta}} \sum_{u\in\mathcal{M}(\gamma^+;\boldsymbol{\delta},\gamma,\boldsymbol{\zeta})} \varepsilon_0(u)K(\gamma)\varepsilon_1(u)$$

for $K(\gamma_i) = a_i$. This coincide with the usual definition of homotopic augmentations using $\varepsilon_0 - \varepsilon_1$-derivations from the Chekanov-Eliashberg algebra. (Compare also [Ng+15, Section 5.3] and [BG19]).

Perspective 8. Homotopic augmentations induces quasi-isomorphic object in $\mathcal{A}ug(\Lambda, A)$, this follows from [FHT01, Lemma 26.3] similarly as in [Ng+15, Proposition 5.16]. This would be interesting to extend this homotopy definition to higher homotopies building a quasi-category of augmentations that conjecturally would recover the homotopy nerves of $\mathcal{A}ug(\Lambda, A)$ (which would be functorial under cobordisms as speculated in Perspective 7).

3.3 Positive category and relation with sheaf category.

We now proceed to give a description of the category $\mathcal{A}ug_+(\Lambda; A)$ as introduced in [Ng+15] extended in [Cha+16] and used in [CNS19] to compare it to sheaves with micro-support on Λ with any rank. This category is a modification of $\mathcal{A}ug(\Lambda; A)$ which allows to see the unit geometrically. In [Ng+15] it is defined only for Legendrian link in $\mathbb{R}^3$ because some explicit deformation of the n-copy Legendrian link is needed. However we believe that using an idea of Seidel in [Sei13, Lecture 10] using localisation to define various Fukaya types categories (we will recall one instance of such a category in Section 4.5) can be used to construct $\mathcal{A}ug_+(\Lambda)$ in any dimension. We describe this procedure here.

We assume that Λ is in general position so that it has a discrete set of Reeb chords. We denote by $2c$ the length of the smallest Reeb chords. Fix $\delta > 0$ so that $\sum_{i\in\mathbb{N}}\delta^i < c$. We choose a family $f_i : \Lambda \to \mathbb{R}$ of Morse functions inductively such that:

1. For each $j > i$ the function $f_i - f_j$ is Morse.

2. The C^0-norm of f_i is less than $\delta^i/2$.

Given such a family we denote by Λ_i the deformation of $\phi_{R_\alpha}^{\delta^i}(\Lambda)$ by the function f_i in its Weinstein neighbourhood.

We now denote by $\mathcal{O}$ the $\mathcal{A}_\infty$ category whose objects are given by pairs (ε, i) for ε an augmentation of Λ and $i \in \mathbb{N}$. The morphisms spaces are given by:

$$\hom_{\mathcal{O}}((\varepsilon, i), (\varepsilon', j)) = \begin{cases} LCC(\Lambda_i, \Lambda_j; \varepsilon, \varepsilon') & \text{if } i < j, \\ A & \text{if } i = j \text{ and } \varepsilon = \varepsilon', \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathcal{A}_\infty$-operations are given on all non-zero chains of morphisms by the restriction of the operation in $\mathcal{A}ug(\Lambda_{i_1} \cup \Lambda_{i_2} \cup \cdots \Lambda_{i_k}, A)$ (with $i_1 < i_2 < \cdots < i_k$) to the appropriate mixed chords. The rest of the operation is determined by requiring $\mathcal{O}$ to be strictly unital.

Remark 3.3.1. *It follows from [EES09, Theorem 5.5] that on a two copy the class c_m coming from the minima of the function f is always a cycle in $LCC(\Lambda, \Lambda'; \varepsilon, \varepsilon)$ and that multiplication by c_m always induces isomorphisms on the cohomology groups. This what should become the unit in $\mathcal{A}ug_+(\Lambda; A)$.*

We denote by C the set of all morphisms $c_{ij} \in \hom_{\mathcal{O}}((\varepsilon, i), (\varepsilon', j))$ given by the sum of all minima of $f_j - f_i$, we define $\mathcal{A}ug_+(\Lambda; A)$ to be $\mathcal{O}[C^{-1}]$.

Given another family of function g_i we have another category $\mathcal{O}'$. Applying a global contactomorphism one arrange the data associated to $\mathcal{O}$ to be transverse to those associated to $\mathcal{O}'$. This gives a bigger category $\mathcal{O}''$ into which both $\mathcal{O}$ and $\mathcal{O}'$ include fully and faithfully. Quotienting by C'' induces a functor from both $\mathcal{O}[C^{-1}]$ and $\mathcal{O}'[(C')^{-1}]$ into $\mathcal{O}''[(C'')^{-1}]$. Those functors are cohomologically full and faithful and as elements in C'' become quasi-isomorphisms. The functors induced in homology become essentially surjective. Hence both categories are quasi-isomorphic to a common category, this implies:

Proposition 3.3.2. *The quasi-isomorphism type of $\mathcal{A}ug_+(\Lambda; A)$ is well defined and is invariant under Legendrian isotopy of Λ.*

This description of $\mathcal{A}ug_+(\Lambda; A)$ has the advantage to work in any dimension. However the quotient procedure make the hom-spaces huge which can make explicit computations hard. Note at least that $H^\bullet(\hom(\varepsilon_0, \varepsilon_1)) = LCH^\bullet(\Lambda; \varepsilon_0, \varepsilon_1)$ (we dropped the index i for object in the quotient category has by definition (ε, i) and (ε, j) are canonically quasi-isomorphic). In [Ng+15] when $A = \mathbb{Z}$ an explicit description of this category and of the $\mathcal{A}_\infty$-product. From [Cha+16], in [CNS19] we extends this explicit description to the case when A is a matrix algebra $M_{n\times n}(\mathbb{F}_2)$ and explicitly compute it on the Legendrian whose Lagrangian projections are given on Figure 3.2.

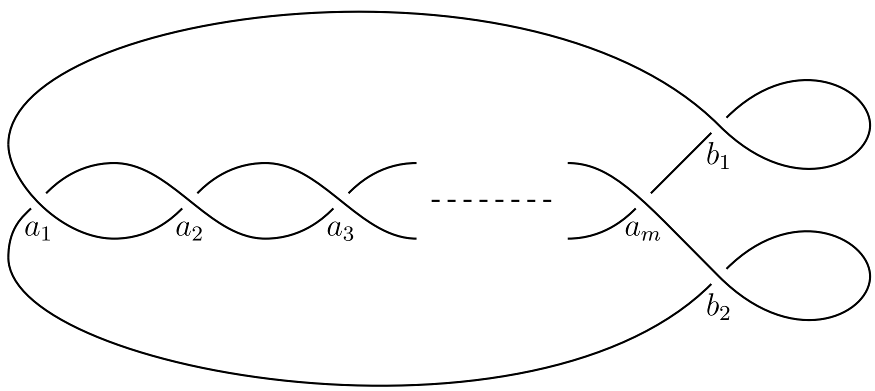

Figure 3.2: A Lagrangian projection of the Legendrian $(2, m)$ torus link Λ_m.

The objects and morphisms spaces are given by the following proposition:

Proposition 3.3.3. *An augmentation of Λ_m in A is given by a n-uple $(A_1, \ldots, A_m)$ of $n \times n$ matrices so that $P_m(A_1, \ldots, A_m)$ is invertible where P_m is the standard continuant polynomial.*

Let $\rho = (A_1, \ldots, A_m)$ and $\rho' = (A'_1, \ldots, A'_m)$ be two objects in $\mathcal{A}ug_+(\Lambda_m, A)$. Then we have:

$$H^0 \operatorname{Hom}(\rho, \rho') \cong \{(u_1, u_2) \in (\mathrm{Mat}_n(\mathbb{F}_2))^2 \mid$$
$$u_1 A'_j = A_j u_2 \text{ for } j \text{ odd and } A_j u_1 = u_2 A'_j \text{ for } j \text{ even}\};$$

$$H^1\operatorname{Hom}(\rho,\rho') \cong (\mathrm{Mat}_n(\mathbb{F}_2))^m \,/ \\ \{(u_1A_1' - A_1u_2, u_2A_2' - A_2u_1, u_1A_3' - A_3u_2, \dots, u_1A_m' - A_mu_2) \mid u_1, u_2 \in \mathrm{Mat}_n(\mathbb{F}_2)\}$$

if m is odd, and

$$H^1\operatorname{Hom}(\rho,\rho') \cong (\mathrm{Mat}_n(\mathbb{F}_2))^m \,/ \\ \{(u_1A_1' - A_1u_2, u_2A_2' - A_2u_1, u_1A_3' - A_3u_2, \dots, u_2A_m' - A_mu_1) \mid \\ u_1, u_2 \in \mathrm{Mat}_n(\mathbb{F}_2)\}$$

if m is even; and $H^i\operatorname{Hom}(\rho,\rho') = 0$ for $i \geq 2$.

We also give an explicit description of the $\mathcal{A}_\infty$-compositions which allows us to compare the so-called category $\mathrm{Sh}_n(\Lambda, \mathbb{F}_2)$. We will not describe it here but just say it is the category of (complexes of) sheaves whose sections can change only in the direction normals to the front of Λ quotiented by acyclic complexes. In [CNS19] we computes its homology category using combinatorial description from [STZ17] to verify:

Theorem 3.3.4. *For $m \geq 1$, let Λ_m be the Legendrian $(2, m)$ torus link whose Lagrangian projection is shown in Figure 3.2, equipped with its standard binary Maslov potential. Then the cohomology categories $H^\bullet\mathcal{A}ug_+(\Lambda_m, A)$ and $H^\bullet\,\mathrm{Sh}_n(\Lambda_m, \mathbb{F}_2)$ are equivalent.*

Perspective 9. The previous theorem is to support some more general conjecture generalising the results from [Ng+15] proving an $\mathcal{A}_\infty$-equivalence between the two categories for $n = 1$ (for Legendrian knots in $\mathbb{R}^3$). A third category that relates to those is the infinitesimally wrapped category of Λ (i.e. the subcategory of the partially wrapped category stopped at Λ generated by objects asymptotic to Λ). The general picture that still have to be fully understood is that given a filling L of a Legendrian Λ in $S(T^*Q)$ the infinitesimally wrapped homology of L corresponds to the morphism space in the sheaf category, the Legendrian contact homology corresponds to the high energy part of the wrapped homology of L. When the full wrapped homology vanishes those two are isomorphic leaving ground to speculate on an equivalence of categories. For knots in $\mathbb{R}^3$ (i.e. in space of contact element with no vertical tangencies) this vanishing is guaranteed because such objects do not meet a generator of the partially wrapped category of $T^*(D^2)$ (see Section 4.5). Note that in this direction, forthcoming work of Bourgeois and Viterbo (building on the construction [Vit19]) associates to an augmentation in $\mathbb{F}_2$ a simple sheaf with the correct singular support.

Perspective 10. The functoriality under cobordisms of the category $\mathcal{A}ug_+$ is not as clear as the one of $\mathcal{A}ug(\Lambda)$. In [Ng+15] for dimension 1 a formula is given but follows from the fact that in this dimension $\mathcal{A}ug_+$ is characterised by $\mathcal{A}ug$. This is also confirmed by the work of [Pan17]. It gives a geometric description of the functorial property using the Cthulhu complex associated to a deformation of a single cobordisms. But note that in this situation also a low dimensional property is used to make the map going in the correct direction. In general it follows from the long exact sequence (4.3) that the Cthulhu complex gives a natural map from $LCH(\Lambda^+)$ to $LCH(\Lambda^-)$, i.e. it goes in the opposite direction than expected by pullback of augmentations (we emphasise that we should not expect in general to be able to push augmentation forward). This is interesting to compare with the difficulties people have to find functorial properties of the sheaf category under Lagrangian cobordisms.

3.4 Product structure in Cthulhu complex and augmentation category.

We now briefly exposed some of the results from [Leg18] which studies product structures on the complex $CF_{-\infty}(\Sigma_0, \Sigma_1)$. Increasing the zoology of holomorphic curves used to define the differential the first result from [Leg18] is

Theorem 3.4.1. *Given a $d+1$-uple of Lagrangian cobordisms $\Sigma_0, \cdots, \Sigma_d$ in the symplectisation of a contactisation. There is a product:*

$$\mu^d : CF_{\infty}(\Sigma_{d-1}, \Sigma_d)) \otimes \cdots \otimes CF_{-\infty}(\Sigma_0, \Sigma_1) \to CF_{-\infty}(\Sigma_0, \Sigma_d).$$

All together these operations satisfy the $\mathcal{A}_\infty$-equation.

Extending the terms $(d_{+0^+}\ d_{+-}\ d_{+0^-})$ to higher order maps to $LCC(\Lambda_0^+, \Lambda_d^+)$ satisfying the $\mathcal{A}_\infty$ functor relation, the second Theorem from [Leg18] is:

Theorem 3.4.2. *Given Legendrians Λ^+, Λ^- there is an $\mathcal{A}_\infty$-category whose object are cobordisms from Λ^- to Λ^+ and the operations are given by the μ^d of the previous theorem. There is a cohomologically full and faithful function from this category to $\mathcal{A}ug(\Lambda^+)$ whose linear term is $(d_{+0^+}\ d_{+-}\ d_{+0^-})$.*

Remark 3.4.3. *In the case of filling, this theorem upgrade Seidel's isomorphism to an $\mathcal{A}_\infty$-functor which is a quasi-equivalence when wrapped homology vanishes. This allows Lagrangian filling to be a possible intermediate step into proving an equivalence between sheaves and augmentations as in Perspective 9.*

Chapter 4

Applications.

We now turns to applications of the theory developed in the previous chapters. In order to use the Cthulhu complex one finds it useful to first start to play with some geometric constructions to simplify its filtration. Indeed as it is constructed it is naturally a triple cone (acyclic in the case of contactisation of Liouville manifold) but if it were a double cone it would lead to exact triangles and if it were a simple cone it would induce some quasi-isomorphisms. So we illustrate here how some geometric inputs allows us to show that some generators of the complex do not exists. This leads to the following situations:

- **No negative chords:** this is for instance the case of Lagrangian filling (actually only one of the two cobordisms have to be a filling). In this situation this leads to a slight generalisation the isomorphisms of Theorem 2.1.19: its consequence holds not in case of filling but also when no mixed chords on the negative end exist.

- **No positive intersection points.** In this situation the Cthulhu complex induces an exact sequence involving a map from the Legendrian contact cohomology of the bottom to the one of the top. Such situation arises in two ways: when perturbing a single Legendrian by a small negative Hamiltonian (this leads to exact sequence involving the homology of the cobordisms as in Section 4.1) and when intersecting a trivial cobordism with one coming from a positive isotopy (this is developed in Section 4.4).

- **No negative intersection points.** This is similar to the previous case but in this situation the map between the Legendrian contact cohomology groups goes from the top to the bottom (see Perspective 10 on that matter). This is also discussed in Section 4.1.

- **No intersection points at all.** In this situation the Cthulhu complex becomes a simple cone and thus induces a quasi-isomorphism between the Legendrian contact cohomology groups of both ends. A simple situation when this occurs is by lifting exact Lagrangian in Liouville manifold and study cobordisms between them. Applied to surgery cobordisms this is one of the key ingredients of the proof of the generation of the wrapped Fukaya category (Theorem 0.0.8) and this is described in details in Section 4.5.

- **No chords on the positive ends.** This situation arises for instane when we consider what is known as Lagrangian caps. They do exists in abundance when the negative ends of the cobordism is flexible (hence admits no augmentations). In symplectisations of contactisations caps with negative ends admitting augmentations do not exist: this is proven in [Dim15] and also in [Cha+15b, Corollary 1.4]. In more general situations those can arise, for instance core of a critical point in a Weinstein cobordism is a nice example. Though we will not discuss more this case here.

In the next sections we details this and deduce some consequences on the topology of Lagrangian cobordisms (Section 4.2) and some obstructions to the existence of some Lagrangian cobordisms (Section 4.3). We conclude this chapter with Section 4.5 where we give an overview of the arguments from [Cha+17] proving that co-core generates the wrapped Fukaya category of a Weinstein sector.

4.1 Exact sequences associated to Lagrangian cobordisms.

In this section we study the Cthulhu complex associated to some pairs built out of a single cobordisms.

Let Σ be a single cobordism from Λ^- and Λ^+ in the symplectisation of a Liouville manifold $P \times \mathbb{R}$. Let ε_0^- be an augmentation of Λ^- with value in a ring A (with a fixed map $\iota : \mathbb{F}_2[\pi(\Sigma)] \to A$) and ε_1^- an augmentation in $\mathbb{F}_2$.

Long exact sequence of triple. Let Σ_{--} be a small deformation of Σ such that, in the Weinstein neighbourhood of Σ, Σ_{--} is the graph of the differential of a Morse function with critical point only in $\overline{\Sigma}$ and such that over the negative and positive and Σ_{--} is the cylinder over a deformation $\Lambda^{\pm}_{\text{def}}$ of $\Lambda^{\pm} - \delta\frac{\partial}{\partial z}$ by some small Morse function $f^{\pm}$. (We refer the reader to [Cha+15b, Section 7.2] for formulas of such an Hamiltonian deformation and remain here on the high-level picture).

For sufficiently small deformations one can arrange for intersection points to have small negative actions. This implies that the complex $CF_+(\Sigma, \Sigma_{--})$ is 0. Furthermore a standard argument going back to [Flo88] then shows that

$$H^k_-(CF(\Sigma, \Sigma_{--}), d_{00}) = H^k(\overline{\Sigma}, \Lambda^+; R).$$

(The only things we want to point out is that curves involved in the computation of d_{00} do not have any extra asymptotics toward Reeb chords as those chords have too big action).

Furthermore the complex $CF_{-\infty}(\Sigma, \Sigma_{--})$ is the sum

$$LCC(\Lambda^-, \Lambda^-_{\text{def}}) \oplus CF_-(\Sigma, \Sigma_{--})$$

with differential $\begin{pmatrix} d_{--} & d_{-0^-} \\ 0 & d_{0^-0^-} \end{pmatrix}$, i.e. it is the cone of the map d_{-0^-}. Combining the exact sequence associated to the cone together with the isomorphism given by the Cthulhu complex give the following exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & LCH^{k-1}_{\text{rel}}(\Lambda^+; \varepsilon^+_0, \varepsilon^+_1) \arrow[d] & & & \\ & H^{k-1}(\overline{\Sigma}, \partial_+\overline{\Sigma}; R) \arrow[r, "d_{-0}"] & LCH^k_{\text{rel}}(\Lambda^-; \varepsilon^-_0, \varepsilon^-_1) \arrow[r, "d_{+-}"] & LCH^k_{\text{rel}}(\Lambda^+; \varepsilon^+_0, \varepsilon^+_1) \arrow[r] & \cdots \end{tikzcd} , \tag{4.1}$$

This prove Theorem 0.0.4. (The fact that the map between the linearised contact homology group require some tracking of the quasi-isomorphism given by the Cthulhu complex).

Exact sequence of pair. Consider now a deformation Σ_{+-} of Σ similar to the one in the previous paragraph but for which the negative end is a deformation of $\Lambda^+ + \delta\frac{\partial}{\partial z}$. This time it is $CF_-(\Sigma_0, \Sigma_1)$ that vanishes therefore $CF_{-\infty}(\Sigma_0, \Sigma_1)$ decomposes as

$$LCC(\Lambda^-, \Lambda^-_{\text{def}}) \oplus CF_+(\Sigma, \Sigma_{+-}).$$

The homology of the complex $CF(\Sigma, \Sigma_{+-})$ is this time $H^\bullet(\overline{\Sigma})$ and we get the exact sequence

$$\begin{array}{l}\cdots \to LCH^{k-1}(\Lambda^+; \varepsilon_0^+, \varepsilon_1^+) \\ \qquad\quad \downarrow G_{\varepsilon_0,\varepsilon_1} \\ \qquad\quad H^{k-1}(\overline{\Sigma}; A) \xrightarrow{d_{-0}} LCH^k_{\mathrm{rel}}(\Lambda^-; \varepsilon_0^-, \varepsilon_1^-) \to LCH^k(\Lambda^+; \varepsilon_0^+, \varepsilon_1^+) \to \cdots ,\end{array} \tag{4.2}$$

This prove Theorem 0.0.5. The map $G_{\varepsilon_0,\varepsilon_1}$ in the previous exact sequence shows some interesting behaviour as stated in the following:

Theorem 4.1.1. *Assume that $\varepsilon_0^+ = \varepsilon_1^+ = \varepsilon$. Let c_m be the minimum class in $LCH^0(\Lambda^+, \varepsilon)$. Then $G_{\varepsilon,\varepsilon}(c_m)$ is the generator of $H^0(\overline{\Sigma}; A)$.*

Sketch of Proof. It suffices to notice that c_m cannot be in the image of d_{+-}. Indeed it follows from equation a curve with a negative asymptotic toward a generator of $LCC_{rel}(\Lambda^-; \varepsilon_0^-, \varepsilon_1^-)$ can only have asymptotic toward a long chord of the link $\Lambda^+ \cup \Lambda^+_{\mathrm{def}}$. As c_m is generated only by small chords we conclude the result. □

Other exact sequences. There are other exact sequences we can get considering deformation we would denote Σ_{++} and Σ_{+-}. For instance for Σ_{++} the term d_{-0^-} vanishes which leads to a sequence:

$$\begin{array}{l}\cdots \to LCH^{k-1}(\Lambda^-; \varepsilon_0^-, \varepsilon_1^-) \\ \qquad\quad \downarrow \\ \qquad\quad H^k(\overline{\Sigma}, \partial_-\overline{\Sigma}; A) \longrightarrow LCH^k(\Lambda^+; \varepsilon_0^+, \varepsilon_1^+) \to LCH^k(\Lambda^-; \varepsilon_0^-, \varepsilon_1^-) \to \cdots\end{array} \tag{4.3}$$

We let the reader figure out what the last of this simple constructions leads to.

4.2 Restrictions on the topology of cobordisms.

We now exploit these exact sequences to deduce some restrictions on the topology of some cobordisms.

Homology of endocobordisms.

Theorem 4.2.1. *Let Λ be a $\mathbb{F}_2$-homology sphere. Let Σ be an exact Lagrangian cobordism from Λ to Λ. If Λ admits an augmentation with value in $\mathbb{F}_2$, then*

$$H^{\bullet}(\overline{\Sigma}, \Lambda; \mathbb{F}_2) = 0,$$

i.e. the inclusion of Λ in Σ is a $\mathbb{F}_2$-homology equivalence.

Proof. Since the negative end of Σ equals its positive end we can form $\Sigma^{\odot k}$, the cobordism obtain by the concatenation of k copies of Σ. Since Λ is a $\mathbb{F}_2$-homology sphere we have that the rank $\dim_{\mathbb{F}_2} H(\Sigma^{\odot k}) = k \cdot \dim_{\mathbb{F}_2} H(\Sigma; \mathbb{F})$. However the long exact sequence (1) imply that $\dim_{\mathbb{F}_2} H(\Sigma^{\odot k}) \leq 2\#\mathcal{R}(\Lambda)$. Thus for all $k \in \mathbb{N}$ $k \cdot \dim_{\mathbb{F}_2} H(\Sigma; \mathbb{F}_2) 2\#\mathcal{R}(\Lambda)$ which imply $H^{\bullet}(\overline{\Sigma}, \Lambda; \mathbb{F}_2) = 0$ □

Remark 4.2.2. *Note that we used only the triangle associated to the LES* (1)*, this is one example of a theorem that holds verbatim when considering ungraded cobordisms.*

Verification 5. A natural question that we can ask is: do we really need homology sphere? In [Cha+15b, Theorem 1.6] where using the Mayer-Vietoris type exact sequence that we choose not to reproduce here we show that in general the total rank of the homology of an endocobordism equal the total rank of the Legendrian sub-manifolds at his end. There exists though example of endocobordisms that are not homology cylinder: for instance the Legendrian surface on Figure A.5 that admits a cobordisms from and to the Whitney sphere: thus there is an endocobordism that is not an homology cylinder. The reason is that when concatenating the two cobordisms the handles are in cancelling position and thus the rank of the homology is not increasing.

In [Cha+15b, Theorem 9.1] the preceding theorem is proved to hold even when we introduce signs and thus with no restriction on the field of coefficient. This has the following consequence:

Theorem 4.2.3. *If Λ is a homology sphere which admits an augmentation over $\mathbb{Z}$, then any exact Lagrangian cobordism Σ from Λ to itself is a homology cylinder (i.e. $H^{\bullet}(\Sigma, \Lambda) = 0$).*

Sketch of proof. Since Λ is assumed to have an augmentation over $\mathbb{Z}$ it admits an augmentation over $\mathbb{Q}$ as well. And thus it follows from Theorem 4.2.1 that $H^\bullet(\Sigma, \Lambda; \mathbb{Q}) = 0$ and thus that $H^\bullet(\Sigma, \Lambda; \mathbb{Z})$ is torsion. The augmentation over $\mathbb{Z}$ also induces an augmentation over any finite field, and thus Theorem 4.2.1 implies that $H^\bullet(\Sigma, \Lambda; \mathbb{Z})$ has no p-torsion for any prime p. Thus $H^\bullet(\Sigma, \Lambda; \mathbb{Z}) = 0$. □

Such a result raise the question if in addition of being able to propagate trivial homology through endocobordisms we are also able to propagate simple connectedness. This lead to study of augmentation with coefficient in $\mathbb{F}[\pi_1(\Lambda)]$ to extract information on the fundamental group of a cobordisms. This the subject of the next paragraph.

Fundamental group of cobordisms. In this paragraph we want to highlight two results which allows to show that some cobordism are simply connected. The first one is similar in spirit to Theorem 4.2.1. Unfortunately we have to state in the theorem some conditions which guarantee that we can work outside of $\mathbb{F}_2$ and use signs so we can only be sketchy here for the proof.

Theorem 4.2.4. *Let Λ be a simply connected Legendrian sub-manifold which is relatively Pin, and let Σ be an exact Lagrangian cobordism from Λ to itself. If Λ admits an augmentation over $\mathbb{C}$, then Σ is simply connected as well.*

Remark 4.2.5. *In the statement of the theorem we do not ask for the cobordism to be relatively Pin. This because in [Cha+15b] we show that vanishing of some characteristic classes of a Legendrian propagates to endocobordisms ([Cha+15b, Theorem 1.9]) this implies that orientability and relatively Pin properties of the ends guarantee the same properties on an endocbordism.*

The complete proof of this theorem requires an extension of the Cthulhu complex to coefficients in an L^2 completion of $\mathbb{C}[\pi_1(\Sigma)]$. This completion allows us to speak about L^2-rank of homology (as introduced in [Ati76]) and do the same type of arguments as in the proof of Theorem 4.2.1. In order to do so we must introduce the language of L^2-complexes and L^2-cohomology. We were brief on that in [Cha+15b, Section 8.5] and we feel it would be useless to summarise this here. Instead we refer to the book of [Lüc02]. Nevertheless the reader can have a good idea on how to prove the result with the following sketch of proof:

Elements of proof. As before we denote by $\Sigma^{\odot k}$ the concatenation of k-copies of Σ. As mentioned in the previous remark all cobordisms $\Sigma^{\odot k}$ are relatively Pin and thus we can consider the Cthulhu complex with coefficient in $\mathbb{C}[\pi_1(\Sigma)]$. We denote by $\widetilde{\Sigma}$. We denote by $\widetilde{\Sigma}^{\odot}$ the cover of $\Sigma^{\odot k}$ the universal cover of Σ obtained as a concatenation of k copies of $\widetilde{\Sigma}$ glued $\pi_1(\Sigma)$-equivariantly (since Λ is simply connected the the negative and positive end of $\widetilde{\Sigma}$ are made each of $\pi_1(\Sigma)$ copies of Λ. In other words this is the cover corresponding to the kernel of the natural map $\iota : \pi_1(\Sigma^{\odot k}) = \pi_1(\Sigma) * \pi_1(\Sigma) \cdots * \pi_1(\Sigma) \to \pi_1(\Sigma)$. Using the exact sequence of pair and the fact that Λ is simply connected we compute that $H^1(\widetilde{\Sigma}, \partial_- \widetilde{\Sigma}) \simeq \mathbb{C}^{|\pi_1(\Sigma)|-1}$. Using Mayer-Vietoris exact sequence we then find

$$H^1(\widetilde{\Sigma}^{\odot k}, \partial_- \widetilde{\Sigma}^{\odot k}) \simeq \mathbb{C}^{k(|\pi_1(\Sigma)|-1)}. \tag{4.4}$$

The exact triangle associated to the long exact sequence (1) with coefficient in $\mathbb{C}[\pi_1(\Sigma)]$ (via the morphism ι) becomes

$$\begin{tikzcd}
LCH^\bullet(\Lambda; \varepsilon, \varepsilon) \arrow[rr] & & LCH^\bullet(\Lambda; \widetilde{\varepsilon}_k, \varepsilon_k) \arrow[dl] \\
& H_\bullet(\widetilde{\Sigma}^{\odot k}, \partial_- \widetilde{\Sigma}^{\odot k}; \mathbb{C}). \arrow[ul] &
\end{tikzcd} \tag{4.5}$$

If $\pi_1(\Sigma)$ is finite the we deduce

$$\dim_{\mathbb{C}} LCH^\bullet(\Lambda; \varepsilon_0, \varepsilon_1) \leq |\pi_1(\Sigma)||\mathcal{R}(\Lambda)|$$

.

Then Equation (4.4) forces $|\pi_1(\Lambda)|-1 = 0$ and thus Σ is simply connected.

To complete the proof we must find a contradiction assuming $\pi_1(\Sigma)$ to be infinite. The dimension argument fails though, this is why we use L^2 completion and a result of [CG85] to construct a sequence analogue to (4.5) in this context. The contradiction is similar using L^2-rank instead of dimension. □

Perspective 11. The fact that we choose not to give definition of L^2 completion and L^2-cohomology should not be understood that we think it is something that should be undermined. We actually find that many things can be explored using this languages. As L^2-cohomology computes a kernel module the closure of an image it tends to vanish more than regular cohomology should. This makes it bad for estimate of number of periodic orbits or chords. But this makes it good to find isomorphisms when complexes arise as cone of maps. This is the case for the Cthulhu complex. Being acyclic it might be interesting to study torsion in this context.

Theorem 0.0.6 becomes now a corollary of Theorem 4.2.4 and 4.2.3 together with the h-cobordism Theorem of [Sma62] for $n \geq 4$. For $n = 1$ this is the classification of surfaces (it also follows more easily from the trivial remark in [Cha10, Theorem 1.3] that the genus of a cobordism is given by the difference of the Thurston-Bennequin of its ends). For $n = 2$ this follows from Perelman's proof of the Poincaré conjecture [Per]. The case $n = 3$ follows from [Fre82]. Finally $n = 4$ is a consequence of [KM63].

A second result allowing to show that some cobordisms are simply connected is given by the following.

Theorem 4.2.6. *Let Σ be a graded exact Lagrangian cobordism from Λ^- to Λ^+. Assume that Λ^- admits and augmentation over $\mathbb{F}_2$ and that Λ^+ has no Reeb chords in degree zero. If Λ^- and Λ^+ both are simply connected, then Σ is simply connected as well.*

Outline of proof. The condition on Λ^+ guarantees that Λ^+ has a unique augmentation over any ring A. This imply that the pullback ε_1^+ and ε_0^+ of an augmentation ε of Λ^- in $\mathbb{F}_2[\pi_1(\Sigma)]$ coincides (note having this unicity of augmentation we might end up with a twisted augmentation for ε_0^+). However as Λ^+ is simply connected, $LCH(\Lambda^+, \varepsilon) = LCH(\Lambda^+, \varepsilon_R) \otimes R[\pi_1(\Sigma)]$. On the other end $H^0(\overline{\Sigma}, R[\pi_1(\Sigma)]) = R$ with the trivial $\pi_1(\Sigma)$ action. From the fact that $G_\varepsilon(c_m) \neq 0$ (see Theorem 4.1.1) and that the action of $\pi_1(\Sigma)$ on c_m is free we deduce $\pi_1(\Sigma) = 1$. □

Remark 4.2.7. *The seemingly unnatural condition that Λ_+ has no Reeb chords in degree zero is used to ensure that A has* at most *one augmentation in A for every unital R-algebra A.*

This condition is clearly not invariant under Legendrian isotopy, but the conclusion of Theorem 4.2.6 can be extended to every Legendrian submanifold which is Legendrian isotopic to Λ^+ because Legendrian isotopies induce Lagrangian cylinders.

Verification 6. 1. The hypothesis of Theorem 4.2.6 seems essential for our proof to work. It is actually not only a fact of the proof but a fact of life.

Indeed Appendix A we find a Legendrian sphere which admits a non simply connected filling (hence is cobordant to the Whitney sphere). This sphere has an essential chord of degree 1 as the conclusion of the theorem does not hold. This chords is the one that appears when we do the surgery.

2. One might wonder also where the hypothesis that Λ^- was simply connected came in. This is when we allows ourselves to see an augmentation of Λ^- in $\mathbb{F}_2$ as an augmentation in $\mathbb{F}_2[\pi_1(\Sigma)]$. For instance in Figure A.5 we see a non-simply connected Legendrian surface. In Appendix A it is explained that it is fillable and hence admit an augmentation. This augmentation does not extend to an augmentation in $\mathbb{F}_2[\pi(\Lambda)]$ (note that at best the augmentation induced by the filling takes values with coefficient in the group ring of the solid torus). As described in Appendix A there is a surgery cobordism Σ to the (simply connected) Whitney sphere. As it is constructed with only one handle that cobordism is not simply connected. One could imagine some similar theorem with some hypothesis making the proof works but none of them are fully satisfactory: either we lack of example satisfying the hypothesis, or we would a priori ask to know what part of the fundamental group of the negative end is killed in the cobordism.

4.3 Obstruction to the existence of cobordisms.

One of the first applications of the augmentation category was to show that some Lagrangian concordances could not be reversed. The first occurrence appeared in [Cha15b] where we showed that there was no concordances from the Legendrian knot shown on Figure A.3 to the trivial Legendrian knot. However on Figure A.4 one see a Lagrangian concordance going the other way. Our original argument was showing that Λ was using composition in the category $\mathcal{A}ug(\Lambda)$.

Note though that the exact sequence (1) has a corollary which allow to simplify the argument. Indeed Theorem 0.0.4 has the following

Corollary 4.3.1. *Let Λ^- and Λ^+ be two graded Legendrian sub-manifolds and let ε_0^- and ε_1^- be two augmentations of Λ^-. If there is a Lagrangian concordance C from Λ^- to Λ^+ then $LCH^\bullet_{\mathrm{rel}}(\Lambda^-;\varepsilon_0^-,\varepsilon_1^-) \simeq LCH^\bullet_{\mathrm{rel}}(\Lambda^+;\varepsilon_0^+,\varepsilon_1^+)$.*

Remark 4.3.2. *Of course using the long exact sequence* (4.3) *gives a similar result for $LCH^\bullet$ instead of $LCH^\bullet_{\text{rel}}$.*

Indeed in this situation first the concordance is necessarily graded (actually one need only to ask that Λ^- is graded) and $H^\bullet(\overline{C}, \partial_-\overline{C}) = 0$.

In the situation of the knot Λ from Figure A.3 the computation of [Cha15b] show that there are two augmentation ε_0 and ε_1 of Λ so that $LCH_{\text{rel}}(\Lambda, \varepsilon_0, \varepsilon_1) = \mathbb{Z}_2[0] + \mathbb{Z}_2^2[1]$. However the Whitney sphere admits only one augmentation, for which $LCH_{\text{rel}}(\Lambda_0; \varepsilon) = \mathbb{Z}_2[1]$. This implies that no concordance can exist from Λ to Λ_0. (Note that if an oriented cobordism between the two exists it is necessarily a concordance because they have the same Thurston-Bennequin number).

Using a Kunneth type formula giving the LCH of a Legendrian spinning proved in [Cha+15a] we can use this criterion to generalise this examples to higher dimension.

4.4 Positive Legendrian isotopies.

Here we outline how the Cthulhu complex can be used to obstruct the existence of certain positive isotopies. We first recall the definition of positive isotopy.

Definition 4.4.1. Let $(Y, \xi = \ker \alpha)$ be a co-oriented contact manifold. Let Λ_0 be a Legendrian sub-manifold of Y. A Legendrian isotopy $\{\Lambda_t\}_{t \in [0,1]}$ is *positive* if

$$H(t_0, q) := \alpha(\frac{d}{dt}|_{t=t_0}\Lambda_t(q)) > 0$$

for all $t_0 \in [0, 1]$.

If $\Lambda_1 = \Lambda_0$ then the isotopy is called a *positive Legendrian loop.*

They arise naturally as propagations of Legendrian sub-manifolds along the Reeb flow of a contact form. They have been introduced in [CFP16] where first obstructions to existence of such positive isotopies between some Legendrian sub-manifolds has been given.

Positive isotopies are related to orderability of contact manifolds as introduced in [EP00] which in turns is related to some non-squeezing type phenomenon in contact geometry. Indeed if a contact manifold is not orderable then any Legendrian sub-manifolds is in a positive Legendrian loop.

Thus obstructing existence of such loop for a particular Legendrian shows orderability of the ambient contact manifold.

Remark 4.4.2. *There are different variations on the notion of orderability of the contact manifold wether we want to find an invariant order on the group of contactomorphisms or on its universal cover. Criterions to prove orderability using positive isotopy goes via obstructing positive Legendrian loops or positive Legendrian loop which are contractible in the space of Legendrian loops.*

The relation between positive Legendrian isotopy and Lagrangian cobordisms uses the following construction from [EG98, Lemma 4.2]: let $\{\Lambda_t\}$ be a positive Legendrian isotopy in a contact manifold $(Y, \ker \alpha)$, and let $H(t, q)$ be the function from definition 4.4.1. Then the map $C_T : [0, 1] \times \Lambda \to \mathbb{R} \times Y$ defined by $C_T(t, q) = (Tt - \ln H(t, q), \Lambda_t(q))$ satisfies $C_T^*(e^t\alpha) = e^{Tt}dt = d(\frac{e^{Tt}}{T})$. Hence C_T is an exact Lagrangian immersion and for T big enough it is an embedding. This can be smoothed and extended so that C_T becomes a Lagrangian concordance from Λ_0 to Λ_1 with the following crucial property: the function f such that $C_T^*(e^t\alpha) = df$ which vanishes near $-\infty$ is strictly positive outside the part where C_T is the trivial cylinder over Λ_0.

Remark 4.4.3. *This is this properties that makes the cylinder from [EG98] much more interesting in our opinion than the one from [Cha10].*

Perspective 12. The constant at ∞ given by the value of f could be related in a way that is not clear to us yet to some metric properties on the space of Legendrian sub-manifolds (as in [CS15]) and to some capacities or size of Lagrangian cobordisms as in [ST17] and [DS16].

Now assume Λ_0 is connected, graded and has an augmentation. Let C_1 be a trivial cylinder over a Legendrian Λ' which admits an augmentation ε' and consider the complex $\mathrm{Cth}(C_T, C_1, \varepsilon, \varepsilon)$. Note that the positivity of the function f implies that $CF_+(C_T, C_1) = 0$ and thus when the Cthulhu complex is acyclic then this leads to a long exact sequence similar to (1) that reads as:

$$\begin{array}{l} \cdots \to LCH^{k-1-\mu}(\Lambda_1, \Lambda'; \varepsilon^+, \varepsilon') \\ \qquad\qquad\downarrow \\ \qquad HF^{k-1}(C_T, C_1; \varepsilon) \longrightarrow LCH^k(\Lambda_0, \Lambda'; \varepsilon, \varepsilon') \to LCH^{k-\mu}(\Lambda_1, \Lambda'; \varepsilon^+, \varepsilon') \to \cdots , \end{array} \tag{4.6}$$

Remark 4.4.4. *The confusing μ degree shift comes from the fact that we think of Λ_0 at the positive end of the cobordisms as being graded by the same grading as at the negative end. However the propagation of the grading of Λ_0 to the cobordisms might induce a shift of the grading at the top, μ accounts of this shift. If the loop is contractible through Legendrian loop then $\mu = 0$.*

This might sound strange that we spoke on the conditional about the acyclicity of the Cthulhu complex, as the only one we have encountered is acyclic. Of course in this discussion we are speculating along the lines of Perspective 6 to see a general picture emerging. In the following we will state actual theorems for which we develop the rigorous framework in [CCD19].

Let's look at a simple situation: if Λ' is in the contactisation of a Liouville manifold P so that it maximum height in the z direction is smaller then the minimal height of Λ_0. Then in this situation C_T and C_1 can be arranged to have no intersection point and therefore $CF^{\bullet}(C_T, C_1) = 0$. In this situation the map $LCH^k(\Lambda_0, \Lambda'; \varepsilon, \varepsilon') \to LCH^{k-\mu}(\Lambda_1, \Lambda'; \varepsilon^+, \varepsilon')$ is an isomorphism. Assuming those group are not 0 leads to a contradiction playing the following game: the map counts holomorphic strips with boundary on C_T and C_1. But since C_T has positive potential the action of a mixed Reeb chord at the positive end satisfies $\mathfrak{a}(\gamma) = e^T l(\gamma) - c$ where $c > 0$. Therefore if γ^- and γ^+ are the ends of such a strip then (1.8) implies that $e^T l(\gamma^+) - l(\gamma^-) > c > 0$. We deduce a contradiction of the surjectivity of the map looking at γ^+ representing the smallest action and modifying the extremities of C_1 (not changing its potential at infinities) so that this action gets as small as we want by pushing Λ' up (note that this does not change the fact that CF_- vanishes). The actual argument goes trough the introduction of some spectral invariant $c_{\varepsilon_0, \varepsilon_1}$ associated to augmentations (see [CCD19, Section 4.4]). These considerations allow us to prove

Theorem 4.4.5. *Let Λ_0 and Λ' be Legendrian sub-manifolds which admit some augmentations ε and ε'. Assume that $LCH(\Lambda_0, \Lambda'; \varepsilon, \varepsilon') \neq 0$ and that*

$$\min_{\Lambda_0} z > \max_{\Lambda'} z$$

then then Λ_0 is not contained inside any *positive loop of Legendrians.*

Of course the non-acyclicity of the LCC complex is far from being granted, for instance if Λ_0 and Λ' are horizontally displaceable it is always acyclic. However a simple situation when this occurs is when Λ_0 and Λ' are two lift of the same exact Lagrangian sub-manifold L of P (hence both of them admits a unique augmentation). In this situation $LCH^\bullet(\Lambda', \Lambda_0) = H^\bullet(L)$ thus Theorem 4.4.5 has the following immediate

Corollary 4.4.6. *The Legendrian lift $\Lambda_L \subset (P \times \mathbb{R}, \alpha_{\mathrm{std}})$ of an exact Lagrangian embedding $L \subset (P, d\theta)$ is not contained in a positive loop of Legendrians.*

This implies that contactisation of Liouville manifolds are orderable as soon as the Liouville manifolds admits an exact Lagrangian.

Following the same stream of idea, let's assume that we are in a situation where the Cthulhu complex is well defined and invariant. Let C_T be a Lagrangian cylinder is built out of a **contractible** positive loop. From the contractibility of the loop we see that C_T is Hamiltonian isotopic to the trivial cylinder via a compactly supported Hamiltonian isotopy. It follows from the invariance that the complex $\mathrm{Cth}(C_T, C; \varepsilon_0, \varepsilon_1)$ is quasi-equivalent to the complex which is the cone of the identity map on $LCC(\Lambda_0^-, \Lambda_1^-)$, which is acyclic. Applied when C_1 is the trivial cylinder over a small deformation of Λ_0 in the positive direction (as in Section 4.1) we obtain the following exact sequence:

$$\begin{tikzcd}
\cdots \arrow[r] & LCH^{k-1}(\Lambda_0; \varepsilon, \varepsilon^+) \arrow[d, "G_{\varepsilon_0,\varepsilon_1}"] & & & \\
& HF^{k-1}(C_1, C_T) \arrow[r, "d_{-0}"] & LCH^k(\Lambda_0; \varepsilon) \arrow[r, "d_{+-}"] & LCH^k(\Lambda_0; \varepsilon, \varepsilon^+) \arrow[r] & \cdots ,
\end{tikzcd} \tag{4.7}$$

This time with no shift μ thanks to the contractibility of the loop. It would be a mistake to think that $HF^\bullet(C_1, C_T) = 0$ thanks to the fact that C_T is equivalent to the trivial cylinder. Indeed some holomorphic curves might slides to the negative ends to chords along the deformation. One situation where it vanishes is when we have nowhere to slide to: if Λ_0 has no chords which a contractible. In [CCD19] we extend the construction of the Cthulhu complex in the hyper-tight world and the same consideration on classes with small action allows us to prove the following:

Theorem 4.4.7. *A hyper-tight Legendrian sub-manifold $\Lambda \subset (M, \alpha)$ of a closed hyper-tight contact manifold is not contained inside a contractible positive loop of Legendrians.*

When we are not able to guarantee vanishing of $HF^\bullet(C_T, C_1)$ we can still hope to find some contradiction if we are able to show that the minimal class is natural under the cobordisms map. For instance this minimal class corresponds to the unit of Wrapped Floer homology of a filling L when we represent it by some high energy classes. This suggested us to prove the following:

Theorem 4.4.8. *If a Legendrian $\Lambda \subset (M, \xi)$ admits an exact Lagrangian filling $L \subset (X, \omega)$ inside a Liouville domain with contact boundary $(\partial X = M, \xi)$, such that the wrapped Floer cohomology of L is non vanishing, then Λ is not contained inside a contractible positive loop of Legendrians.*

We will define wrapped Floer homology in the next section. For this theorem we use only the Cthulhu complex as an inspiration and reformulated all ideas in terms of wrapping as in [CO18]. Combining this result with formula for wrapped Floer homology of a Lagrangian co-core in [BEE12] we obtain:

Corollary 4.4.9. *Let (M_+, ξ_+) obtained by performing a contact surgery along a Legendrian link $\Lambda \subset (M_-, \xi_-)$ of spheres where*

- *(M_-, ξ_-) is the boundary of a subcritical Weinstein domain,*
- *the Legendrian contact homology DGA of each component of Λ is not acyclic.*

Then there is no contractible positive Legendrian loops containing a Legendrian co-core sphere inside (M_+, ξ_+) created by the surgery.

We obtain many variations of these criterions. All those results show that the ambient contact manifolds are orderable. This recover many of previously known example and introduce new one. For instance applying the previous theorem to the diagonal in the product of a Weinstein manifolds shows:

Theorem 4.4.10. *If $(M, \xi = \ker \alpha)$ is the contact boundary of a Liouville domain $(W, \omega = d\alpha)$ whose symplectic cohomology does not vanish, i.e. $SH^\bullet(W, \omega) \neq 0$, then (M, ξ) is strongly orderable.*

4.5 Generation of the wrapped Fukaya categories.

In this last section we present the results from [Cha+17] showing that the wrapped Fukaya category of a Weinstein sector has a finite collection of generators. We begin by briefly recalling what is the wrapped Fukaya category of a Weinstein sector.

Remark 4.5.1. *As Weinstein manifold are particular cases of sectors we focus on the latter and deduce the results for manifold case from it. There are several description of the wrapped Fukaya category using various of deformation of Lagrangians in the cylindrical boundary, namely the so-called linear setting from [AS10] or the quadratic setting for instance used in [Abo10]. In the former paper $\mathcal{A}_\infty$-products used a lot of extra-data on the Stasheff polyedra due to object not being necessarily in transverse position. The language of localisation from [Sei13] that is used in [GPS17] makes things easier. This is the one we will be focusing on but in [Cha+17] we explain how our methods works in any of those setting.*

Wrapped Fukaya category. Let $(S, \mathfrak{f}, \lambda)$ be a Weinstein sector. Let I be a countable set of proper graded Lagrangian exact Lagrangian sub-manifolds of the interior of S cylindrical at infinity. We assume that:

1. Any proper exact Lagrangian sub-manifold of S cylindrical outside of a compact set is Hamiltonian isotopic to an element of I.

2. Any k-uple of distinct element of I have generic intersections (transverse with only double points).

For each $L \in I$ we choose a family $L^{(i)}$ of exact Lagrangian Hamiltonian isotopic to L such that the Hamiltonian isotopy outside of a compact set is the lift of a positive Legendrian isotopy of $\partial_+ L$ and the value of the function $H(t, q)$ defined by the positive isotopy tends to infinity. (This guarantee that the family $\{L^{(i)}\}$ is co-final in the wrapping category of L as defined in [GPS17, Section 3.4]). We assume that all k-uple of element $L_k^{(i_k)}$ have generic intersections.

Given two Lagrangians $L_1^{(i_1)}$ and $L_2^{(i_2)}$ consider their Floer complex is $CF^\bullet(L_1^{(i_1)}, L_2^{(i_2)})$ (already described in Section 2.1.3).

Those are the objects of a strictly unital $\mathcal{A}_\infty$-category $\mathcal{O}$ defined by:

- $\mathcal{O}b(\mathcal{O}) = \{L_i^{(i_k)} | L_i \in I, k \in \mathbb{Z}\}$
- $\hom(L^{(i)}, K^{(j)}) = \begin{cases} CF(L^{(i)}, K^{(j)}) & \text{if } i > j, \\ \mathbb{F} & \text{if } L^{(i)} = K^{(j)}, \\ 0 & \text{otherwise.} \end{cases}$
- Differential and higher compositions for d objects is given by the operation μ^d in $\mathcal{A}ug_-(\Lambda_1^{(i_1)} \cup \Lambda_2^{(i_2)} \cdots \Lambda_d^{(i_d)})$ when $i_1 < i_2 < \cdots i_d$. The other composition being characterised by the strict unitality of $\mathcal{O}$.

We will be primarily interested by a quotient of $\mathcal{O}$ but note that with this one can immediately enlarge $\mathcal{O}$ to a category $\mathcal{O}'$ where the index set I' contains exact Lagrangian immersions with augmentations of their Legendrian lifts.

When L' is a small Hamiltonian deformation of L which is positive on the cylindrical ends then $HF^k(L, L') = H^k(L)$ (we've encounters similar fact in Section 4.1 and in Theorem 2.1.19). Thus $HF^k(L, L')$ has a particular element $c_{L,L'}$ which correspond to the unit under this isomorphism. We call $c_{L,L'}$ the continuation element of L to L'. For $i < j$ $HF(L^{(i)}, L^{(j)})$ has a continuation element which is defined as the composition of all continuation element for a decomposition of the isotopy from $L^{(i)}$ to $L^{(j)}$ into small one.

Remark 4.5.2. *If the isotopy was negative on the cylindrical end then we find $HF^k(L, L') = H^k(L, \partial_+ L)$ which contains no unit this is why no continuation element exists in this situation.*

We denote by C the set of all continuation element arising in $\mathcal{O}$ this way and define the wrapped Fukaya of S by the localisation $\mathcal{W}F(S) := \mathcal{O}[C^{-1}]$.

Remark 4.5.3. *Of course this rushing through the definition. We did not show that it was well defined (in particular independent on the choices of I). We refer to [Sei13], [Aur18] and [GPS17, Section 3.4] for more details and discussions.*

On the level of morphisms space the homology is given by $HW(L, K) := H^\bullet(\hom_{\mathcal{W}F}(L, K)) = \lim_{i \to \infty} HF(L, K^{(i)})$ where the maps in the direct system is given by the composition with the continuation elements.

Note that the collection $\{D_i\}$ of co-core and spreading of co-core of the boundary can be seen as object of $WF(S)$. The main result from [Cha+17] is to show that:

Theorem 4.5.4. *Any object L of $WF(S)$ is isomorphic in $Tw\mathcal{W}F(S)$ to an twisted complex build out of the $\{D_i\}$.*

To prove that objects are isomorphic we use the fact that the cone of a quasi-isomorphism represents the 0 object. We then use a geometric characterisation of these 0 objects given by the following theorem:

Theorem 4.5.5. *Let (S, θ, I) be a Liouville sector and let K and L be exact Lagrangian sub-manifolds of S with cylindrical ends. We allow K to be immersed, and in that case we assume its Legendrian lift admits an augmentation ε. If the Liouville flow of (S, θ) displaces K from every compact set of S, then $HW(L, (K, \varepsilon)) = 0$.*

When L and K are embedded there are many occurrences of this fact in the literature, see for instance [CO18, Theorem 9.8] or [AS10, Section 4k]. The proof relies on estimate on the the displacement energy by the Liouville flow (which induces Hamiltonian isotopies of exact Lagrangian). In [Cha+17] we extended this estimate to the immersed case to prove this result.

Cobordisms and wrapped Floer homology. The relevance of the Cthulhu complex to prove Theorem 4.5.4 goes with the following construction. Let Σ be a graded exact Lagrangian cobordisms from Λ^- to Λ^+. Let ε^- be an augmentation of Λ^-. For an exact Lagrangian L of S we choose a Legendrian lift Λ_P such that the cylinder of Λ_P is contained under Σ in the z-direction. The Cthulhu complex has now the simple description as $\mathrm{Cth}(C_{\Lambda_p}, \Sigma) = CF(P, (\pi(\Lambda^+))[1] \oplus CF(P, (\pi(\Lambda^-), \varepsilon))$ and the differential is the cone of a map from $CF(P, (\pi(\Lambda^-), \varepsilon))$ to $CF(P, (\pi(\Lambda^+), \varepsilon^+)[1]$. Its acyclicity implies that the homology groups $HF(P, (\pi(\Lambda^-), \varepsilon))$ and $HF(P, (\pi(\Lambda^+), \varepsilon^+))$ are isomorphic. This isomorphism commutes with the product μ^2 (in particular with composition with continuation elements), this as immediate from [Leg18] (see Theorem 3.4.2). Therefore:

Theorem 4.5.6. *Let Σ be a graded exact Lagrangian cobordisms from Λ^- to Λ^+ and let ε^- be an augmentation of Λ^-. Then for any exact Lagrangian T*

$$WF(T, (L^+, \varepsilon^+) \simeq WF(T, (L^-, \varepsilon^-))$$

(where $L^\pm$ denotes the Lagrangian projection of $\Lambda^\pm$).

Remark 4.5.7. *Applied to the surgery cobordism this theorem states that certain Lagrangian surgeries represent some iterated cones in the Fukaya category. There have been several instance of phenomenon similar to these in the past: see [Sei03] for the case of spheres intersecting in one point, [BC13] and [BC14a] for surgeries leading to embedded Lagrangian and the recent paper [PW19] for a comprehensive treatment of surgery of one point in the monotone case.*

We want to use this theorem to use Lagrangian cobordisms to go from an immersed Lagrangian to a Lagrangian which is displaced from the skeleton (and hence is the 0 object). More precisely we want to use some surgery to remove some intersection points with the skeleton. Note however that the surgery cobordism described in Section 1.3.6 goes from the surgered manifold to the original one. This is the opposite direction that augmentation goes. Hence we need a criterion that tells us when we can push forward augmentation instead of pulling them back. This can be achieved by the following result from [Dim16a, Theorem 1.1]:

Theorem 4.5.8. *Let Λ be a graded Legendrian and $a_1, \cdots a_k$ some contractible chords of Λ. Let Σ be the Lagrangian cobordisms from $\Lambda(a_1, \cdots, a_k)$ to Λ. Let ε be an augmentation of Λ such that $\varepsilon(a_i) = 1$ for all $i = 1 \cdots k$. Then there exist an augmentation $\overline{\varepsilon}$ of $\Lambda(a_1, \cdots, a_k)$ such that $\varepsilon = \phi_\Sigma \overline{\varepsilon}$.*

Twisted complexes and augmentation. Consider now k exact Lagrangian sub-manifolds L_k and fix Legendrian lifts Λ_k of each L_k. Let ε be an augmentation of the link $\cup_k \Lambda_k$ such that $\varepsilon(\gamma)$ if γ is a chord from Λ_i to L_j with $i \leq j$. Then the element $X = \sum \varepsilon(\gamma)\gamma$ satisfies the Maurer-Cartan equation and

$$WF(T, (\cup_k L_k, \varepsilon)) \simeq WF(T, (\{L_i\}, X)).$$

Where the homology on the left corresponds to the homology of the morphisms spaces in $Tw\mathcal{W}F(S)$. This statement is almost a tautology has on the nose the differential on the complex as defined on the right corresponds to the differential in the twisted complex. This is of course before localisation but it goes down to the localisation as everything tautologically commutes with products with continuation elements.

All of this implies the following

Theorem 4.5.9. *Let $L_1, \ldots, L_m$, be embedded exact Lagrangian sub-manifolds with cylindrical ends. If there exist a Legendrian lift $\mathbb{L}^+$ of $\mathbb{L} = L_1 \cup \ldots \cup L_m$,* *an augmentation ε of the Chekanov-Eliashberg algebra of $\mathbb{L}^+$ and a set of contractible Reeb chords $\{a_1, \ldots, a_k\}$ such that:*

(1) $\varepsilon(a_i) = 1$ for $i = 1, \ldots, k$, and

(2) $\varepsilon(q) = 0$ if q is a Reeb chord from L_i^+ to L_j^+,

then there exist a twisted complex $\mathfrak{L}$ built from $L_1, \ldots, L_m$ and an augmentation $\overline{\varepsilon}$ of the Chekanov–Eliashberg algebra of $\mathbb{L}(a_1, \ldots, a_k)^+$ such that, for any other exact Lagrangian sub-manifold with cylindrical end T there is an isomorphism

$$HW(T, (\mathbb{L}(a_1, \ldots, a_k), \overline{\varepsilon})) \cong H \hom_{\mathcal{W}F}(T, \mathfrak{L}).$$

Construction of the augmentation and generation criterion. All of this leads to the following plan to prove generation by co-cores and their spreadings. Start with a Lagrangian L and assume it intersects transversely the skeleton of S in points $a_1, \cdots a_k$. Then through each intersection points we take a copy of the corresponding co-core or spreading D_{a_i}. Then lift L and each of the D_{a_i} such that each chords corresponding to an intersection point a_i is contractible (note that this forces a choice of grading for each D_{a_i} once L is graded). Then the surgery $(L \cup_i D_{a_i})(a_1, \cdots, a_k)$ is an exact Lagrangian immersion disjoint from the skeleton and thus if it admits an appropriate augmentation it represents a twisted complex built out of the D_i and L that is a 0 object.

Of course all is fine up to that last bit about the appropriate augmentation. Indeed its existence is not granted if we do not modify our configuration. In [Cha+17, Section 9] we explain how wrapping the object D_i near the skeleton allow to ensure the existence of such an augmentation. We will not reproduce it here but instead we would like to illustrate this procedure on an explicit example that we find relevant for the understanding of the global argument. The simple case of the cotangent of S^1 is actually rich enough to understand most of what is involved in the construction of the augmentation. For instance lets try to generate the 0-section in T^*S^1. We first deform it so that it is transverse to the skeleton (i.e. itself) as in Figure 4.1.

We then normalise its intersections with the skeleton so that around each intersection point the triple of Lagrangians given by the skeleton, the object we want to generate and the co-core (here the cotangent fibre) are always equivalent (see Figure 4.2). We then lift the Lagrangian and the copies of

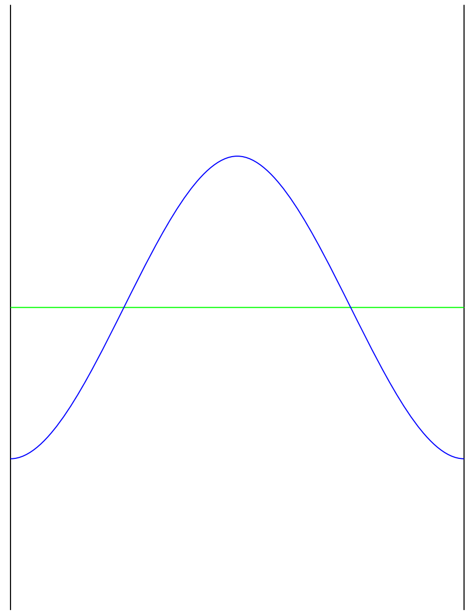

Figure 4.1: A deformation of the 0-section.

the co-core to a Legendrian link such that each co-core are slightly under the object we want to generate above the intersection points with the skeleton. This leads to contractible Reeb chords γ_i that we can surger which would have the effect to remove intersection points with the skeleton (see right hand side of Figure 4.2). However there are other Reeb chords from the co-core D_i to the object: this might obstructs the existence of an augmentation sending the chord γ_i to 1. For instance on Figure 4.2 the augmentation equation (2.1) applied to δ_1 (or δ_2) forces $\varepsilon(\gamma_1) = 0$: there is only one punctured disks out of δ_1 and it has negative end only at γ_1.

To resolve this problem with start by wrapping co-cores in the neighbourhood of the skeleton (increasing the z coordinated of their Legendrian lifts) so that all chords outside a neighbourhood of the skeleton are going from the object to co-cores (or from co-cores to co-cores with higher index) those are not problematic for defining a twisted complex. There is most likely still chords from co-cores to the object but they happen near an intersection point with the skeleton where the object is a small deformation of co-cores. So for any such chords δ_{ij} near γ_j there is a twin chord ζ_{ij} from the corresponding co-core and the triple formed by those two chords and the intersection with the skeleton are asymptotics of a holomorphic trian-

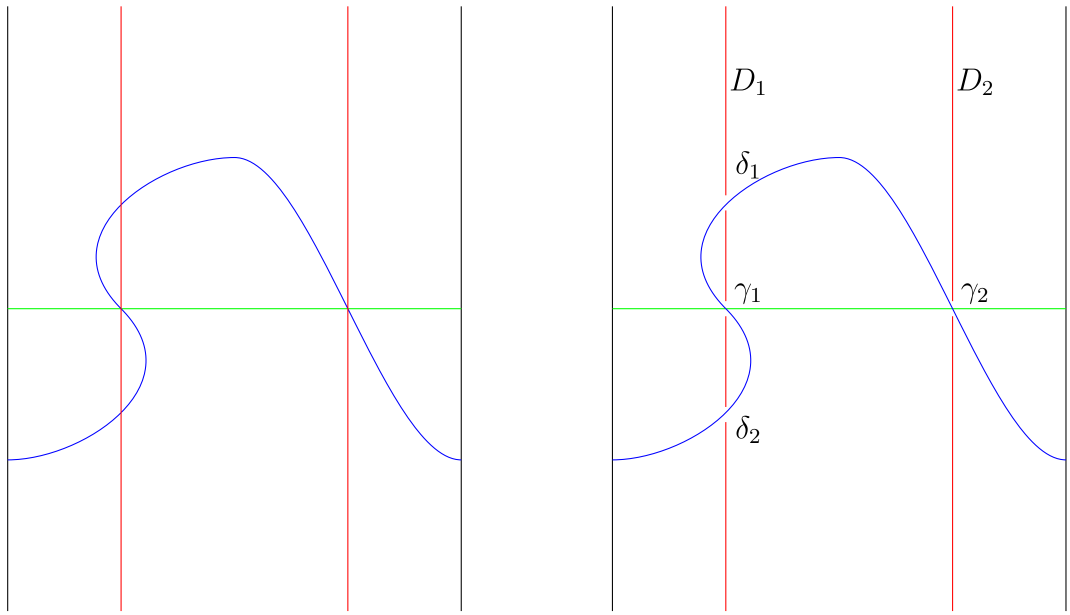

Figure 4.2: Normalising intersection near the skeleton.

gles in $\mathcal{M}(\delta_{ij}; \gamma_j, \zeta_{ij})$. For instance on Figure 4.3 the chords δ_{12} still have a punctured disks toward γ_1 but also one to (γ_2, ζ_{12}) so now the augmentation equation become $\varepsilon(\gamma_1)+\varepsilon(\gamma_2)\varepsilon(\zeta_{12}) = 0$ which we can solved mapping γ_1 and γ_2 to 1 (as needed to push the augmentation forward). A similar argument works for the chords on the opposite side of the figure. We build in this a way an augmentation that we can push-forward along the surgery cobordism. The surgery leads to a Lagrangian disjoint from the skeleton (see right hand side of Figure 4.3). It represents the 0-object by Theorem 4.5.5. Theorem 4.5.9 and Lemma 3.1.10 implies thus that the 0-section is generated by two copies of the cotangent fibre.

The general argument works similarly: we order the copies of the co-cores according to the height of the lift of the Lagrangian L we want to generate. And we wrap similarly in neighbourhoods of the skeleton that we nest so that new modifications do not alter previous ones until all a priori problematic intersection points comes with a triangle allowing us to construct the augmentation inductively by action.

Remark 4.5.10. *In the case of sectors it is possible that this wrapping brings us dangerously close to the boundary of the sector but in that situation we*

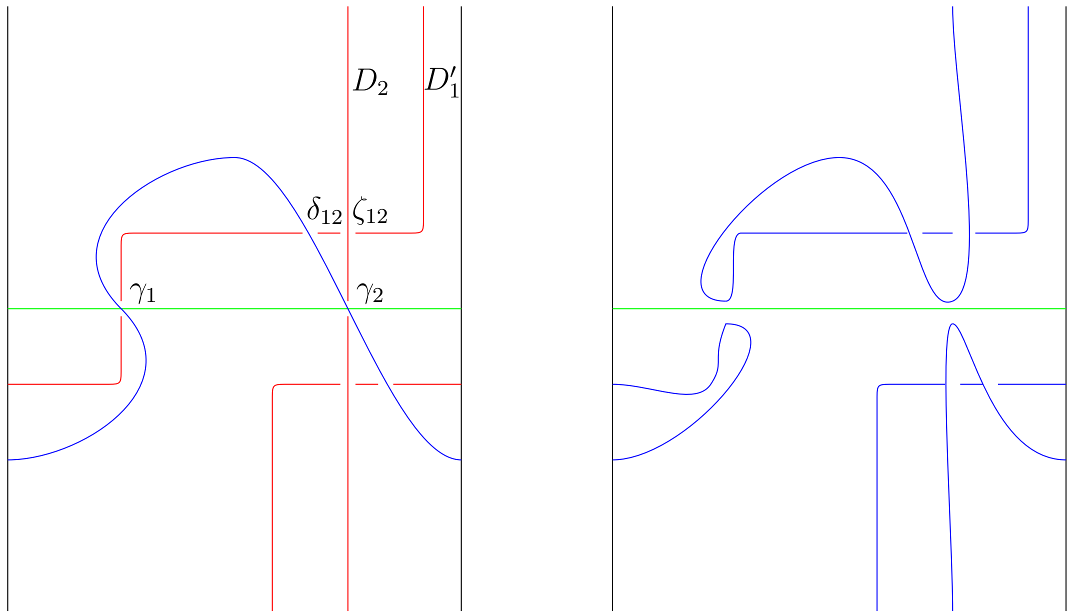

Figure 4.3: Wrapping to construct the correct augmentation.

can just stop wrapping as we know that our object is away from the boundary and thus no problematic intersection points can exists.

Appendix A

Some useful examples of Legendrians and cobordisms.

1-jets of functions. After conormals of sub-manifolds which are Legendrians in spaces of contact element. The most natural family of example of Legendrian sub-manifolds are graph 1-jets of functions. Given a function $f : M \to \mathbb{R}$ its one jet is $j^1(f)(q) = (q, [f]_q) = (q, df_q, f(q))$ in $\mathcal{J}^1(M) = T^*M \times \mathbb{R}$.

For any critical point q of f there is a Reeb chord $(q, 0, t)$ going from the 0-section to $j^1(f)$ if $f(q) > 0$ and from $j^1(f)$ to the 0-section if $f(q) < 0$. If $f(q) = 0$ then $j^1(f)$ intersects the 0-section.

The Whitney sphere. The first useful Legendrian sub-manifold which is not the jet of a function or a co-normal is the Whitney sphere. Consider the sphere given in coordinates by $S^n = \{(q, z) \in \mathbb{R}^n \times \mathbb{R} | |q|^2 + z^2 = 1\}$. It admits a Legendrian embedding in $\mathcal{J}^1(\mathbb{R}^n)$ by

$$(q, z) \to (q, z \cdot q, -\frac{z^3}{3}).$$

On Figure A.1 we see the projection to $\mathbb{R} \times \mathbb{R}$ (on the left) and to $T^*\mathbb{R}$ (on the right) of this Legendrian when $n = 1$. In any dimension it admits a single Reeb chords $\gamma(t) = \frac{1}{3}(0, \cdots, 0, t)$ $t \in [-1, 1]$.

It is the boundary of a Lagrangian disc in $\mathbb{R} \times \mathcal{J}^1(\mathbb{R})^n$ whose front is half of the Whitney sphere 1-dimension higher (completed conically as in Section 1.3.3).

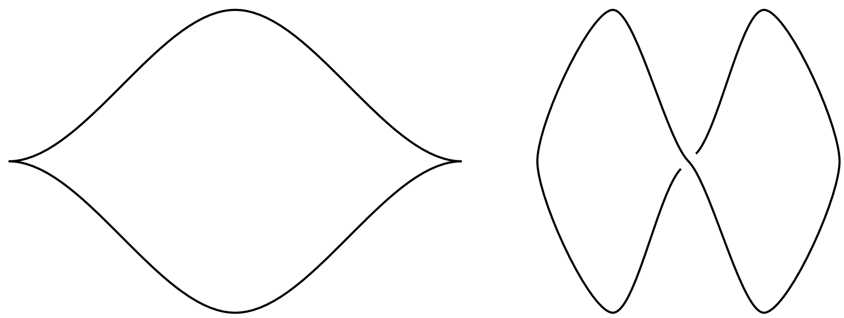

Figure A.1: Front and Lagrangian projections of the trivial Legendrian knot.

The Legendrian trefoil. On Figure A.2 we see the Lagrangian projection of a Legendrian knot Λ in $\mathcal{J}^1(\mathbb{R})$ which has the the topological type of a right-handed trefoil knot. It admits 5 Reeb chords. The star denotes a point outside the capping paths of each of those chords (this determines uniquely the homotopy type of those path).

All chords b_i's are contractible and we can perform the surgery cobordism to each of them. Depending on the order we perform the surgery this leads to 5 different genus 1 cobordisms from Λ_0 to Λ. As we fill cap Λ_0 with a disk this gives 5 Lagrangian fillings of Λ by punctured torus.

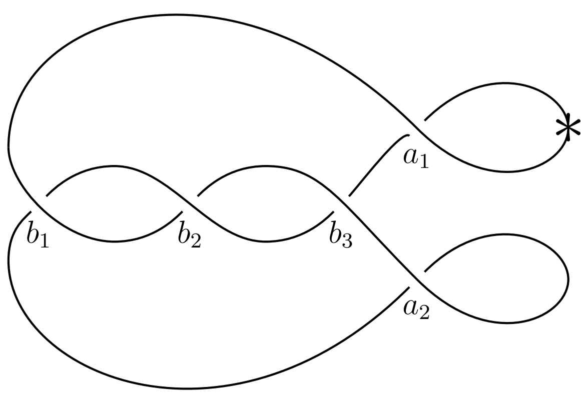

Figure A.2: A Lagrangian projections of the Legendrian trefoil.

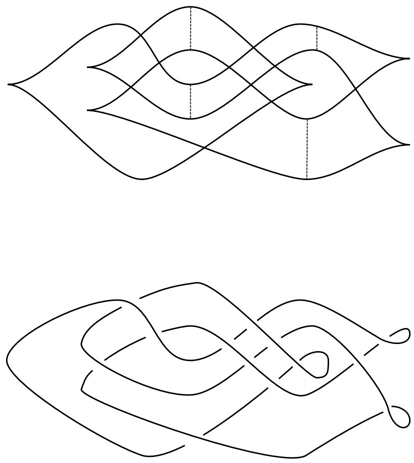

Figure A.3: The front and Lagrangian projection of a Legendrian representative of the knot $m(9_{46})$.

The $m(9_{46})$ Legendrian knots. On Figure A.3 we see the front and Lagrangian projections of a Legendrian knots. It is relevant because it is the simplest Legendrian knot after Λ_0 in $\mathcal{J}^1(\mathbb{R})$ that admits a Lagrangian disk filling. On Figure A.4 we see the movie of a concordance from Λ_0 to Λ that leads to this filling once we fill Λ_0. This concordance is not unique and we can find other by surgery on the chords highlighted in Figure A.3.

Spinnings. Given a Legendrian sub-manifold $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$, the so called front spinning construction produces a Legendrian embedding of $\Lambda \times S^1$ inside $(\mathbb{R}^{2(n+1)+1}, \xi_{st})$, as described by Ekholm, Etnyre and Sullivan [EES05]. In [Gol14] this construction was extended to the S^m-front spinning, which produces a Legendrian embedding of $\Lambda \times S^m$ inside $(\mathbb{R}^{2(n+m)+1}, \xi_{st})$. It was also shown that this construction extends to exact Lagrangian cobordisms.

The spinning of 1-dimensional Legendrian knots along a circle leads to a Legendrian surface whose 2-dimensional front is simply the rotation of the 1-dimensional front. For instance the spinning of the Whitney sphere is the torus depicted on Figure A.5: it has two circle of cusps. It can be shown that it is obtain by subcritical surgery on the 2-dimensional Whitney sphere, hence there is an exact cobordism from Λ_0 to this Legendrian spun. Note also that we can do surgery along the inner circle of cusp which lead to a

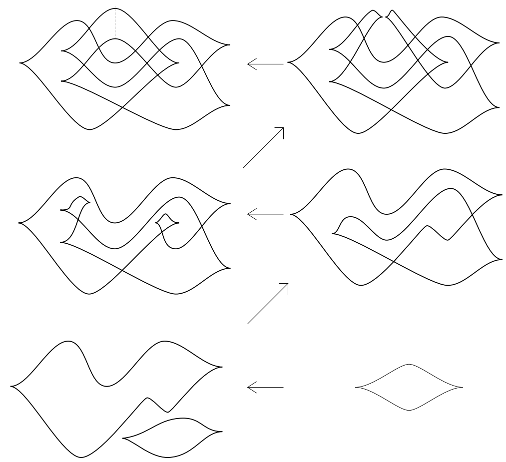

Figure A.4: A concordance from the Whitney sphere to the $m(9_46)$ Legendrian knot.

cobordisms from the spinning to the Whitney sphere.

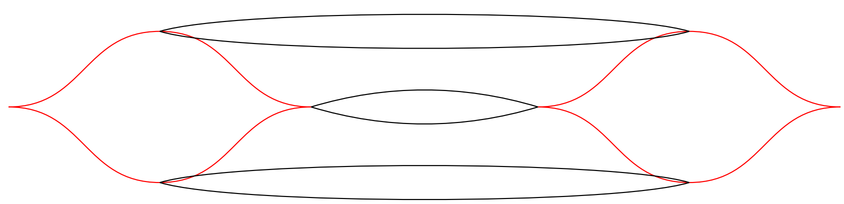

Figure A.5: Front of the spinning of the Whitney sphere.

A non simply connected filling of a Legendrian sphere. We start with a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ which admits a non-simply connected Lagrangian filling Σ. For instance, we can take the Legendrian right handed trefoil knot and one of its exact Lagrangian filling diffeomorphic to a punctured torus. It follows that $\Sigma_{S^m}\Lambda \subset (\mathbb{R}^{2(m+1)+1}, \xi_{\mathrm{std}})$ is a Legendrian $S^m \times S^1$ which admits an exact Lagrangian filling $\Sigma_{S^m}\Sigma$ diffeomorphic to $S^m \times \Sigma$; this filling is of course also not simply connected.

The Legendrian ambient surgery along a cusp-edge in the class $S^m \times \{p\}$ for $p \in \Lambda$ corresponding to the left-most cusp edge of $\Lambda \subset (\mathbb{R}^3, \xi_{\mathrm{std}})$ as described above produces a Legendrian sphere Λ', and concatenating $\Sigma_{S^m}\Sigma$

with the corresponding elementary Lagrangian $(m+1)$-handle provides a non-simply connected filling Σ' of Λ'.

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Index

Résumé

Ce mémoire fait état des travaux de l'auteur sur l'étude des invariants des sous-variétés legendriennes provenant des augmentations de celles-ci. Après un chapitre introduisant les notions de bases, la notions d'augmentations est introduite dans le chapitre 2. Il y est décrit comment de tels objets permettent de définir un complexe de Floer pour les cobordismes lagrangiens entre sous-variétés legendriennes. Dans le chapitre 3 nous discutons comment les augmentations peuvent être organisées en une catégorie $\mathcal{A}_\infty$, la catégorie d'augmentations. Dans le chapitre 4 nous présentons diverses applications des chapitres précédents à la topologie des cobordismes lagrangiens, les isotopies legendriennes positives et la génération des catégories de Fukaya enroulées des secteurs de Weinstein.

Abstract

This memoir presents an overview of the author's contribution to the study of invariants of Legendrian sub-manifolds arising from augmentations. After a chapter introducing basic notions of symplectic and contact topology, augmentations are defined in Chapter 2. It is explained how augmentations are used to define a Floer complex associated to Lagrangian cobordisms between Legendrian sub-manifolds. In Chapter 3 we explain how augmentations can be organised in an $\mathcal{A}_\infty$-category, the augmentation category. In Chapter 4 we present various application of the preceding chapter to the topology of Lagrangian cobordisms, positive Legendrian isotopies and the generation of the wrapped Fukaya category of a Weinstein sector.