Augmentations of Legendrian submanifolds:

Two applications to the study of Lagrangian submanifolds

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December 10th 2019

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Main results.

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Let (W, λ, f) a Weinstein manifold and $(P = W \times \mathbb{R}, dz + \lambda)$ its contactisation. Then

Theorem (C.–Dimitroglou-Rizell–Ghiggini–Golovko)
The wrapped Fukaya category of W is generated by its Lagrangian cocores.

Theorem (CDRGG)
Any exact Lagrangian cobordism from a Legendrian sphere Λ in P to itself is a cylinder if Λ admits an augmentation.

Plan

Main results.

Symplectic geometry.

Fukaya categories.

Generation.

Augmentation of Legendrian sub-manifolds and Floer theory.

Proofs

Perspectives

Symplectic geometry.

Symplectic form.

Let

$$X_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

an Hamiltonian vector field for some function $H(q_1, \cdots, q_n, p_1, \cdots, p_n)$.

Symplectic form.

If ϕ_t^H is the flow of this vector field (i.e. $\forall q,\ \phi_t^H(q)$ is the solution of the equation starting at q), then Poincaré observes that for any two tangent vectors X, Y on T^*Q:

$$\frac{d}{dt}\sum_{i=1}^{n}\det(d\phi_t(\pi_i(X)), d\phi_t(\pi_i(Y))) = 0,$$

where π_i is the projection to the plane $\mathbb{R}^2_{q_i,p_i}$.

In other words $\phi_t^*\omega_0 = \omega_0$ where $\omega_0 = \sum dq_i \wedge dp_i$.

Symplectic form.

Note that the bilinear antisymmetric pairing ω_0 is non degenerate and that X_H is caracterised by $\omega_0(X_H, Y) = dH(Y)$.

Thus a function G is constant along trajectories of X_H ($dG(X_H) = 0$) iff $\omega_0(X_H, X_G) = 0$. This leads us to study linear subspace E for which $\omega(X, Y)$ for any $X, Y \in E$. Such a subspace of dimension n is called Lagrangian. A submanifold $L \subset T^*Q$ is called *Lagrangian* if for any $(q, p) \in L$ then $T_{(q,p)}L$ is a Lagrangian subspace of $T_{(q,p)}T^*Q$.

Lagrangian submanifolds.

In other word $i : L \to T^*Q$ is Lagrangian if it is of dimension n and $i^*\omega_0 = 0$.This implies that $i^* \sum_i p_i dq_i$ is closed. We say that L is *exact* if $i^* \sum_i p_i dq_i = df$.

Conjecture (Nearby Lagrangian conjecture)
Let Q be a compact manifold. If L is a compact exact Lagrangian of T^*Q then L is Hamiltonian equivalent to $Q_0 = \{(q, 0)\} \in T^*Q$.

Toward symplectic algebraic topology.

Classify up to Hamiltonian isotopy Lagrangian submanifolds of (T^*Q, ω_0): **HARD!**

Replace with an algebraic set-up: define a category whose objects are Lagrangian submanifolds and morphisms are chain complexes (called Floer complexes).

But first let's enlarge our context a little.

Symplectic manifolds.

A symplectic manifold (M, ω) is a space with a bilinear pairing ω on it tangent space that locally look like $(\mathbb{R}^{2n}, \omega_0)$.

A cotangent bundle as an extra feature: the vector field $p\frac{\partial}{\partial p}$ expands ω_0. It is "gradient" for $|p|^2$ and its negative attractor is Q_0 (which is Lagrangian). It is a Weinstein manifold.

Weinstein manifolds.

An exact symplectic manifold $(W, d\lambda)$ is *Weinstein* if there is a complete vector field V and a proper function $\mathfrak{f}$ bounded from below such that:

- $V \iota d\lambda = \lambda$ ($\Rightarrow \mathcal{L}_V d\lambda = d\lambda$)
- V is "gradient" for $\mathfrak{f}$.

Weinstein manifolds.

Examples:

- $(T^*Q, d\lambda)$ with $\mathfrak{f} = p^2$ and $V = p\frac{\partial}{\partial p}$.
- $X \hookrightarrow \mathbb{C}^n$ holomorphically, $\theta = \frac{1}{2i}(zd\overline{z} - \overline{z}dz)$, $V = \operatorname{grad} \mathfrak{f}$ for $\mathfrak{f} = \frac{1}{2}d(\mathrm{pt}, \cdot)^2$.
- Any $M \setminus \Sigma$ with (M, ω) symplectic and $\Sigma = PD(k[\omega])$ (Donaldson-Giroux).

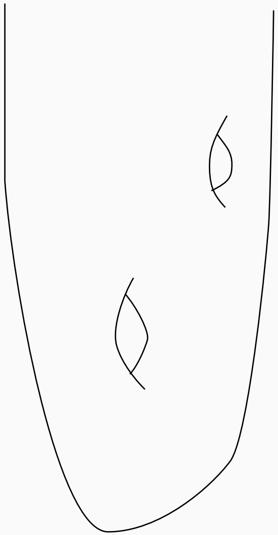
Weinstein manifolds.

We deform the structure so that V is Morse-Smale (i.e. critical points are Morse and all ascending and descending manifolds intersects transversely).

We say that $(W, d\theta, \mathfrak{f})$ is of finite type if $\mathfrak{f}$ has a finite number of critical points. From now on assume that all Weinstein manifolds are of finite type.

The descending manifolds of critical points of V gives a handle decomposition of W.

Weinstein manifolds.



Weinstein manifolds

For a critical point q of V we denote by

$$H_q = \{x| \lim_{t\to+\infty} \phi_V^t(x) = q\}$$

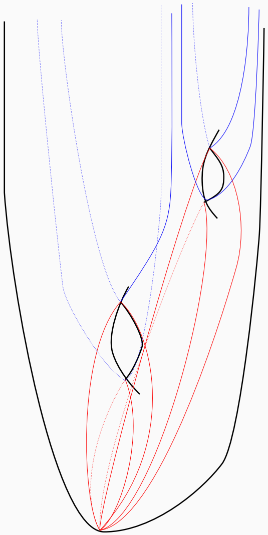
and

$$D_q = \{x| \lim_{t\to-\infty} \phi_V^t(x) = q\}$$

One can check that H_q is isotropic and D_q is co-isotropic ($V\iota\omega|_{T_{D_q}} = 0 \Rightarrow V \in T_{D_q}$).

In particular, $ind_q V \leq n$, and if $ind_q V = n$ then D_q and H_q are Lagrangian disks.

Weinstein manifolds.



Weinstein manifolds.

We have the exact same notion of Lagrangian and exact Lagrangian as in the cotangent. For instance an embedding $i : L \to W$ is an exact Lagrangian submanifold if $i^*\lambda = df$. Keeping track of the values of f gives

$$\widetilde{i} : L \to W \times \mathbb{R},$$

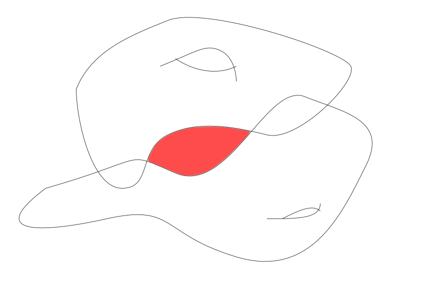
given by $\widetilde{i} = (i(q), -f(q))$.

The relation $i^*\lambda = df$ implies $\widetilde{i}^*(dz + \lambda) = 0$. This means that $\widetilde{i}$ a Legendrian embedding in the contact space $(W \times \mathbb{R}, \ker(dz + \lambda))$.

Fukaya categories.

Floer complex.

Given two exact Lagrangian L_1 and L_2 of W Floer defined a complex $(CF(L_1, L_2), d)$ whose underlying vector space is generated by $L_1 \pitchfork L_2$. The differential counts "holomorphic strips":



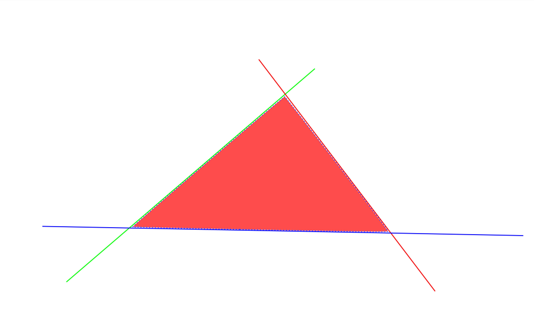
Using Gromov's compactness of holomorphic curves, Floer proved that $d^2 = 0$ and that $HF(L, L) \simeq H_*(L)$.

Fukaya category.

Organise this into a category:

- Objects are exact Lagrangian.
- Morphisms are the $CF(L_1, L_2)$ (chain complex version of the Donaldson category).

Compositions are given by counting "holomorphic triangles".



Fukaya category.

Problem 1:

This composition is not associative.It is up to homotopy counting “holomorphic squares”. This leads to define an $\mathcal{A}_\infty$-category where operation μ^k counts holomorphic $(k+1)$-gones. These operations satisfy:

$$\sum \mu^{d-i}(a_d, \ldots, a_{k+i+1}, \mu^i(a_{k+i}, \ldots, a_k) \ldots, a_0) = 0.$$

Fukaya category.

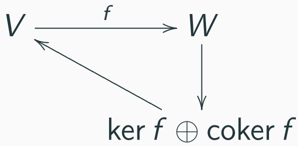
Problem 2:

0-objects are represented geometrically by objects we can move to "infinity". But Floer's result that $HF(L, L) = H_\bullet(L)$ implies that no such 0-object exists. But having a 0-object allows us to detect isomorphisms.

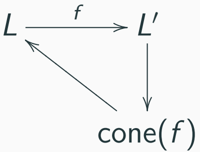
$f : V \to W$ is an isomorphism iff $\ker f \oplus \operatorname{coker} f = 0$.

Fukaya category

$\ker f \oplus \operatorname{coker} f$ is the only vector spaces that makes



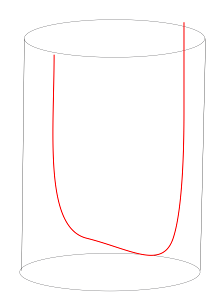
exact. In an $\mathcal{A}_\infty$, the cone of f is an object that makes

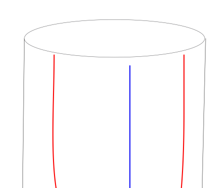


exact.

Wrapped Fukaya category.

We enlarge the class of object allowing non-compact Lagrangian tangent to V outside a compact set.





Main results.

Let (W, λ, f) a Weinstein manifold and $(P = W \times \mathbb{R}, dz + \lambda)$ its contactisation. Then

Theorem (C.–Dimitroglou-Rizell–Ghiggini–Golovko)
The wrapped Fukaya category of W is generated by its Lagrangian cocores.

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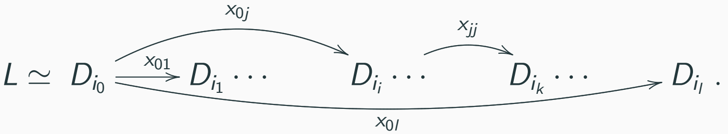
Generation.

Generation.

Note that the Lagrangian disks D_q are objects of this category.

We say that a collection of objects $\{D_i\}$ *generates* an $\mathcal{A}_\infty$-category if for any other objects L:



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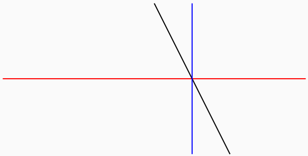
Generation

- This implies that to understand a Lagrangian L one needs to understand how it intersects all D_q's.
- To compare with other categories one needs to check only on generators.
- Gives restrictions on the topology of Lagrangians.

Tentative geometric proof.

We want to remove intersection with the skeleton $\cup_q H_q$.



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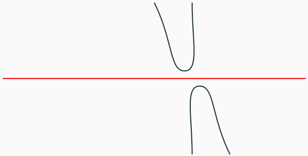
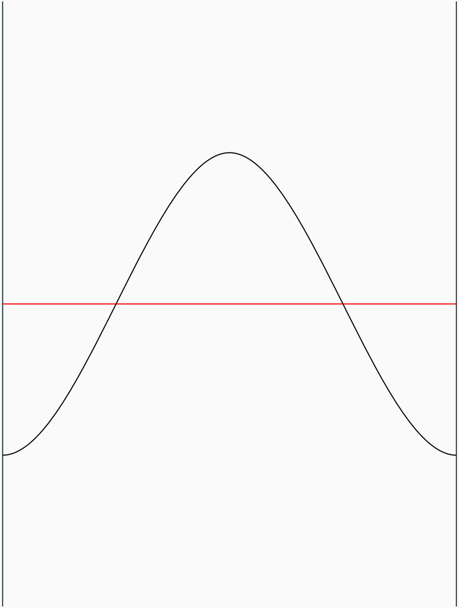


Tentative geometric proof.

Applying this to the 0-section in T^*S^1 we immediately run into trouble.

Tentative geometric proof.



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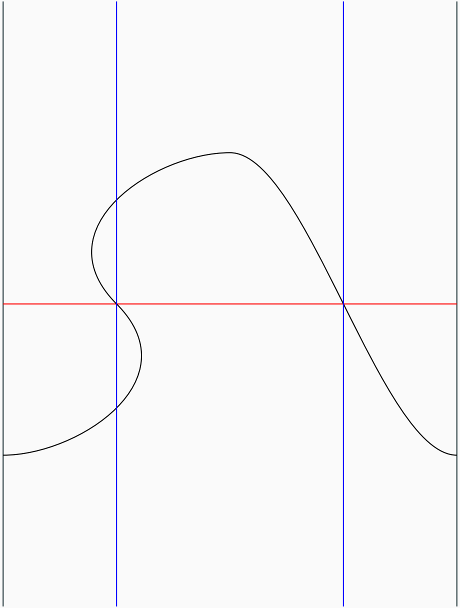
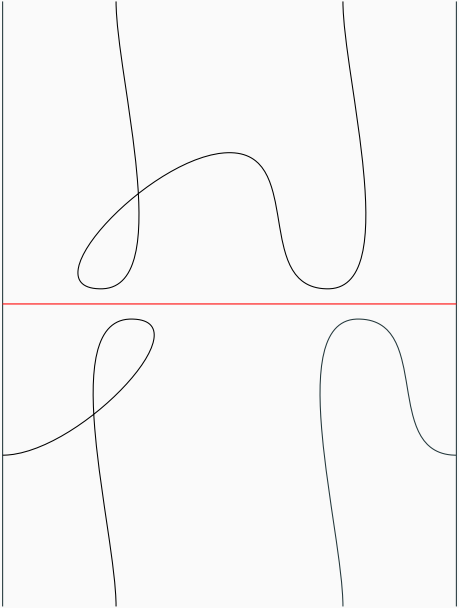


Tentative geometric proof.



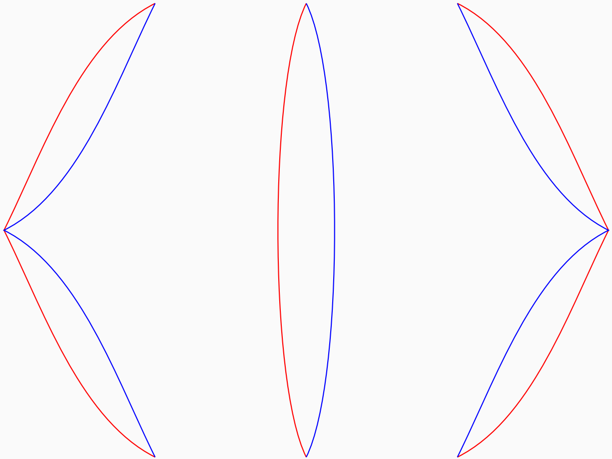
Tentative geometric proof.

The Lagrangian surgery forces us to consider immersed Lagrangian. So we lift the picture to the contactisation and consider Legendrian submanifolds.

Augmentation of Legendrian sub-manifolds and Floer theory.

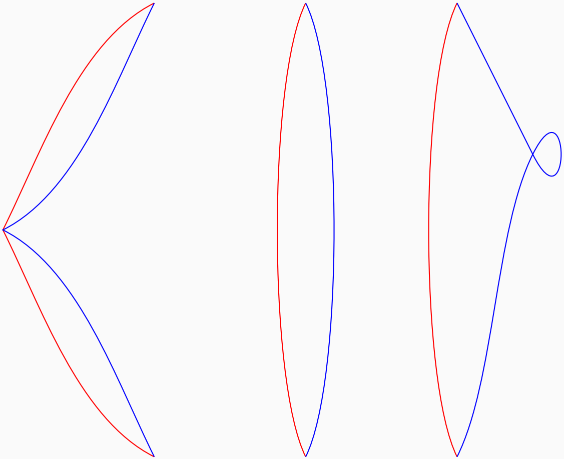
Augmentations.

The fact that $d^2 = 0$ from Floer homology follows from the fact that 1-parameter families of holomorphic strips degenerates to broken strips.



Augmentations

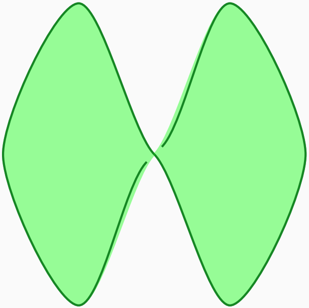
But if one of the Lagrangian is immersed then other degenerations can occur.



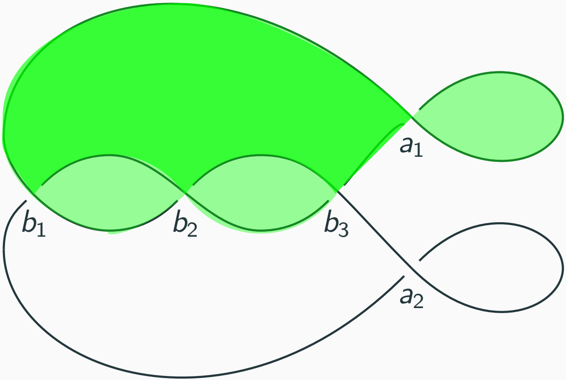
Augmentations.

An augmentation is a compensation of such degenerations. Observed that if the number of *teardrops* is zero the we still have $d^2 = 0$.

Augmentations.

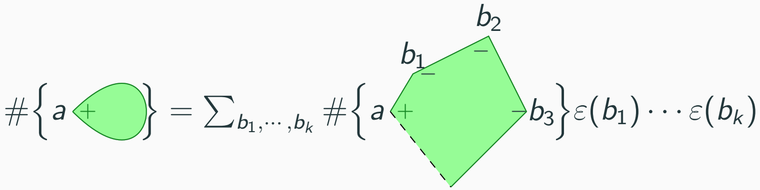


Augmentations.



Augmentations

An *augmentation* is a map $\varepsilon : \mathcal{R}(\Lambda) \to R$ such that for any a:



One then uses augmentations of Λ_1 and Λ_2 to define $LCC(\Lambda_1, \Lambda_2)$ generated by chords from Λ_1 to Λ_2 where the differential counts weighted (generalised) strips (Chekanov, Ekholm-Etnyre-Sullivan).

Main results.

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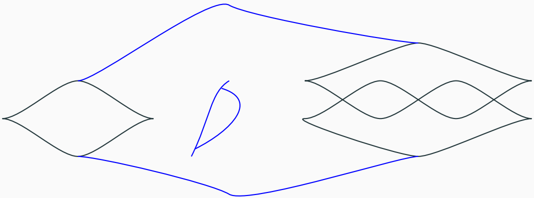
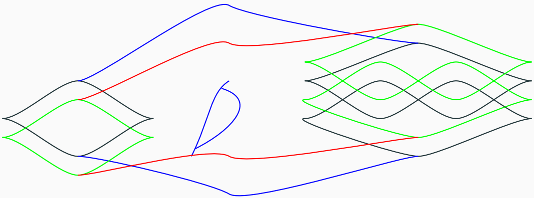
Cobordisms

An *exact lagrangian cobordisms* from a Legendrian Λ^- to another Legendrian Λ^+ is an embedding $\Sigma \to \mathbb{R} \times P$ such that $\Sigma^*(e^t(dz + \lambda)) = df$.



Floer theory.

We can use augmentation to define Floer homology for a pair of cobordisms. Generators of the Floer complex are intersection points plus chords at the top and the bottom of the pair of cobordisms.



Floer theory.

As we can displace all cobordisms for any other, the homology of the complex is 0. In some favorable cases this leads to exact sequences of the form:

$$\begin{array}{ccc} LCH(\Lambda_0^-, \Lambda_1^-) & \longrightarrow & LCH(\Lambda_0^+, \Lambda_1^+) \\ & \nwarrow & \downarrow \\ & & HF(\Sigma_0, \Sigma_1) \end{array}$$

Proofs

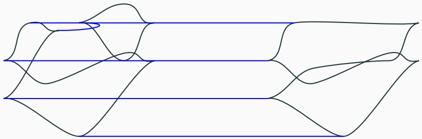
Proof of generation criterion.

To correct our tentative proof we want to make sure of two things:

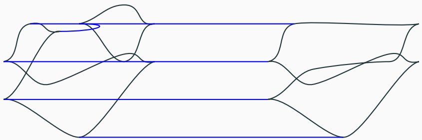
1. The Lagrangian surgery is an iterated cone.
2. The surgery procedure gives a Legendrian submanifold that admits an augmentation.

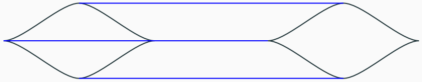
Proof of generation criterion.

Both things are solved by seeing that there is a cobordism from the surgery to the original configuration.



Proof of generation criterion.



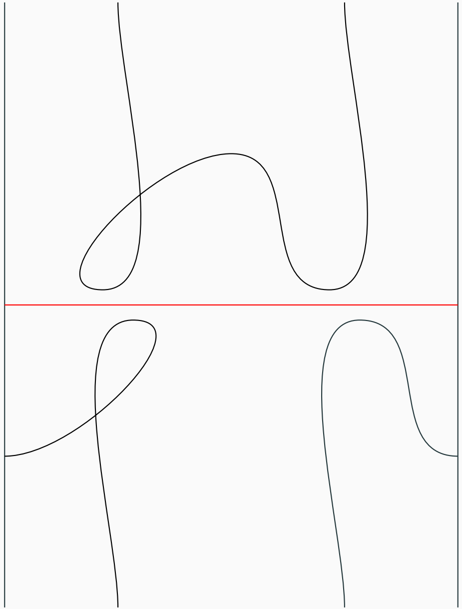


Using any test object T far under the cobordisms gives:

$$HF(T, \Lambda_{\mathrm{surg}}) \simeq HF(T, \Lambda).$$

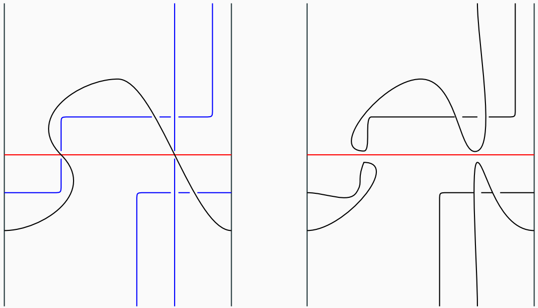
If Λ_{surg} admits an augmentation!

Tentative geometric proof.



Proof of generation criterion.

To make sure that an augmentation exists we wrap the picture near the skeleton.



Proof of generation criterion.

We end up with an iterated cone starting with L and involving only copies of D_q's that is the zero object.

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Proof of the triviality of the endocobordisms.

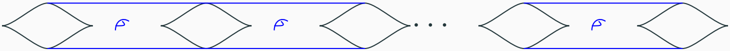
We start with a cobordisms Σ.

This leads to the exact sequence:

$$\begin{array}{ccc} LCH_{\mathrm{rel}}(\Lambda) & \longrightarrow & LCH_{\mathrm{rel}}(\Lambda) \\ & \nwarrow & \downarrow \\ & & H_\bullet(\Sigma, \Lambda) \end{array}$$

Proof of the triviality of the endocobordisms.

We concatenate Σ with itself to obtain exact sequences:



This leads to the exact sequence:

$$\begin{array}{ccc} LCH_{\mathrm{rel}}(\Lambda) & \longrightarrow & LCH_{\mathrm{rel}}(\Lambda) \\ & \nwarrow & \downarrow \\ & & H_\bullet(\Sigma_k, \Lambda) \end{array}$$

But the topology of Σ_k explodes if Σ has some topology.
Thus $H_\bullet(\Sigma, \Lambda) = 0$.

Proof of the triviality of the endocobordisms.

Studying the complex with coefficient in $\mathbb{C}[\pi_1(\Sigma)]$ allows us to show that $\pi_1(\Sigma) = 1$. (This uses an L^2 completion of $\mathbb{C}[\pi_1(\Sigma)]$ and L^2-betti numbers). The h-cobordism theorem allows us to conclude.

Perspectives

Some perspectives.

- Use generation criterion to find restriction on Lagrangian embeddings.
- Study L^2 torsion of the Floer complex.
- Study the complex for Lagrangian cobordisms in non-trivial Liouville cobordisms.
- Organise those into a category.
- Find generators of this category.
- Gluing formula (Mayer-Vietoris for Fukaya categories).