

Micro-Macro Monte Carlo approach for collisional kinetic equations

Anaïs Crestetto¹, Nicolas Crouseilles², Giacomo Dimarco³
and Mohammed Lemou⁴



CEA-SMAI/GAMNI seminar
January 24, 2022

¹University of Nantes, LMJL.

²Inria & University of Rennes 1, IRMAR.

³University of Ferrara, Dept of Mathematics and Computer Science.

⁴University of Rennes 1, IRMAR & CNRS & Inria.

Outline

- 1 Multiscale BGK problem
- 2 Monte Carlo / FV discretization
- 3 Numerical results
- 4 Towards the Boltzmann operator

Multiscale BGK Problem

Radiative transport equation in the diffusive scaling

$$\partial_t f + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2} Q(f, f), \quad Q(f, f) = (\rho M - f) \quad (1)$$

- $\mathbf{x} \in \Omega \subset \mathbb{R}^{d_x}$, $\mathbf{v} \in V = \mathbb{R}^{d_v}$,
- charge density $\rho(t, \mathbf{x}) = \int_V f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$,
- $M(\mathbf{v}) = \frac{1}{(2\pi)^{d_v/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right)$,
- periodic conditions in $\mathbf{x}$ and initial conditions.

Main difficulty:

- Knudsen number ε may be of order 1 or tend to 0 in the diffusive scaling. The asymptotic diffusion equation being

$$\partial_t \rho - \Delta_{\mathbf{x}} \rho = 0. \quad (2)$$

Objectives

- Construction of an Asymptotic Preserving (AP) scheme ^[5].
- Reduction of the numerical cost at the limit $\varepsilon \rightarrow 0$.

Tools

- Micro-macro decomposition ^[6,7] for this model. Previous work with a grid in $\mathbf{v}$ for the micro part ^[8], cost was constant w.r.t. ε .
- Particle method for the micro part since few information in $\mathbf{v}$ is necessary at the limit ^[9].
- Monte Carlo techniques ^[10].

⁵Jin, SISC 1999.

⁶Lemou, Mieussens, SIAM SISC 2008.

⁷Liu, Yu, CMP 2004.

⁸Crouseilles, Lemou, KRM 2011.

⁹C., Crouseilles, Lemou, CMS 2018.

¹⁰Degond, Dimarco, Pareschi, IJNMF 2011.

Micro-macro decomposition

- Micro-macro decomposition:

$$f = \rho M + g$$

with g the perturbation.

- $\mathcal{N} = \text{Span}\{M\} = \{f = \rho M\}$ null space of the BGK operator $Q(f) = \rho M - f$.
- Π orthogonal projection onto $\mathcal{N}$:

$$\Pi h := \langle h \rangle M, \quad \langle h \rangle := \int_V h \, d\mathbf{v}.$$

- Applying Π to (1) $\implies$ macro equation on ρ

$$\partial_t \rho + \frac{1}{\varepsilon} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0. \quad (3)$$

- Applying $(I - \Pi)$ to (1) $\implies$ micro equation on g

$$\partial_t g + \frac{1}{\varepsilon} [\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M] = -\frac{1}{\varepsilon^2} g. \quad (4)$$

Equation (1) $\Leftrightarrow$ micro-macro system:

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0, \\ \partial_t g + \frac{1}{\varepsilon} \mathcal{F}(\rho, g) = -\frac{1}{\varepsilon^2} g, \end{cases} \quad (5)$$

where $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

Difficulties

- Stiff terms in the micro equation (4) on g .
- In previous works ^[11,12], stiffest term (of order $1/\varepsilon^2$) considered implicit in time $\implies$ transport term (of order $1/\varepsilon$) stabilized.

But here:

- use of particles for the micro part
- $\implies$ splitting between the transport term and the source term,
 $\implies$ not possible to use the same strategy.

Idea?

- Suitable reformulation of the model.

¹¹Lemou, Mieussens, SIAM SISC 2008.

¹²Crouseilles, Lemou, KRM 2011.

- Strategy ^[13]:

1. rewrite (4) $\partial_t g + \frac{1}{\varepsilon} \mathcal{F}(\rho, g) = -\frac{1}{\varepsilon^2} g$ as

$$\partial_t (e^{t/\varepsilon^2} g) = -\frac{e^{t/\varepsilon^2}}{\varepsilon} \mathcal{F}(\rho, g),$$

2. integrate in time between two times t^n and $t^{n+1} = t^n + \Delta t$:

$$e^{t^{n+1}/\varepsilon^2} g^{n+1} = e^{t^n/\varepsilon^2} g^n + \int_{t^n}^{t^{n+1}} -\frac{e^{t/\varepsilon^2}}{\varepsilon} \mathcal{F}(\rho, g) dt,$$

3. use rectangle method for $\mathcal{F}(\rho, g)$ and multiply by $e^{-t^{n+1}/\varepsilon^2}/\Delta t$:

$$\frac{g^{n+1} - g^n}{\Delta t} = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g^n - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho^n, g^n) + \mathcal{O}(\Delta t),$$

4. approximate up to terms of order $\mathcal{O}(\Delta t)$ by:

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho, g).$$

- No more stiff terms and consistent with initial micro equation (4).

¹³Lemou, CRAS 2010.

New micro-macro model

The new micro-macro model writes

$$\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0, \quad (6)$$

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho, g), \quad (7)$$

with $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

We propose the following hybrid discretization:

- macro equation (6): Finite Volume method,
- micro equation (7): Monte Carlo technique.

Discretization of the micro equation

- Model: considering at each time step N^n particles, with position $\mathbf{x}_k^n$, velocity $\mathbf{v}_k^n$ and constant weight ω_k , $k = 1, \dots, N^n$, g is approximated by ^[14]

$$g_{N^n}(t^n, \mathbf{x}, \mathbf{v}) = \sum_{k=1}^{N^n} \omega_k \delta(\mathbf{x} - \mathbf{x}_k^n) \delta(\mathbf{v} - \mathbf{v}_k^n).$$

- For the coupling with the macro equation, we need a grid in $\mathbf{x}$. For $d_x = 1$, we define for $i = 0, \dots, N_x - 1$

$$\mathbf{x}_i = x_{\min} + i\Delta x, \quad \mathbf{x}_{i\pm 1/2} = \mathbf{x}_i \pm \frac{\Delta x}{2}.$$

- How to define/compute ω_k , N^n , $\mathbf{x}_k^n$, $\mathbf{v}_k^n$?

¹⁴Crouseilles, Dimarco, Lemou, KRM 2017.

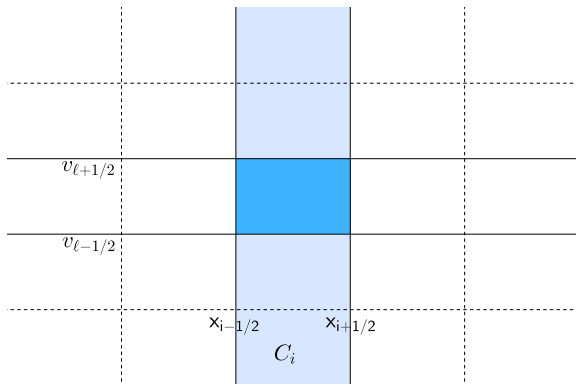
Initialization

- Choose the characteristic weight m_p or the characteristic number of particles N_p necessary to sample the full distribution function f , and link them with

$$m_p = \frac{1}{N_p} \int_{\mathbb{R}^{d_x}} \int_{\mathbb{R}^{d_v}} f(t=0, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x}.$$

- Now, we want to sample $g(t=0, \mathbf{x}, \mathbf{v})$, that has no sign.
- We impose $\omega_k \in \{m_p, -m_p\}$.
- For velocities, we impose $\mathbf{v}_k^n$ on a cartesian grid in $\mathbb{R}^{d_v}$.
For $d_v = 1$, we have $\mathbf{v}_k^n \in \{v_\ell, \ell = 0, \dots, N_v - 1\}$
 $\forall k = 1, \dots, N^n$, where $v_\ell = v_{\min} + \ell \Delta v$, $\ell = 0, \dots, N_v - 1$
and $v_{\ell \pm 1/2} = v_\ell \pm \frac{\Delta x}{2}$.

Let us introduce the notations in 1D...



Let us introduce the notations in 1D...

- The number of initial positive (resp. negative) particles having the velocity $v_k = v_\ell$ in the strip

$C_i = [x_{i-1/2}, x_{i+1/2}] \times \mathbb{R}$ is given by

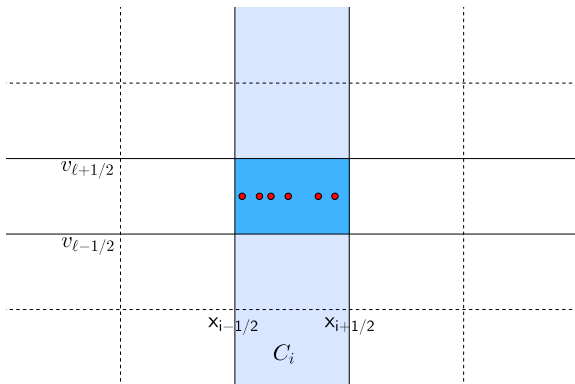
$$N_{i,\ell}^{0,\pm} = \lfloor \pm \frac{\Delta x \Delta v}{m_p} g^\pm(t=0, x_i, v_\ell) \rfloor,$$

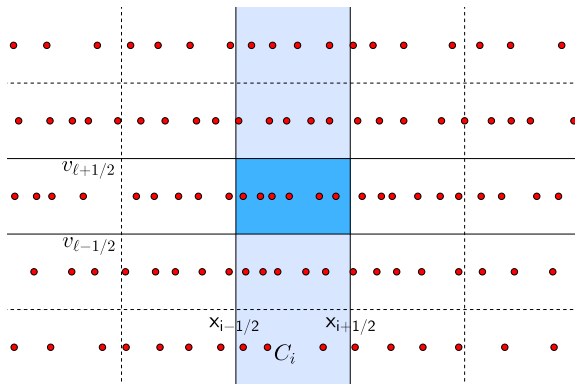
that is an approximation of

$$N_{i,\ell}^{0,\pm} = \pm \frac{1}{m_p} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{v_{\ell-1/2}}^{v_{\ell+1/2}} g^\pm(t=0, x, v) dv dx,$$

with $g^\pm = \frac{g \pm |g|}{2}$ the positive and negative parts of g .

- Positions of these $N_{i,\ell}^{0,\pm}$ particles are taken uniformly in $[x_{i-1/2}, x_{i+1/2}]$.
- At time $t=0$, we have $N^0 = \sum_i \left(\sum_\ell N_{i,\ell}^{0,+} + \sum_\ell N_{i,\ell}^{0,-} \right)$.





From t^n to t^{n+1}

Solve the micro equation (7) by Monte Carlo technique.

- **Splitting** between the transport part

$$\partial_t g + \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v} \cdot \nabla_{\mathbf{x}} g = 0,$$

and the interaction part

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} (\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M).$$

- Solve the **transport part** thanks to motion equation:

$$\frac{d\mathbf{x}_k}{dt}(t) = \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v}_k, \quad \mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \varepsilon (1 - e^{-\Delta t/\varepsilon^2}) \mathbf{v}_k^n.$$

Remark that $\mathbf{v}_k^{n+1} = \mathbf{v}_k^n$.

- Solve **interaction part** by writing

$$g^{n+1} = e^{-\Delta t/\varepsilon^2} \tilde{g}^n + (1 - e^{-\Delta t/\varepsilon^2}) \varepsilon [-\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g} \rangle^n M]$$

where $\tilde{g}^n$ is the function after the transport part.

Apply a **Monte Carlo technique**:

- with probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change,
- with probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by

$$\varepsilon [-\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g} \rangle^n M].$$

In practice (1)

“With probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change.”

→ In each strip C_i

- we choose randomly $e^{-\Delta t/\varepsilon^2} \tilde{N}_i^n$ particles and keep them unchanged (with $\tilde{N}_i^n$ the number of particles in C_i after the transport part),
- we discard the others.

In practice (2)

“With probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by $\varepsilon[-\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g} \rangle^n M]$.”

→ We define the function

$$\mathcal{P}^{n,\pm}(\mathbf{x}, \mathbf{v}) = \varepsilon[-\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g} \rangle^n M]^{\pm}.$$

→ In each strip C_i

- we sample a corresponding number $M_i^{n,\pm}$ of new particles with weights $\pm m_p$ from $(1 - e^{-\Delta t/\varepsilon^2})\mathcal{P}^{n,\pm}(\mathbf{x}_i, \mathbf{v})$,
- in 1D, we have

$$M_{i,\ell}^{n,\pm} = \frac{1}{m_p} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{v_{\ell-1/2}}^{v_{\ell+1/2}} \pm (1 - e^{-\Delta t/\varepsilon^2}) \mathcal{P}^{n,\pm}(x, v) dv dx,$$

- these $M_{i,\ell}^{n,\pm}$ created particles are such that $\mathbf{v}_k^n = v_\ell$ and $\mathbf{x}_k^n$ are uniformly distributed in $[x_{i-1/2}, x_{i+1/2}]$.

Asymptotically Complexity Diminishing Property

- At the end of the time step, we have in each strip C_i

$$N_i^{n+1} = e^{-\Delta t/\varepsilon^2} \tilde{N}_i^n + \sum_{\ell} \left(M_{i,\ell}^{n,+} + M_{i,\ell}^{n,-} \right)$$

particles.

- The number of particles automatically diminishes with ε .
- Reduction of the computational complexity when approaching equilibrium: **Asymptotically Complexity Diminishing Property**.

Macro equation

- Equation $\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0$.
- First proposition:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g^{n+1} \rangle = 0,$$

discretized in space by a Finite Volume method.

- Problem: g^{n+1} suffers from numerical noise inherent to particles method. This noise, amplified by $\frac{1}{\varepsilon}$, will damage ρ^{n+1} .
- Use the expression of g^{n+1} and plug it into the macro equation

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t / \varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t / \varepsilon^2}) \Delta_{\mathbf{x}} \rho^n = 0.$$

- To avoid the parabolic CFL condition of type $\Delta t \leq C\Delta x^2$, take the **diffusion term implicit**:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t/\varepsilon^2} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^{n+1} = 0.$$

- **No more stiffness**, the numerical noise does not damage ρ .
- **AP property**: for fixed $\Delta t > 0$, the scheme degenerates when $\varepsilon \rightarrow 0$ to an implicit discretization of the diffusion equation $\partial_t \rho - \Delta_{\mathbf{x}} \rho = 0$.

Space discretization in 2D

In 2D, we use an Alternating Direction Implicit (ADI) method ^[15]:

- 1) Starting from ρ^n , solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{xx} \rho = 0,$$

using a Crank-Nicolson time discretization to get ρ^* .

- 2) Starting from ρ^* , solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{yy} \rho = 0,$$

using a Crank-Nicolson time discretization to get ρ^{n+1} .

¹⁵Peaceman, Rachford, J. Soc. Indust. Appl. Math. 1955.

Nice properties

- Only 1D systems of size N_x or N_y .
- ADI method unconditionally stable in 2D.
- Straightforward extension in 3D: a priori conditionally stable, but better extensions have been derived ^[16].
- Right asymptotic behaviour.

¹⁶Sharma, Hammett, JCP 2011.

Test 1 - 2Dx2D, constant ε , $g(t = 0, \mathbf{x}, \mathbf{v}) = 0$

Initialization:

$$f(t = 0, \mathbf{x}, \mathbf{v}) = \rho_0(\mathbf{x})M(\mathbf{v}), \quad \mathbf{x} \in [0, 4\pi]^2, \quad \mathbf{v} \in \mathbb{R}^2$$

with

$$\rho_0(\mathbf{x}) = 1 + \frac{1}{2} \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right),$$

$$M(\mathbf{v}) = \frac{1}{2\pi} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right),$$

so that

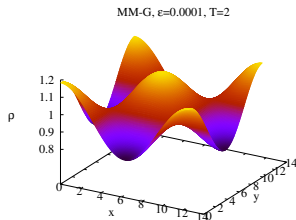
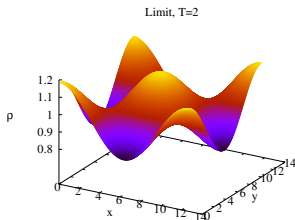
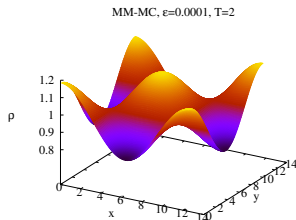
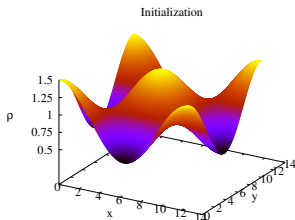
$$g(t = 0, \mathbf{x}, \mathbf{v}) = 0.$$

Periodic boundary conditions in space.

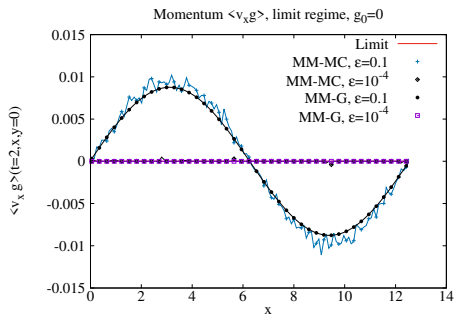
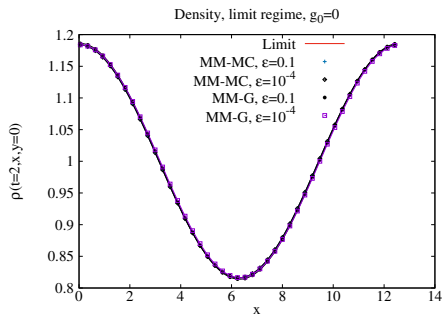
Asymptotic behaviour, $\varepsilon = 10^{-4}$

MM-MC: the presented Micro-Macro Monte Carlo scheme.

MM-G: a Micro-Macro Grid code, considered as reference.



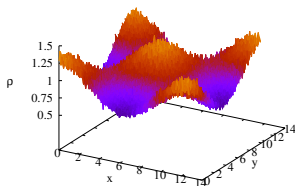
Slices of the density $\rho(T = 2, x, y = 0)$ and of the momentum $\langle v_x g \rangle(T = 2, x, y = 0)$.



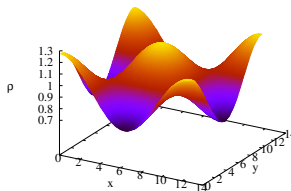
Kinetic regime, $\varepsilon = 1$

Full PIC: standard particle method on f .

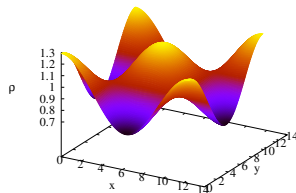
Full PIC, $\varepsilon=1$, $T=2$



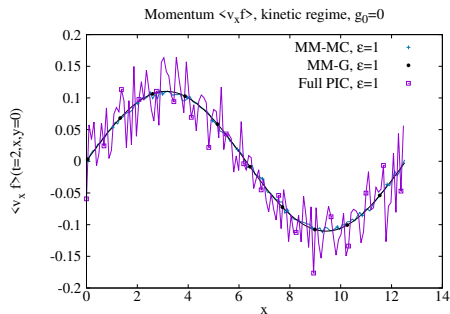
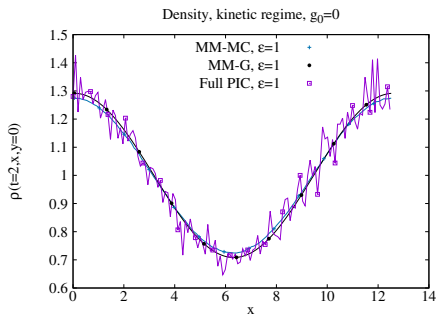
MM-MC, $\varepsilon=1$, $T=2$



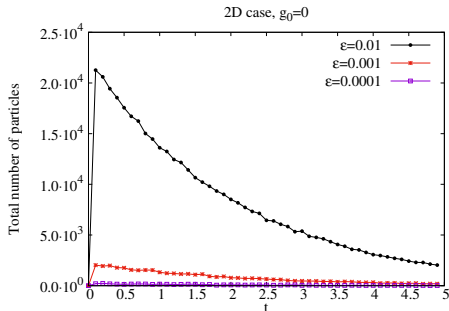
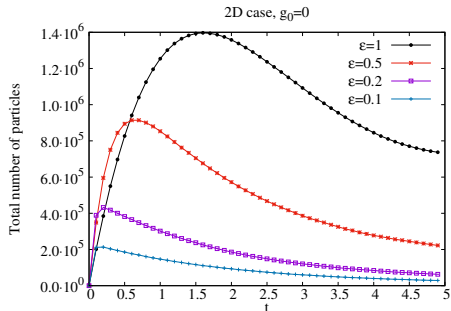
MM-G, $\varepsilon=1$, $T=2$



Slices of the density $\rho(T = 2, x, y = 0)$ and of the momentum $\langle v_x f \rangle(T = 2, x, y = 0)$.



Time evolution of the number of particles



Test 3 - 3Dx3D, constant ε , $g(t = 0, \mathbf{x}, \mathbf{v}) \neq 0$

Initialization:

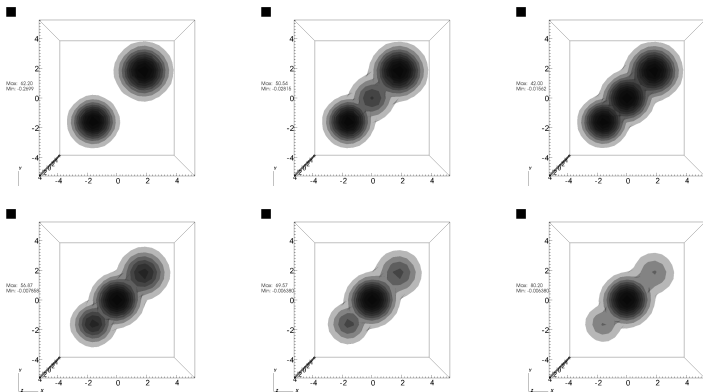
$$f(0, \mathbf{x}, \mathbf{v}) = \frac{1}{2(2\pi)^{3/2}} \left[\exp\left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{2}\right) + \exp\left(-\frac{|\mathbf{v} + \mathbf{u}|^2}{2}\right) \right] \rho_0(\mathbf{x}),$$

with $\mathbf{u} = (2, 2, 2)$,

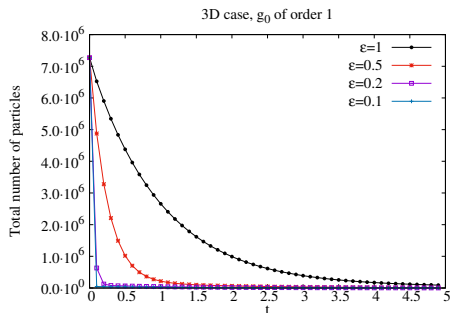
$$\rho_0(\mathbf{x}) = 1 + \frac{1}{2} \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) \cos\left(\frac{z}{2}\right),$$

$$\mathbf{x} = (x, y, z) \in [0, 4\pi]^3, \mathbf{v} = (v_x, v_y, v_z) \in \mathbb{R}^3.$$

Integral of the distribution function in space $\int_{\mathbf{x}} f(T, \mathbf{x}, \mathbf{v}) d\mathbf{x}$ for $\varepsilon = 1$ and different times ($T=0, 0.2, 0.4, 0.6, 0.8, 1$).



Time evolution of the number of particles



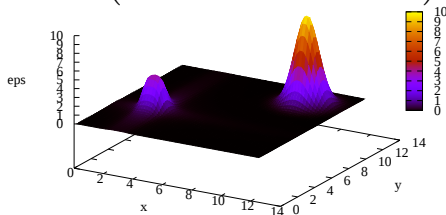
Test 4 - 2Dx2D, $\varepsilon(\mathbf{x})$, $g(t = 0, \mathbf{x}, \mathbf{v}) \neq 0$

Modified model:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2(\mathbf{x})}(\rho M - f),$$

where $(\mathbf{x}, \mathbf{v}) \in [0, 4\pi]^2 \times \mathbb{R}^2$,

$$\varepsilon(\mathbf{x}) = 10 \left[\text{atan}\left(2(y - 5)\right) + \text{atan}\left(-2(y - 5)\right) \right] \\ \times \exp\left(- (x - 10)^2 - (y - 10)^2\right) + 10^{-3}.$$



Initialization:

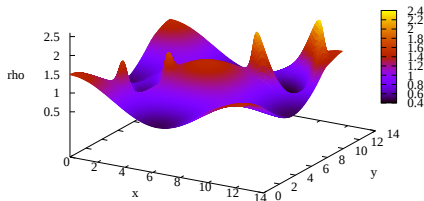
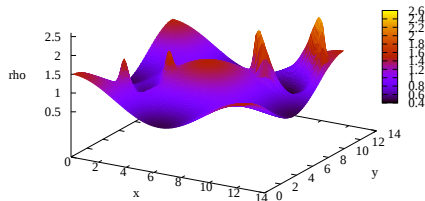
$$f(t=0, \mathbf{x}, \mathbf{v}) = \frac{1}{4\pi} \left(\exp \left(-\frac{|\mathbf{v} - \mathbf{u}|^2}{2} \right) + \exp \left(-\frac{|\mathbf{v} + \mathbf{u}|^2}{2} \right) \right) \rho_0(\mathbf{x}),$$

with

$$\mathbf{x} \in [0, 4\pi]^2, \quad \mathbf{v} \in \mathbb{R}^2, \quad \mathbf{u} = (2, 2)$$

$$\rho_0(\mathbf{x}) = 1 + \frac{1}{2} \cos \left(\frac{x}{2} \right) \cos \left(\frac{y}{2} \right).$$

Density profile $\rho(T=1, x, y)$. Left: MM-MC, right: MM-G.

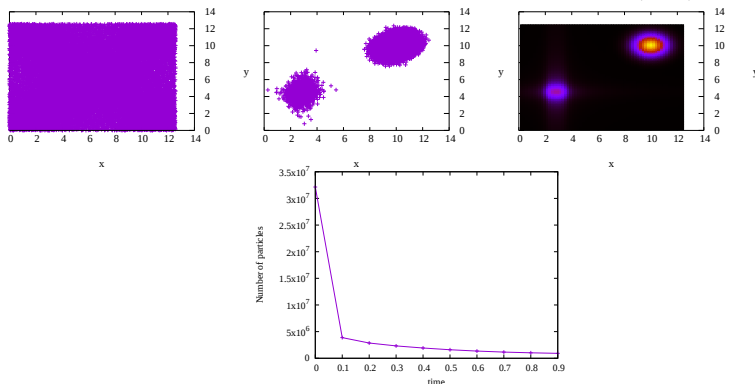


Asymptotically Complexity Diminishing Property

Top: position of the particles in $\mathbf{x}$.

Left: at $T = 0$; middle: at $T = 1$.

Right: $\varepsilon(\mathbf{x}, y)$.



Bottom: time evolution of the number of particles.

Space homogeneous Boltzmann Problem

Space homogeneous Boltzmann equation in the Maxwell molecules case

$$\partial_t f = \frac{1}{\varepsilon} Q(f, f), \quad (8)$$

- $f := f(t, \mathbf{v}), \mathbf{v} \in \mathbb{R}^{d_v},$
- $Q(f, f)(\mathbf{v}) = \int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v-1}} \underbrace{B(|\mathbf{v} - \mathbf{v}_\star|, \omega)}_{b(\omega)=C>0} (f(\mathbf{v}')f(\mathbf{v}'_\star) - f(\mathbf{v})f(\mathbf{v}_\star)) d\omega d\mathbf{v}_\star,$
- $\omega = \frac{\mathbf{v}' - \mathbf{v}'_\star}{|\mathbf{v}' - \mathbf{v}'_\star|}$ vector of the unitary sphere $\mathbb{S}^{d_v-1} \subset \mathbb{R}^{d_v},$
- $(\mathbf{v}', \mathbf{v}'_\star)$ pre and $(\mathbf{v}, \mathbf{v}_\star)$ post-collisional velocities linked by $\mathbf{v} = \frac{1}{2} (\mathbf{v}' + \mathbf{v}'_\star + |\mathbf{v}' - \mathbf{v}'_\star|\omega), \mathbf{v}_\star = \frac{1}{2} (\mathbf{v}' + \mathbf{v}'_\star - |\mathbf{v}' - \mathbf{v}'_\star|\omega).$

We can write $Q(f, f) = P(f, f) - \mu f$

with

- $P(f, f)(\mathbf{v}) = \int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v-1}} b(\omega) f(\mathbf{v}') f(\mathbf{v}'_\star) d\omega d\mathbf{v}_\star$
the (bilinear) gain term,
- $\mu f(\mathbf{v}) = f(\mathbf{v}) \int_{\mathbb{R}^{d_v}} f(\mathbf{v}_\star) d\mathbf{v}_\star \int_{\mathbb{S}^{d_v-1}} b(\omega) d\omega$
the loss term (mass preservation $\Rightarrow \mu$ constant).

We use the micro-macro decomposition

$f(t, \mathbf{v}) = M(\mathbf{v}) + g(t, \mathbf{v})$, where M is the gaussian function such that $\int_{\mathbb{R}^{d_v}} \phi(\mathbf{v}) f(t, \mathbf{v}) d\mathbf{v} = \int_{\mathbb{R}^{d_v}} \phi(\mathbf{v}) M(\mathbf{v}) d\mathbf{v}$,
 $\phi(\mathbf{v}) = (1, \mathbf{v}, \mathbf{v}^2/2)^T$, and write

$$\partial_t g = \frac{1}{\varepsilon} (P(M + g, M + g) - \mu M) - \frac{\mu}{\varepsilon} g$$

or

$$\partial_t (g e^{\mu t/\varepsilon}) = \frac{1}{\varepsilon} (P(g, g) + P(M, g) + P(g, M)) e^{\mu t/\varepsilon}.$$

Our micro-macro Monte Carlo method

We use a first-order exponential scheme:

$$g^{n+1} = e^{-\frac{\mu\Delta t}{\varepsilon}} g^n + \frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}} \left(\frac{P(g^n, g^n) + P(M, g^n) + P(g^n, M)}{\mu} \right).$$

Monte Carlo interpretation: g represented by particles and

- with probability $e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles are not modified,
- with probability $\frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles collide,
- with probability $1 - e^{-\frac{\mu\Delta t}{\varepsilon}} - \frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles are discarded.

How to perform collisions?

At time t^n , you have a set of N_+^n positive particles and a set of N_-^n negative particles.

Sample $P(g^n, g^n)/\mu$:

- Select $\frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}} (N_+^n + N_-^n)$ particles.
- For each one (k), select randomly a second one (j).
Compute the new v_k^{n+1} thanks to collision rules.
- If particles k and j were both positives or both negatives, the new particle k belongs to the positive category. Else it belongs to the negative category.

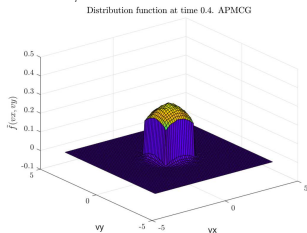
Sample $P(g^n, M)/\mu$ or $P(M, g^n)/\mu$:

- Same idea but instead of colliding two particles representing g , use one of g and one representing M .

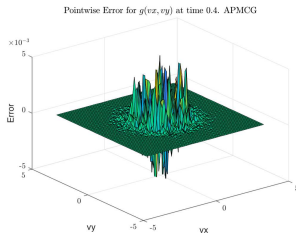
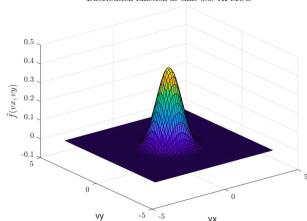
Test - 2D, f initialized as an indicator function

Left: distribution function f , right: error on g .

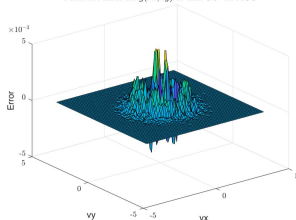
Top: $T=0.4$, bottom: $T=3.6$.



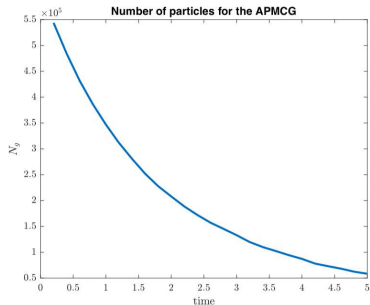
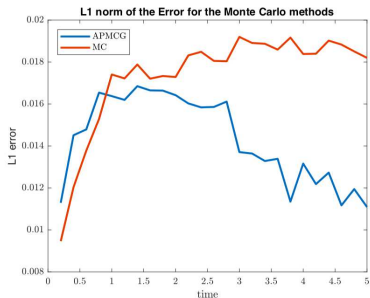
Distribution function at time 3.6. APMCG



Pointwise Error for $g(vx, vy)$ at time 3.6. APMCG



Time evolution of error and particles number



Thank you for your attention!