Numerical approximation of non classical solutions of Riemann problems

Christophe Berthon[1], Marianne Bessemoulin-Chatard[2], Anaïs Crestetto[1], Françoise Foucher[3]

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[1]LMJL, Nantes Université.
[2]CNRS & LMJL, Nantes Université.
[3]École Centrale Nantes & LMJL, Nantes Université.

Introduction

Hyperbolic system of conservation laws (in 1D):

$$\begin{aligned} \partial_t \mathbf{u}(x,t) + \partial_x \mathbf{f}(\mathbf{u}(x,t)) &= 0, && x \in \mathbb{R},\ t > 0, \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), && x \in \mathbb{R}, \end{aligned}$$

with $\mathbf{u} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^d$ vector of state quantities and $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$ given flux function.
Some examples:

- linear transport equation: $\partial_t \rho(x,t) + a(x,t)\partial_x \rho(x,t) = 0$,
- Burgers equation: $\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0$,
- Euler equations in fluid dynamics, *etc.*

- Existence of smooth solutions in short time.
- Nonlinearity and long time $\Rightarrow$ lost of smoothness.
- Work with weak solutions... Not unique! Impose an entropy criterion to select one admissible solution (a physical one).

Some references:

- Godlewski, Raviart, *Hyperbolic Systems of Conservation Laws* (1991)
- Leveque, *Numerical Methods for Conservation Laws* (1992)
- Serre, *Systèmes de lois de conservation, tomes 1, 2* (1996)

Classical numerical approach: Finite Volumes

- Consider a bounded domain $[x_{min}, x_{max}]$ and add boundary conditions.
- Introduce mesh points $x_i = x_{min} + i\Delta x$ and consider approximation at time $t^n = n\Delta t$ given by

$$\mathbf{u}_i^n \approx \frac{1}{\Delta x} \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} \mathbf{u}(t^n, x) dx.$$

- From time t^n to time t^{n+1}, solve on each cell a Riemann problem

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \qquad t \in [t^n, t^{n+1}],$$
$$\mathbf{u}(x, t^n) = \begin{cases} \mathbf{u}_i^n & \text{if } x < x_i + \Delta x/2, \\ \mathbf{u}_{i+1}^n & \text{if } x > x_i + \Delta x/2. \end{cases}$$

- Construct schemes by defining approximations of flux derivative.

Our objective

- Approximate the Riemann solutions of systems of conservation laws in the form:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1}$$

completed with an initial data given by

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L & \text{if } x < 0, \\ \mathbf{u}_R & \text{if } x > 0, \end{cases} \tag{2}$$

where
  - $\mathbf{u}_L$ and $\mathbf{u}_R$ are two given constant states in $\Omega \subset \mathbb{R}^d$,
  - $\mathbf{u}(x,t) \in \Omega$ is the unknown state vector,
  - $\mathbf{f} : \Omega \to \mathbb{R}^d$ is a given smooth flux function.
- Contrary to the usual approach, we do not enforce entropy criterion, so that we will be able to see non classical solutions.

Non classical solutions?

- In the scalar classical case (*i.e.* with entropy criterion):
  - if f is convex (or concave): solution of Riemann problem (1)-(2) is a shock, or a rarefaction wave,
  - if f is neither convex nor concave: we may have a composite wave (two appended waves).
- Here, we are interested in solutions that violate entropy conditions: non classical solutions composed of two separated waves.

Outline

Diffusive-dispersive system of conservation laws

- Solution $\mathbf{u} \in \Omega$ assumed to be the limit solution [see 4], when $\varepsilon \to 0$, of

$$\partial_t \mathbf{u}^\varepsilon + \partial_x \mathbf{f}(\mathbf{u}^\varepsilon) = \varepsilon\beta\partial_{xx}\mathbf{D}(\mathbf{u}^\varepsilon) + \varepsilon^2\gamma\partial_{xxx}\mathbf{D}(\mathbf{u}^\varepsilon), \qquad (3)$$

  with $\mathbf{D} : \Omega \to \mathbb{R}^d$ a given smooth function.
- Competition between diffusion that makes the solution smoother and dispersion that creates oscillations.
- Solutions $\mathbf{u}^\varepsilon$ of (3) may depend on the ratio β/γ [see 5].
- If f is neither convex nor concave: waves may be separated, solutions may be non classical.

[4] LeFloch, Rohde 2001 (IUMJ)
[5] LeFloch 2002 (Book)

Dafermos viscosity approach

- Riemann solutions $\mathbf{u}$ of (1)-(2) are self-similar: only depend on the variable $\xi = x/t$.
- Solutions $\mathbf{u}^\varepsilon$ of diffusive-dispersive problem (3) are not self-similar.
- Reformulation of the diffusive-dispersive system according to the Dafermos viscosity approach [see 6]:

$$\partial_t \mathbf{u}^\varepsilon + \partial_x \mathbf{f}(\mathbf{u}^\varepsilon) = \varepsilon t \beta \partial_{xx} \mathbf{D}(\mathbf{u}^\varepsilon) + \varepsilon^2 t^2 \gamma \partial_{xxx} \mathbf{D}(\mathbf{u}^\varepsilon).$$

[6] Dafermos 2010 (Book)

ODE problem

- Change of variables given by

$$\mathbf{u}^\varepsilon(x,t) = \mathbf{u}(\xi) \quad \text{with } \xi = \frac{x}{t}.$$

- ODE system

$$-\xi\mathbf{u}' + \mathbf{f}(\mathbf{u})' = \varepsilon\beta\mathbf{D}(\mathbf{u})'' + \varepsilon^2\gamma\mathbf{D}(\mathbf{u})'''.$$

- Limit boundary conditions

$$\lim_{\xi\to-\infty}\mathbf{u}(\xi) = \mathbf{u}_L \quad \text{and} \quad \lim_{\xi\to+\infty}\mathbf{u}(\xi) = \mathbf{u}_R$$

replaced by

$$\mathbf{u}(-\ell) = \mathbf{u}_L \quad \text{and} \quad \mathbf{u}(\ell) = \mathbf{u}_R,$$

with $\ell > 0$ large enough [see 7].

[7] Joseph, LeFloch 2007 (P ROY SOC EDINB A)

Numerical method

- 4th order Finite Difference scheme (N mesh points).
- For fixed $\varepsilon > 0$: existence of the discrete solution is proven.
- For fixed $\varepsilon > 0$: solving the nonlinear system in $\mathbb{R}^N$ with a Newton method.
- Decrease ε to approach the limit solution and use the solution given by a higher ε as initial guess to make the convergence of the Newton method easier.

From scalar conservation law to ODE

- Diffusive-dispersive scalar equation

$$\partial_t w^\varepsilon + \partial_x f(w^\varepsilon) = \varepsilon t \beta \partial_{xx} w^\varepsilon + \varepsilon^2 t^2 \gamma \partial_{xxx} w^\varepsilon, \qquad x \in \mathbb{R},\ t > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function.

- Change of variables

$$w^\varepsilon(x,t) = w(\xi) \quad \text{with } \xi = \frac{x}{t}.$$

- w solution of

$$-\xi w' + f(w)' = \varepsilon \beta w'' + \varepsilon^2 \gamma w''', \tag{4}$$

with boundary conditions given by

$$w(-\ell) = w_L \quad \text{and} \quad w(\ell) = w_R, \tag{5}$$

with $\ell > 0$ large enough.

Numerical scheme - Mesh

- Interval $[-\ell, \ell]$ discretized with $N+1$ cells of length $\Delta\xi = \frac{2\ell}{N+1}$: $|\xi_i, \xi_{i+1})$, $\xi_i = -\ell + i\Delta\xi$.
- Notation $w_i \approx w(\xi_i)$.
- Need of ghost cells for the boundary conditions: $i = -2, \ldots, N+3$.

Numerical scheme - 4th order Finite Differences

For a given vector $(X_i)_{i=-2,\dots,N+3}$ and a smooth function $g(X)$, we consider the following discrete operators, for $i = 1, \dots, N$:

$$\overline{g(X)}'_i = \frac{g(X_{i-2}) - 8g(X_{i-1}) + 8g(X_{i+1}) - g(X_{i+2})}{12\Delta\xi},$$

$$\bar{X}''_i = \frac{-X_{i-2} + 16X_{i-1} - 30X_i + 16X_{i+1} - X_{i+2}}{12\Delta\xi^2},$$

$$\bar{X}'''_i = \frac{X_{i-3} - 8X_{i-2} + 13X_{i-1} - 13X_{i+1} + 8X_{i+2} - X_{i+3}}{8\Delta\xi^3}.$$

As soon as $U_i = U(\xi_i)$, where $U(\xi)$ denotes a smooth function, we get

$$\overline{g(U)}'_i = g(U)'(\xi_i) + \mathcal{O}(\Delta\xi^4),$$
$$\bar{U}''_i = U''(\xi_i) + \mathcal{O}(\Delta\xi^4),$$
$$\bar{U}'''_i = U'''(\xi_i) + \mathcal{O}(\Delta\xi^4).$$

Numerical scheme - 4th order scheme

The finite difference scheme applied to our problem (4) and (5) writes:

$$-\xi_i \overline{w}_i' + \overline{f(w)}_i' = \varepsilon\beta\overline{w}_i'' + \varepsilon^2\gamma\overline{w}_i''', \quad i = 1, \dots, N, \tag{6}$$

supplemented by the following boundary conditions [see 8]:

$$w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R. \tag{7}$$

[8] Schecter, Plohr, Marchesin 2004 (DCDS)

Existence result

We are able to state the following result [see 9].

Theorem

Let $\varepsilon > 0$ be given and assume the existence of

$$M_{f'} := \sup_{w \in \mathbb{R}} |f'(w)|.$$

Then there exists $\Delta\xi_0 \leq \sqrt{\varepsilon\beta}$ depending on β, ε, ℓ and $M_{f'}$ such that for $\Delta\xi \leq \Delta\xi_0$, there exists a solution $w^{\Delta} = (w_i)_{i=1,\dots,N}$ to the scheme (6)–(7).

[9] Berthon, Bessemoulin-Chatard, AC, Foucher 2019 (Calcolo)

Idea of the proof

- We rewrite the problem as

$$\mathcal{E}(w^\Delta) = 0,$$

where

$$\mathcal{E}(w^\Delta)_i = \varepsilon\beta\overline{w}_i'' + \varepsilon^2\gamma\overline{w}_i''' + \xi_i\overline{w}_i' - \overline{f(w)}_i', \quad i = 1, \dots, N,$$

with

$$w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R.$$

- A change of variable let us rewrite this problem as

$$\tilde{\mathcal{E}}(\tilde{w}^\Delta) = 0,$$

with

$$\tilde{w}_{-2} = \tilde{w}_{-1} = \tilde{w}_0 = \tilde{w}_{N+1} = \tilde{w}_{N+2} = \tilde{w}_{N+3} = 0.$$

Key lemma: Zeros of a vector field

We will use the following lemma [see 10], which is a consequence of Brouwer's fixed point theorem.

Lemma

Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying

$$F(x) \cdot x \leq 0 \quad \text{if } \|x\| = k,$$

for some $k > 0$. Then there exists a point $x \in B(0,k)$ such that $F(x) = 0$.

A technical study of the scalar product $\tilde{\mathcal{E}}(\tilde{w}^{\Delta}).\tilde{w}^{\Delta}$ and Poincaré inequality give us the proof of our existence theorem.

[10] Evans 1998 (Book)

Continuation-type Newton method

- For $\varepsilon = \mathcal{O}(1)$, the nonlinear system

$$\mathcal{E}(w_\varepsilon^\Delta) = 0,$$

where

$$\mathcal{E}(w_\varepsilon^\Delta)_i = \varepsilon\beta\overline{w}_i'' + \varepsilon^2\gamma\overline{w}_i''' + \xi_i\overline{w}_i' - \overline{f(w)}_i', \quad i = 1,\dots,N,$$

with

$$w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R,$$

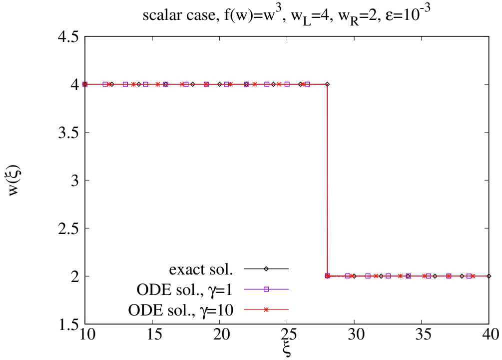
is easily solvable with a (damping [see 11]) Newton method.

- When $\varepsilon \to 0$, it is essential to have a good initial guess to make the Newton method converge.
- The idea is to decrease ε step by step: w_ε^Δ is the initial guess of the Newton method when solving $\mathcal{E}(w_{\varepsilon-\Delta\varepsilon}^\Delta) = 0$.

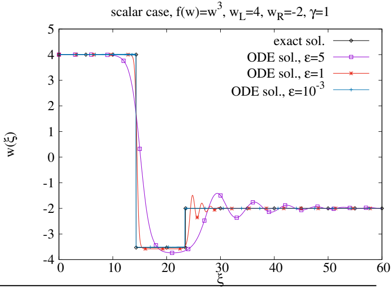
[11] Ralph 1994 (MOR)

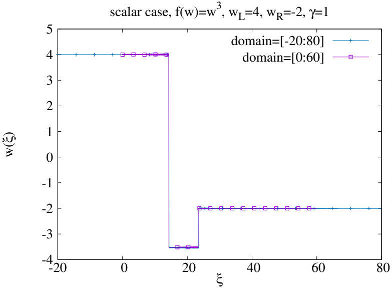
Numerical tests with $f(w) = w^3$

- Classical case: $w_L = 4$, $w_R = 2$.
- Exact solution: shock at $\xi = 28 = \frac{f(w_L)-f(w_R)}{w_L - w_R}$.
- Domain $\mathcal{D} = [10, 40]$, $N = 1000$ $\beta = 1$ and $\varepsilon = 10^{-3}$.
- Two values of γ (1 and 10) are considered to verify that γ has no influence on the solution.



- Non classical case: $w_L = 4$, $w_R = -2$, $\beta = 1$ and $\gamma = 1$.
- Exact solution [see 12]: intermediate state given by $w^\star = -w_L + \frac{\sqrt{2}}{3} \approx -3.5286$ and two shocks, one at $\xi = \frac{f(w_L)-f(w^\star)}{w_L-w^\star} \approx 14.3366$, the other at $\xi = \frac{f(w_R)-f(w^\star)}{w_R-w^\star} \approx 23.5082$.
- Domain $\mathcal{D} = [0, 60]$ or $[-20, 80]$, $N = 2000$. Different values of ε.



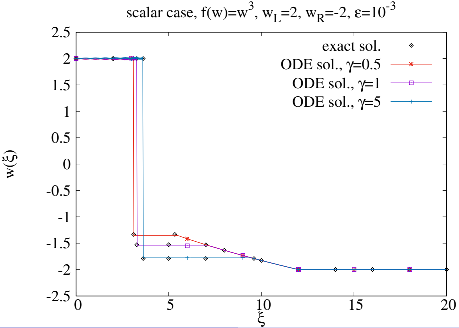


[12] Ernest, LeFloch, Mishra 2013 (SAM)

- Non classical case: $w_L = 2$, $w_R = -2$, $\beta = 1$.
- Exact solution [see 12]: intermediate state given by $w^\star = -w_L + \frac{\sqrt{2}}{3\sqrt{\gamma}}$. A shock and a rarefaction wave given by

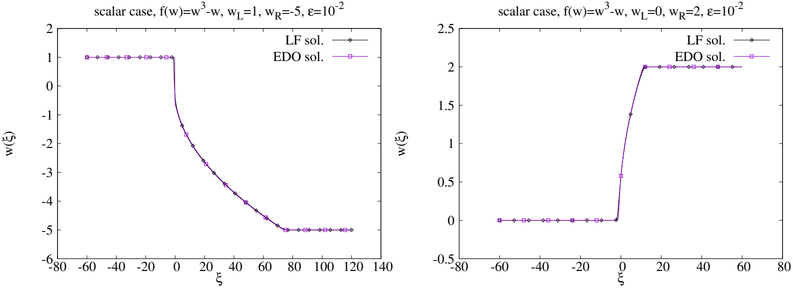
$$\begin{cases} w^\star, & \xi \leq f'(w^\star), \\ f'^{-1}(\xi), & f'(w^\star) \leq \xi \leq f'(w_R), \\ w_R, & f'(w_R) \leq \xi. \end{cases}$$

- Domain $\mathcal{D} = [0, 60]$, $N = 2000$, $\varepsilon = 10^{-3}$.



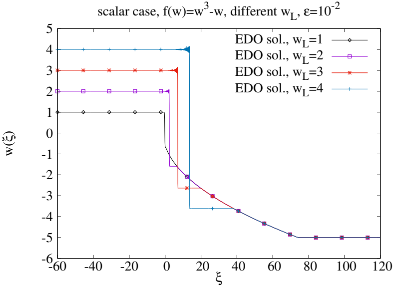
Numerical tests with $f(w) = w^3 - w$

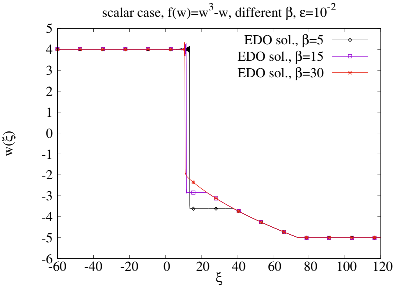
- Classical solutions [see 13]: $w_L = 1$, $w_R = -5$ (Left) or $w_L = 0$, $w_R = 2$ (Right), $\beta = 5$ and $\gamma = 37.5$.
- Our approach: $\mathcal{D} = [-60, 120]$ (Left), $\mathcal{D} = [-60, 60]$ (Right), $N = 2000$, $\varepsilon = 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 5 \times 10^4$, $\Delta t = 2 \times 10^{-5}$.



[13] Chalons, LeFloch 2001 (Numer. Math.)

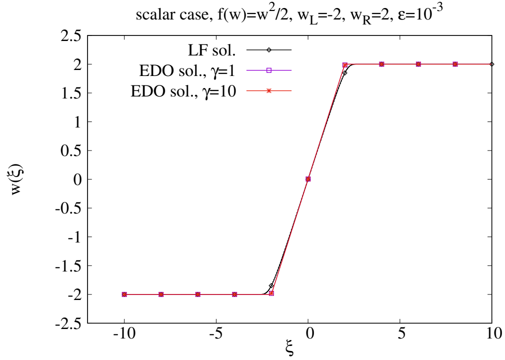
- From classical to non classical solutions [see 13].
- Left: $w_R = -5$, w_L varying from 1 to 4, $\beta = 5$, $\gamma = 37.5$, $\mathcal{D} = [-60, 120]$, $N = 2000$, $\varepsilon = 10^{-2}$.
- Right: $w_R = -5$, $w_L = 4$, β varying from 5 to 30, $\gamma = 37.5$, $\mathcal{D} = [-60, 120]$, $N = 2000$, $\varepsilon = 10^{-2}$.



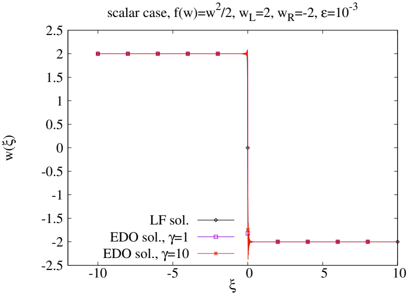


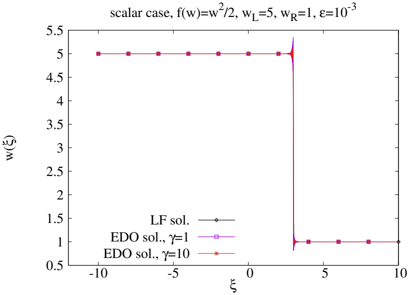
Numerical tests with $f(w) = \frac{w^2}{2}$ (Burgers)

- This problem only admits classical solutions.
- Case of a rarefaction wave: $w_L = -2$, $w_R = 2$, $\beta = 1$, $\gamma = 1$ or 10.
- Our approach: $\mathcal{D} = [-10, 10]$, $N = 2000$, $\varepsilon = 10^{-3}$.
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



- Case of a stationary shock (Left): $w_L = 2$, $w_R = -2$.
- Case of a non stationary shock (Right): $w_L = 5$, $w_R = 1$.
- Other parameters are unchanged.





From p-system to ODE

- Considering the 2×2 diffusive-dispersive system, introduced by Joseph and LeFloch [see 14]

$$\begin{cases} \partial_t w^\varepsilon - \partial_x v^\varepsilon = 0, & x \in \mathbb{R},\ t > 0, \\ \partial_t v^\varepsilon - \partial_x p(w^\varepsilon) = \varepsilon t \beta \partial_{xx} v^\varepsilon + \varepsilon^2 t^2 \gamma \partial_{xxx} w^\varepsilon, \end{cases}$$

  where $p : \mathbb{R} \to \mathbb{R}$ is a given smooth function.
- Focus on Riemann problems and so on self-similar solutions.
- Change of variables

$$w^\varepsilon(x,t) = w(\xi) \quad \text{and} \quad v^\varepsilon(x,t) = v(\xi) \quad \text{with } \xi = \frac{x}{t}.$$

[14] Joseph, LeFloch 2007 (P ROY SOC EDINB A)

- We obtain

$$\begin{cases} -\xi w' - v' = 0, \\ -\xi v' - p(w)' = \varepsilon\beta v'' + \varepsilon^2 \gamma w''', \end{cases}$$

supplemented by boundary conditions

$$(v, w)(-\ell) = (v_L, w_L) \quad \text{and} \quad (v, w)(\ell) = (v_R, w_R),$$

with $\ell > 0$ large enough.

- Using $v' = -\xi w'$, w is governed by a nonlinear equation independent of v

$$\left(\xi^2 + \varepsilon\beta\right) w' - p(w)' = -\varepsilon\beta\xi w'' + \varepsilon^2 \gamma w''', \tag{8}$$

supplemented by the boundary conditions

$$w(-\ell) = w_L \quad \text{and} \quad w(\ell) = w_R. \tag{9}$$

- Problem (8)-(9) contains the full structure of the expected Riemann solution.

- Analysis of (8) [see 15] shows some degeneracy for $\xi = 0$.
- Must be studied into the two regions $[-\ell, 0)$ and $(0, \ell]$ separately.
- The viscous term governed by $\varepsilon\beta\xi$ is of prime importance, but the viscosity vanishes as soon as $\xi = 0$. From a numerical point of view, we have to avoid $\xi = 0$.
- We consider two intervals $[-\ell, -\xi^\star]$ and $[\xi^\star, \ell]$ separately where $\xi^\star > 0$ is a given constant small enough.
- We impose boundary conditions

$$w(-\xi^\star) = w^\star \quad \text{and} \quad w(\xi^\star) = w^\star,$$

where $\xi^\star$ has to be fixed and $w^\star$ has to be determined.

[15] Joseph, LeFloch 2007 (P ROY SOC EDINB A)

Numerical scheme - Finite differences

Assume that the state $w^\star \in \mathbb{R}$ is given. We use 4th order Finite Differences on the right interval $[\xi^\star, \ell]$ (resp. on the left interval $[-\ell, -\xi^\star]$).

- Interval $[\xi^\star, \ell]$ is discretized with $N+1$ cells $[\xi_i, \xi_{i+1})$ of size $\Delta\xi = (\ell - \xi^\star)/(N+1)$: $\xi_i = \xi^\star + i\Delta\xi$.
- For boundary conditions, we define ξ_i for $i = -2, \ldots, N+3$.
- We denote w_i an approximation of $w(\xi_i)$ for $i = 1, \cdots, N$ and define the scheme

$$(\xi_i^2 + \varepsilon\beta)\overline{w}_i' - \overline{p(w)}_i' = -\varepsilon\beta\xi_i\overline{w}_i'' + \varepsilon^2\gamma\overline{w}_i''', \quad i = 1, \cdots, N, \tag{10}$$

completed with the following boundary conditions

$$w_{-2} = w_{-1} = w_0 = w^\star, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R. \tag{11}$$

Determination of $w^\star$ - Mass conservation

- Conservation law: total mass of w must be preserved.
- Initial mass of w given by

$$M_0 = \ell(w_L + w_R).$$

- Total mass of the approximated solution w^Δ depends on $w^\star$ and reads

$$M(w^\star) = \Delta\xi w_L + \sum_{i=1}^{N} \Delta\xi w_i(w_L, w^\star) + 2\xi^\star w^\star + \sum_{i=1}^{N} \Delta\xi w_i(w^\star, w_R) + \Delta\xi w_R.$$

- $w^\star$ must be solution of the following nonlinear equation:

$$M(w^\star) = M_0.$$

Determination of $w^\star$ - Dichotomy technique

- We initialize the dichotomy algorithm as follows:

$$(w_{\inf}, w_{\sup}) = \begin{cases} (w_L, w_R) \text{ if } M(w_L) < M(w_R), \\ (w_R, w_L) \text{ elsewhere}, \end{cases}$$

$$w_0^\star = \frac{1}{2}(w_L + w_R).$$

- For iterations $k \geq 1$, we compute the left and right solutions $w^\Delta(w_L, w_{k-1}^\star)$ and $w^\Delta(w_{k-1}^\star, w_R)$ and we deduce $M_k = M(w_{k-1}^\star)$.
- If $M_k < M_0$ then $w_{\inf} = w_{k-1}^\star$, else $w_{\sup} = w_{k-1}^\star$, and we compute the new iterate value $w_k^\star = \frac{1}{2}(w_{\inf} + w_{\sup})$.
- In practice, we accept a small mass error of 10^{-6}.

Existence result

We are able to state the following result [see 16].

Theorem

Let $\varepsilon > 0$ be given and $\xi^\star > 2\Delta\xi$. Assume the existence of

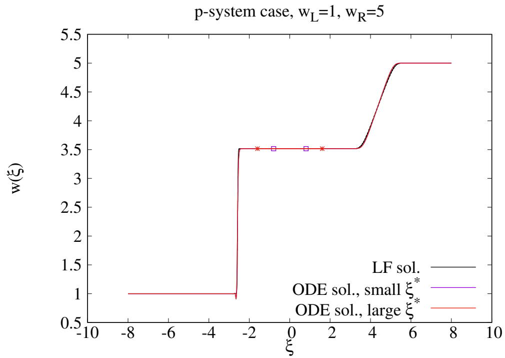
$$M_{p'} := \sup_{w\in\mathbb{R}} |p'(w)|.$$

Then there exists $\Delta\xi_0 \leq \sqrt{\varepsilon\beta}$ depending on β, ε, ℓ, $\xi^\star$ and $M_{p'}$ such that for $\Delta\xi \leq \Delta\xi_0$, there exists a solution $w^\Delta = (w_i)_{i=1,\ldots,N}$ to the scheme (10) with the boundary conditions (11).

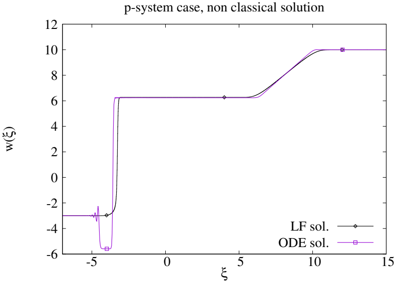
[16] Berthon, Bessemoulin-Chatard, AC, Foucher 2019 (Calcolo)

Numerical tests with $p(w) = \frac{w^3}{3} + w$

- Classical case: $w_L = 1$, $w_R = 5$, $\beta = 1$, $\gamma = 1$.
- Our approach: $\ell = 8$, $\xi^\star$: 0.8 or 1.6, $N = 1000$, $\varepsilon = 4 \times 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



- Non classical case: $w_L = -3$, $w_R = 10$, $\beta = 1$, $\gamma = 20$.
- Our approach: $\ell = 20$, $N = 1000$, $\varepsilon = 2.5 \times 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



Conclusions

- A new numerical approach to approximate (non classical) Riemann solutions of system of conservation laws.
- Solutions seen as limit solutions of diffusive-dispersive problem.
- Use self-similarity and Dafermos technique to obtain an ODE problem.
- Gives quite good numerical results on classical and non classical tests.
- Future works: more complex system problems as Shallow-Water equations.

Thank you for your attention!