Coupling of an Asymptotic-Preserving scheme with the Limit model for highly anisotropic-elliptic problems

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Physical context

- Ionosphere made of partially ionized plasma, submitted to a strong magnetic field.
- Ratio of the collision and cyclotron frequencies, parameterized by ε, considerably varying in the domain.
- Application: communication with satellites.
- Model[3]: quasi-neutral fluid description of the plasma. Study of a highly anisotropic elliptic equation for the electric potential.

[3] Besse, Claudel, Degond, Deluzet, Gallice, Tessieras, MM & MAS 2004.

Objectives

General difficulties in multiscale problems:

- develop a numerical scheme efficient in each regime:
  - spatial coupling of two schemes, with an interface,
  - asymptotic-preserving (AP) scheme[4],
- preserve a good precision/cost ratio.

Objectives of our work:

- couple the (AP)-scheme developed in previous works[5] for the 2D problem to its 1D limit model,
- preserve the accuracy in the whole domain,
- reduce computational time.

[4] Jin, SIAM JSC 1999.
[5] Degond *et al.*, SIAM MMS 2010 - CMS 2012 - Besse *et al.*, SIAM JSC 2013.

Outline

Singular perturbation problem

- Domain: $\Omega = \Omega_x \times \Omega_z = [x_-, x_+] \times [z_-, z_+]$.
- Considered (P)-problem:

$$(P) \begin{cases} -\partial_x \left(A_x \partial_x u_\varepsilon\right) - \partial_z \left(\frac{A_z}{\varepsilon(z)} \partial_z u_\varepsilon\right) = f, & \text{for } (x,z) \in \Omega_x \times \Omega_z, \\ \frac{A_z(x,z_\pm)}{\varepsilon(z_\pm)} \partial_z u_\varepsilon \left(x, z_\pm\right) = g_\pm \left(x\right), & \text{for } x \in \Omega_x, \\ u_\varepsilon \left(x_\pm, z\right) = 0, & \text{for } z \in \Omega_z, \end{cases}$$

u_ε the solution,
$A_x(x,z)$, $A_z(x,z)$ of the same order of magnitude, in order to have existence and uniqueness of a solution u_ε for $\varepsilon > 0$.

Numerical problem at the $\varepsilon \to 0$ limit

- ε-constant case, let formally ε tend to zero in (P):

$$(R)\begin{cases} -\partial_z\left(A_z\partial_z u\right) = 0, & \text{for } (x,z) \in \Omega_x \times \Omega_z, \\ \partial_z u\left(x, z_\pm\right) = 0, & \text{for } x \in \Omega_x, \\ u\left(x_\pm, z\right) = 0, & \text{for } z \in \Omega_z. \end{cases}$$

- Non-unique solution: functions constant along the z-coordinate and satisfying $u(x,z) = 0$ on $\partial\Omega_x \times \Omega_z$.
- Ill-conditioned (P)-model when $\varepsilon \ll 1$.

Remark: unique solution if Dirichlet or periodic conditions on $\partial\Omega_z$.

Limit model

- Notations:

$$\overline{f}(x) := \frac{1}{L_z}\int_{\Omega_z} f(x,z)\ \mathrm{d}z, \qquad f' = f - \overline{f}.$$

- Properties:

$$\overline{f'} = 0, \qquad \overline{\left(\frac{\partial f}{\partial x}\right)} = \frac{\partial \overline{f}}{\partial x}, \qquad \overline{fg} = \overline{f}\,\overline{g} + \overline{f'g'},$$

$$\frac{\partial f}{\partial z} = \frac{\partial f'}{\partial z}, \qquad \left(\frac{\partial f}{\partial x}\right)' = \partial\frac{f'}{\partial x}, \qquad (fg)' = f'g' - \overline{f'g'} + \overline{f}g' + f'\overline{g}.$$

- Integrating (P) along the z-coordinate and assuming $u_0 = u_0(x)$ (valid on the limit) gives the limit model:

$$(L)\begin{cases} -\partial_x\left(\overline{A_x}\partial_x u_0\right) = \overline{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z}, & \text{for } x \in \Omega_x, \\ u_0(x_\pm) = 0. \end{cases}$$

(AP)-reformulation

- Decomposition: $u(x,z) = \overline{u}(x) + u'(x,z)$.
- (AP)-reformulation[6,7]:

$$(\overline{AP}) \begin{cases} -\partial_x\left(\overline{A_x}\partial_x\overline{u}_\varepsilon\right) = \overline{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z} + \partial_x\left(\overline{A'_x\partial_x u'_\varepsilon}\right), & \text{for } x \in \Omega_x, \\ \overline{u}_\varepsilon(x_\pm) = 0. \end{cases}$$

$$(AP') \begin{cases} -\partial_x\left(A_x\partial_x u'_\varepsilon\right) - \partial_z\left(\frac{A_z}{\varepsilon(z)}\partial_z u'_\varepsilon\right) \\ \qquad = f + \partial_x\left(A_x\partial_x\overline{u}_\varepsilon\right), & \text{for } (x,z) \in \Omega_x \times \Omega_z, \\ \frac{A_z(x,z_\pm)}{\varepsilon(z_\pm)}\partial_z u'_\varepsilon(x,z_\pm) = g_\pm(x), & \text{for } x \in \Omega_x, \\ u'_\varepsilon(x_\pm, z) = 0, & \text{for } z \in \Omega_z, \\ \overline{u'}_\varepsilon = 0, & \text{for } x \in \Omega_x \text{ (constraint)}. \end{cases}$$

[6] Degond, Deluzet, Negulescu, SIAM MMS 2010.
[7] Besse, Deluzet, Negulescu, Yang, SIAM JSC 2013.

Motivations

- The (AP)-scheme is accurate whatever the value of ε, but more costly than the (P)-model or the 1D (L)-model.
- In ionospheric plasma, $\varepsilon \ll 1$ in a large part of the domain. The use of the 1D (L)-model is sufficient in this region.
- With a (P-L)-coupling, it may be difficult to find an adequate interface position, because their domains of accuracy do not always intersect[8].

$\rightarrow$ Develop an (AP-L)-coupling.

[8] Degond, Deluzet, Lozinski, Narski, Negulescu, CMS 2012.

Coupling strategy

- Assumptions:
  - in a large region of the computational domain, $\varepsilon \ll 1$,
  - the domain can be decomposed in the z-direction into two subdomains, delimited by an interface on $z_\iota \in [z_-, z_+]$:

$$\Omega_z = \Omega_z^1 \cup \Omega_z^2$$

where $\Omega_z^2 = [z_-, z_\iota]$ and $\Omega_z^1 = [z_\iota, z_+]$.

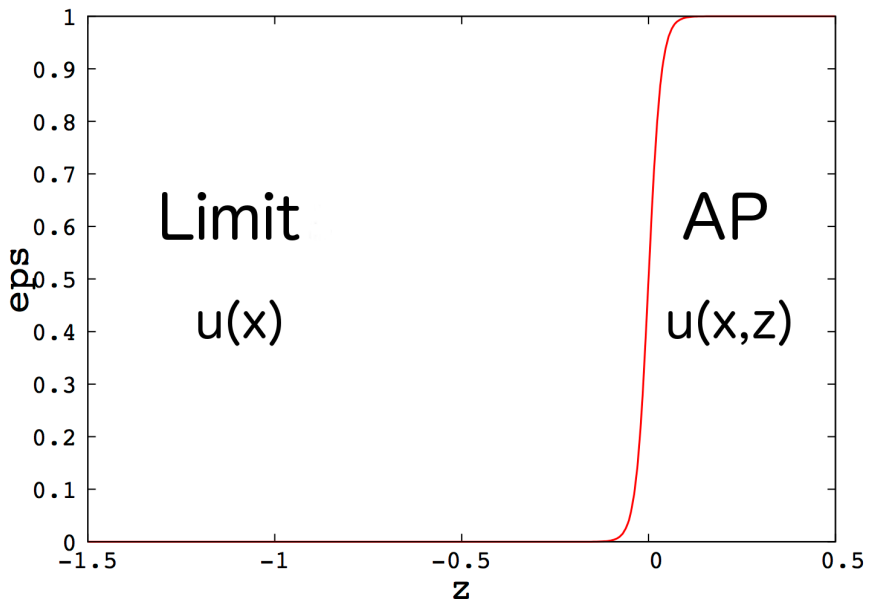
- Decomposition

$$u'_{|\Omega_z^1}(x,z) = u'_1(x,z), \quad u'_{|\Omega_z^2}(x,z) = u'_2(x,z)$$

and coupling via Dirichlet-to-Neumann boundary conditions

$$\partial_z u'_1(x,z_\iota) = \partial_z u'_2(x,z_\iota), \quad u'_2(x,z_\iota) = u'_1(x,z_\iota).$$

- Use of the limit model in Ω_z^2: u_2' does not depend on z.



- (AP-L)-formulation: $\left(\overline{AP}\right) - \left(AP_1'\right) - \left(L\right)$ with

$$
\left(\overline{AP}\right)\begin{cases}
-\partial_x\left(\overline{A_x}\partial_x\overline{u}\right) = \overline{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z} \\
\quad + \frac{1}{L_z}\partial_x\left(\int_{\Omega_z^1} A_x'\partial_x u_1'\mathrm{d}z + \int_{\Omega_z^2} A_x'\partial_x u_2'\mathrm{d}z\right), & \text{for } x \in \Omega_x, \\
\overline{u}\left(x_\pm\right) = 0,
\end{cases}
$$

$$
\left(AP_1'\right)\begin{cases}
-\partial_x\left(A_x\partial_x u_1'\right) - \partial_z\left(\frac{A_z}{\varepsilon(z)}\partial_z u_1'\right) \\
\quad = f + \partial_x\left(A_x\partial_x\overline{u}\right), & \text{for } (x,z) \in \Omega_1, \\
\frac{A_z(x,z_+)}{\varepsilon(z_+)}\partial_z u_1'\left(x,z_+\right) = g_+\left(x\right), & \text{for } x \in \Omega_x, \\
u_1'\left(x_\pm, z\right) = 0, & \text{for } z \in \Omega_z^1, \\
\partial_z u_1'\left(x, z_\iota\right) = 0, & \text{for } x \in \Omega_x, \\
\int_{\Omega_z^1} u_1'\left(x,z\right)\mathrm{d}z + L_z^2 u_2'\left(x\right) = 0, & \text{for } x \in \Omega_x \text{ (constraint)},
\end{cases}
$$

$$
\left(L\right)\quad\left\{\; u_2'\left(x\right) = u_1'\left(x, z_\iota\right), \qquad \text{for } x \in \Omega_x.\right.
$$

Mathematical study

- Regularity hypothesis
  - $A_x,\ A_z \in W^{1,\infty}(\Omega)$, satisfying
    $0 < m_x \leq A_x(x,z) \leq M_x, \quad 0 < m_z \leq A_z(x,z) \leq M_z,$
  - $f \in L^2(\Omega)$, $g_\pm \in L^2(\Omega_x)$,
  - $\varepsilon \in L^\infty(\Omega_z)$, satisfying $0 < \varepsilon_{\min} \leq \varepsilon(z) \leq \varepsilon_{\max}$,

  with $m_x,\ m_z,\ M_x,\ M_z,\ \varepsilon_{\min},\ \varepsilon_{\max}$ some given positive constants.

- Introdution of the Dirichlet-to-Neumann boundary conditions
  The obtained $(\overline{AP})$-(AP_1')-(AP_2')-model is equivalent to the $(\overline{AP})$-(AP')-one. It admits a unique solution.

- Error coming from the use of the limit model (assumption $u_2' = u_2'(x)$)
  This error tends to zero in $H^1(\Omega)$ as $\sqrt{\varepsilon(z_\iota)}$ tends to zero.

Finite element discretization

- Discretization of $\Omega_x \times \Omega_z$ by

$$\begin{aligned} x_i &= i\Delta x, \qquad & i = 0, \dots, N_x + 1, \\ z_k &= k\Delta z, \qquad & k = 0, \dots, N_z + 1. \end{aligned}$$

- Interface put on z_ι (we assume $\exists k \in \{1, \dots, N_z\}$ s.t. $z_\iota = z_k$).
- $\mathbb{P}_1$ hat functions $\chi_i(x)$ and $\kappa_k(z)$, with

$$\chi_i(x) = \begin{cases} \frac{x - x_{i-1}}{\Delta x}, & x \in [x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{\Delta x}, & x \in [x_i, x_{i+1}) \\ 0, & \text{elsewhere} \end{cases}, \quad i = 1, \dots, N_x,$$

$$\chi_0(x) = \begin{cases} \frac{x_1 - x}{\Delta x}, & x \in [x_0, x_1) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \chi_{N_x+1}(x) = \begin{cases} \frac{x - x_{N_x}}{\Delta x}, & x \in [x_{N_x}, x_{N_x+1}) \\ 0, & \text{elsewhere} \end{cases},$$

and $\kappa_k(z)$ of the same form.

- Test functions

$$\begin{aligned} &\chi_i, && i = 1, \ldots, N_x, \\ &\chi_i \kappa_k, && i = 1, \ldots, N_x, \\ & && k = \iota, \ldots, N_z + 1. \end{aligned}$$

- Unknowns approximated by

$$\begin{aligned} \overline{u}_h(x) &= \textstyle\sum_{i=1}^{N_x} \alpha_i \chi_i(x), \\ u'_{1h}(x,z) &= \textstyle\sum_{i=1}^{N_x} \sum_{k=\iota}^{N_z+1} \beta_{ik} \chi_i(x) \kappa_k(z), \\ \overline{P}_h(x) &= \textstyle\sum_{i=0}^{N_x+1} \gamma_i \chi_i(x). \end{aligned}$$

- Weak formulation approximated thanks to a three-point Gauss quadrature formula.

- System

$$\begin{pmatrix} A_{xa} & \frac{1}{L_z}(C_{a1} + C_{a2}) & 0 \\ C_{f1} & A_{xf1} + A_{z1} & B_{l1} \\ 0 & B_{c1} + B_{c2} & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} F_{\overline{u}} \\ F_{u'_1} \\ 0 \end{pmatrix}.$$

ε-variable test case

- Domain $\Omega_x \times \Omega_z = [0,1] \times \left[-\frac{3}{2}, \frac{1}{2}\right]$.
- Test case

$$\varepsilon(z) = \frac{1}{2}\left(\varepsilon_{\max}\left(1+\tanh(rz)\right) + \varepsilon_{\min}\left(1-\tanh(rz)\right)\right),$$

with $r,\ \varepsilon_{\min},\ \varepsilon_{\max} \in \mathbb{R}^{+\star}$,

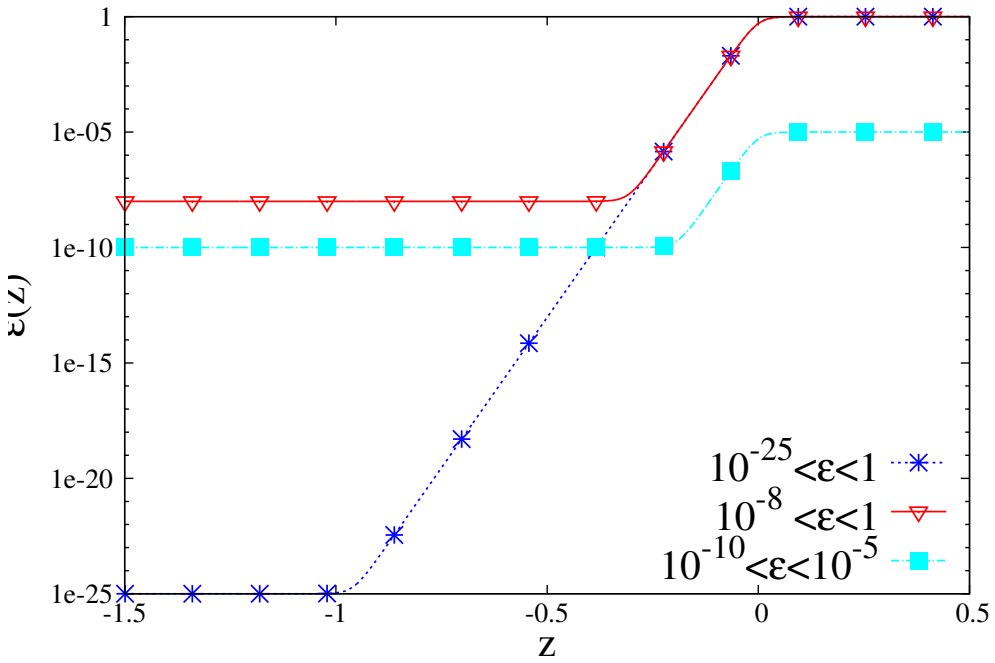
$$A_x(x,z) = L_z + xz^2, \quad A_z(x,z) = L_z + xz.$$

- Exact solution

$$u_{ex}(x,z) = \sin\left(\frac{2\pi}{L_x}x\right)\left(1+\varepsilon(z)\sin\left(\frac{2\pi}{L_z}z\right)\right).$$

f, g_+ and g_- are computed by injecting the exact solution into the equations.

Anisotropy ratio ε as a function of z, for different values of $\varepsilon_{\min}$ and $\varepsilon_{\max}$ with $r = 30$.

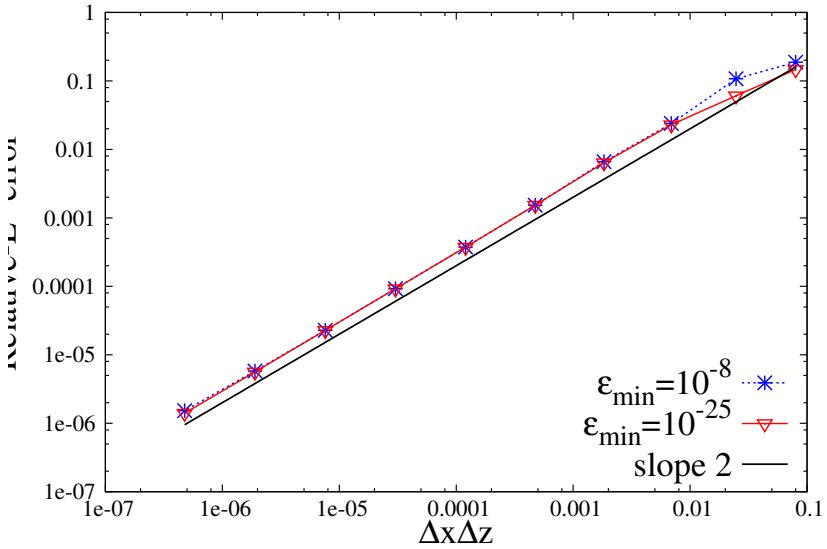


About the choice of the interface

- As soon as ε is small in one part of the domain, it is always possible to put the interface, anywhere in the accuracy domain of the (L)-model.
- Here, we fixed the interface somewhere in the domain of validity of the (L)-model.
- To improve the performance of the coupling, it would be interesting to compute automatically the position z_ι such that the error introduced by using the (L)-model in Ω_2 is of the same order as the discretization error.

Scheme order

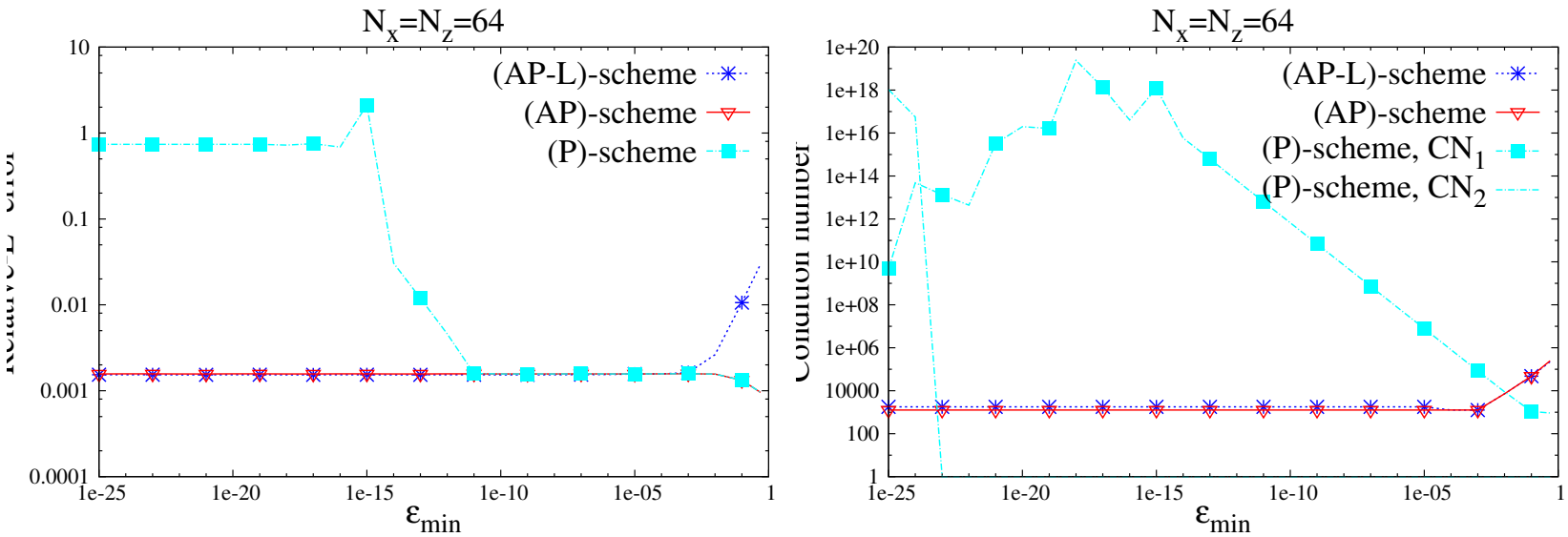
L^2 relative error between the exact solution and its numerical approximation as a function of the squared mesh size $\Delta x \Delta z$.

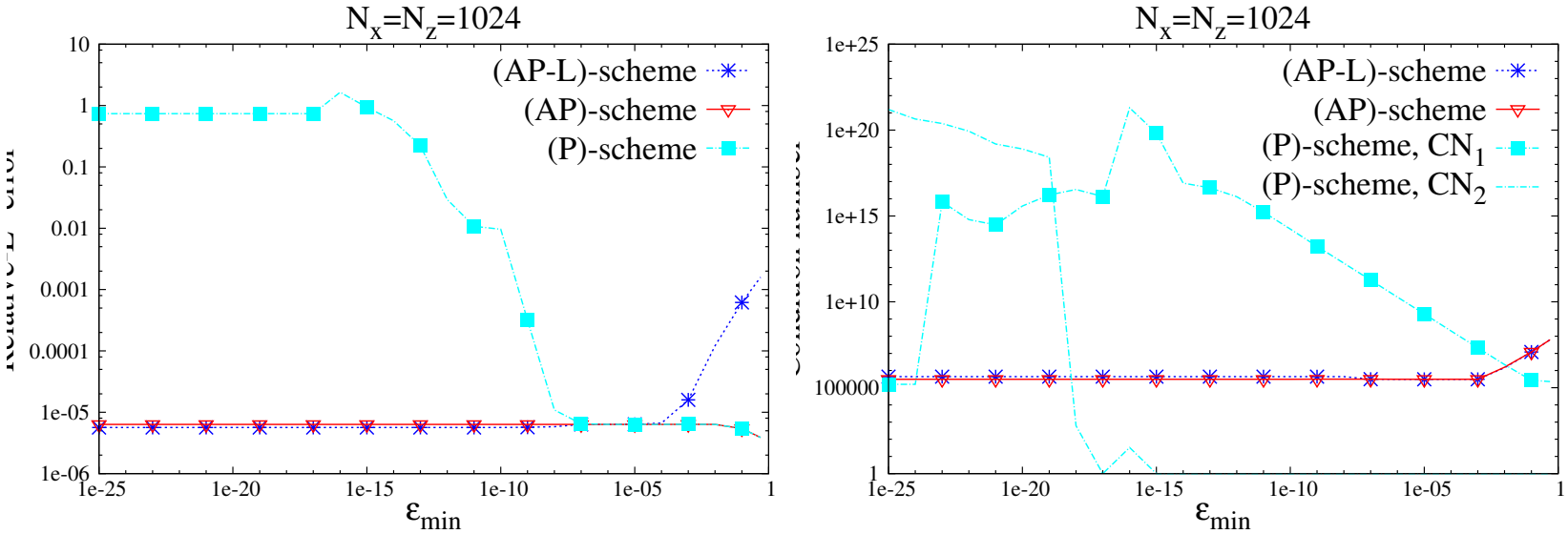


→ 2nd-order scheme.

AP property

L^2 relative error between the exact solution and its numerical approximations and condition numbers of the linear systems (estimated by MUMPS) as a functions of ε_{min} (with $\varepsilon_{\text{max}} = 1$).





→ Condition number as well as precision of the (AP-L)-coupling independent of the anisotropy strength.

→ AP scheme.

Computational time

With $|\Omega_z^1| = \frac{2}{5}\,|\Omega_z|$ and $|\Omega_z^2| = \frac{3}{5}\,|\Omega_z|$,
$r = 30$, $\varepsilon_{\max} = 1$ and $\varepsilon_{\min} = 10^{-8}$.

Scheme	$N_x = N_z$	Time	#rows	#non zeros	L^2-error
AP-L	250	72%	26 000	533 324	1.06×10^{-4}
AP	250	187%	63 500	1 318 724	1.06×10^{-4}
P	250	100%	63 000	563 992	1.06×10^{-4}
AP-L	2000	43%	1 608 000	33 666 774	1.54×10^{-6}
AP	2000	146%	4 008 000	84 049 974	1.66×10^{-6}
P	2000	100%	4 004 000	36 011 992	8.88×10^{-5}

$\rightarrow$ Time reduced compared to the (AP)-scheme.

Conclusions. . .

→ (AP) and (L)-models naturally coupled via Dirichlet-to-Neumann boundary conditions.

→ Always possible to find an adequate interface position.

→ Same precision as the (AP)-model.

→ Computational time reduced compared to the (AP)-formulation!

. . . and perspectives

- Computation of a Scharfetter-Gummel[9] scheme to improve the accuracy in case of larger gradients.
- Extension of this strategy in 3D, coupling a 3D (AP)-model to the 2D (L)-model.
- For the 3D framework, replace the direct solver by an iterative one.

[9] Saito, PJA 2006.

- **C. Besse, J. Claudel, P. Degond, F. Deluzet, G. Gallice, C. Tessieras**: *A model hierarchy for ionospheric plasma modeling*, Mathematical Models & Methods in Applied Sciences **14** (2004).
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Thank you for your attention!