Coupling of an Asymptotic-Preserving scheme with the Limit model for highly anisotropic-elliptic problems

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Physical context

- Ionosphere made of partially ionized plasma, submitted to a strong magnetic field.

- Ratio of the collision and cyclotron frequencies, parameterized by $\varepsilon$, considerably varying in the domain.

- **Application**: communication with satellites.

- **Model**: quasi-neutral fluid description of the plasma. Study of a highly anisotropic elliptic equation for the electric potential.

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General difficulties in multiscale problems:

- develop a numerical scheme efficient in each regime:
  - spatial coupling of two schemes, with an interface,
  - asymptotic-preserving (AP) scheme\footnote{Jin, SIAM JSC 1999.},
- preserve a good precision/cost ratio.

Objectives of our work:

- couple the (AP)-scheme developed in previous works\footnote{Degond et al., SIAM MMS 2010 - CMS 2012 - Besse et al., SIAM JSC 2013.} for the 2D problem to its 1D limit model,
- preserve the accuracy in the whole domain,
- reduce computational time.
Outline

1. The elliptic problem and its AP reformulation
2. The (AP-L)-coupling
3. The numerical discretization and some results
Singular perturbation problem

- Domain: \( \Omega = \Omega_x \times \Omega_z = [x_-, x_+] \times [z_-, z_+] \).
- Considered (P)-problem:

\[
\begin{cases}
- \partial_x (A_x \partial_x u_\varepsilon) - \partial_z \left( \frac{A_z}{\varepsilon(z)} \partial_z u_\varepsilon \right) = f, & \text{for } (x, z) \in \Omega_x \times \Omega_z, \\
\frac{A_z(x, z_\pm)}{\varepsilon(z_\pm)} \partial_z u_\varepsilon (x, z_\pm) = g_\pm(x), & \text{for } x \in \Omega_x, \\
u_\varepsilon(x_\pm, z) = 0, & \text{for } z \in \Omega_z,
\end{cases}
\]

\( u_\varepsilon \) the solution, 
\( A_x(x, z), A_z(x, z) \) of the same order of magnitude, in order to have existence and uniqueness of a solution \( u_\varepsilon \) for \( \varepsilon > 0 \).
Numerical problem at the $\varepsilon \rightarrow 0$ limit

- $\varepsilon$-constant case, let formally $\varepsilon$ tend to zero in (P):

$$
(R) \begin{cases} 
- \partial_z (A_z \partial_z u) = 0, & \text{for } (x, z) \in \Omega_x \times \Omega_z, \\
\partial_z u(x, z_\pm) = 0, & \text{for } x \in \Omega_x, \\
u(x_\pm, z) = 0, & \text{for } z \in \Omega_z.
\end{cases}
$$

- Non-unique solution: functions constant along the $z$-coordinate and satisfying $u(x, z) = 0$ on $\partial \Omega_x \times \Omega_z$.

- Ill-conditioned (P)-model when $\varepsilon \ll 1$.

Remark: unique solution if Dirichlet or periodic conditions on $\partial \Omega_z$. 
Limit model

- Notations:

\[ \bar{f}(x) := \frac{1}{L_z} \int_{\Omega_z} f(x, z) \, dz, \quad f' = f - \bar{f}. \]

- Properties:

\[ f' = 0, \quad \left( \frac{\partial f}{\partial x} \right) = \frac{\partial \bar{f}}{\partial x}, \quad \bar{fg} = \bar{f}\bar{g} + \bar{f}'\bar{g}', \]
\[ \frac{\partial f}{\partial z} = \frac{\partial f'}{\partial z}, \quad \left( \frac{\partial f}{\partial x} \right)' = \partial \frac{f'}{\partial x}, \quad (fg)' = f'g' - \bar{f}'\bar{g}' + \bar{f}g' + f'\bar{g}. \]

- Integrating (P) along the z-coordinate and assuming \( u_0 = u_0(x) \) (valid on the limit) gives the limit model:

\[
\begin{cases}
-\partial_x \left( A_x \partial_x u_0 \right) = \bar{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z}, & \text{for } x \in \Omega_x, \\
u_0(x_\pm) = 0.
\end{cases}
\]
(AP)-reformulation

- Decomposition: \( u(x, z) = \overline{u}(x) + u'(x, z) \).
- (AP)-reformulation\(^6,7\):

\[
\begin{aligned}
\text{(AP)}: & \quad -\partial_x \left( A_x \partial_x \overline{u}_\varepsilon \right) = \overline{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z} + \partial_x \left( \overline{A}'_x \partial_x u'_\varepsilon \right), \quad \text{for } x \in \Omega_x, \\
& \quad \overline{u}_\varepsilon (x_\pm) = 0.
\end{aligned}
\]

\[
\begin{aligned}
\text{(AP')}: & \quad -\partial_x \left( A_x \partial_x u'_\varepsilon \right) - \partial_z \left( \frac{A_z}{\varepsilon(z)} \partial_z u'_\varepsilon \right) \\
& \quad \quad = f + \partial_x \left( A_x \partial_x \overline{u}_\varepsilon \right), \quad \text{for } (x, z) \in \Omega_x \times \Omega_z, \\
& \quad \frac{A_z(x, z_\pm)}{\varepsilon(z_\pm)} \partial_z u'_\varepsilon (x, z_\pm) = g_\pm (x), \quad \text{for } x \in \Omega_x, \\
& \quad u'_\varepsilon (x_\pm, z) = 0, \quad \text{for } z \in \Omega_z, \\
& \quad \overline{u}'_\varepsilon = 0, \quad \text{for } x \in \Omega_x \text{ (constraint)}.
\end{aligned}
\]

\(^6\)Degond, Deluzet, Negulescu, SIAM MMS 2010.
\(^7\)Besse, Deluzet, Negulescu, Yang, SIAM JSC 2013.
The (AP)-scheme is accurate whatever the value of $\varepsilon$, but more costly than the (P)-model or the 1D (L)-model.

In ionospheric plasma, $\varepsilon \ll 1$ in a large part of the domain. The use of the 1D (L)-model is sufficient in this region.

With a (P-L)-coupling, it may be difficult to find an adequate interface position, because their domains of accuracy do not always intersect$^8$.

→ Develop an (AP-L)-coupling.

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$^8$Degond, Deluzet, Lozinski, Narski, Negulescu, CMS 2012.
Coupling strategy

- **Assumptions:**
  - in a large region of the computational domain, $\varepsilon \ll 1$,
  - the domain can be decomposed in the $z$-direction into two subdomains, delimited by an interface on $z_ι ∈ [z_-, z_+]$:

  $$\Omega_z = \Omega^1_z \cup \Omega^2_z$$

  where $\Omega^2_z = [z_-, z_ι]$ and $\Omega^1_z = [z_ι, z_+]$.

- **Decomposition**

  $$u'_{|\Omega^1_z} (x, z) = u'_1 (x, z), \quad u'_{|\Omega^2_z} (x, z) = u'_2 (x, z)$$

  and coupling via Dirichlet-to-Neumann boundary conditions

  $$\partial_z u'_1 (x, z_ι) = \partial_z u'_2 (x, z_ι), \quad u'_2 (x, z_ι) = u'_1 (x, z_ι).$$
Use of the limit model in $\Omega_z^2$: $u_2'$ does not depend on $z$. 
(AP-L)-formulation: \((\overline{AP}) - (AP_1') - (L)\) with

\[
\begin{align*}
\overline{AP} & \quad \left\{ \begin{array}{l}
- \partial_x \left( \overline{A}_x \partial_x \overline{u} \right) = \overline{f} + \frac{g_+}{L_z} - \frac{g_-}{L_z} \\
& + \frac{1}{L_z} \partial_x \left( \int_{\Omega_z^1} A_x' \partial_x u_1' \, dz + \int_{\Omega_z^2} A_x' \partial_x u_2' \, dz \right), \quad \text{for } x \in \Omega_x,
\end{array} \right.
\end{align*}
\]

\[
\overline{u} (x_{\pm}) = 0,
\]

\[
\begin{align*}
AP_1' & \quad \left\{ \begin{array}{l}
- \partial_x (A_x \partial_x u_1') - \partial_z \left( \frac{A_z}{\varepsilon (z)} \partial_z u_1' \right) \\
& = f + \partial_x (A_x \partial_x \overline{u}), \\
A_z(x,z_+) \varepsilon (z_+) \partial_z u_1' (x, z_+) = g_+ (x), \quad \text{for } (x, z) \in \Omega_1, \\
u_1' (x_{\pm}, z) = 0, \quad \text{for } x \in \Omega_x, \\
\partial_z u_1' (x, z_{\ell}) = 0, \quad \text{for } x \in \Omega_x, \\
\int_{\Omega_z^1} u_1' (x, z) \, dz + L_z^2 u_2' (x) = 0, \quad \text{for } x \in \Omega_x \text{ (constraint),}
\end{array} \right.
\end{align*}
\]

\[
L \quad \left\{ \begin{array}{l}
u_2' (x) = u_1' (x, z_{\ell}), \quad \text{for } x \in \Omega_x.
\end{array} \right.
\]

\[\text{(AP-L)-coupling for anisotropic-elliptic problems}\]
Mathematical study

- Regularity hypothesis
  - \( A_x, \ A_z \in W^{1,\infty} (\Omega) \), satisfying
    \[
    0 < m_x \leq A_x(x, z) \leq M_x, \quad 0 < m_z \leq A_z(x, z) \leq M_z,
    \]
  - \( f \in L^2 (\Omega), \ g_{\pm} \in L^2 (\Omega_x) \),
  - \( \varepsilon \in L^\infty (\Omega_z) \), satisfying
    \[
    0 < \varepsilon_{\text{min}} \leq \varepsilon(z) \leq \varepsilon_{\text{max}},
    \]
  with \( m_x, \ m_z, \ M_x, \ M_z, \ \varepsilon_{\text{min}}, \ \varepsilon_{\text{max}} \) some given positive constants.

- Introduction of the Dirichlet-to-Neumann boundary conditions
  The obtained (\( \overline{AP}_1 \))-\( (AP'_1) \)-\( (AP'_2) \)-model is equivalent to the
  (\( \overline{AP} \))-\( (AP') \)-one. It admits a unique solution.

- Error coming from the use of the limit model (assumption
  \( u'_2 = u'_2 (x) \))
  This error tends to zero in \( H^1 (\Omega) \) as \( \sqrt{\varepsilon (z_t)} \) tends to zero.
Finite element discretization

Discretization of $\Omega_x \times \Omega_z$ by

\[ x_i = i\Delta x, \quad i = 0, \ldots, N_x + 1, \]
\[ z_k = k\Delta z, \quad k = 0, \ldots, N_z + 1. \]

Interface put on $z_l$ (we assume $\exists k \in \{1, \ldots, N_z\}$ s.t. $z_l = z_k$).

$\mathbb{P}_1$ hat functions $\chi_i(x)$ and $\kappa_k(z)$, with

\[
\chi_i(x) = \begin{cases} 
\frac{x-x_{i-1}}{\Delta x}, & x \in [x_{i-1}, x_i) \\
\frac{x_{i+1}-x}{\Delta x}, & x \in [x_i, x_{i+1}) \\
0, & \text{elsewhere}
\end{cases}, \quad i = 1, \ldots, N_x,
\]

\[
\chi_0(x) = \begin{cases} 
\frac{x_1-x}{\Delta x}, & x \in [x_0, x_1) \\
0, & \text{elsewhere}
\end{cases}
\] and

\[
\chi_{N_x+1}(x) = \begin{cases} 
\frac{x-x_{N_x}}{\Delta x}, & x \in [x_{N_x}, x_{N_x+1}) \\
0, & \text{elsewhere}
\end{cases}
\]

and $\kappa_k(z)$ of the same form.
Test functions

\[ \chi_i, \quad i = 1, \ldots, N_x, \]
\[ \chi_i \kappa_k, \quad i = 1, \ldots, N_x, \]
\[ k = \nu, \ldots, N_z + 1. \]

Unknowns approximated by

\[ \bar{u}_h (x) = \sum_{i=1}^{N_x} \alpha_i \chi_i (x), \]
\[ u'_{1h} (x, z) = \sum_{i=1}^{N_x} \sum_{k=\nu}^{N_z+1} \beta_{ik} \chi_i (x) \kappa_k (z), \]
\[ \bar{P}_h (x) = \sum_{i=0}^{N_x+1} \gamma_i \chi_i (x). \]

Weak formulation approximated thanks to a three-point Gauss quadrature formula.

System

\[
\begin{pmatrix}
A_{xa} & \frac{1}{L_z} \left( C_{a1} + C_{a2} \right) & 0 \\
C_{r1} & A_{xf1} + A_{z1} & B_{l1} \\
0 & B_{c1} + B_{c2} & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
= 
\begin{pmatrix}
F_{\bar{u}} \\
F_{u'_{11}} \\
0
\end{pmatrix}.
\]
\( \varepsilon \)-variable test case

- Domain \( \Omega_x \times \Omega_z = [0, 1] \times [-\frac{3}{2}, \frac{1}{2}] \).

- Test case

\[
\varepsilon (z) = \frac{1}{2} (\varepsilon_{\text{max}} (1 + \tanh (rz)) + \varepsilon_{\text{min}} (1 - \tanh (rz))),
\]

with \( r, \varepsilon_{\text{min}}, \varepsilon_{\text{max}} \in \mathbb{R}^{+*} \),

\[
A_x (x, z) = L_z + xz^2, \quad A_z (x, z) = L_z + xz.
\]

- Exact solution

\[
u_{\text{ex}} (x, z) = \sin \left( \frac{2\pi}{L_x} x \right) \left( 1 + \varepsilon (z) \sin \left( \frac{2\pi}{L_z} z \right) \right).
\]

\( f, g^+ \) and \( g^- \) are computed by injecting the exact solution into the equations.
Anisotropy ratio $\varepsilon$ as a function of $z$, for different values of $\varepsilon_{\text{min}}$ and $\varepsilon_{\text{max}}$ with $r = 30$. 

![Anisotropy ratio graph](image)
About the choice of the interface

- As soon as $\varepsilon$ is small in one part of the domain, it is *always possible to put the interface*, anywhere in the accuracy domain of the (L)-model.

- Here, we fixed the interface somewhere in the domain of validity of the (L)-model.

- To improve the performance of the coupling, it would be interesting to compute automatically the position $z_\iota$ such that the error introduced by using the (L)-model in $\Omega_2$ is of the same order as the discretization error.
Scheme order

Let's consider the $L^2$ relative error between the exact solution and its numerical approximation as a function of the squared mesh size $\Delta x\Delta z$.

![Graph showing relative error vs. mesh size with a slope of 2 indicating a 2nd-order scheme.]

$\rightarrow$ 2nd-order scheme.
L² relative error between the exact solution and its numerical approximations and condition numbers of the linear systems (estimated by MUMPS) as a functions of $\varepsilon_{\text{min}}$ (with $\varepsilon_{\text{max}} = 1$).
Condition number as well as precision of the (AP-L)-coupling independent of the anisotropy strength.

AP scheme.
Computational time

With $|\Omega_1^z| = \frac{2}{5} |\Omega_z|$ and $|\Omega_2^z| = \frac{3}{5} |\Omega_z|$, $r = 30$, $\varepsilon_{\text{max}} = 1$ and $\varepsilon_{\text{min}} = 10^{-8}$.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$N_x = N_z$</th>
<th>Time</th>
<th>#rows</th>
<th>#non zeros</th>
<th>$L^2$-error</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP-L</td>
<td>250</td>
<td>72%</td>
<td>26 000</td>
<td>533 324</td>
<td>$1.06 \times 10^{-4}$</td>
</tr>
<tr>
<td>AP</td>
<td>250</td>
<td>187%</td>
<td>63 500</td>
<td>1 318 724</td>
<td>$1.06 \times 10^{-4}$</td>
</tr>
<tr>
<td>P</td>
<td>250</td>
<td>100%</td>
<td>63 000</td>
<td>563 992</td>
<td>$1.06 \times 10^{-4}$</td>
</tr>
<tr>
<td>AP-L</td>
<td>2000</td>
<td>43%</td>
<td>1 608 000</td>
<td>33 666 774</td>
<td>$1.54 \times 10^{-6}$</td>
</tr>
<tr>
<td>AP</td>
<td>2000</td>
<td>146%</td>
<td>4 008 000</td>
<td>84 049 974</td>
<td>$1.66 \times 10^{-6}$</td>
</tr>
<tr>
<td>P</td>
<td>2000</td>
<td>100%</td>
<td>4 004 000</td>
<td>36 011 992</td>
<td>$8.88 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

→ Time reduced compared to the (AP)-scheme.
Conclusions...

→ (AP) and (L)-models naturally coupled via Dirichlet-to-Neumann boundary conditions.
→ Always possible to find an adequate interface position.
→ Same precision as the (AP)-model.
→ Computational time reduced compared to the (AP)-formulation!

...and perspectives

- Computation of a Scharfetter-Gummel scheme to improve the accuracy in case of larger gradients.
- Extension of this strategy in 3D, coupling a 3D (AP)-model to the 2D (L)-model.
- For the 3D framework, replace the direct solver by an iterative one.

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9 Saito, PJA 2006.
The elliptic problem and its AP reformulation
The (AP-L)-coupling
The numerical discretization and some results


Thank you for your attention!