Multi-water-bag model and method of moments for the Vlasov equation

Anaïs Crestetto[1], Philippe Helluy[2], Nicolas Besse[3]

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[1]INRIA Nancy - Grand Est & University of Strasbourg - IRMA
crestetto@math.unistra.fr (PhD thesis supported by DGA)
[2]University of Strasbourg - IRMA
[3]University Henri Poincaré Nancy I - IECN & IJL

Summary:

- Vlasov-Poisson 1D system:

$$\begin{cases} \partial_t f(x,v,t) + v\partial_x f(x,v,t) + \frac{q}{m} E(x,t)\, \partial_v f(x,v,t) = 0 \\ \\ \partial_x E(x,t) = \frac{q}{m}\left(\int f(x,v,t)\ dv - n_0(x)\right), \end{cases}$$

where n_0 is the constant density of ions.

- Difficulties:
  - in 3D $\Longrightarrow$ 7 variables: 3 positions, 3 velocities and time $\Longrightarrow$ very heavy computations,
  - no dissipation $\Longrightarrow$ the solution may become turbulent (filaments, vortices, etc.).

- Objectives:
  - reduce the dimension of the problem,
  - keep enough information about the solution.

Multi-water-bag model

We consider f piecewise constant in v:

$$f(x,v,t) = \sum_{j=1}^{k} A_j, \text{for } v \in \left[v_k^-(x,t), v_k^+(x,t)\right]$$

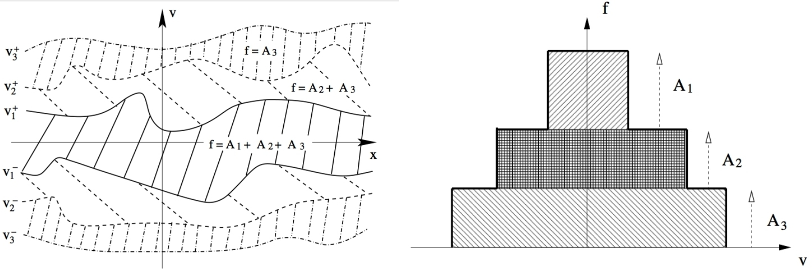
$$\Longleftrightarrow$$

$$f(x,v,t) = \sum_{j=1}^{N} A_j \left(H\left(v_j^+(x,t) - v\right) - H\left(v_j^-(x,t) - v\right)\right),$$

where H is the Heaviside function:

$$H(u) = \begin{cases} 0 \text{ if } u < 0, \\ 1 \text{ if } u > 0, \end{cases}$$

N is an integer, $v_j^{\pm}(x,t)$ are velocities and A_j are constants, for $j = 1, \ldots, N$.



3 pairs of bags model in phase space (left) and corresponding distribution function (right).

Idea of the model: evolve the $v_j^{\pm}(x,t)$ instead of f.
Resolution of transport equations for the $v_j^{\pm}$ by a finite volume method (Godunov for example):

$$\partial_t v_j^{\pm} + v_j^{\pm} \partial_x v_j^{\pm} - \frac{q}{m} E = 0, \quad \forall j = 1, \ldots, N.$$

Method of moments

- Definition: moment of order k, in v, of a function f integrable in v:

$$M_k(x,t) = \int_{-\infty}^{+\infty} v^k f(x,v,t)\ dv.$$

- Moments system (by taking the $2N$ first moments of the Vlasov equation):

$$\begin{cases} \partial_t M_0 + \partial_x M_1 = 0 \\ \partial_t M_k + \partial_x M_{k+1} - k\dfrac{q}{m} E M_{k-1} = 0, \text{ for } k = 1, \dots, 2N-1 \\ \partial_t E(x,t) = \dfrac{q}{m}\left(M_1(0,t) - M_1(x,t)\right). \end{cases}$$

- Remarks:
  - v no longer involved $\Longrightarrow$ one dimension less,
  - N can be chosen according to the expected precision and the computation capabilities,
  - unknown: $M_0, \ldots, M_{2N-1}$, and $M_{2N} \Longrightarrow$ need of a closure relation.

- Closure relation: we have to choose a representation of f, for example by the multi-water-bag model. Moments are then written:

$$M_k(x,t) = \sum_{j=1}^{N} A_j \frac{{v_j^+}^{k+1}(x,t) - {v_j^-}^{k+1}(x,t)}{k+1}, \ \forall\ k = 0, \ldots, 2N$$

$\Longrightarrow$ expression of M_{2N} as a function of $v_j^{\pm}$.

Algorithm

- Moments, E and $v_j^{\pm}(x,t)$ known at time n.

- Computation of moments and E at time $n+1$ with a kinetic scheme (natural expression of fluxes):
  - upwind discretization for the Vlasov equation:

$$\frac{f_i^{n+1}-f_i^n}{\Delta t}+\frac{1}{\Delta x}\left(v^+\left(f_i^n-f_{i-1}^n\right)+v^-\left(f_{i+1}^n-f_i^n\right)\right)-E_i^n\partial_v f_i^n=0,$$

  - multiplication by v^k and integration in v:

$$\begin{aligned}&\frac{1}{\Delta t}\int v^k\left(f_i^{n+1}-f_i^n\right)\,dv\\&+\frac{1}{\Delta x}\int\left({v^+}^{k+1}\left(f_i^n-f_{i-1}^n\right)+{v^-}^{k+1}\left(f_{i+1}^n-f_i^n\right)\right)\,dv\\&-E_i^n\int v^k\partial_v f_i^n\,dv=0,\end{aligned}$$

- finite volume scheme:

$$\frac{M_{k,i}^{n+1} - M_{k,i}^{n}}{\Delta t} + \frac{1}{\Delta x}\left(\mathcal{F}_{k,i}^{n} - \mathcal{F}_{k,i-1}^{n}\right) - kE_i^n M_{k-1,i}^{n} = 0,$$

- expression of the flux:

$$\begin{aligned}\mathcal{F}_{k,i}^{n} = \mathcal{F}\left(M_{k,i}^{n}, M_{k,i+1}^{n}\right) &= \sum_{j=1}^{N} A_j \left({v_j^+}^+ \frac{{v_j^+}^{k+1}}{k+2} - {v_j^-}^+ \frac{{v_j^-}^{k+1}}{k+2} \right)_i^n \\ &+ \sum_{j=1}^{N} A_j \left({v_j^+}^- \frac{{v_j^+}^{k+1}}{k+2} - {v_j^-}^- \frac{{v_j^-}^{k+1}}{k+2} \right)_{i+1}^n,\end{aligned}$$

where $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$,

- scheme for the electric field:

$$\frac{E_i^{n+1} - E_i^n}{\Delta t} = M_{1,i}^n - M_{1,0}^n.$$

- Computation of $v_j^{\pm}(x,t)$ at time $n+1$ by:
  - the Newton-Raphson method, iterative and hard to initialize,
  - the algorithm of Gosse and Runborg when $A_j = \pm 1$, unstable when two bags are close to each other.

Validation

- Validation of the method on Landau damping and two-stream instability, by comparing it to the Particle In Cell (PIC) method.

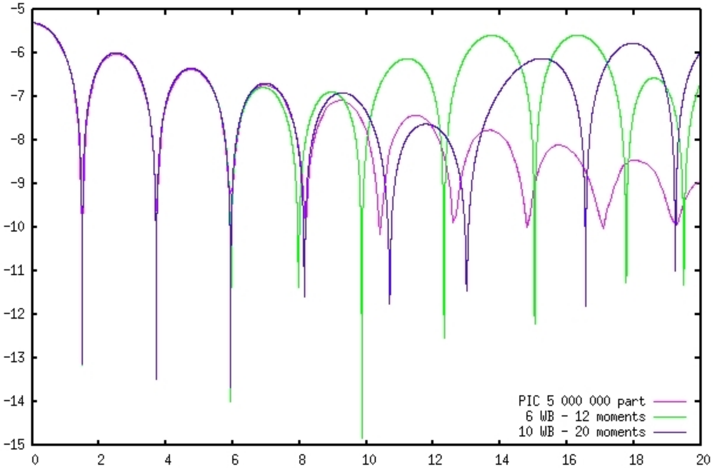
- Initial condition of the Landau damping:

$$\begin{cases} f(x,v,t=0) = (1+\epsilon\cos(kx))\dfrac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}} \\ E(x,t=0) = \dfrac{q}{m}\dfrac{\epsilon}{k}\sin(kx). \end{cases}$$

- Initial condition of the two-stream instability:

$$\begin{cases} f(x,v,t=0) = (1+\epsilon\cos(kx))\dfrac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(v-v_0)^2}{2}} + e^{-\frac{(v+v_0)^2}{2}}\right) \\ E(x,t=0) = \dfrac{q}{m}\dfrac{\epsilon}{k}\sin(kx). \end{cases}$$

Landau damping



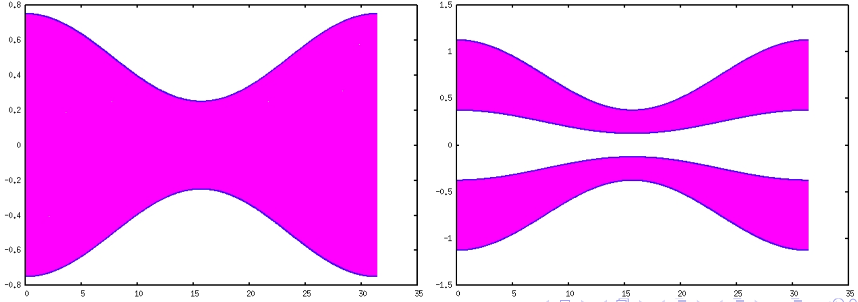
Landau damping: k=0.5, $\epsilon = 0.001$. Electrical energy (logarithmic scale) as a function of time.

Studied test cases

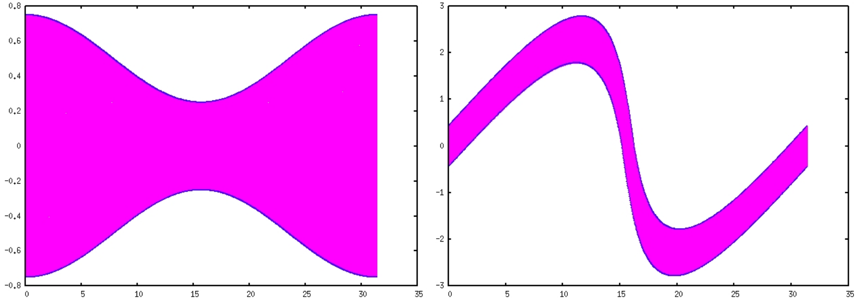
Initial conditions such that:

- $f(x, v, t = 0)$ is exactly described by one or two pairs of bags,
- the solution destabilizes: creation of filaments, vortices, etc.

Examples at initial time:

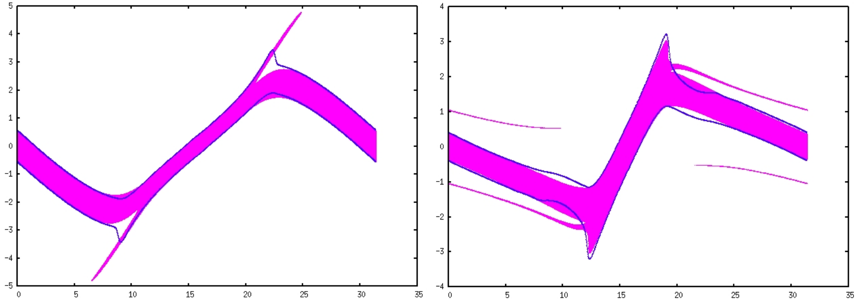


First test case: 1 pair of bags



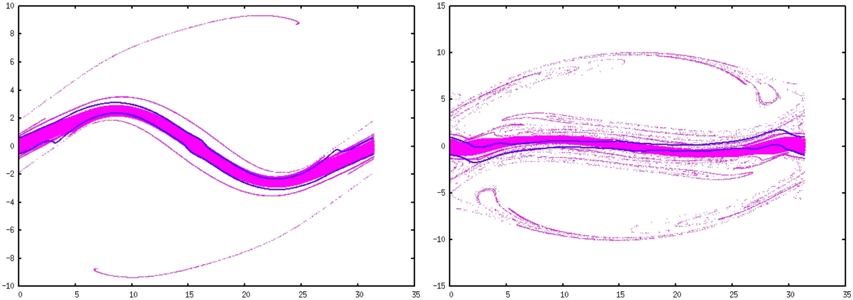
Phase space (v as a function of x) at times 0 and 2:

- particles (PIC method),
- v_1^+ and v_1^- (water-bag).



Phase space (v as a function of x) at times 5 and 10:

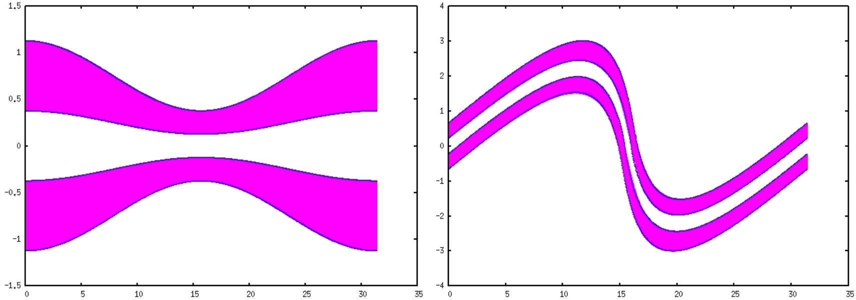
- particles (PIC method),
- v_1^+ and v_1^- (water-bag).



Phase space (v as a function of x) at times 20 and 50:

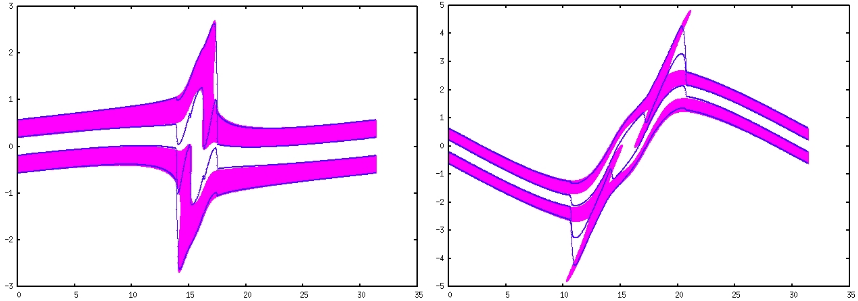
- particles (PIC method),
- v_1^+ and v_1^- (water-bag).

Second test case: 2 pairs of bags



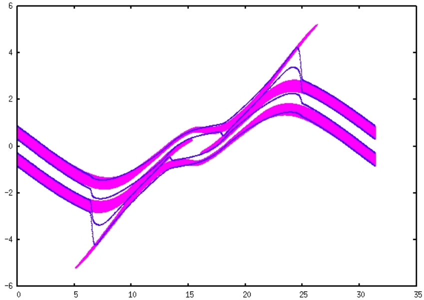
Phase space (v as a function of x) at times 0 and 2:

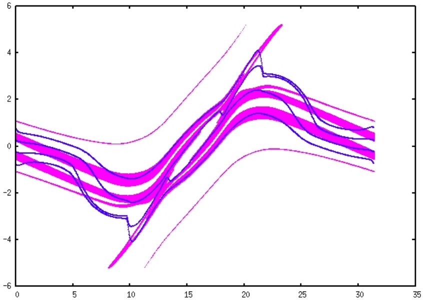
- particles (PIC method),
- v_1^+, v_1^-, v_2^+ and v_2^- (water-bag).



Phase space (v as a function of x) at times 3 and 4:

- particles (PIC method),
- v_1^+, v_1^-, v_2^+ and v_2^- (water-bag).



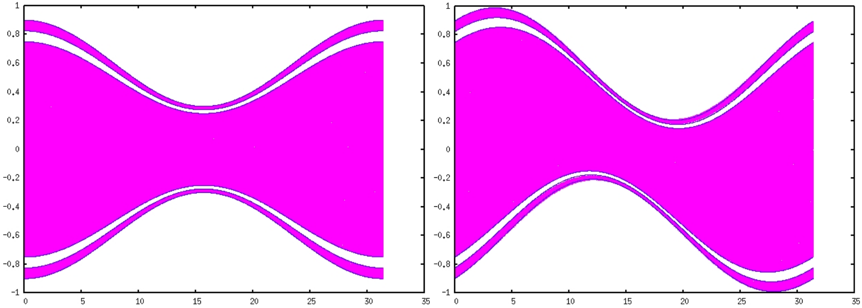


Phase space (v as a function of x) at times 5 and 10:

- particles (PIC method),
- v_1^+, v_1^-, v_2^+ and v_2^- (water-bag).

- Conclusions:
  - monovalued solution correctly described,
  - $v_j^{\pm}(x,t)$ cannot be multivalued $\Longrightarrow$ loss of information when filaments appear,
  - used algorithm unstable when at least two bags are close to each other (ill-conditioned matrix).

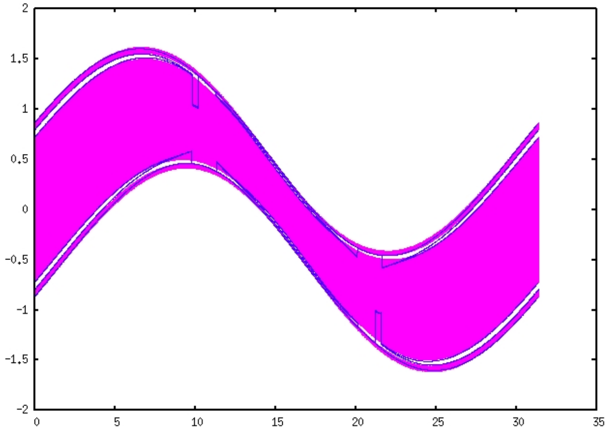
- Work in progress in order to improve the representation of the filaments:
  - stabilize the algorithm of Gosse and Runborg when several bags are close to each other,
  - first idea: merge these bags.



Phase space (v as a function of x) at times 0 and 0.1:

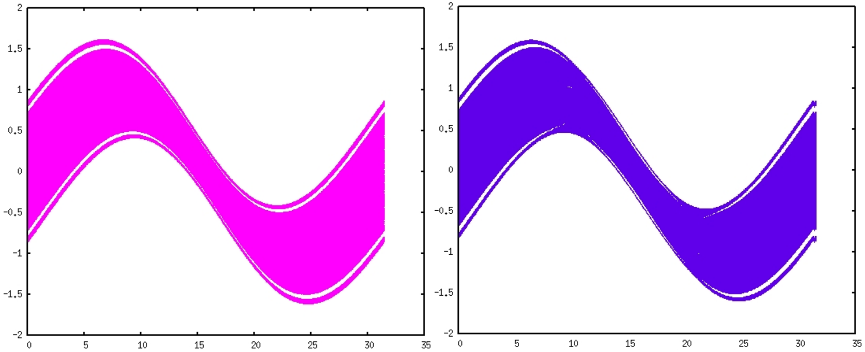
particles (PIC method),

v_1^+, v_1^-, v_2^+, v_2^-, v_3^+ and v_3^- (water-bag).



Phase space (v as a function of x) at time 0.4:

- particles (PIC method),
- v_1^+, v_1^-, v_2^+, v_2^-, v_3^+ and v_3^- (water-bag).



Phase space (v as a function of x) at time 0.4:

- left: f obtained by the PIC method (particles),
- right: f reconstructed by the method of moments/multi-water-bag.

- Perspectives:
  - generalize the algorithm of Gosse and Runborg when the A_j are different of ± 1,
  - couple our method to a PIC method: create particles when filaments appear.

Thank you for your attention!

References

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