Un schéma micro-macro pour les équations cinétiques en limite de diffusion dont le coût diminue à l'approche de l'équilibre

Anaïs Crestetto[1], Nicolas Crouseilles[2], Giacomo Dimarco[3] et Mohammed Lemou[4]

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[1] University of Nantes, LMJL.

[2] INRIA Rennes - Bretagne Atlantique & University of Rennes 1, IRMAR.

[3] University of Ferrara, Department of Mathematics and Computer Science.

[4] CNRS & University of Rennes 1, IRMAR & INRIA Rennes - Bretagne Atlantique.

Outline

Numerical simulation of particle systems

We are interested in

- the numerical simulation of collisional kinetic Problems$_\varepsilon$,
- different scales: collisions parameterized by the Knudsen number $\varepsilon(t, \mathbf{x})$,
- the development of schemes that are efficient in both kinetic and fluid regimes.

There are two main strategies for multiscale problems:

- domain decomposition methods,
- asymptotic preserving (AP) schemes.

Asymptotic Preserving approach[5]: develop a model suitable in any region.

$$\begin{tikzcd}
\boxed{\text{Problem}_{\varepsilon}} \arrow[d, "\varepsilon \to 0"] & \boxed{\text{Discretized Problem}_{h,\varepsilon}} \arrow[l, "h \to 0"'] \arrow[d, dashed, "\varepsilon \to 0"] \\
\boxed{\text{Limit}} & \boxed{\text{Discretized limit}_{h}} \arrow[l, "h \to 0"']
\end{tikzcd}$$

h: space step Δx or time step Δt.

Prop.: Stability and consistency $\forall \varepsilon$, particularly when $\varepsilon \to 0$.

:-(Standard schemes: constraint $h = \mathcal{O}(\varepsilon)$.

Aim: Construct a scheme for which h is independent of ε.

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[5] Jin, SISC 1999.

Our Problem$_\varepsilon$

Radiative transport equation in the diffusive scaling

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2}(\rho M - f) \qquad (1)$$

- $\mathbf{x} \in \Omega \subset \mathbb{R}^{d_x}$, $\mathbf{v} \in V = \mathbb{R}^{d_v}$,
- charge density $\rho(t,\mathbf{x}) = \int_V f(t,\mathbf{x},\mathbf{v})d\mathbf{v}$,
- $M(\mathbf{v}) = \frac{1}{(2\pi)^{d_v/2}}\exp\left(-\frac{|\mathbf{v}|^2}{2}\right)$,
- periodic conditions in $\mathbf{x}$ and initial conditions.

Main difficulty:

- Knudsen number ε may be of order 1 or tend to 0 in the diffusive scaling. The asymptotic diffusion equation being

$$\partial_t \rho - \Delta_{\mathbf{x}}\rho = 0. \qquad (2)$$

Objectives

- Construction of an AP scheme.
- Reduction of the numerical cost at the limit $\varepsilon \to 0$.

Tools

- Micro-macro decomposition[6,7] for this model. Previous work with a grid in v for the micro part[8], cost was constant w.r.t. ε.
- Particle method for the micro part since few information in v is necessary at the limit[9].
- Monte Carlo techniques[10,11,12,13].

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[6] Lemou, Mieussens, SIAM SISC 2008.

[7] Liu, Yu, CMP 2004.

[8] Crouseilles, Lemou, KRM 2011.

[9] C., Crouseilles, Lemou, CMS 2018.

[10] Degond, Dimarco, Pareschi, IJNMF 2011.

[11] Degond, Dimarco, JCP 2012.

[12] Crouseilles, Dimarco, Lemou, KRM 2017

[13] Dimarco, Pareschi, Samaey, SIAM SISC 2018.

Micro-macro decomposition

- Micro-macro decomposition[14,15]: $f = \rho M + g$ with g the rest.
- $\mathcal{N} = \mathrm{Span}\left\{M\right\} = \left\{f = \rho M\right\}$ null space of the BGK operator $Q\left(f\right) = \rho M - f$.
- Π orthogonal projection onto $\mathcal{N}$:

$$\Pi h := \langle h \rangle M, \quad \langle h \rangle := \int h \, \mathrm{d}\mathbf{v}.$$

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[14] Lemou, Mieussens, SIAM JSC 2008.

[15] Crouseilles, Lemou, KRM 2011.

- Applying Π to (1) $\Longrightarrow$ macro equation on ρ

$$\partial_t \rho + \frac{1}{\varepsilon}\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g\rangle = 0. \tag{3}$$

- Applying $(I-\Pi)$ to (1) $\Longrightarrow$ micro equation on g

$$\partial_t g + \frac{1}{\varepsilon}\left[\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho M + \mathbf{v}\cdot\nabla_{\mathbf{x}} g - \langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g\rangle M\right] = -\frac{1}{\varepsilon^2} g. \tag{4}$$

Equation (1) $\Leftrightarrow$ micro-macro system:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g\rangle = 0, \\ \partial_t g + \dfrac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\dfrac{1}{\varepsilon^2} g, \end{cases} \tag{5}$$

where $\mathcal{F}(\rho, g) = \mathbf{v}\cdot\nabla_{\mathbf{x}}\rho M + \mathbf{v}\cdot\nabla_{\mathbf{x}} g - \langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g\rangle M.$

Difficulties

- Stiff terms in the micro equation (4) on g.
- In previous works[16,17], stiffest term (of order $1/\varepsilon^2$) considered implicit in time $\Longrightarrow$ transport term (of order $1/\varepsilon$) stabilized.

But here:

- use of particles for the micro part

$\Rightarrow$ splitting between the transport term and the source term,

$\Rightarrow$ not possible to use the same strategy.

Idea?

- Suitable reformulation of the model.

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[16] Lemou, Mieussens, SIAM SISC 2008.

[17] Crouseilles, Lemou, KRM 2011.

- Strategy of Lemou[18]:
  1. rewrite (4) $\partial_t g + \frac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\frac{1}{\varepsilon^2} g$ as

$$\partial_t(e^{t/\varepsilon^2} g) = -\frac{e^{t/\varepsilon^2}}{\varepsilon}\mathcal{F}(\rho, g),$$

  2. integrate in time between t^n and t^{n+1} and multiply by $e^{-t^{n+1}/\varepsilon^2}$:

$$\frac{g^{n+1} - g^n}{\Delta t} = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g^n - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho^n, g^n) + \mathcal{O}(\Delta t),$$

  3. approximate up to terms of order $\mathcal{O}(\Delta t)$ by:

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho, g). \qquad (6)$$

- No more stiff terms and consistent with the initial micro equation (4).

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[18] Lemou, CRAS 2010.

New micro-macro model

The new micro-macro model writes

$$\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0, \tag{7}$$

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho, g), \tag{8}$$

with $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

We propose the following hybrid discretization:

- macro equation (7): Eulerian method,
- micro equation (8): Monte Carlo technique.

Discretization of the micro equation

- Model: considering at each time step N^n particles, with position $\mathbf{x}_\mathbf{k}^\mathbf{n}$, velocity $\mathbf{v}_\mathbf{k}^\mathbf{n}$ and constant weight ω_k, $k = 1, \ldots, N^n$, g is approximated by[19]

$$g_{N^n}\left(t^n, \mathbf{x}, \mathbf{v}\right) = \sum_{k=1}^{N^n} \omega_k \delta\left(\mathbf{x} - \mathbf{x}_\mathbf{k}^\mathbf{n}\right) \delta\left(\mathbf{v} - \mathbf{v}_\mathbf{k}^\mathbf{n}\right).$$

- For the coupling with the macro equation, we need a grid in $\mathbf{x}$. For $d_x = 1$, we define $\mathrm{x}_i = x_{\min} + i\Delta x$, $i = 0, \ldots, N_x - 1$.
- How to define ω_k, N^n, $\mathbf{x}_\mathbf{k}^\mathbf{n}$, $\mathbf{v}_\mathbf{k}^\mathbf{n}$?

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[19] Crouseilles, Dimarco, Lemou, KRM 2017.

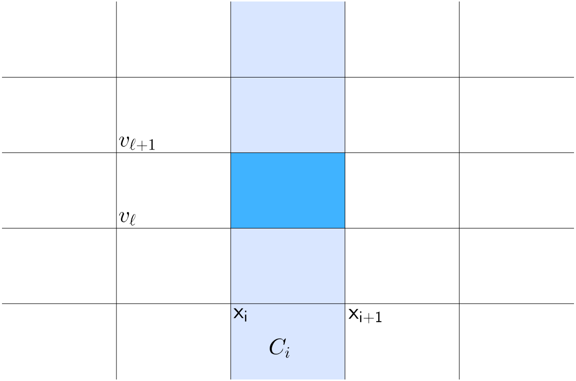
Initialization

- Choose the characteristic weight m_p or the characteristic number of particles N_p necessary to sample the full distribution function f, and link them with

$$m_p = \frac{1}{N_p} \int_{\mathbb{R}^{d_x}} \int_{\mathbb{R}^{d_v}} f(t = 0, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x}.$$

- Now, we want to sample $g(t = 0, \mathbf{x}, \mathbf{v})$, that has no sign.
- We impose $\omega_k \in \{m_p, -m_p\}$.
- For velocities, we impose $\mathbf{v_k^n}$ on a cartesian grid in $\mathbb{R}^{d_v}$.
For $d_v = 1$, it writes $v_k^n \in \{\mathrm{v}_\ell,\ \ell = 0, \dots, N_v - 1\}$
$\forall k = 1, \dots, N^n$, where $\mathrm{v}_\ell = v_{\min} + \ell \Delta v$, $\ell = 0, \dots, N_v - 1$.

Let us introduce the notations in 1D...



Let us introduce the notations in 1D...

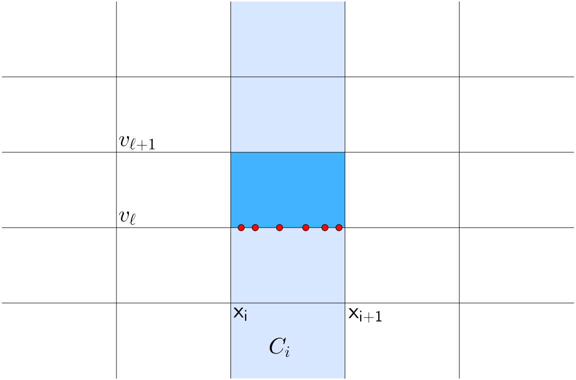
- The number of initial positive (resp. negative) particles having the velocity $v_k = \mathrm{v}_\ell$ in the cell $C_i = [\mathrm{x}_i, \mathrm{x}_{i+1}] \times \mathbb{R}$ is given by

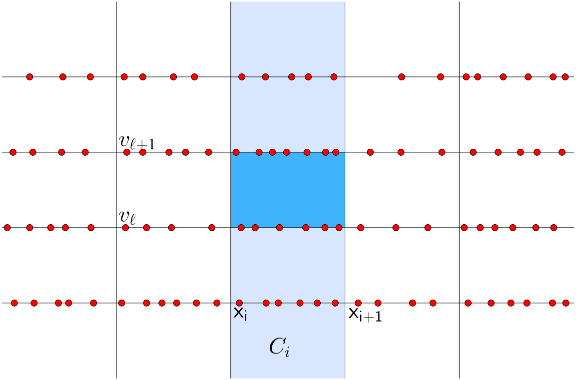
$$N_{i,\ell}^{0,\pm} = \lfloor \pm \frac{\Delta x \Delta v}{m_p} g^{\pm}(t=0, \mathrm{x}_i, \mathrm{v}_\ell) \rfloor,$$

  that is an approximation of

$$N_{i,\ell}^{0,\pm} = \pm \frac{1}{m_p} \int_{\mathrm{x}_i}^{\mathrm{x}_{i+1}} \int_{\mathrm{v}_\ell}^{\mathrm{v}_{\ell+1}} g^{\pm}(t=0, x, v) dv dx,$$

  with $g^{\pm} = \frac{g \pm |g|}{2}$ the positive (resp. negative) part of g.
- Positions of these $N_{i,\ell}^{0,\pm}$ particles are taken uniformly in $[\mathrm{x}_i, \mathrm{x}_{i+1}]$.
- At time $t = 0$, we have $N^0 = \sum_i \left(\sum_\ell N_{i,\ell}^{0,+} + \sum_\ell N_{i,\ell}^{0,-} \right)$.


$v_{\ell+1}$
v_ℓ
x_i
x_{i+1}
C_i


$v_{\ell+1}$
v_ℓ
x_i
x_{i+1}
C_i

From t^n to t^{n+1}

Solve the micro equation (8) by Monte Carlo technique.

- Splitting between the transport part

$$\partial_t g + \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v} \cdot \nabla_{\mathbf{x}} g = 0,$$

  and the interaction part

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \left(\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M \right).$$

- Solve the transport part by shifting particles:

$$\frac{d\mathbf{x}_k}{dt}(t) = \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v}_k, \qquad \mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \varepsilon(1 - e^{-\Delta t/\varepsilon^2}) \mathbf{v}_k^n.$$

  Remark that $\mathbf{v}_k^{n+1} = \mathbf{v}_k^n$.

- Solve interaction part by writing

$$g^{n+1} = e^{-\Delta t/\varepsilon^2}\tilde{g}^n + (1 - e^{-\Delta t/\varepsilon^2})\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big]$$

where $\tilde{g}^n$ is the function after the transport part.

Apply a Monte Carlo technique:

- with probability $e^{-\Delta t/\varepsilon^2}$, the distribution g^{n+1} does not change,
- with probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g^{n+1} is replaced by a new distribution given by $\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big]$.

- Solve interaction part by writing

  $$g^{n+1} = e^{-\Delta t/\varepsilon^2}\tilde{g}^n + (1 - e^{-\Delta t/\varepsilon^2})\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big]$$

  where $\tilde{g}^n$ is the function after the transport part.

In practice:

- In each cell C_i, we keep $e^{-\Delta t/\varepsilon^2}\tilde{N}_i^n$ particles unchanged (with $\tilde{N}_i^n$ the number of particles in C_i after the transport part) and discard the others.
- Create new particles to sample

  $$(1 - e^{-\Delta t/\varepsilon^2})\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big]^{\pm},$$

  as in the initialization stage. Let us denote by M_i^n the number of created particles in C_i.

Time-Diminishing Property

- At the end of the time step, we have in each cell C_i

  $$N_i^{n+1} = e^{-\Delta t/\varepsilon^2} \tilde{N}_i^n + M_i^n$$

  particles.
- The number of particles automatically diminishes with ε.
- Reduction of the computational complexity when approaching equilibrium: Time-Diminishing Property.

Macro equation

- Equation $\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0$.
- First proposition:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g^{n+1} \rangle = 0.$$

- Problem: g^{n+1} suffers from numerical noise inherent to particles method. This noise, amplified by $\frac{1}{\varepsilon}$, will damage ρ^{n+1}.

Correction of the macro discretization

- Use the expression of g^{n+1} and write

$$\begin{aligned}\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g^{n+1}\rangle = e^{-\Delta t/\varepsilon^2}&\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde{g}^n\rangle\\ &-\varepsilon(1-e^{-\Delta t/\varepsilon^2})\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\left(\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M\right)\rangle\\ &+\varepsilon(1-e^{-\Delta t/\varepsilon^2})\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\left(\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde{g}\rangle^n M\right)\rangle,\end{aligned}$$

  or after simplifications

$$\langle \mathbf{v}\cdot\nabla_{\mathbf{x}} g^{n+1}\rangle = e^{-\Delta t/\varepsilon^2}\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde{g}^n\rangle-\varepsilon(1-e^{-\Delta t/\varepsilon^2})\Delta_{\mathbf{x}}\rho^n.$$

- Plug it into the macro equation

$$\frac{\rho^{n+1}-\rho^n}{\Delta t}+\frac{1}{\varepsilon}e^{-\Delta t/\varepsilon^2}\langle \mathbf{v}\cdot\nabla_{\mathbf{x}}\tilde{g}^n\rangle-(1-e^{-\Delta t/\varepsilon^2})\Delta_{\mathbf{x}}\rho^n=0.$$

- To avoid the parabolic CFL condition of type $\Delta t \leq C \Delta x^2$, take the diffusion implicit:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t/\varepsilon^2} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^{n+1} = 0.$$

- No more stiffness, the numerical noise does not damage ρ.
- As $\varepsilon \to 0$, implicit discretization of the diffusion equation $\partial_t \rho - \Delta_{\mathbf{x}} \rho = 0$.

Space discretization in 2D

In 2D, we use an Alternating Direction Implicit (ADI) method[20]:

1) Starting from ρ^n, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{xx} \rho = 0,$$

using a Crank-Nicolson time discretization to get $\rho^\star$.

2) Starting from $\rho^\star$, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{yy} \rho = 0,$$

using a Crank-Nicolson time discretization to get ρ^{n+1}.

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[20] Peaceman, Rachford, J. Soc. Indust. Appl. Math. 1955.

Nice properties

- Only 1D systems of size N_x and N_y.
- ADI method unconditionally stable in 2D.
- Straightforward extension in 3D: a priori conditionally stable, but better extensions have been derived[21].
- Right asymptotic behaviour.

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[21] Sharma, Hammett, JCP 2011.

1. Problem and objectives
2. Micro-macro model
3. Monte Carlo / Eulerian discretization
4. **Numerical results**
    - Test 1 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})=0$
    - Test 2 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$
    - Test 3 - 3Dx3D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$
    - Test 4 - 2Dx2D, $\varepsilon(\mathbf{x})$, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Test 1 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})=0$

Initialization:

$$f(t=0,\mathbf{x},\mathbf{v})=\rho(t=0,\mathbf{x})M(\mathbf{v}),\ \mathbf{x}\in[0,4\pi]^2,\ \mathbf{v}\in\mathbb{R}^2$$

with

$$\rho(t=0,\mathbf{x})=1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right),$$

$$M(\mathbf{v})=\frac{1}{2\pi}\exp\left(-\frac{|\mathbf{v}|^2}{2}\right),$$

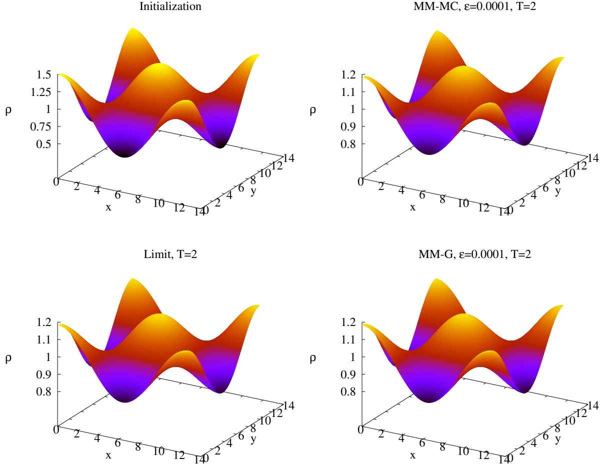
so that

$$g(t=0,\mathbf{x},\mathbf{v})=0.$$

Periodic boundary conditions in space.

Asymptotic behaviour, $\varepsilon = 10^{-4}$

MM-MC: the presented Micro-Macro Monte Carlo scheme.
MM-G: a Micro-Macro Grid code, considered as reference.



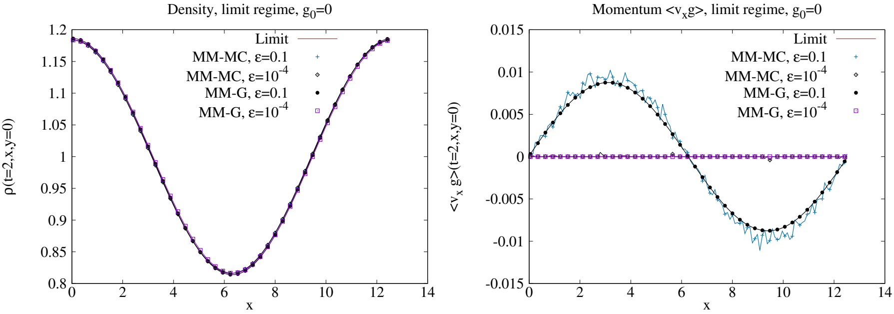
Initialization

MM-MC, ε=0.0001, T=2

Limit, T=2

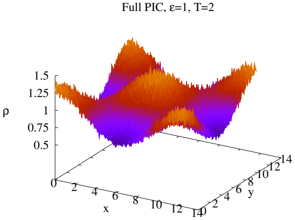
MM-G, ε=0.0001, T=2

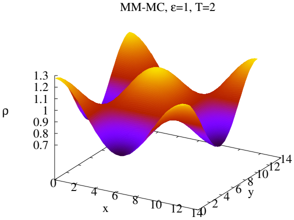
Slices of the density $\rho(T=2,x,y=0)$ and of the momentum $\langle v_x g\rangle(T=2,x,y=0)$.

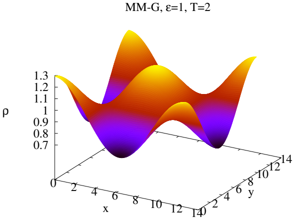


Kinetic regime, $\varepsilon = 1$

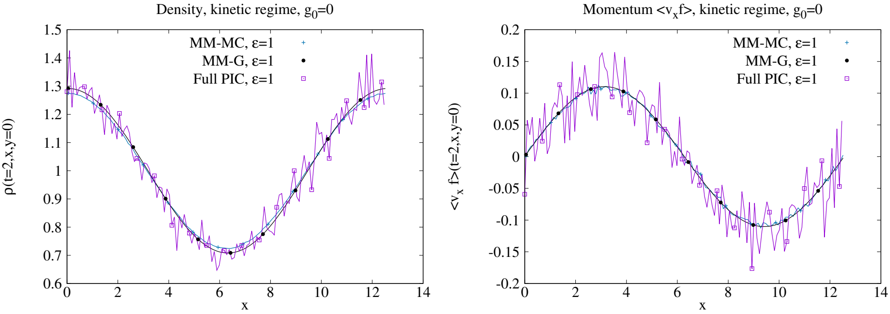
Full PIC: standard particle method on f.


Full PIC, ε=1, T=2

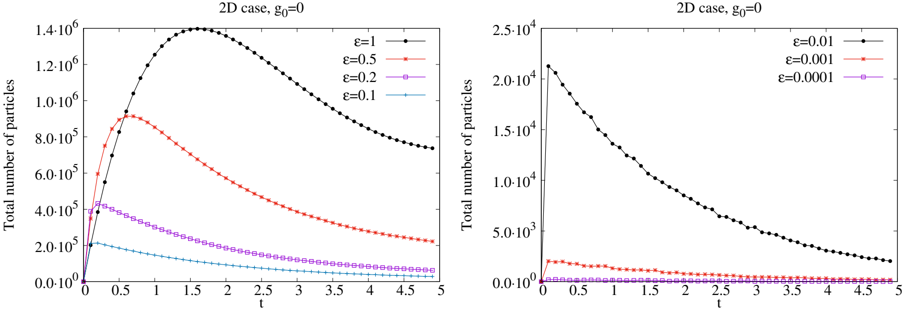

MM-MC, ε=1, T=2


MM-G, ε=1, T=2

Slices of the density $\rho(T=2,x,y=0)$ and of the momentum $\langle v_x g\rangle(T=2,x,y=0)$.



Time evolution of the number of particles



Test 2 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Initialization:

$$f(t=0,\mathbf{x},\mathbf{v})=\frac{1}{4\pi}\left(\exp\left(-\frac{|\mathbf{v}-2|^2}{2}\right)+\exp\left(-\frac{|\mathbf{v}+2|^2}{2}\right)\right)\rho(t=0,\mathbf{x}),$$

with

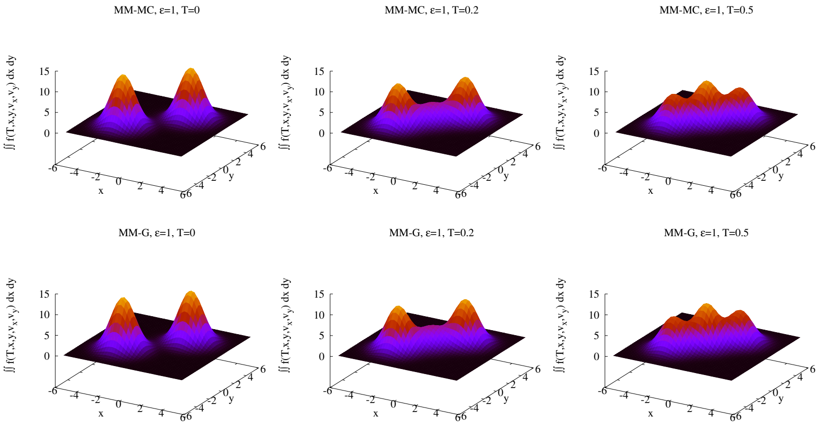
$$\mathbf{x}\in[0,4\pi]^2,\quad \mathbf{v}\in\mathbb{R}^2,$$

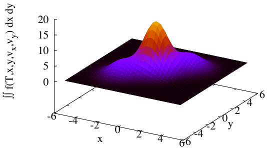
$$\rho(t=0,\mathbf{x})=1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right),$$

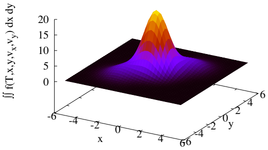
so that

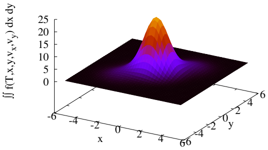
$$g(t=0,\mathbf{x},\mathbf{v})=\rho(t=0,\mathbf{x})M(\mathbf{v})-f(t=0,\mathbf{x},\mathbf{v})\neq 0.$$

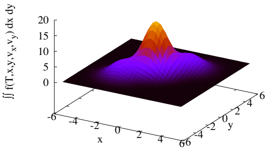
Integral of the distribution function in space $\int f(T,\mathbf{x},\mathbf{v})d\mathbf{x}$.

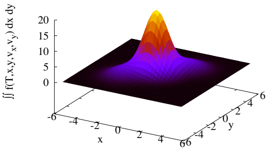


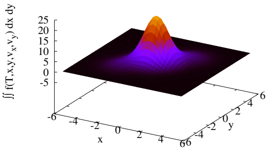

MM-MC, ε=1, T=1


MM-MC, ε=1, T=1.5

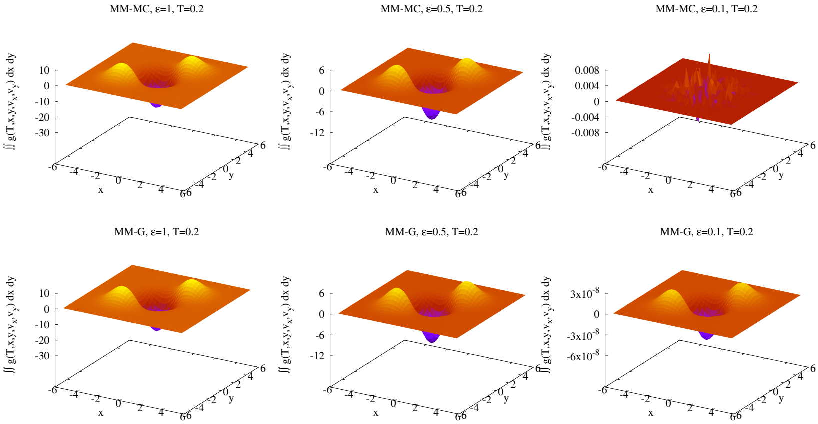

MM-MC, ε=1, T=2


MM-G, ε=1, T=1

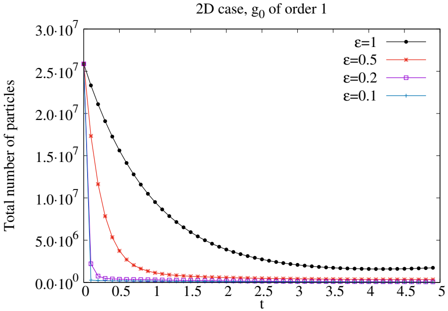

MM-G, ε=1, T=1.5


MM-G, ε=1, T=2

Integral of the perturbation in space $\int g(T=0.2,\mathbf{x},\mathbf{v})d\mathbf{x}$ for different ε.


MM-MC, ε=1, T=0.2
MM-MC, ε=0.5, T=0.2
MM-MC, ε=0.1, T=0.2
MM-G, ε=1, T=0.2
MM-G, ε=0.5, T=0.2
MM-G, ε=0.1, T=0.2

Time evolution of the number of particles



Test 3 - 3Dx3D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Initialization:

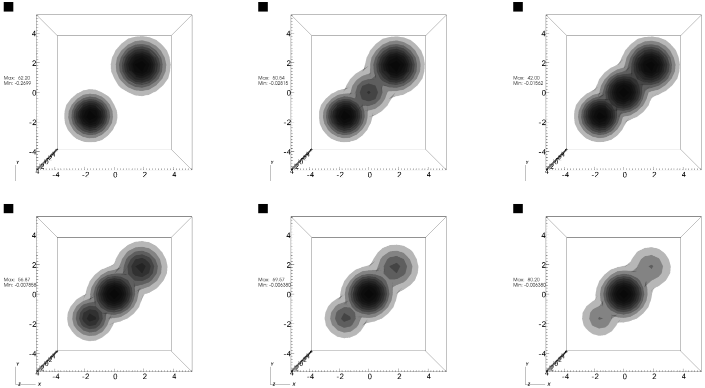
$$f_0(\mathbf{x},\mathbf{v}) = \frac{1}{2\,(2\pi)^{3/2}}\left[\exp\Big(-\frac{|\mathbf{v}-u|^2}{2}\Big)+\exp\Big(-\frac{|\mathbf{v}+u|^2}{2}\Big)\right]\rho(0,\mathbf{x}),$$

with $u=(2,2,2)$,

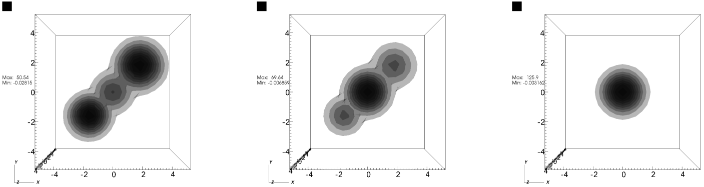
$$\rho(0,\mathbf{x}) = 1+\frac{1}{2}\cos(\frac{x}{2})\cos(\frac{y}{2})\cos(\frac{z}{2}),$$

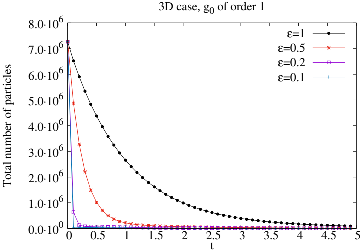
$\mathbf{x}=(x,y,z)\in[0,4\pi]^3$, $\mathbf{v}=(v_x,v_y,v_z)\in\mathbb{R}^3$.

Integral of the distribution function in space $\int_{\mathbf{x}} f(T,\mathbf{x},\mathbf{v})d\mathbf{x}$ for $\varepsilon = 1$ and different times (T=0, 0.2, 0.4, 0.6, 0.8, 1).



Top: integral of the distribution function in space $\int_{\mathbf{x}} f(T,\mathbf{x},\mathbf{v})d\mathbf{x}$ for $T=0.2$ and different ε (1, 0.5, 0.1).





Bottom: time evolution of the number of particles.

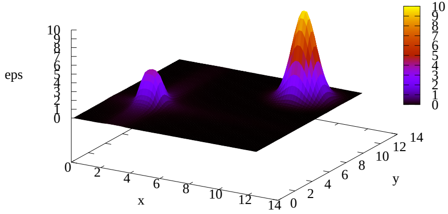
Test 4 - 2Dx2D, $\varepsilon(\mathbf{x})$, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Modified model:

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2(\mathbf{x})}(\rho M - f),$$

where $(\mathbf{x},\mathbf{v})\in[0,4\pi]^2\times\mathbb{R}^2$,

$$\varepsilon(\mathbf{x}) = 10\left[\operatorname{atan}\Big(2(y-5)\Big)+\operatorname{atan}\Big(-2(y-5)\Big)\right] \times \exp\Big(-(x-10)^2-(y-10)^2\Big)+10^{-3}.$$



Initialization:

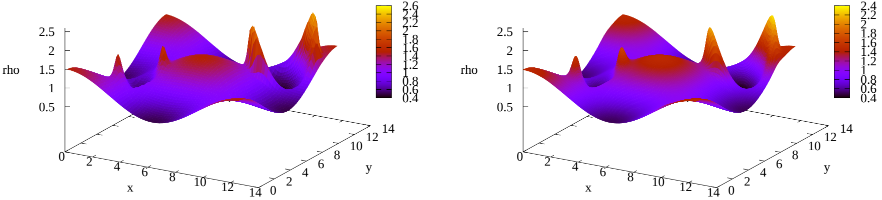
$$f(t=0,\mathbf{x},\mathbf{v})=\frac{1}{4\pi}\left(\exp\left(-\frac{|\mathbf{v}-2|^2}{2}\right)+\exp\left(-\frac{|\mathbf{v}+2|^2}{2}\right)\right)\rho(t=0,\mathbf{x}),$$

with

$$\mathbf{x}\in[0,4\pi]^2,\quad \mathbf{v}\in\mathbb{R}^2,$$

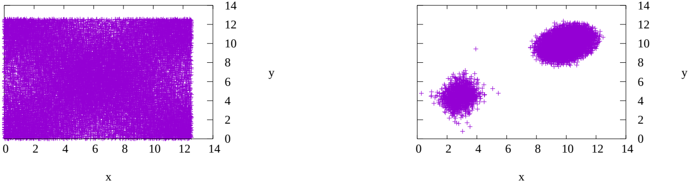
$$\rho(t=0,\mathbf{x})=1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right).$$

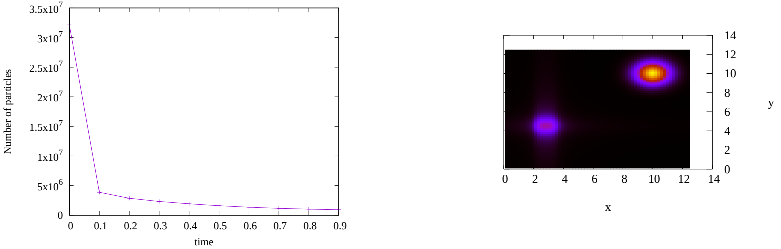
Density profile $\rho(T=1,x,y)$. Left: MM-MC, right: MM-G.



Time-Diminishing Property

Top: position of the particles in $\mathbf{x}$. Left: at $T=0$; right: at $T=1$.





Bottom: left: time evolution of the number of particles; right: $\varepsilon(x,y)$.

Conclusions

- Right asymptotic behaviour.
- Computational cost diminishes as the equilibrium is approached.
- Numerical noise smaller than a standard particle method on f.
- Implicit treatment of the diffusion term.
- Suitable for multi-dimensional testcases.

Possible extensions

- More 3D-3D testcases, more physical relevance.
- Boltzmann operator.
- Second-order in time scheme.
- Add an electromagnetic field $\Rightarrow$ v_k no constant anymore.

Merci pour votre attention !

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smai2019@smai.emath.fr
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CONFÉRENCES PLÉNIÈRES

Hermine Biermé (Université de Poitiers)
Yves Coudière (Université de Bordeaux)
Anne-Laure Dalibard (Sorbonne Universités)
François Delarue (Université Nice Sophia Antipolis)
Jimmy Lamboley (Sorbonne Universités)
Eva Löcherbach (Université Paris 1 Panthéon Sorbonne)
Quentin Mérigot (Université Paris Sud)
Marcelo Pereyra (Heriot Watt University)
Marcela Szopos (Université Paris Descartes)

Organisé par la SMAI et le Laboratoire de Mathématiques Jean Leray