Particle Micro-Macro schemes for collisional kinetic equations in the diffusive scaling

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Outline

Numerical simulation of particle systems

We are interested in

- the numerical simulation of collisional kinetic Problems$_\varepsilon$,
- different scales: collisions parameterized by the Knudsen number $\varepsilon(t, \mathbf{x})$,
- the development of schemes that are efficient in both kinetic and fluid regimes.

Kinetic regime $\varepsilon(t, \mathbf{x}) = \mathcal{O}(1)$

- Particles represented by a distribution function $f(t, \mathbf{x}, \mathbf{v})$.
- Solving a Boltzmann or Vlasov-type equation

$$\partial_t f + \mathcal{A}(\mathbf{v}, \varepsilon) \cdot \nabla_{\mathbf{x}} f + \mathcal{B}(\mathbf{v}, \mathbf{E}, \mathbf{B}, \varepsilon) \cdot \nabla_{\mathbf{v}} f = \mathcal{S}(\varepsilon).$$

:-) Accurate and necessary far from thermodynamical equilibrium.

:-(In 3D $\Longrightarrow$ 7 variables $\Longrightarrow$ heavy computations.

Fluid regime $\varepsilon(t, \mathbf{x}) \ll 1$

- Moment equations on physical quantities linked to f (density ρ, mean velocity u, temperature T, etc.).

:-(Lost of precision.

:-) Small cost and sufficient at thermodynamical equilibrium.

There are two main strategies for multiscale problems:

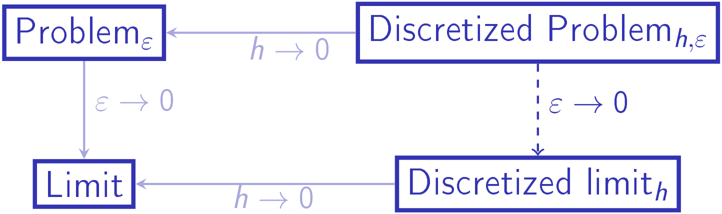
- domain decomposition methods,
- asymptotic preserving (AP) schemes.

- Domain decomposition approach: use the suitable model in the appropriate region.
  - :-) Very efficient in each region.
  - :-(How to determine the validity of each model and let them communicate?

- Domain decomposition approach: use the suitable model in the appropriate region.
  - :-) Very efficient in each region.
  - :-(How to determine the validity of each model and let them communicate?

- Asymptotic Preserving approach: develop a model suitable in any region.
  - :-) Only one model.
  - :-(Generally have the less favorable cost.

Asymptotic Preserving approach[5]



h: space step Δx or time step Δt.

Prop.: Stability and consistency $\forall \varepsilon$, particularly when $\varepsilon \to 0$.

:-(Standard schemes: constraint $h = \mathcal{O}(\varepsilon)$.

Aim: Construct a scheme for which h is independent of ε.

[5] Jin, SISC 1999.

Our Problem$_\varepsilon$

Radiative transport equation in the diffusive scaling

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2}(\rho M - f) \qquad (1)$$

- $\mathbf{x} \in \Omega \subset \mathbb{R}^{d_x}$, $\mathbf{v} \in V = \mathbb{R}^{d_v}$,
- charge density $\rho(t,\mathbf{x}) = \int_V f(t,\mathbf{x},\mathbf{v})d\mathbf{v}$,
- $M(\mathbf{v}) = \frac{1}{(2\pi)^{d_v/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right)$,
- periodic conditions in $\mathbf{x}$ and initial conditions.

Main difficulty:

- Knudsen number ε may be of order 1 or tend to 0 in the diffusive scaling. The asymptotic diffusion equation being

$$\partial_t \rho - \Delta_{\mathbf{x}}\rho = 0. \qquad (2)$$

Objectives

- Construction of an AP scheme.
- Reduction of the numerical cost at the limit $\varepsilon \to 0$.

Tools

- Micro-macro decomposition[6,7] for this model. Previous work with a grid in v for the micro part[8], cost was constant w.r.t. ε.
- Particle method for the micro part since few information in v is necessary at the limit[9].
- Monte Carlo techniques[10].

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[6] Lemou, Mieussens, SIAM SISC 2008.
[7] Liu, Yu, CMP 2004.
[8] Crouseilles, Lemou, KRM 2011.
[9] C., Crouseilles, Lemou, CMS 2018.
[10] Degond, Dimarco, Pareschi, IJNMF 2011.

Micro-macro decomposition

- Micro-macro decomposition:

$$f = \rho M + g$$

  with g the perturbation.
- $\mathcal{N} = \mathrm{Span}\,\{M\} = \{f = \rho M\}$ null space of the BGK operator $Q\,(f) = \rho M - f$.
- Π orthogonal projection onto $\mathcal{N}$:

$$\Pi h := \langle h \rangle M, \quad \langle h \rangle := \int_V h \,\mathrm{d}\mathbf{v}.$$

- Applying Π to (1) $\Longrightarrow$ macro equation on ρ

$$\partial_t \rho + \frac{1}{\varepsilon}\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0. \tag{3}$$

- Applying $(I - \Pi)$ to (1) $\Longrightarrow$ micro equation on g

$$\partial_t g + \frac{1}{\varepsilon}\left[\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M\right] = -\frac{1}{\varepsilon^2} g. \tag{4}$$

Equation (1) $\Leftrightarrow$ micro-macro system:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0, \\ \partial_t g + \dfrac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\dfrac{1}{\varepsilon^2} g, \end{cases} \tag{5}$$

where $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M.$

Difficulties

- Stiff terms in the micro equation (4) on g.
- In previous works[11,12], stiffest term (of order $1/\varepsilon^2$) considered implicit in time $\Longrightarrow$ transport term (of order $1/\varepsilon$) stabilized.

But here:

- use of particles for the micro part

$\Rightarrow$ splitting between the transport term and the source term,

$\Rightarrow$ not possible to use the same strategy.

Idea?

- Suitable reformulation of the model.

[11] Lemou, Mieussens, SIAM SISC 2008.
[12] Crouseilles, Lemou, KRM 2011.

- Strategy of Lemou[13]:
  1. rewrite (4) $\partial_t g + \frac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\frac{1}{\varepsilon^2} g$ as
  $$\partial_t (e^{t/\varepsilon^2} g) = -\frac{e^{t/\varepsilon^2}}{\varepsilon}\mathcal{F}(\rho, g),$$
  2. integrate in time between two times t^n and $t^{n+1} = t^n + \Delta t$:
  $$e^{t^{n+1}/\varepsilon^2} g^{n+1} = e^{t^n/\varepsilon^2} g^n + \int_{t^n}^{t^{n+1}} -\frac{e^{t/\varepsilon^2}}{\varepsilon}\mathcal{F}(\rho, g)\mathrm{d}t,$$
  3. use left-rectangle method for $\mathcal{F}(\rho, g)$ and multiply by $e^{-t^{n+1}/\varepsilon^2}/\Delta t$:
  $$\frac{g^{n+1} - g^n}{\Delta t} = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g^n - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho^n, g^n) + \mathcal{O}(\Delta t),$$
  4. approximate up to terms of order $\mathcal{O}(\Delta t)$ by:
  $$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho, g).$$
- No more stiff terms and consistent with initial micro equation (4).

[13] Lemou, CRAS 2010.

New micro-macro model

The new micro-macro model writes

$$\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0, \tag{6}$$

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho, g), \tag{7}$$

with $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

We propose the following hybrid discretization:

- macro equation (6): Finite Volume method,
- micro equation (7): Monte Carlo technique.

Discretization of the micro equation

- Model: considering at each time step N^n particles, with position $\mathbf{x_k^n}$, velocity $\mathbf{v_k^n}$ and constant weight ω_k, $k = 1, \dots, N^n$, g is approximated by[14]

$$g_{N^n}(t^n, \mathbf{x}, \mathbf{v}) = \sum_{k=1}^{N^n} \omega_k \delta(\mathbf{x} - \mathbf{x_k^n}) \delta(\mathbf{v} - \mathbf{v_k^n}).$$

- For the coupling with the macro equation, we need a grid in $\mathbf{x}$. For $d_x = 1$, we define for $i = 0, \dots, N_x - 1$

$$\mathrm{x}_i = x_{\min} + i\Delta x, \quad \mathrm{x}_{i\pm1/2} = \mathrm{x}_i \pm \frac{\Delta x}{2}.$$

- How to define/compute ω_k, N^n, $\mathbf{x_k^n}$, $\mathbf{v_k^n}$?

[14] Crouseilles, Dimarco, Lemou, KRM 2017.

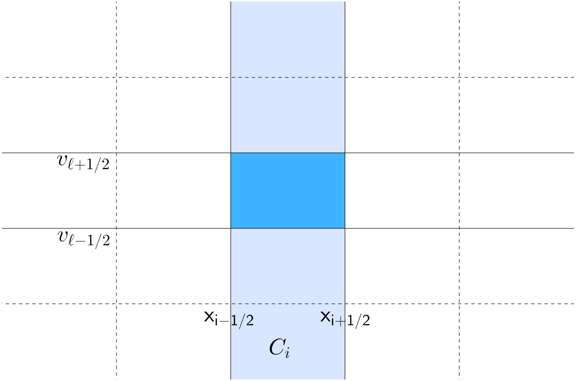
Initialization

- Choose the characteristic weight m_p or the characteristic number of particles N_p necessary to sample the full distribution function f, and link them with

$$m_p = \frac{1}{N_p} \int_{\mathbb{R}^{d_x}} \int_{\mathbb{R}^{d_v}} f(t=0, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x}.$$

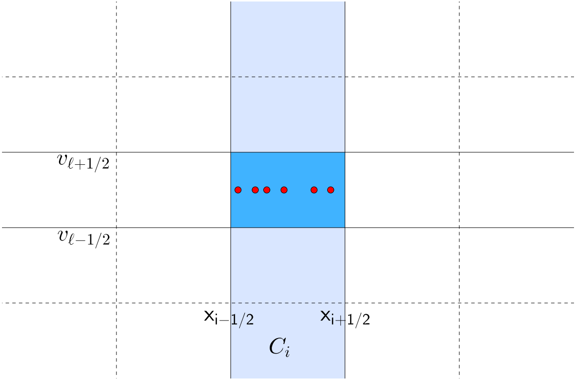
- Now, we want to sample $g(t=0, \mathbf{x}, \mathbf{v})$, that has no sign.
- We impose $\omega_k \in \{m_p, -m_p\}$.
- For velocities, we impose $\mathbf{v}_\mathbf{k}^\mathbf{n}$ on a cartesian grid in $\mathbb{R}^{d_v}$. For $d_v = 1$, it writes $v_k^n \in \{\mathrm{v}_\ell,\ \ell = 0, \ldots, N_v - 1\}$ $\forall k = 1, \ldots, N^n$, where $\mathrm{v}_\ell = v_{\min} + \ell \Delta v$, $\ell = 0, \ldots, N_v - 1$.

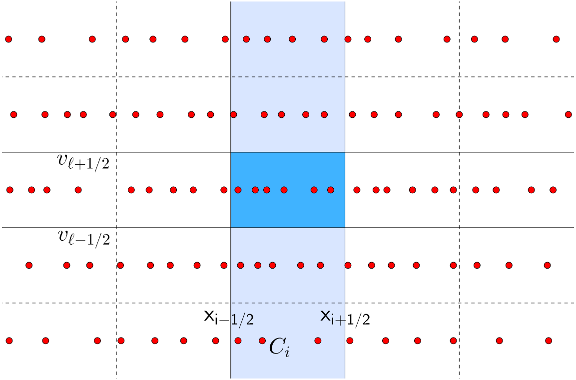
Let us introduce the notations in 1D...



Let us introduce the notations in 1D...

- The number of initial positive (resp. negative) particles having the velocity $v_k = \mathrm{v}_\ell$ in the strip $C_i = \left[\mathrm{x}_{i-1/2}, \mathrm{x}_{i+1/2}\right] \times \mathbb{R}$ is given by
$$N_{i,\ell}^{0,\pm} = \lfloor \pm \frac{\Delta x \Delta v}{m_p} g^{\pm}(t=0, \mathrm{x}_i, \mathrm{v}_\ell) \rfloor,$$
that is an approximation of
$$N_{i,\ell}^{0,\pm} = \pm \frac{1}{m_p} \int_{\mathrm{x}_i - \Delta x/2}^{\mathrm{x}_i + \Delta x/2} \int_{\mathrm{v}_\ell - \Delta v/2}^{\mathrm{v}_\ell + \Delta v/2} g^{\pm}(t=0, x, v) dv dx,$$
with $g^{\pm} = \frac{g \pm |g|}{2}$ the positive (resp. negative) part of g.
- Positions of these $N_{i,\ell}^{0,\pm}$ particles are taken uniformly in $\left[\mathrm{x}_{i-1/2}, \mathrm{x}_{i+1/2}\right]$.
- At time $t=0$, we have $N^0 = \sum_i \left(\sum_\ell N_{i,\ell}^{0,+} + \sum_\ell N_{i,\ell}^{0,-} \right)$.


$v_{\ell+1/2}$
$v_{\ell-1/2}$
$x_{i-1/2}$
$x_{i+1/2}$
C_i


$v_{\ell+1/2}$
$v_{\ell-1/2}$
$x_{i-1/2}$
$x_{i+1/2}$
C_i

From t^n to t^{n+1}

Solve the micro equation (7) by Monte Carlo technique.

- Splitting between the transport part

$$\partial_t g + \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v} \cdot \nabla_{\mathbf{x}} g = 0,$$

and the interaction part

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \left(\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M \right).$$

- Solve the transport part thanks to motion equation:

$$\frac{d\mathbf{x}_k}{dt}(t) = \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v}_k, \qquad \mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \varepsilon (1 - e^{-\Delta t/\varepsilon^2}) \mathbf{v}_k^n.$$

Remark that $\mathbf{v}_k^{n+1} = \mathbf{v}_k^n$.

- Solve interaction part by writing

$$g^{n+1} = e^{-\Delta t/\varepsilon^2}\tilde{g}^n + (1 - e^{-\Delta t/\varepsilon^2})\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\tilde{g}\rangle^n M\big]$$

where $\tilde{g}^n$ is the function after the transport part.

Apply a Monte Carlo technique:

- with probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change,
- with probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by

$$\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\tilde{g}\rangle^n M\big].$$

In practice (1)

"With probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change."

$\rightarrow$ In each strip C_i

- we choose randomly $e^{-\Delta t/\varepsilon^2}\tilde{N}_i^n$ particles and keep them unchanged (with $\tilde{N}_i^n$ the number of particles in C_i after the transport part),
- we discard the others.

In practice (2)

"With probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by $\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\tilde{g}\rangle^n M\big]$."

$\rightarrow$ We define the function

$$\mathcal{P}^{n,\pm}(\mathbf{x},\mathbf{v}) = \varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle\mathbf{v}\tilde{g}\rangle^n M\big]^{\pm}.$$

$\rightarrow$ In each strip C_i

- we sample a corresponding number $M_i^{n,\pm}$ of new particles with weights $\pm m_p$ from $(1 - e^{-\Delta t/\varepsilon^2})\mathcal{P}^{n,\pm}(\mathbf{x}_i,\mathbf{v})$,
- in 1D, we have

$$M_{i,\ell}^{n,\pm} = \frac{1}{m_p}\int_{x_i-\Delta x/2}^{x_i+\Delta x/2}\int_{v_\ell-\Delta v/2}^{v_\ell+\Delta v/2} \pm(1-e^{-\Delta t/\varepsilon^2})\mathcal{P}^{n,\pm}(x,v)\,dv\,dx,$$

- these $M_{i,\ell}^{n,\pm}$ created particles are such that $v_k^n = v_\ell$ and x_k^n are uniformly distributed in $\left[\mathrm{x}_{i-1/2},\mathrm{x}_{i+1/2}\right]$.

Asymptotically Complexity Diminishing Property

- At the end of the time step, we have in each strip C_i

$$N_i^{n+1} = e^{-\Delta t/\varepsilon^2} \tilde{N}_i^n + \sum_{\ell} \left(M_{i,\ell}^{n,+} + M_{i,\ell}^{n,-} \right)$$

particles.

- The number of particles automatically diminishes with ε.
- Reduction of the computational complexity when approaching equilibrium: Asymptotically Complexity Diminishing Property.

Macro equation

- Equation $\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0$.
- First proposition:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g^{n+1} \rangle = 0,$$

discretized by a Finite Volume method. For example in 1D:

$$\rho_i^n \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \rho(t^n, x) \mathrm{d}x,$$

$$\langle v g^n \rangle_i = \frac{1}{\Delta x} \sum_{x_k^n \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]} v_k \omega_k \approx \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \langle v g \rangle (t^n, x) \mathrm{d}x.$$

- Problem: g^{n+1} suffers from numerical noise inherent to particles method. This noise, amplified by $\frac{1}{\varepsilon}$, will damage ρ^{n+1}.

Correction of the macro discretization

- Use the expression of g^{n+1} and write

$$\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g^{n+1} \rangle = e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - \varepsilon(1 - e^{-\Delta t/\varepsilon^2}) \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \left(\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M \right) \rangle + \varepsilon(1 - e^{-\Delta t/\varepsilon^2}) \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \left(\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g} \rangle^n M \right) \rangle,$$

or after simplifications

$$\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g^{n+1} \rangle = e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - \varepsilon(1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^n.$$

- Plug it into the macro equation

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^n = 0.$$

- To avoid the parabolic CFL condition of type $\Delta t \leq C \Delta x^2$, take the diffusion term implicit:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t/\varepsilon^2} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^{n+1} = 0.$$

- No more stiffness, the numerical noise does not damage ρ.
- AP property: for fixed $\Delta t > 0$, the scheme degenerates when $\varepsilon \to 0$ to an implicit discretization of the diffusion equation $\partial_t \rho - \Delta_{\mathbf{x}} \rho = 0$.

Space discretization

- Macro equation is solved between transport and source parts of micro scheme.
- In 1D, solve

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\frac{1}{\varepsilon}e^{-\Delta t/\varepsilon^2}\frac{\langle v\tilde{g}^n\rangle_{i+1}-\langle v\tilde{g}^n\rangle_{i-1}}{2\Delta x}$$
$$-(1-e^{-\Delta t/\varepsilon^2})\frac{\rho_{i+1}^{n+1}-2\rho_i^{n+1}+\rho_{i-1}^{n+1}}{\Delta x^2}=0.$$

Space discretization in 2D

In 2D, we use an Alternating Direction Implicit (ADI) method[15]:

1) Starting from ρ^n, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{xx} \rho = 0,$$

using a Crank-Nicolson time discretization to get $\rho^\star$.

2) Starting from $\rho^\star$, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{yy} \rho = 0,$$

using a Crank-Nicolson time discretization to get ρ^{n+1}.

[15] Peaceman, Rachford, J. Soc. Indust. Appl. Math. 1955.

Nice properties

- Only 1D systems of size N_x or N_y.
- ADI method unconditionally stable in 2D.
- Straightforward extension in 3D: a priori conditionally stable, but better extensions have been derived[16].
- Right asymptotic behaviour.

[16] Sharma, Hammett, JCP 2011.

Test 1 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})=0$

Initialization:

$$f(t=0,\mathbf{x},\mathbf{v})=\rho(t=0,\mathbf{x})M(\mathbf{v}),\ \mathbf{x}\in[0,4\pi]^2,\ \mathbf{v}\in\mathbb{R}^2$$

with

$$\rho(t=0,\mathbf{x})=1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right),$$

$$M(\mathbf{v})=\frac{1}{2\pi}\exp\left(-\frac{|\mathbf{v}|^2}{2}\right),$$

so that

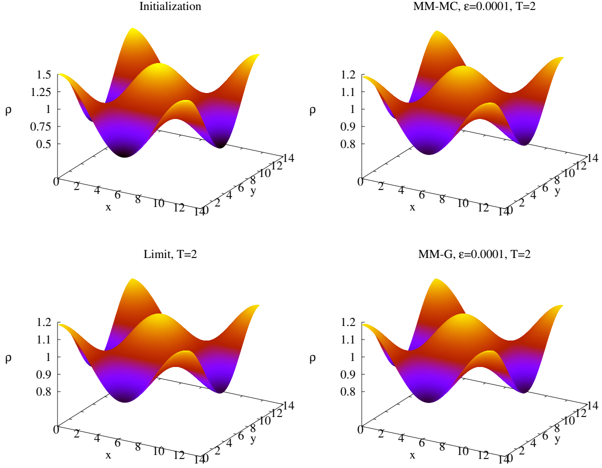
$$g(t=0,\mathbf{x},\mathbf{v})=0.$$

Periodic boundary conditions in space.

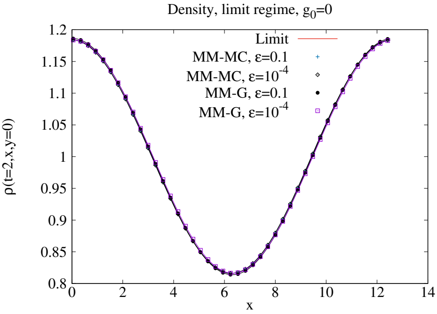
Asymptotic behaviour, $\varepsilon = 10^{-4}$

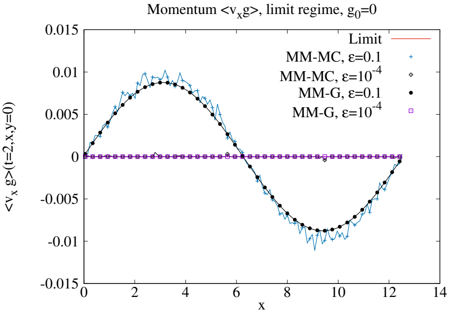
MM-MC: the presented Micro-Macro Monte Carlo scheme.

MM-G: a Micro-Macro Grid code, considered as reference.



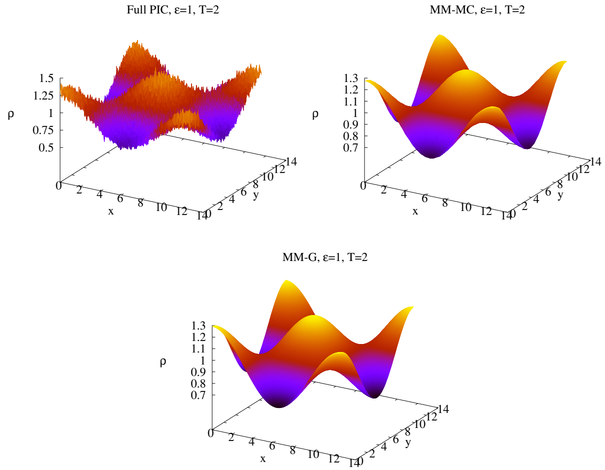
Slices of the density $\rho(T=2,x,y=0)$ and of the momentum $\langle v_x g\rangle(T=2,x,y=0)$.



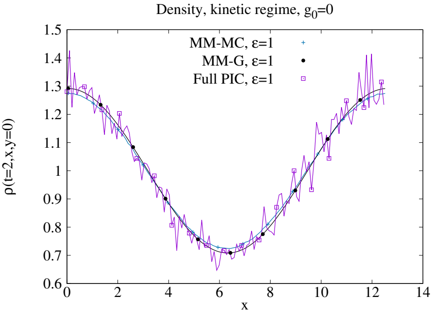


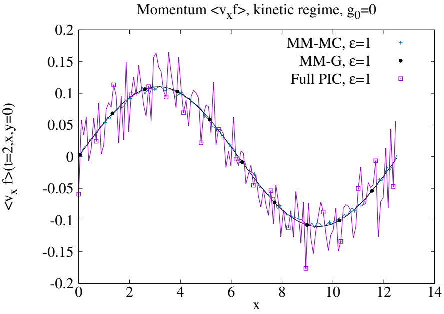
Kinetic regime, $\varepsilon = 1$

Full PIC: standard particle method on f.

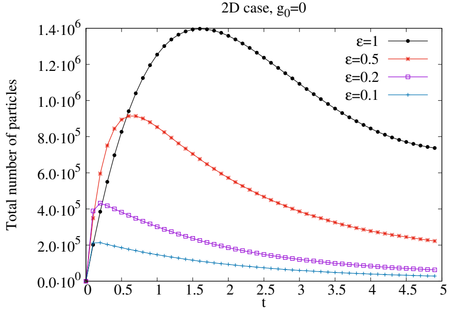


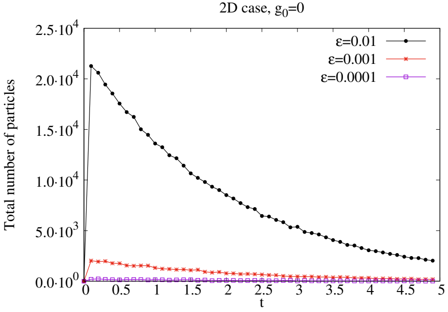
Slices of the density $\rho(T=2,x,y=0)$ and of the momentum $\langle v_x g\rangle(T=2,x,y=0)$.





Time evolution of the number of particles





Test 2 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Initialization:

$$f(t=0,\mathbf{x},\mathbf{v})=\frac{1}{4\pi}\left(\exp\left(-\frac{|\mathbf{v}-2|^2}{2}\right)+\exp\left(-\frac{|\mathbf{v}+2|^2}{2}\right)\right)\rho(t=0,\mathbf{x}),$$

with

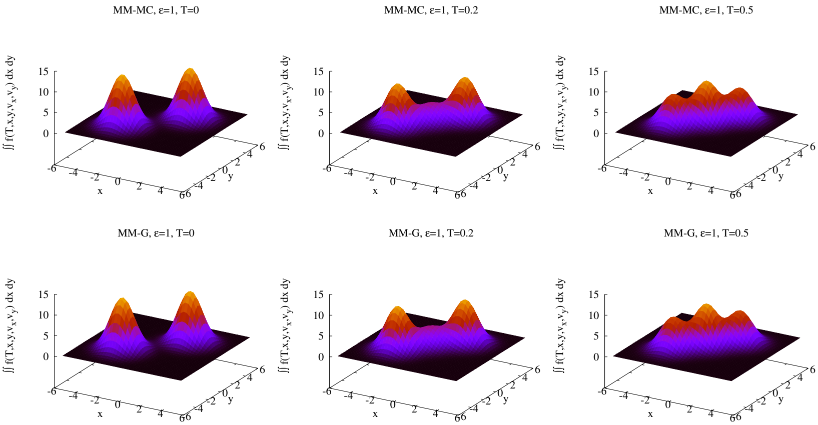
$$\mathbf{x}\in[0,4\pi]^2,\quad \mathbf{v}\in\mathbb{R}^2,$$

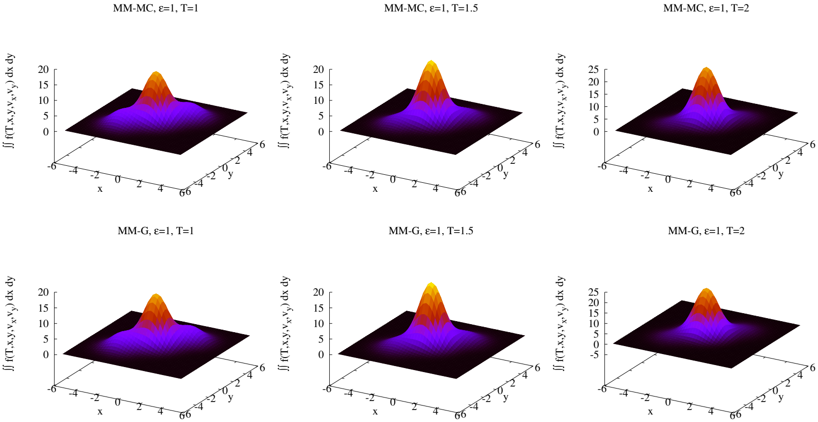
$$\rho(t=0,\mathbf{x})=1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right),$$

so that

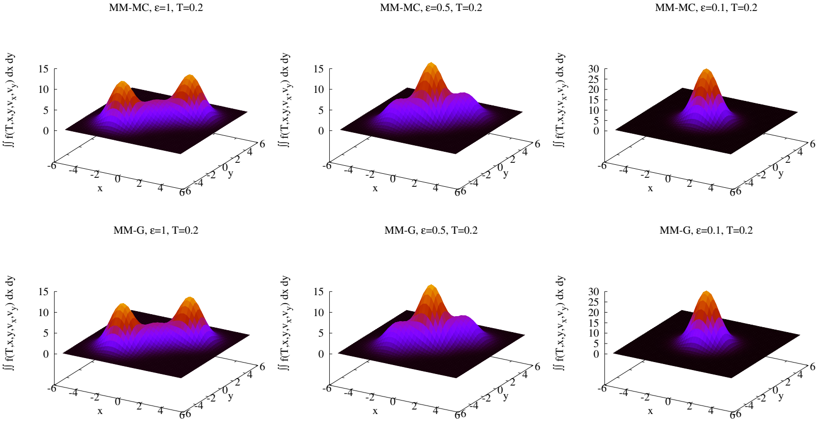
$$g(t=0,\mathbf{x},\mathbf{v})=\rho(t=0,\mathbf{x})M(\mathbf{v})-f(t=0,\mathbf{x},\mathbf{v})\neq 0.$$

Integral of the distribution function in space $\int f(T,\mathbf{x},\mathbf{v})dx$.

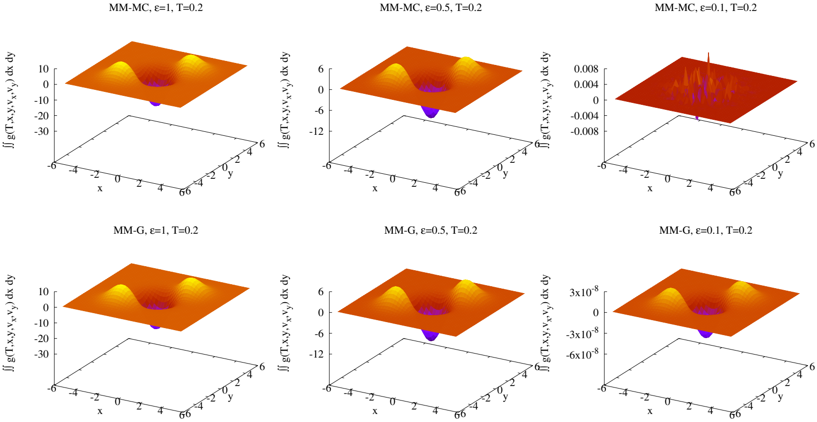



MM-MC, ε=1, T=1
MM-MC, ε=1, T=1.5
MM-MC, ε=1, T=2
MM-G, ε=1, T=1
MM-G, ε=1, T=1.5
MM-G, ε=1, T=2
∬ f(T,x,y,v_x,v_y) dx dy
x
y

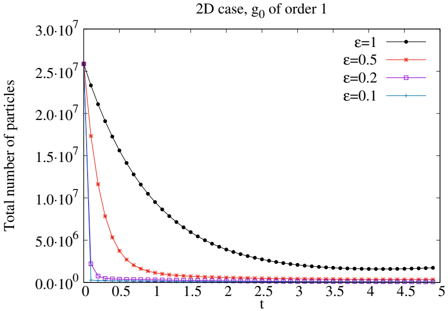
Influence of ε



Integral of the perturbation in space $\int g(T=0.2,\mathbf{x},\mathbf{v})d\mathbf{x}$ for different ε.



Time evolution of the number of particles



Test 3 - 3Dx3D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Initialization:

$$f_0(\mathbf{x},\mathbf{v}) = \frac{1}{2\,(2\pi)^{3/2}}\left[\exp\Big(-\frac{|\mathbf{v}-u|^2}{2}\Big)+\exp\Big(-\frac{|\mathbf{v}+u|^2}{2}\Big)\right]\rho(0,\mathbf{x}),$$

with $u=(2,2,2)$,

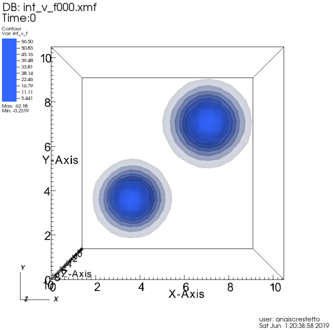
$$\rho(0,\mathbf{x}) = 1+\frac{1}{2}\cos(\frac{x}{2})\cos(\frac{y}{2})\cos(\frac{z}{2}),$$

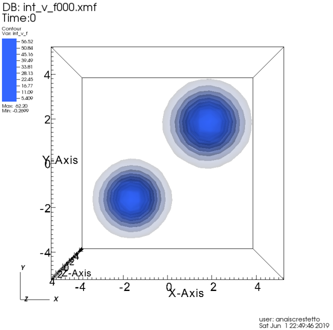
$\mathbf{x}=(x,y,z)\in[0,4\pi]^3$, $\mathbf{v}=(v_x,v_y,v_z)\in\mathbb{R}^3$.

Full $d_x = d_v = 3$ case

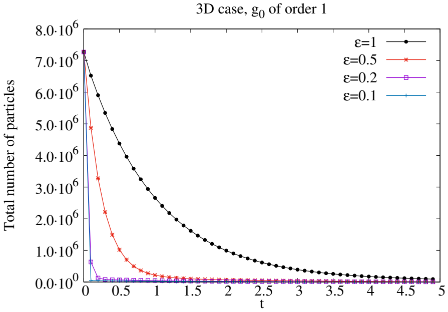
Integral of the distribution function in space $\int_{\mathbf{x}} f(T,\mathbf{x},\mathbf{v})dx$

Left: $\varepsilon = 1$, right: $\varepsilon = 0.5$, $T = 1$.





Time evolution of the number of particles



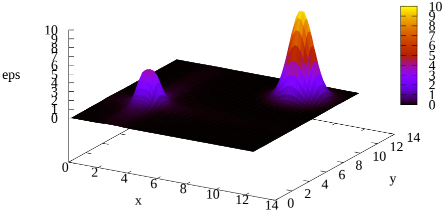
Test 4 - 2Dx2D, $\varepsilon(\mathbf{x})$, $g(t=0,\mathbf{x},\mathbf{v})\neq 0$

Modified model:

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2(\mathbf{x})}(\rho M - f),$$

where $(\mathbf{x},\mathbf{v})\in[0,4\pi]^2\times\mathbb{R}^2$,

$$\varepsilon(\mathbf{x}) = 10\left[\mathrm{atan}\Big(2(y-5)\Big)+\mathrm{atan}\Big(-2(y-5)\Big)\right]$$
$$\times\exp\Big(-(x-10)^2-(y-10)^2\Big)+10^{-3}.$$



Initialization:

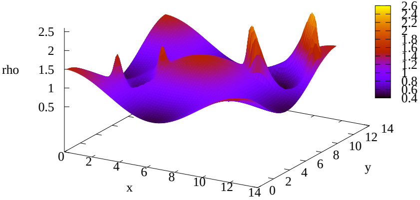
$$f(t=0,\mathbf{x},\mathbf{v}) = \frac{1}{4\pi}\left(\exp\left(-\frac{|\mathbf{v}-2|^2}{2}\right)+\exp\left(-\frac{|\mathbf{v}+2|^2}{2}\right)\right)\rho(t=0,\mathbf{x}),$$

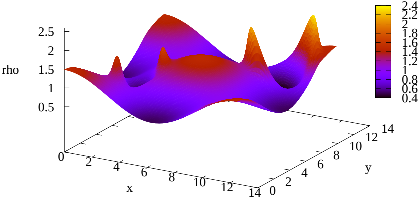
with

$$\mathbf{x}\in[0,4\pi]^2,\quad \mathbf{v}\in\mathbb{R}^2,$$

$$\rho(t=0,\mathbf{x}) = 1+\frac{1}{2}\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right).$$

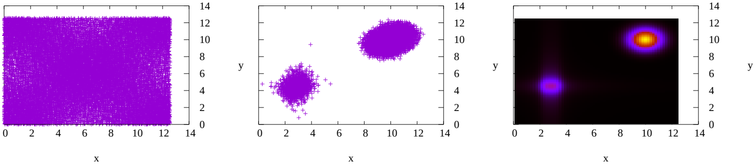
Density profile $\rho(T=1,x,y)$. Left: MM-MC, right: MM-G.

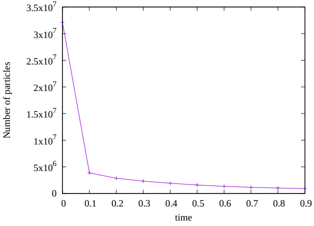




Asymptotically Complexity Diminishing Property

Top: position of the particles in $\mathbf{x}$.
Left: at $T=0$; middle: at $T=1$. Right: $\varepsilon(x,y)$.





Bottom: time evolution of the number of particles.

Conclusions

- Right asymptotic behaviour.
- Computational cost diminishes as the equilibrium is approached.
- Numerical noise smaller than a standard particle method on f.
- Implicit treatment of the diffusion term.
- Suitable for multi-dimensional testcases.
- Somehow an automatic domain decomposition method without imposing any artificial transition to pass from the microscopic to the macroscopic model.

Possible extensions

- More 3D-3D testcases, more physical relevance.
- Boltzmann operator.
- Second-order in time scheme.
- Add an electromagnetic field $\Rightarrow v_k$ no constant anymore.

Thank you for your attention!