Micro-Macro Monte Carlo approach for collisional kinetic equations of Boltzmann type

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Outline

Our first multi-scale BGK Problem

Radiative transport equation in the diffusive scaling

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2}Q(f,f), \quad Q(f,f) = (\rho M - f) \tag{1}$$

- $\mathbf{x} \in \Omega \subset \mathbb{R}^{d_x}$, $\mathbf{v} \in V = \mathbb{R}^{d_v}$,
- charge density $\rho(t,\mathbf{x}) = \int_V f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$,
- $M(\mathbf{v}) = \frac{1}{(2\pi)^{d_v/2}}\exp\left(-\frac{|\mathbf{v}|^2}{2}\right)$,
- periodic conditions in $\mathbf{x}$ and initial conditions.

Main difficulty:

- Knudsen number ε may be of order 1 or tend to 0 in the diffusive scaling. The asymptotic diffusion equation being

$$\partial_t \rho - \Delta_{\mathbf{x}}\rho = 0. \tag{2}$$

Objectives

- Construction of an Asymptotic Preserving (AP) scheme [5].
- Reduction of the numerical cost at the limit $\varepsilon \to 0$.

Tools

- Micro-macro decomposition [6,7] for this model. Previous work with a grid in v for the micro part [8], cost was constant w.r.t. ε.
- Particle method for the micro part since few information in v is necessary at the limit [9].
- Monte Carlo techniques [10].

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[5] Jin, SISC 1999.
[6] Lemou, Mieussens, SIAM SISC 2008.
[7] Liu, Yu, CMP 2004.
[8] Crouseilles, Lemou, KRM 2011.
[9] C., Crouseilles, Lemou, CMS 2018.
[10] Degond, Dimarco, Pareschi, IJNMF 2011.

Micro-macro decomposition

- Micro-macro decomposition:

$$f = \rho M + g$$

  with g the perturbation.
- $\mathcal{N} = \mathrm{Span}\,\{M\} = \{f = \rho M\}$ null space of the BGK operator $Q\,(f) = \rho M - f$.
- Π orthogonal projection onto $\mathcal{N}$:

$$\Pi h := \langle h \rangle M, \quad \langle h \rangle := \int_V h \ \mathrm{d}\mathbf{v}.$$

- Applying Π to (1) $\Longrightarrow$ macro equation on ρ

$$\partial_t \rho + \frac{1}{\varepsilon}\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0. \tag{3}$$

- Applying $(I - \Pi)$ to (1) $\Longrightarrow$ micro equation on g

$$\partial_t g + \frac{1}{\varepsilon}\left[\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M\right] = -\frac{1}{\varepsilon^2} g. \tag{4}$$

Equation (1) $\Leftrightarrow$ micro-macro system:

$$\begin{cases} \partial_t \rho + \dfrac{1}{\varepsilon}\langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle = 0, \\ \partial_t g + \dfrac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\dfrac{1}{\varepsilon^2} g, \end{cases} \tag{5}$$

where $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

Difficulties

- Stiff terms in the micro equation (4) on g.
- In previous works [11,12], stiffest term (of order $1/\varepsilon^2$) considered implicit in time $\Longrightarrow$ transport term (of order $1/\varepsilon$) stabilized.

But here:

- use of particles for the micro part

$\Rightarrow$ splitting between the transport term and the source term,

$\Rightarrow$ not possible to use the same strategy.

Idea?

- Suitable reformulation of the model.

[11] Lemou, Mieussens, SIAM SISC 2008.

[12] Crouseilles, Lemou, KRM 2011.

- Strategy [13]:
  1. rewrite (4) $\partial_t g + \frac{1}{\varepsilon}\mathcal{F}(\rho, g) = -\frac{1}{\varepsilon^2} g$ as
  $$\partial_t(e^{t/\varepsilon^2} g) = -\frac{e^{t/\varepsilon^2}}{\varepsilon}\mathcal{F}(\rho, g),$$
  2. integrate in time between two times t^n and $t^{n+1} = t^n + \Delta t$:
  $$e^{t^{n+1}/\varepsilon^2} g^{n+1} = e^{t^n/\varepsilon^2} g^n + \int_{t^n}^{t^{n+1}} -\frac{e^{t/\varepsilon^2}}{\varepsilon}\mathcal{F}(\rho, g)\mathrm{d}t,$$
  3. use rectangle method for $\mathcal{F}(\rho, g)$ and multiply by $e^{-t^{n+1}/\varepsilon^2}/\Delta t$:
  $$\frac{g^{n+1} - g^n}{\Delta t} = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g^n - \varepsilon\frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho^n, g^n) + \mathcal{O}(\Delta t),$$
  4. approximate up to terms of order $\mathcal{O}(\Delta t)$ by:
  $$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon\frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t}\mathcal{F}(\rho, g).$$
- No more stiff terms and consistent with initial micro equation (4).

[13] Lemou, CRAS 2010.

New micro-macro model

The new micro-macro model writes

$$\partial_t \rho + \frac{1}{\varepsilon} \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0, \tag{6}$$

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathcal{F}(\rho, g), \tag{7}$$

with $\mathcal{F}(\rho, g) = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M + \mathbf{v} \cdot \nabla_{\mathbf{x}} g - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M$.

We propose the following hybrid discretization:

- macro equation (6): Finite Volume method,
- micro equation (7): Monte Carlo technique.

Discretization of the micro equation

- Model: considering at each time step N^n particles, with position $\mathbf{x}_k^n$, velocity $\mathbf{v}_k^n$ and constant weight ω_k, $k = 1, \dots, N^n$, g is approximated by [14]

$$g_{N^n}\left(t^n, \mathbf{x}, \mathbf{v}\right) = \sum_{k=1}^{N^n} \omega_k \delta\left(\mathbf{x} - \mathbf{x}_k^n\right) \delta\left(\mathbf{v} - \mathbf{v}_k^n\right).$$

- For the coupling with the macro equation, we need a grid in $\mathbf{x}$. For $d_x = 1$, we define for $i = 0, \dots, N_x - 1$

$$\mathrm{x}_i = x_{\min} + i\Delta x, \quad \mathrm{x}_{i\pm 1/2} = \mathrm{x}_i \pm \frac{\Delta x}{2}.$$

- How to define/compute ω_k, N^n, $\mathbf{x}_k^n$, $\mathbf{v}_k^n$?

[14] Crouseilles, Dimarco, Lemou, KRM 2017.

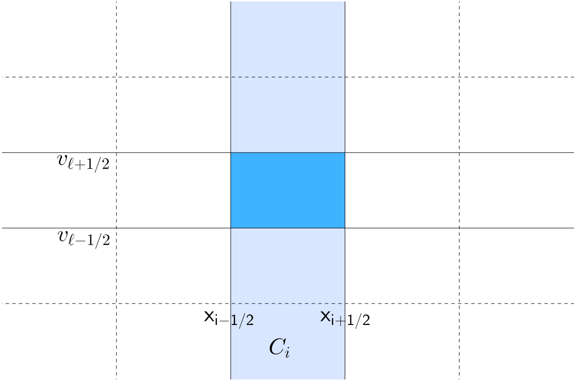
Initialization

- Choose the characteristic weight m_p or the characteristic number of particles N_p necessary to sample the full distribution function f, and link them with

$$m_p = \frac{1}{N_p} \int_{\mathbb{R}^{d_x}} \int_{\mathbb{R}^{d_v}} f(t=0, \mathbf{x}, \mathbf{v}) d\mathbf{v} d\mathbf{x}.$$

- Now, we want to sample $g(t=0, \mathbf{x}, \mathbf{v})$, that has no sign.
- We impose $\omega_k \in \{m_p, -m_p\}$.
- For velocities, we impose $\mathbf{v}_k^n$ on a cartesian grid in $\mathbb{R}^{d_v}$. For $d_v = 1$, we have $\mathrm{v}_k^n \in \{v_\ell,\ \ell = 0, \dots, N_v - 1\}$ $\forall k = 1, \dots, N^n$, where $v_\ell = v_{\min} + \ell \Delta v$, $\ell = 0, \dots, N_v - 1$ and $v_{\ell \pm 1/2} = v_\ell \pm \frac{\Delta x}{2}$.

Let us introduce the notations in 1D...



Let us introduce the notations in 1D...

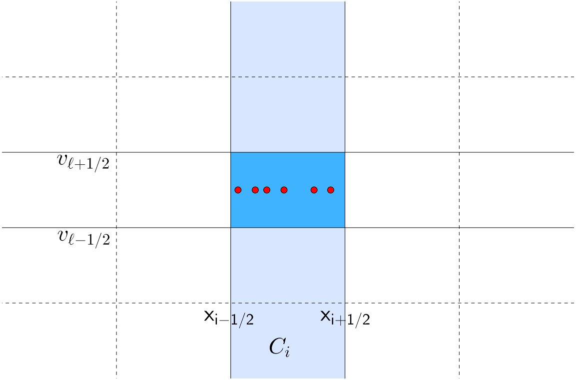
- The number of initial positive (resp. negative) particles having the velocity $\mathrm{v}_k = v_\ell$ in the strip $C_i = \left[\mathrm{x}_{i-1/2}, \mathrm{x}_{i+1/2}\right] \times \mathbb{R}$ is given by

$$N_{i,\ell}^{0,\pm} = \lfloor \pm \frac{\Delta x \Delta v}{m_p} g^{\pm}(t=0, \mathrm{x}_i, v_\ell) \rfloor,$$

  that is an approximation of

$$N_{i,\ell}^{0,\pm} = \pm \frac{1}{m_p} \int_{\mathrm{x}_{i-1/2}}^{\mathrm{x}_{i+1/2}} \int_{v_{\ell-1/2}}^{v_{\ell+1/2}} g^{\pm}(t=0, x, v) dv dx,$$

  with $g^{\pm} = \frac{g \pm |g|}{2}$ the positive and negative parts of g.
- Positions of these $N_{i,\ell}^{0,\pm}$ particles are taken uniformly in $\left[\mathrm{x}_{i-1/2}, \mathrm{x}_{i+1/2}\right]$.
- At time $t = 0$, we have $N^0 = \sum_i \left(\sum_\ell N_{i,\ell}^{0,+} + \sum_\ell N_{i,\ell}^{0,-} \right)$.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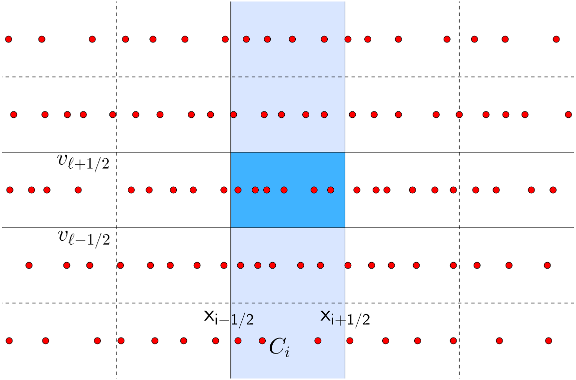
$v_{\ell-1/2}$
$x_{i-1/2}$
$x_{i+1/2}$
C_i


$v_{\ell+1/2}$
$v_{\ell-1/2}$
$x_{i-1/2}$
$x_{i+1/2}$
C_i

From t^n to t^{n+1}

Solve the micro equation (7) by Monte Carlo technique.

- Splitting between the transport part

$$\partial_t g + \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v} \cdot \nabla_{\mathbf{x}} g = 0,$$

and the interaction part

$$\partial_t g = \frac{e^{-\Delta t/\varepsilon^2} - 1}{\Delta t} g - \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \left(\mathbf{v} \cdot \nabla_{\mathbf{x}} \rho M - \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} g \rangle M \right).$$

- Solve the transport part thanks to motion equation:

$$\frac{d\mathbf{x}_k}{dt}(t) = \varepsilon \frac{1 - e^{-\Delta t/\varepsilon^2}}{\Delta t} \mathbf{v}_k, \qquad \mathbf{x}_k^{n+1} = \mathbf{x}_k^n + \varepsilon (1 - e^{-\Delta t/\varepsilon^2}) \mathbf{v}_k^n.$$

Remark that $\mathbf{v}_k^{n+1} = \mathbf{v}_k^n$.

- Solve interaction part by writing

$$g^{n+1} = e^{-\Delta t/\varepsilon^2}\tilde{g}^n + (1 - e^{-\Delta t/\varepsilon^2})\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big]$$

where $\tilde{g}^n$ is the function after the transport part.

Apply a Monte Carlo technique:

- with probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change,
- with probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by

$$\varepsilon\big[-\mathbf{v}\cdot\nabla_{\mathbf{x}}\rho^n M + \nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}\rangle^n M\big].$$

In practice (1)

"With probability $e^{-\Delta t/\varepsilon^2}$, the distribution g does not change."

$\rightarrow$ In each strip C_i

- we choose randomly $e^{-\Delta t/\varepsilon^2}\tilde{N}_i^n$ particles and keep them unchanged (with $\tilde{N}_i^n$ the number of particles in C_i after the transport part),
- we discard the others.

In practice (2)

"With probability $(1 - e^{-\Delta t/\varepsilon^2})$, the distribution g is replaced by a new distribution given by $\varepsilon\big[- \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}\tilde{g} \rangle^n M \big]$."

→ We define the function

$$\mathcal{P}^{n,\pm}(\mathbf{x}, \mathbf{v}) = \varepsilon\big[- \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho^n M + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v}\tilde{g} \rangle^n M \big]^{\pm}.$$

→ In each strip C_i

- we sample a corresponding number $M_i^{n,\pm}$ of new particles with weights $\pm m_p$ from $(1 - e^{-\Delta t/\varepsilon^2})\mathcal{P}^{n,\pm}(\mathbf{x}_i, \mathbf{v})$,
- in 1D, we have

$$M_{i,\ell}^{n,\pm} = \frac{1}{m_p} \int_{\mathrm{x}_{i-1/2}}^{\mathrm{x}_{i+1/2}} \int_{v_{\ell-1/2}}^{v_{\ell+1/2}} \pm(1 - e^{-\Delta t/\varepsilon^2})\mathcal{P}^{n,\pm}(x, v) dv dx,$$

- these $M_{i,\ell}^{n,\pm}$ created particles are such that $\mathrm{v}_k^n = v_\ell$ and x_k^n are uniformly distributed in $\big[\mathrm{x}_{i-1/2}, \mathrm{x}_{i+1/2}\big]$.

Asymptotically Complexity Diminishing Property

- At the end of the time step, we have in each strip C_i

$$N_i^{n+1} = e^{-\Delta t/\varepsilon^2} \tilde{N}_i^n + \sum_{\ell} \left(M_{i,\ell}^{n,+} + M_{i,\ell}^{n,-} \right)$$

particles.

- The number of particles automatically diminishes with ε.
- Reduction of the computational complexity when approaching equilibrium: Asymptotically Complexity Diminishing Property.

Macro equation

- Equation $\partial_t \rho + \frac{1}{\varepsilon}\nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g \rangle = 0$.
- First proposition:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon}\nabla_{\mathbf{x}} \cdot \langle \mathbf{v} g^{n+1} \rangle = 0,$$

  discretized in space by a Finite Volume method.
- Problem: g^{n+1} suffers from numerical noise inherent to particles method. This noise, amplified by $\frac{1}{\varepsilon}$, will damage ρ^{n+1}.
- Use the expression of g^{n+1} and plug it into the macro equation

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \Delta_{\mathbf{x}} \rho^n = 0.$$

- To avoid the parabolic CFL condition of type $\Delta t \leq C\Delta x^2$, take the diffusion term implicit:

$$\frac{\rho^{n+1}-\rho^n}{\Delta t}+\frac{1}{\varepsilon}e^{-\Delta t/\varepsilon^2}\nabla_{\mathbf{x}}\cdot\langle \mathbf{v}\tilde{g}^n\rangle-(1-e^{-\Delta t/\varepsilon^2})\Delta_{\mathbf{x}}\rho^{n+1}=0.$$

- No more stiffness, the numerical noise does not damage ρ.
- AP property: for fixed $\Delta t > 0$, the scheme degenerates when $\varepsilon \to 0$ to an implicit discretization of the diffusion equation $\partial_t \rho - \Delta_{\mathbf{x}}\rho = 0$.

Space discretization in 2D

In 2D, we use an Alternating Direction Implicit (ADI) method [15]:

1) Starting from ρ^n, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{xx} \rho = 0,$$

using a Crank-Nicolson time discretization to get $\rho^\star$.

2) Starting from $\rho^\star$, solve over a time step Δt

$$\partial_t \rho + \frac{1}{2\varepsilon} e^{-\Delta t/\varepsilon^2} \langle \mathbf{v} \cdot \nabla_{\mathbf{x}} \tilde{g}^n \rangle - (1 - e^{-\Delta t/\varepsilon^2}) \partial_{yy} \rho = 0,$$

using a Crank-Nicolson time discretization to get ρ^{n+1}.

[15] Peaceman, Rachford, J. Soc. Indust. Appl. Math. 1955.

Nice properties

- Only 1D systems of size N_x or N_y.
- ADI method unconditionally stable in 2D.
- Straightforward extension in 3D: a priori conditionally stable, but better extensions have been derived [16].
- Right asymptotic behaviour.

[16] Sharma, Hammett, JCP 2011.

Test 1 - 2Dx2D, constant ε, $g(t=0,\mathbf{x},\mathbf{v})=0$

Initialization:

$$f(t=0,\mathbf{x},\mathbf{v}) = \rho_0(\mathbf{x})M(\mathbf{v}),\ \ \mathbf{x}\in[0,4\pi]^2,\ \ \mathbf{v}\in\mathbb{R}^2$$

with

$$\rho_0(\mathbf{x}) = 1+\frac{1}{2}\cos\left(\frac{\mathrm{x}}{2}\right)\cos\left(\frac{\mathrm{y}}{2}\right),$$

$$M(\mathbf{v}) = \frac{1}{2\pi}\exp\left(-\frac{|\mathbf{v}|^2}{2}\right),$$

so that

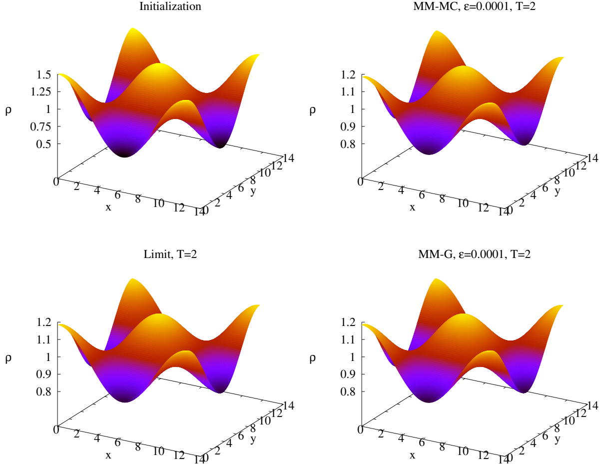
$$g(t=0,\mathbf{x},\mathbf{v}) = 0.$$

Periodic boundary conditions in space.

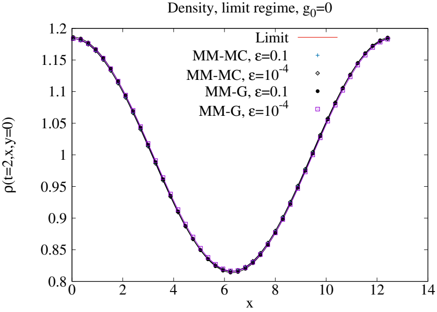
Asymptotic behaviour, $\varepsilon = 10^{-4}$

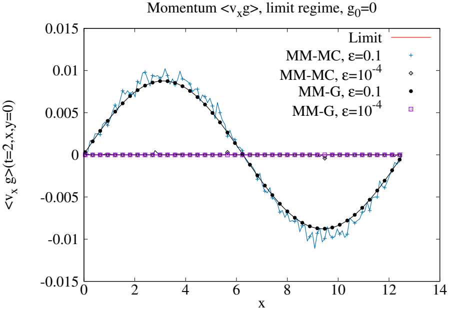
MM-MC: the presented Micro-Macro Monte Carlo scheme.

MM-G: a Micro-Macro Grid code, considered as reference.



Slices of the density $\rho(T=2,\mathrm{x},\mathrm{y}=0)$ and of the momentum $\langle \mathrm{v_x} g\rangle(T=2,\mathrm{x},\mathrm{y}=0)$.

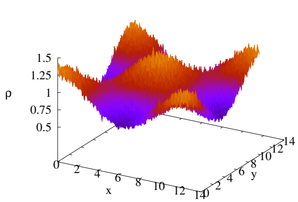




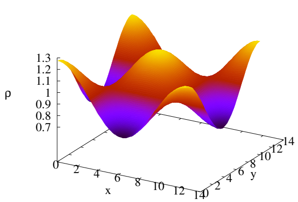
Kinetic regime, $\varepsilon = 1$

Full PIC: standard particle method on f.

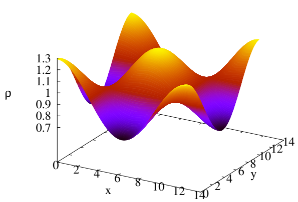
Full PIC, ε=1, T=2



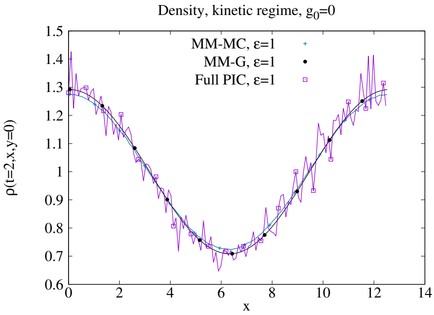
MM-MC, ε=1, T=2

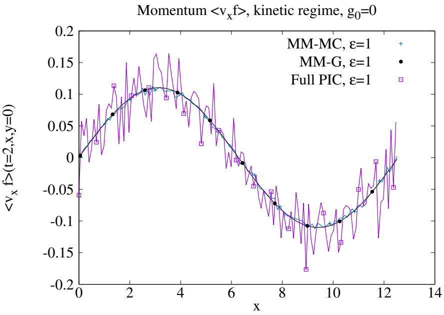


MM-G, ε=1, T=2

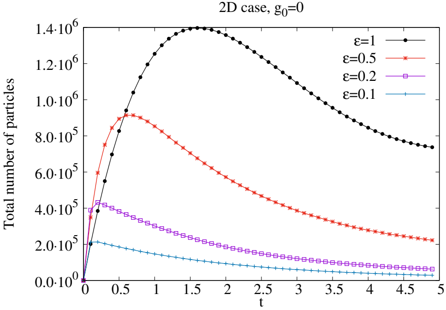


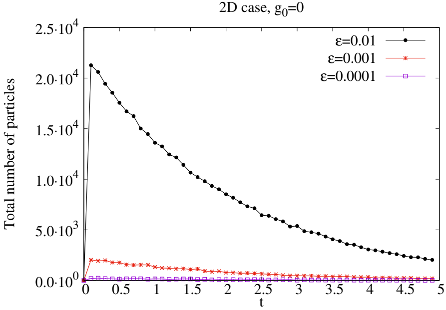
Slices of the density $\rho(T = 2, \mathrm{x}, \mathrm{y} = 0)$ and of the momentum $\langle \mathrm{v_x} f \rangle(T = 2, \mathrm{x}, \mathrm{y} = 0)$.





Time evolution of the number of particles





Test 3 - 3Dx3D, constant ε, $g(t=0,\mathbf{x},\mathbf{v}) \neq 0$

Initialization:

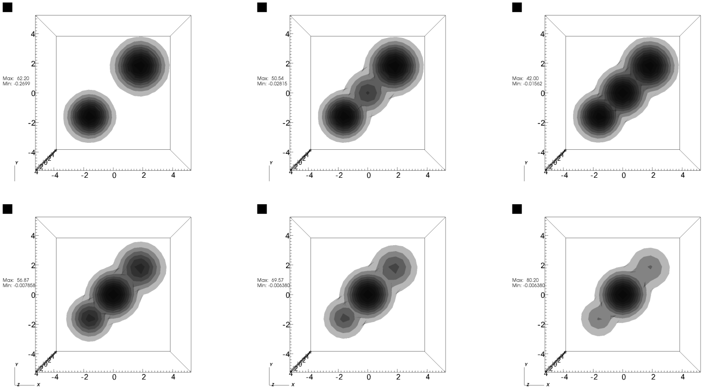
$$f(0,\mathbf{x},\mathbf{v}) = \frac{1}{2\,(2\pi)^{3/2}}\left[\exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2}\right)+\exp\left(-\frac{|\mathbf{v}+\mathbf{u}|^2}{2}\right)\right]\rho_0(\mathbf{x}),$$

with $\mathbf{u}=(2,2,2)$,

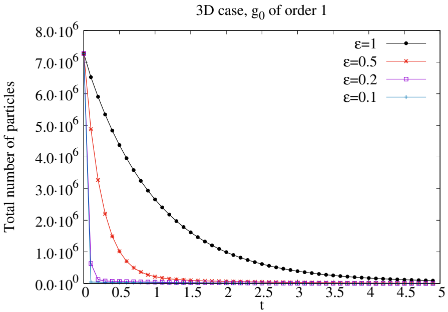
$$\rho_0(\mathbf{x}) = 1+\frac{1}{2}\cos\left(\frac{\mathrm{x}}{2}\right)\cos\left(\frac{\mathrm{y}}{2}\right)\cos\left(\frac{\mathrm{z}}{2}\right),$$

$\mathbf{x}=(\mathrm{x},\mathrm{y},\mathrm{z})\in[0,4\pi]^3$, $\mathbf{v}=(\mathrm{v_x},\mathrm{v_y},\mathrm{v_z})\in\mathbb{R}^3$.

Integral of the distribution function in space $\int_{\mathbf{x}} f(T, \mathbf{x}, \mathbf{v}) d\mathbf{x}$ for $\varepsilon = 1$ and different times (T=0, 0.2, 0.4, 0.6, 0.8, 1).



Time evolution of the number of particles



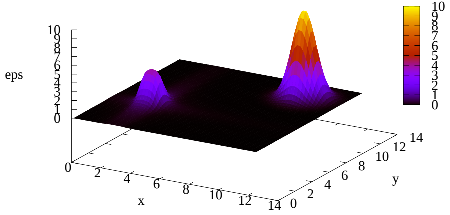
Test 4 - 2Dx2D, $\varepsilon(\mathbf{x})$, $g(t=0,\mathbf{x},\mathbf{v}) \neq 0$

Modified model:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{1}{\varepsilon^2(\mathbf{x})}(\rho M - f),$$

where $(\mathbf{x},\mathbf{v}) \in [0,4\pi]^2 \times \mathbb{R}^2$,

$$\varepsilon(\mathbf{x}) = 10\left[\mathrm{atan}\Big(2(\mathrm{y}-5)\Big) + \mathrm{atan}\Big(-2(\mathrm{y}-5)\Big)\right] \times \exp\Big(-(\mathrm{x}-10)^2 - (\mathrm{y}-10)^2\Big) + 10^{-3}.$$



Initialization:

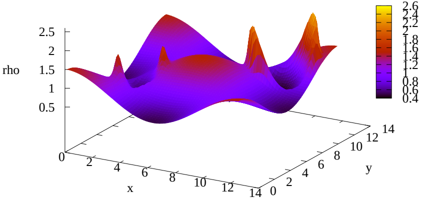
$$f(t=0,\mathbf{x},\mathbf{v}) = \frac{1}{4\pi}\left(\exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2}\right) + \exp\left(-\frac{|\mathbf{v}+\mathbf{u}|^2}{2}\right)\right)\rho_0(\mathbf{x}),$$

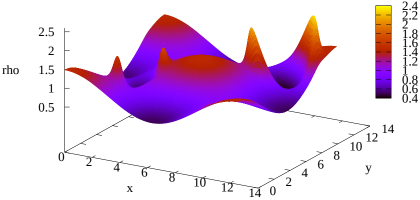
with

$$\mathbf{x}\in[0,4\pi]^2,\quad \mathbf{v}\in\mathbb{R}^2,\quad \mathbf{u}=(2,2)$$

$$\rho_0(\mathbf{x}) = 1+\frac{1}{2}\cos\left(\frac{\mathrm{x}}{2}\right)\cos\left(\frac{\mathrm{y}}{2}\right).$$

Density profile $\rho(T=1,\mathrm{x},\mathrm{y})$. Left: MM-MC, right: MM-G.

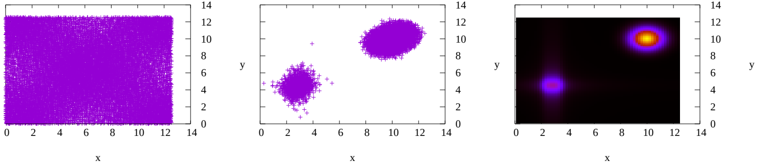


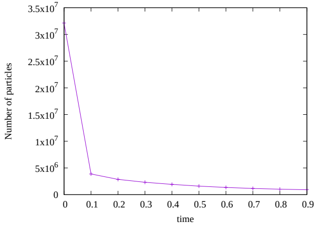


Asymptotically Complexity Diminishing Property

Top: position of the particles in $\mathbf{x}$.

Left: at $T = 0$; middle: at $T = 1$. Right: $\varepsilon(\mathrm{x}, \mathrm{y})$.





Bottom: time evolution of the number of particles.

Our space homogeneous Boltzmann Problem

Space homogeneous Boltzmann equation in the Maxwell molecules case

$$\partial_t f = \frac{1}{\varepsilon} Q(f,f), \tag{8}$$

- $f := f(t,\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^{d_v}$,
- $Q(f,f)(\mathbf{v}) =$
  $\int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v-1}} \underbrace{B\left(|\mathbf{v}-\mathbf{v}_\star|,\omega\right)}_{b(\omega)=C>0} \left(f(\mathbf{v}')f(\mathbf{v}'_\star) - f(\mathbf{v})f(\mathbf{v}_\star)\right) d\omega d\mathbf{v}_\star,$
- $\omega = \frac{\mathbf{v}'-\mathbf{v}'_\star}{|\mathbf{v}'-\mathbf{v}'_\star|}$ vector of the unitary sphere $\mathbb{S}^{d_v-1} \subset \mathbb{R}^{d_v}$,
- $(\mathbf{v}',\mathbf{v}'_\star)$ pre and $(\mathbf{v},\mathbf{v}_\star)$ post-collisional velocities linked by
  $\mathbf{v} = \frac{1}{2}\left(\mathbf{v}'+\mathbf{v}'_\star + |\mathbf{v}'-\mathbf{v}'_\star|\omega\right)$, $\mathbf{v}_\star = \frac{1}{2}\left(\mathbf{v}'+\mathbf{v}'_\star - |\mathbf{v}'-\mathbf{v}'_\star|\omega\right)$.

We can write $$Q(f,f) = P(f,f) - \mu f$$

with

- $P(f,f)(\mathbf{v}) = \int_{\mathbb{R}^{d_v}} \int_{\mathbb{S}^{d_v-1}} b(\omega)\, f(\mathbf{v}') f(\mathbf{v}'_\star) d\omega d\mathbf{v}_\star$
  the (bilinear) gain term,
- $\mu f(\mathbf{v}) = f(\mathbf{v}) \int_{\mathbb{R}^{d_v}} f(\mathbf{v}_\star) d\mathbf{v}_\star \int_{\mathbb{S}^{d_v-1}} b(\omega)\, d\omega$
  the loss term (mass preservation $\Rightarrow \mu$ constant).

We use the micro-macro decomposition
$f(t,\mathbf{v}) = M(\mathbf{v}) + g(t,\mathbf{v})$, where M is the gaussian function such that $\int_{\mathbb{R}^{d_v}} \phi(\mathbf{v}) f(t,\mathbf{v}) d\mathbf{v} = \int_{\mathbb{R}^{d_v}} \phi(\mathbf{v}) M(\mathbf{v}) d\mathbf{v}$,
$\phi(\mathbf{v}) = \left(1, \mathbf{v}, \mathbf{v}^2/2\right)^T$, and write

$$\partial_t g = \frac{1}{\varepsilon}\left(P(M+g, M+g) - \mu M\right) - \frac{\mu}{\varepsilon} g$$

or

$$\partial_t \left(g e^{\mu t/\varepsilon}\right) = \frac{1}{\varepsilon}\left(P(g,g) + P(M,g) + P(g,M)\right) e^{\mu t/\varepsilon}.$$

Our micro-macro Monte Carlo method

We use a first-order exponential scheme:

$$g^{n+1} = e^{-\frac{\mu\Delta t}{\varepsilon}} g^n + \frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}} \left(\frac{P(g^n, g^n) + P(M, g^n) + P(g^n, M)}{\mu} \right).$$

Monte Carlo interpretation: g represented by particles and

- with probability $e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles are not modified,
- with probability $\frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles collide,
- with probability $1 - e^{-\frac{\mu\Delta t}{\varepsilon}} - \frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}}$ particles are discarded.

How to perform collisions?

At time t^n, you have a set of N_+^n positive particles and a set of N_-^n negative particles.

Sample $P(g^n, g^n)/\mu$:

- Select $\frac{\mu\Delta t}{\varepsilon} e^{-\frac{\mu\Delta t}{\varepsilon}} (N_+^n + N_-^n)$ particles.
- For each one (k), select randomly a second one (j). Compute the new v_k^{n+1} thanks to collision rules.
- If particles k and j were both positives or both negatives, the new particle k belongs to the positive category. Else it belongs to the negative category.

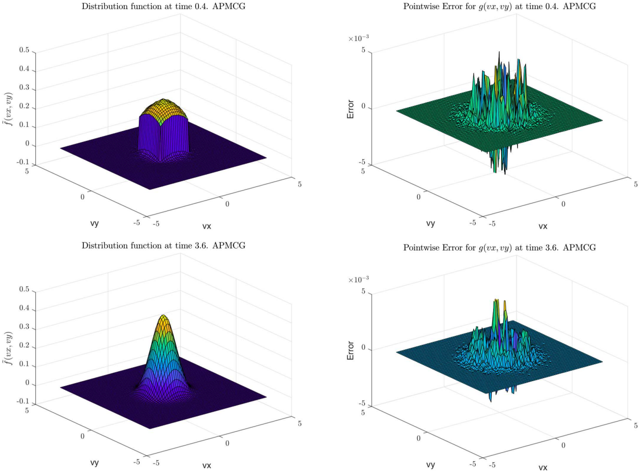
Sample $P(g^n, M)/\mu$ or $P(M, g^n)/\mu$:

- Same idea but instead of colliding two particles representing g, use one of g and one representing M.

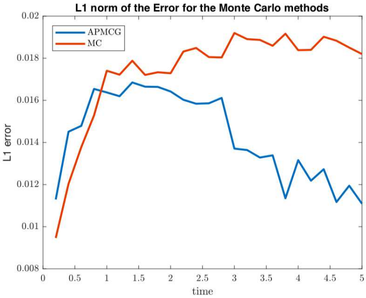
Test - 2D, f initialized as an indicator function

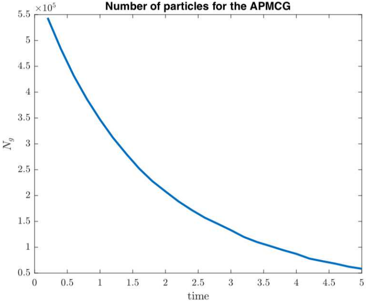
Left: distribution function f, right: error on g.

Top: T=0.4, bottom: T=3.6.



Time evolution of error and particles number





Thank you for your attention!