Numerical approximation of non classical solutions of Riemann problems

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Objective

• Approximate the Riemann solutions of systems of conservation laws in the form:

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \qquad x \in \mathbb{R}, \ t > 0,$$
 (1)

completed with an initial data given by

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_L & \text{if } x < 0, \\ \mathbf{u}_R & \text{if } x > 0, \end{cases}$$
(2)

where

- \mathbf{u}_L and \mathbf{u}_R are two given constant states in $\Omega \subset \mathbb{R}^d$,
- $\mathbf{u}(x,t) \in \Omega$ is the unknown state vector,
- $\mathbf{f}: \Omega \to \mathbb{R}^d$ is a given smooth flux function.
- Contrary to the usual approach, we do not enforce entropy criterion, so that we will be able to see non classical solutions.

Remarks

- In general, non uniqueness of weak solution of (1)-(2).
- Usual approach: entropy conditions are enforced to select one solution.
- Scalar classical case:
 - if f is convex (or concave): a shock, or a rarefaction wave,
 - if f is neither convex nor concave: composite wave (two appended waves).
- Here, we are interested in solutions that violate entropy conditions: non classical solutions (two separated waves).

Outline

1 From PDE to ODE

- 2 Application to a scalar problem
- 3 Application to a *p*-system problem

1 From PDE to ODE

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Diffusive-dispersive system of conservation laws

 Solution u ∈ Ω assumed to be the limit solution ^[see 4], when ε → 0, of

$$\partial_t \mathbf{u}^{\varepsilon} + \partial_x \mathbf{f}(\mathbf{u}^{\varepsilon}) = \varepsilon \beta \partial_{xx} \mathbf{D}(\mathbf{u}^{\varepsilon}) + \varepsilon^2 \gamma \partial_{xxx} \mathbf{D}(\mathbf{u}^{\varepsilon}), \quad (3)$$

with $\mathbf{D}: \Omega \to \mathbb{R}^d$ a given smooth function.

- Competition between diffusion that makes the solution smoother and dispersion that creates oscillations.
- Solutions \mathbf{u}^{ε} of (3) may depend on the ratio β/γ [see 5].
- If f is neither convex nor concave: waves may be separated, solutions may be non classical.

⁴LeFloch, Rohde 2001 (IUMJ) ⁵LeFloch 2002 (Book)

Dafermos viscosity approach

- Riemann solutions **u** of (1)-(2) are self-similar: only depend on the variable $\xi = x/t$.
- Solutions \mathbf{u}^{ε} of diffusive-dispersive problem (3) are not self-similar.
- Reformulation of the diffusive-dispersive system according to the Dafermos viscosity approach ^[see 6]:

$$\partial_t \mathbf{u}^{\varepsilon} + \partial_x \mathbf{f}(\mathbf{u}^{\varepsilon}) = \varepsilon t \beta \partial_{xx} \mathbf{D}(\mathbf{u}^{\varepsilon}) + \varepsilon^2 t^2 \gamma \partial_{xxx} \mathbf{D}(\mathbf{u}^{\varepsilon}).$$

⁶Dafermos 2010 (Book)

ODE problem

• Change of variables given by

$$\mathbf{u}^{\varepsilon}(x,t) = \mathbf{u}(\xi) \quad \text{with } \xi = \frac{x}{t}.$$

• ODE system

$$-\xi \mathbf{u}' + \mathbf{f}(\mathbf{u})' = \varepsilon \beta \mathbf{D}(\mathbf{u})'' + \varepsilon^2 \gamma \mathbf{D}(\mathbf{u})'''.$$

• Limit boundary conditions

$$\lim_{\xi \to -\infty} \mathbf{u}(\xi) = \mathbf{u}_L \quad \text{and} \quad \lim_{\xi \to +\infty} \mathbf{u}(\xi) = \mathbf{u}_R$$

replaced by

$$\mathbf{u}(-\ell) = \mathbf{u}_L$$
 and $\mathbf{u}(\ell) = \mathbf{u}_R$,

with $\ell > 0$ large enough [see 7]. ⁷Joseph, LeFloch 2007 (P ROY SOC EDINB A)

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Non classical Riemann problems

Numerical method

- 4th order Finite Difference scheme (N mesh points).
- For fixed $\varepsilon > 0$: existence of the discrete solution is proven.
- For fixed $\varepsilon > 0$: solving the nonlinear system in \mathbb{R}^N with a Newton method.
- Decrease ε to approach the limit solution and use the solution given by a higher ε as initial guess to make the convergence of the Newton method easier.

1 From PDE to ODE

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From scalar conservation law to ODE

• Diffusive-dispersive scalar equation

$$\partial_t w^\varepsilon + \partial_x f(w^\varepsilon) = \varepsilon t \beta \partial_{xx} w^\varepsilon + \varepsilon^2 t^2 \gamma \partial_{xxx} w^\varepsilon, \qquad x \in \mathbb{R}, \ t > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given smooth function.

• Change of variables

$$w^{\varepsilon}(x,t) = w(\xi)$$
 with $\xi = \frac{x}{t}$.

 $\bullet w$ solution of

$$-\xi w' + f(w)' = \varepsilon \beta w'' + \varepsilon^2 \gamma w''', \qquad (4)$$

with boundary conditions given by

$$w(-\ell) = w_L$$
 and $w(\ell) = w_R$, (5)

with $\ell > 0$ large enough.

Numerical scheme - Mesh

- Interval $[-\ell, \ell]$ discretized with N + 1 cells of length $\Delta \xi = \frac{2\ell}{N+1}$: $|\xi_i, \xi_{i+1}\rangle, \ \xi_i = -\ell + i\Delta \xi.$
- Notation $w_i \approx w(\xi_i)$.
- Need of ghost cells for the boundary conditions: $i = -2, \ldots, N + 3.$

Numerical scheme - 4th order Finite Differences

For a given vector $(X_i)_{i=-2,...,N+3}$ and a smooth function g(X), we consider the following discrete operators, for i = 1, ..., N:

$$\overline{g(X)}_{i}' = \frac{g(X_{i-2}) - 8g(X_{i-1}) + 8g(X_{i+1}) - g(X_{i+2})}{12\Delta\xi},$$

$$\overline{X}_{i}'' = \frac{-X_{i-2} + 16X_{i-1} - 30X_{i} + 16X_{i+1} - X_{i+2}}{12\Delta\xi^{2}},$$

$$\overline{X}_{i}''' = \frac{X_{i-3} - 8X_{i-2} + 13X_{i-1} - 13X_{i+1} + 8X_{i+2} - X_{i+3}}{8\Delta\xi^{3}}.$$

As soon as $U_i = U(\xi_i)$, where $U(\xi)$ denotes a smooth function, we get

$$\overline{g(U)}_{i}' = g(U)'(\xi_{i}) + \mathcal{O}(\Delta\xi^{4}),$$

$$\overline{U}_{i}'' = U''(\xi_{i}) + \mathcal{O}(\Delta\xi^{4}),$$

$$\overline{U}_{i}''' = U'''(\xi_{i}) + \mathcal{O}(\Delta\xi^{4}).$$

Numerical scheme - 4th order scheme

The finite difference scheme applied to our problem (4) and (5) writes:

$$-\xi_i \overline{w}'_i + \overline{f(w)}'_i = \varepsilon \beta \overline{w}''_i + \varepsilon^2 \gamma \overline{w}''_i, \quad i = 1, \dots, N,$$
(6)

supplemented by the following boundary conditions [see 8]:

$$w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R.$$
(7)

⁸Schecter, Plohr, Marchesin 2004 (DCDS)

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Non classical Riemann problems

Existence result

We are able to state the following result [see 9].

Theorem

Let $\varepsilon > 0$ be given and assume the existence of

 $M_{f'} := \sup_{w \in \mathbb{R}} |f'(w)|.$

Then there exists $\Delta \xi_0 \leq \sqrt{\varepsilon\beta}$ depending on β , ε , ℓ and $M_{f'}$ such that for $\Delta \xi \leq \Delta \xi_0$, there exists a solution $w^{\Delta} = (w_i)_{i=1,...,N}$ to the scheme (6)–(7).

⁹Berthon, Bessemoulin-Chatard, AC, Foucher 2019 (Calcolo)

Idea of the proof

• We rewrite the problem as

$$\mathcal{E}(w^{\Delta}) = 0,$$

where

$$\mathcal{E}(w^{\Delta})_{i} = \varepsilon \beta \overline{w}_{i}^{\prime \prime} + \varepsilon^{2} \gamma \overline{w}_{i}^{\prime \prime \prime} + \xi_{i} \overline{w}_{i}^{\prime} - \overline{f(w)}_{i}^{\prime}, \quad i = 1, \dots, N,$$

with

 $w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R.$

• A change of variable let us rewrite this problem as

$$\tilde{\mathcal{E}}(\tilde{w}^{\Delta}) = 0$$

with

$$\tilde{w}_{-2} = \tilde{w}_{-1} = \tilde{w}_0 = \tilde{w}_{N+1} = \tilde{w}_{N+2} = \tilde{w}_{N+3} = 0.$$

Key lemma: Zeros of a vector field

We will use the following lemma $[see \ 10]$, which is a consequence of Brouwer's fixed point theorem.

Lemma

Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function satisfying

$$F(x) \cdot x \le 0 \quad \text{if } \|x\| = k,$$

for some k > 0. Then there exists a point $x \in B(0, k)$ such that F(x) = 0.

A technical study of the scalar product $\tilde{\mathcal{E}}(\tilde{w}^{\Delta}).\tilde{w}^{\Delta}$ and Poincaré inequality give us the proof of our existence theorem.

¹⁰Evans 1998 (Book)

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Continuation-type Newton method

• For $\varepsilon = \mathcal{O}(1)$, the nonlinear system

$$\mathcal{E}(w_{\varepsilon}^{\Delta}) = 0,$$

where

$$\mathcal{E}(w_{\varepsilon}^{\Delta})_{i} = \varepsilon \beta \overline{w}_{i}^{\prime \prime} + \varepsilon^{2} \gamma \overline{w}_{i}^{\prime \prime \prime} + \xi_{i} \overline{w}_{i}^{\prime} - \overline{f(w)}_{i}^{\prime}, \quad i = 1, \dots, N,$$

with

 $w_{-2} = w_{-1} = w_0 = w_L, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R,$

is easily solvable with a (damping [see 11]) Newton method.

- When ε → 0, it is essential to have a good initial guess to make the Newton method converge.
- The idea is to decrease ε step by step: w_{ε}^{Δ} is the initial guess of the Newton method when solving $\mathcal{E}(w_{\varepsilon-\Delta\varepsilon}^{\Delta}) = 0$.

¹¹Ralph 1994 (MOR)

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Numerical tests with $f(w) = w^3$

- Classical case: $w_L = 4, w_R = 2.$
- Exact solution: shock at $\xi = 28 = \frac{f(w_L) f(w_R)}{w_L w_R}$.
- Domain $\mathcal{D} = [10, 40], N = 1000 \ \beta = 1 \text{ and } \varepsilon = 10^{-3}.$
- Two values of γ (1 and 10) are considered to verify that γ has no influence on the solution.



- Non classical case: $w_L = 4$, $w_R = -2$, $\beta = 1$ and $\gamma = 1$.
- Exact solution ^[see 12]: intermediate state given by

$$w^* = -w_L + \frac{\sqrt{2}}{3} \approx -3.5286 \text{ and two shocks, one at}$$

$$\xi = \frac{f(w_L) - f(w^*)}{w_L - w^*} \approx 14.3366, \text{ the other at}$$

$$\xi = \frac{f(w_R) - f(w^*)}{w_R - w^*} \approx 23.5082.$$

• Domain $\mathcal{D} = [0, 60]$ or [-20, 80], N = 2000. Different values of ε .



- Non classical case: $w_L = 2, w_R = -2, \beta = 1.$
- Exact solution ^[see 11]: intermediate state given by

 $w^{\star} = -w_L + \frac{\sqrt{2}}{3\sqrt{\gamma}}$. A shock and a rarefaction wave given by

$$\begin{cases} w^{\star}, & \xi \le f'(w^{\star}), \\ f'^{-1}(\xi), & f'(w^{\star}) \le \xi \le f'(w_R), \\ w_R, & f'(w_R) \le \xi. \end{cases}$$

• Domain $\mathcal{D} = [0, 60], N = 2000, \varepsilon = 10^{-3}.$



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Numerical tests with $f(w) = w^3 - w$

- Classical solutions ^[see 13]: $w_L = 1$, $w_R = -5$ (Left) or $w_L = 0$, $w_R = 2$ (Right), $\beta = 5$ and $\gamma = 37.5$.
- Our approach: $\mathcal{D} = [-60, 120]$ (Left), $\mathcal{D} = [-60, 60]$ (Right), N = 2000, $\varepsilon = 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 5 \times 10^4$, $\Delta t = 2 \times 10^{-5}$.



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- From classical to non classical solutions ^[see 12].
- Left: $w_R = -5$, w_L varying from 1 to 4, $\beta = 5$, $\gamma = 37.5$, $\mathcal{D} = [-60, 120]$, N = 2000, $\varepsilon = 10^{-2}$.
- Right: $w_R = -5$, $w_L = 4$, β varying from 5 to 30, $\gamma = 37.5$, $\mathcal{D} = [-60, 120]$, N = 2000, $\varepsilon = 10^{-2}$.



Numerical tests with $f(w) = \frac{w^2}{2}$ (Burgers)

- This problem only admits classical solutions.
- Case of a rarefaction wave: $w_L = -2$, $w_R = 2$, $\beta = 1$, $\gamma = 1$ or 10.
- Our approach: $\mathcal{D} = [-10, 10], N = 2000, \varepsilon = 10^{-3}.$
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



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- Case of a stationary shock (Left): $w_L = 2, w_R = -2$.
- Case of a non stationary shock (Right): $w_L = 5$, $w_R = 1$.
- Other parameters are unchanged.



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From p-system to ODE

• Considering the 2×2 diffusive-dispersive system, introduced by Joseph and LeFloch ^[see 14]

$$\begin{cases} \partial_t w^{\varepsilon} - \partial_x v^{\varepsilon} = 0, & x \in \mathbb{R}, \ t > 0, \\ \partial_t v^{\varepsilon} - \partial_x p(w^{\varepsilon}) = \varepsilon t \beta \partial_{xx} v^{\varepsilon} + \varepsilon^2 t^2 \gamma \partial_{xxx} w^{\varepsilon}, \end{cases}$$

where $p: \mathbb{R} \to \mathbb{R}$ is a given smooth function.

- Focus on Riemann problems and so on self-similar solutions.
- Change of variables

$$w^{\varepsilon}(x,t) = w(\xi)$$
 and $v^{\varepsilon}(x,t) = v(\xi)$ with $\xi = \frac{x}{t}$.

¹⁴Joseph, LeFloch 2007 (P ROY SOC EDINB A)

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Non classical Riemann problems

• We obtain

$$\begin{cases} -\xi w' - v' = 0, \\ -\xi v' - p(w)' = \varepsilon \beta v'' + \varepsilon^2 \gamma w''', \end{cases}$$

supplemented by boundary conditions

$$(v, w)(-\ell) = (v_L, w_L)$$
 and $(v, w)(\ell) = (v_R, w_R),$

with $\ell > 0$ large enough.

• Using $v' = -\xi w'$, w is governed by a nonlinear equation independent of v

$$\left(\xi^2 + \varepsilon\beta\right)w' - p(w)' = -\varepsilon\beta\xi w'' + \varepsilon^2\gamma w''',\tag{8}$$

supplemented by the boundary conditions

$$w(-\ell) = w_L$$
 and $w(\ell) = w_R$. (9)

• Problem (8)-(9) contains the full structure of the expected Riemann solution.

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- Analysis of (8) ^[see 15] shows some degeneracy for $\xi = 0$.
- Must be studied into the two regions $[-\ell, 0)$ and $(0, \ell]$ separately.
- The viscous term governed by $\varepsilon\beta\xi$ is of prime importance, but the viscosity vanishes as soon as $\xi = 0$. From a numerical point of view, we have to avoid $\xi = 0$.
- We consider two intervals $[-\ell, -\xi^*]$ and $[\xi^*, \ell]$ separately where $\xi^* > 0$ is a given constant small enough.
- We impose boundary conditions

$$w(-\xi^{\star}) = w^{\star}$$
 and $w(\xi^{\star}) = w^{\star}$,

where ξ^* has to be fixed and w^* has to be determined.

¹⁵Joseph, LeFloch 2007 (P ROY SOC EDINB A)

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Numerical scheme - Finite differences

Assume that the state $w^* \in \mathbb{R}$ is given. We use 4th order Finite Differences on the right interval $[\xi^*, \ell]$ (resp. on the left interval $[-\ell, -\xi^*]$).

- Interval $[\xi^*, \ell]$ is discretized with N + 1 cells $[\xi_i, \xi_{i+1})$ of size $\Delta \xi = (\ell \xi^*)/(N + 1)$: $\xi_i = \xi^* + i\Delta \xi$.
- For boundary conditions, we define ξ_i for $i = -2, \ldots, N+3$.
- We denote w_i an approximation of $w(\xi_i)$ for $i = 1, \dots, N$ and define the scheme

$$(\xi_i^2 + \varepsilon\beta)\overline{w}_i' - \overline{p(w)}_i' = -\varepsilon\beta\xi_i\overline{w}_i'' + \varepsilon^2\gamma\overline{w}_i''', \quad i = 1, \cdots, N,$$
(10)

completed with the following boundary conditions

$$w_{-2} = w_{-1} = w_0 = w^*, \quad w_{N+1} = w_{N+2} = w_{N+3} = w_R.$$
(11)

Determination of w^* - Mass conservation

- \bullet Conservation law: total mass of w must be preserved.
- Initial mass of w given by

$$M_0 = \ell(w_L + w_R).$$

• Total mass of the approximated solution w^{Δ} depends on w^{\star} and reads

$$M(w^{\star}) = \Delta \xi w_L + \sum_{i=1}^N \Delta \xi w_i(w_L, w^{\star}) + 2\xi^{\star} w^{\star} + \sum_{i=1}^N \Delta \xi w_i(w^{\star}, w_R) + \Delta \xi w_R.$$

• w^{\star} must be solution of the following nonlinear equation:

$$M(w^{\star}) = M_0.$$

Determination of w^* - Dichotomy technique

• We initialize the dichotomy algorithm as follows:

$$(w_{\text{inf}}, w_{\text{sup}}) = \begin{cases} (w_L, w_R) \text{ if } M(w_L) < M(w_R), \\ (w_R, w_L) \text{ elsewhere,} \end{cases}$$

$$w_0^{\star} = \frac{1}{2}(w_L + w_R).$$

- For iterations $k \geq 1$, we compute the left and right solutions $w^{\Delta}(w_L, w_{k-1}^{\star})$ and $w^{\Delta}(w_{k-1}^{\star}, w_R)$ and we deduce $M_k = M(w_{k-1}^{\star}).$
- If $M_k < M_0$ then $w_{\inf} = w_{k-1}^{\star}$, else $w_{\sup} = w_{k-1}^{\star}$, and we compute the new iterate value $w_k^{\star} = \frac{1}{2}(w_{\inf} + w_{\sup})$.
- In practice, we accept a small mass error of 10^{-6} .

Existence result

We are able to state the following result $[see \ 16]$.

Theorem

Let $\varepsilon > 0$ be given and $\xi^* > 2\Delta \xi$. Assume the existence of

 $M_{p'} := \sup_{w \in \mathbb{R}} |p'(w)|.$

Then there exists $\Delta \xi_0 \leq \sqrt{\varepsilon \beta}$ depending on β , ε , ℓ , ξ^* and $M_{p'}$ such that for $\Delta \xi \leq \Delta \xi_0$, there exists a solution $w^{\Delta} = (w_i)_{i=1,...,N}$ to the scheme (10) with the boundary conditions (11).

¹⁶Berthon, Bessemoulin-Chatard, AC, Foucher 2019 (Calcolo)

Numerical tests with $p(w) = \frac{w^3}{3} + w$

- Classical case: $w_L = 1, w_R = 5, \beta = 1, \gamma = 1.$
- Our approach: $\ell = 8, \xi^*$: 0.8 or 1.6, $N = 1000, \varepsilon = 4 \times 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



- Non classical case: $w_L = -3$, $w_R = 10$, $\beta = 1$, $\gamma = 20$.
- Our approach: $\ell = 20, N = 1000, \varepsilon = 2.5 \times 10^{-2}$.
- FV Lax-Friedrichs scheme: $N = 10^4$, $\Delta t = 10^{-4}$.



Conclusions

- A new numerical approach to approximate (non classical) Riemann solutions of system of conservation laws.
- Solutions seen as limit solutions of diffusive-dispersive problem.
- Use self-similarity and Dafermos technique to obtain an ODE problem.
- Gives quite good numerical results on classical and non classical tests.
- Future works: Shallow-Water, two-layer problems (without dispersive term, only diffusion)...

Thank you for your attention!