Numerical schemes for the collisional Vlasov equation in the finite Larmor radius regime

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Presentation of the model

We consider the collisional Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = \frac{1}{\tau} Q[f]$$

with

- $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ the time, space and velocity variables,
- $f(t, x, v) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ the particle distribution function,
- $E(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ and $B(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ external electric and magnetic fields.

Presentation of the model - collision term

- $Q[f]$ the BGK collision operator[3]:

$$Q[f] = \mathcal{M}[f] - f$$

where $\mathcal{M}[f](t,x,v) = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}$ with

$$n(t,x) = \int_{\mathbb{R}^3} f \mathrm{d}v, \qquad nu(t,x) = \int_{\mathbb{R}^3} f v \mathrm{d}v,$$

$$n\left(\frac{|u|^2}{2} + \frac{3}{2}T\right)(t,x) = \int_{\mathbb{R}^3} f \frac{|v|^2}{2} \mathrm{d}v.$$

- τ the Knudsen number ($\tau \gg 1$ few collisions, $\tau \ll 1$ many collisions).

[3] Remark: Fokker-Planck-Landau in [Bostan, Finot 2019]

Presentation of the model - finite Larmor radius regime

- B does not depend on t, is space-homogeneous and oriented along the x_3-direction only:

$$B = (0, 0, b)\,,$$

- perpendicular dynamics time scale is smaller than the parallel one,
- a rescaling gives

$$\partial_t f + \frac{1}{\varepsilon}\left(v_1 \partial_{x_1} f + v_2 \partial_{x_2} f\right) + v_3 \partial_{x_3} f + E \cdot \nabla_v f + \frac{1}{\varepsilon}\left(v_2 \partial_{v_1} f - v_1 \partial_{v_2} f\right) = \frac{1}{\tau} Q[f],$$

with ε the scaled cyclotronic period.

Context of our work

- Model and asymptotics studied by M. Bostan and A. Finot, *Communications in Contemporary Mathematics*, 2019.
- Our contribution: multiscale schemes with asymptotic properties.

Outline

Multiscale model

$$\partial_t f + \frac{1}{\varepsilon}\left(v_1 \partial_{x_1} f + v_2 \partial_{x_2} f\right) + v_3 \partial_{x_3} f + E \cdot \nabla_v f$$
$$+\frac{1}{\varepsilon}\left(v_2 \partial_{v_1} f - v_1 \partial_{v_2} f\right) = \frac{1}{\tau} Q[f]$$
$$\Longleftrightarrow$$
$$\partial_t f + \frac{1}{\varepsilon} Az \cdot \nabla_z f + h(t,z) \cdot \nabla_z f = \frac{1}{\tau} Q[f], \quad \text{with } z = (x, v).$$

- Red: fast and periodic scale in the perpendicular plane (x_1, x_2).
- Blue: slow scale in the parallel direction x_3.
- Magenta: collisional scale.

Three asymptotics are considered:

- fluid: fixed $\varepsilon > 0$, $\tau \to 0$,
- gyrokinetic: $\varepsilon \to 0$, fixed $\tau > 0$,
- gyrofluid: ε and $\tau \to 0$, $\varepsilon < \tau$.

Fluid asymptotic: fixed $\varepsilon > 0$, $\tau \to 0$

We get a hydrodynamic limit[4]:

- f converges toward a thermodynamical equilibrium given by the Gaussian function $\mathcal{M}[f](t, x, v)$,
- this provides a closure condition for equations on the moments of f associated to 5 collisional invariants,
- the limit model is the 3D-Euler system.

[4] C. Bardos, F. Golse, D. Levermore, *Journal of Statistical Physics*, 1991

Gyrokinetic asymptotic: $\varepsilon \to 0$, fixed $\tau > 0$

We get a highly oscillatory limit:

- strong magnetic field $B = \left(0, 0, \frac{1}{\varepsilon}\right)$ leads to fast oscillations,
- change of variable to filter out the main oscillation: $Z = e^{-\frac{t}{\varepsilon}A} z$, so that $F(t, Z) = f(t, z)$ is solution to

$$\partial_t F + h_{filt}(t, t/\varepsilon, Z) \cdot \nabla_Z F = \frac{1}{\tau} Q_{filt}[F](t, t/\varepsilon, Z),$$

  with $h_{filt}(t, s, Z) = e^{-sA} h\left(t, e^{sA} Z\right)$
  and $Q_{filt}[F](t, s, Z) = Q[F(t, e^{-sA} \cdot)]\left(e^{sA} Z\right)$,
- average with respect to the fast time variable t/ε (considering $\langle \star \rangle = \frac{1}{2\pi} \int_0^{2\pi} \star(t, s, Z) \mathrm{d}s$):

$$\partial_t F + \langle h_{filt} \rangle(t, Z) \cdot \nabla_Z F = \frac{1}{\tau} \langle Q_{filt} \rangle [F](t, Z),$$

- we obtain a collisional Vlasov equation in filtered variables.

Gyrofluid asymptotic: first $\varepsilon \to 0$, then $\tau \to 0$

Collision operator in the gyrokinetic model:

$$\langle Q_{filt} \rangle [F](t, Z) = \frac{1}{2\pi} \int_0^{2\pi} Q_{filt}[F](t, s, Z) \mathrm{d}s$$

- when $\tau \to 0$, F converges toward an equilibrium of $\langle Q_{filt} \rangle$ called a gyromaxwellian $\mathcal{G}[F]$,
- in [Bostan, Finot 2019]: study of its 8 invariants, closure relation for (gyro-)moments of F,
- the limit model is a system of 1D (in X_3) Euler-like equations,
- important point: $\mathcal{G}[F] \neq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}_{filt}[F](t, s, Z) \mathrm{d}s$.

Objectives

Develop numerical schemes for the multiscale problem

$$\partial_t f + \frac{1}{\varepsilon} Az \cdot \nabla_z f + h(t,z) \cdot \nabla_z f = \frac{1}{\tau} Q[f], \quad \text{with } z = (x, v).$$

- 7 variables: 3 in x, 3 in v and time t.
- Follow the oscillations is too costly when $\varepsilon \to 0$.
- Stability of the scheme at the limit $\tau \to 0$.
- Recover right asymptotics (consistency).

UA and AP properties

We are interested in the following properties.

- Uniform Accuracy (UA) in gyrokinetic limit ($\varepsilon \to 0$):
  - the accuracy of the scheme does not depend on ε,
  - rich literature especially by Chartier, Crouseilles, Lemou, Méhats, Zhao (several papers since 2015).

- Asymptotic Preserving (AP) in fluid and gyrofluid limit ($\tau \to 0$):
  - stable and consistent scheme $\forall \tau$, in particular when $\tau \to 0$,
  - rich literature in the fluid hydrodynamic limit, for example [Jin 1999], [Filbet, Jin 2011], [Dimarco, Pareschi 2011], [Lemou, Mieussens 2008], [Coron, Perthame 1991], [C, Crouseilles, Lemou 2012].

Tools

Use efficient ideas from the literature:

- PIC method for the 6D phase-space semi-discretization, leading to a multiscale set of ODEs,
- scale-separation strategy[5] and exponential integrator on t to get UA property in the gyrokinetic limit $\varepsilon \to 0$,
- implicit scheme on weights to get AP property in the fluid limit $\tau \to 0$,
- penalization method[6,7] to get AP property in the gyrofluid limit $\varepsilon, \tau \to 0$, $\varepsilon < \tau$.

[5] P. Chartier, N. Crouseilles, M. Lemou, F. Méhats, *Numerische Mathematik*, 2015

[6] F. Filbet, S. Jin, *Journal of Computational Physics*, 2010

[7] G. Dimarco, L. Pareschi, *SIAM Journal on Numerical Analysis*, 2011

PIC method for f

Considering $N_p \in \mathbb{N}$ numerical particles, of position $x_p(t)$, velocity $v_p(t)$ and weight $\omega_p(t)$, $1 \leq p \leq N_p$, and an averaged volume V_{N_p}, we approximate

$$f(t,z) \approx f_{N_p}(t,z) = \sum_{p=1}^{N_p} \omega_p(t)\delta(z - z_p(t)), \quad \text{with } z_p = (x_p, v_p),$$

and $z_p(0) = z_{p,0}$, $\omega_p(0) = f(0, z_p(0))V_{N_p}$.

- Transport part: motion equations for z_p

$$\dot{z}_p(t) = \frac{1}{\varepsilon} A z_p(t) + h(t, z_p(t)), \quad 1 \leq p \leq N_p,$$

- collisional part: weights evolve taking into account collisions

$$\dot{\omega}_p(t) = \frac{1}{\tau}(m_p(t) - \omega_p(t)), \quad 1 \leq p \leq N_p,$$

  where m_p are weights associated to $\mathcal{M}[f_{N_p}]$ after reconstruction of moments.

PIC method in the filtered variables

As at the continuous level, the main oscillation is filtered out:

- change of variable on particles: $Z_p(t) = e^{-\frac{t}{\varepsilon}A} z_p(t)$,
- transport equation:

$$\dot{Z}_p(t) = h_{filt}(t, t/\varepsilon, Z_p(t)), \quad 1 \leq p \leq N_p,$$

- compute $m_p(t) = \mathcal{M}_{filt}[F_{N_p}](t, t/\varepsilon, Z_p(t)) V_{N_p}$ after reconstructing the moments of F_{N_p},
- evolution of weights:

$$\dot{\omega}_p(t) = \frac{1}{\tau}(m_p(t) - \omega_p(t)), \quad 1 \leq p \leq N_p.$$

This corresponds to a PIC method for our filtered equation

$$\partial_t F + h_{filt}(t, t/\varepsilon, Z) \cdot \nabla_Z F = \frac{1}{\tau} Q_{filt}[F](t, t/\varepsilon, Z).$$

Scale-separation strategy for UA property

To construct an UA scheme, we follow ideas of Chartier and coauthors.

- Consider the fast periodic time scale $s = t/\varepsilon$ independent from the slow time scale t,
- introduce double-scale quantities $\mathcal{Z}_p(t,s)$ and $\mathcal{W}_p(t,s)$ satisfying

$$\mathcal{Z}_p(t,t/\varepsilon) = Z_p(t), \quad \mathcal{W}_p(t,t/\varepsilon) = \omega_p(t),$$

with equations of the form

$$\partial_t \mathcal{Z}_p(t,s) + \frac{1}{\varepsilon}\partial_s \mathcal{Z}_p(t,s) = \mathcal{H}_p(t,s),$$

$$\partial_t \mathcal{W}_p(t,s) + \frac{1}{\varepsilon}\partial_s \mathcal{W}_p(t,s) = \frac{1}{\tau}(\mathcal{M}_p(t,s) - \mathcal{W}_p(t,s)),$$

- additional variable s gives a degree of freedom which is used to get UA property.

Let focus on transport equation (same strategy on weights)

$$\partial_t \mathcal{Z}_p(t,s) + \frac{1}{\varepsilon}\partial_s \mathcal{Z}_p(t,s) = \mathcal{H}_p(t,s),$$

where $\mathcal{H}_p(t,s) = h_{filt}(t,s,\mathcal{Z}_p(t,s))$.

- Fourier transform gives equations for modes $l = -\frac{N_s}{2}, \dots, \frac{N_s}{2} - 1$:

$$\frac{d}{dt}\hat{\mathcal{Z}}_{p,l}(t) + \frac{il}{\varepsilon}\hat{\mathcal{Z}}_{p,l}(t) = \hat{\mathcal{H}}_{p,l}(t),$$

- multiply by $e^{\frac{il}{\varepsilon}t}$ and integrate between two times t^n and $t^{n+1} = t^n + \Delta t$:

$$\hat{\mathcal{Z}}_{p,l}(t^{n+1}) = e^{-\frac{il}{\varepsilon}\Delta t}\hat{\mathcal{Z}}_{p,l}(t^n) + \int_{t^n}^{t^{n+1}} e^{-\frac{il}{\varepsilon}(t^{n+1}-t)}\hat{\mathcal{H}}_{p,l}(t)\mathrm{d}t,$$

- approximate $\hat{\mathcal{H}}_{p,l}(t)$ by $\hat{\mathcal{H}}_{p,l}(t^n)$.

Asymptotic behaviour of scheme A

Scheme A given by

$$\hat{\mathcal{Z}}_{p,l}^{n+1} = e^{-\frac{il}{\varepsilon}\Delta t}\hat{\mathcal{Z}}_{p,l}^{n} + \frac{\varepsilon}{il}\left(1 - e^{-\frac{il}{\varepsilon}\Delta t}\right)\hat{\mathcal{H}}_{p,l}^{n},$$

$$\hat{\mathcal{W}}_{p,l}^{n+1} = e^{-\left(\frac{il}{\varepsilon}+\frac{1}{\tau}\right)\Delta t}\hat{\mathcal{W}}_{p,l}^{n} + \frac{\varepsilon}{il\tau+\varepsilon}\left(1 - e^{-\left(\frac{il}{\varepsilon}+\frac{1}{\tau}\right)\Delta t}\right)\hat{\mathcal{M}}_{p,l}^{\star},$$

- is UA in the gyrokinetic limit ($\varepsilon \to 0$, fixed $\tau > 0$), if we take constant initial datas $\mathcal{Z}_p(0,s) = Z_p(0)$, $\mathcal{W}_p(0,s) = \omega_p(0)$,
- is AP in the fluid limit ($\tau \to 0$, fixed $\varepsilon > 0$), if we consider $\hat{\mathcal{M}}_{p,l}^{\star}$ "semi-implicit" (computed from $\mathcal{Z}_p^{n+1}$ and $\mathcal{W}_p^{n}$).

Penalization approach

- In the gyrofluid limit ($\varepsilon \to 0$ then $\tau \to 0$), equilibrium of collision operator $\langle Q_{filt} \rangle[F]$ is $\mathcal{G}[F] \neq \langle \mathcal{M}_{filt}[F] \rangle$,
- we will enforce the right asymptotic behaviour by modifying weights scheme, starting from

$$\partial_t \mathcal{W}_p + \frac{1}{\varepsilon} \partial_s \mathcal{W}_p = \frac{1}{\tau}(\mathcal{M}_p - \mathcal{W}_p) - \frac{1}{\tau}\mathcal{G}_p + \frac{1}{\tau}\mathcal{G}_p,$$

- Fourier transform in s + integration on $[t^n, t^{n+1}]$ but different quadratures on

$$\frac{1}{\tau} \int_{t^n}^{t^{n+1}} e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)(t^{n+1}-t)} (\hat{\mathcal{M}}_{p,l}(t) - \hat{\mathcal{G}}_{p,l}(t)) \mathrm{d}t$$

and

$$\frac{1}{\tau} \int_{t^n}^{t^{n+1}} e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)(t^{n+1}-t)} \hat{\mathcal{G}}_{p,l}(t) \mathrm{d}t,$$

- for asymptotics in $\varepsilon \to 0$, we focus on mode 0.

Asymptotic behaviour of scheme B

Scheme B whose mode 0 is given by

$$\hat{\mathcal{Z}}_{p,0}^{n+1} = \hat{\mathcal{Z}}_{p,0}^{n} + \Delta t \hat{\mathcal{H}}_{p,0}^{n},$$

$$\hat{\mathcal{W}}_{p,0}^{n+1} = e^{-\frac{\Delta t}{\tau}} \hat{\mathcal{W}}_{p,0}^{n} + \frac{\Delta t}{\tau} e^{-\frac{\Delta t}{\tau}} \left(\hat{\mathcal{M}}_{p,0}^{n} - \hat{\mathcal{G}}_{p,0}^{n} \right) + \left(1 - e^{-\frac{\Delta t}{\tau}} \right) \hat{\mathcal{G}}_{p,0}^{n+1},$$

- is AP in the gyrofluid limit ($\varepsilon \to 0$ then $\tau \to 0$),
- is AP in the gyrokinetic limit ($\varepsilon \to 0$, fixed $\tau > 0$).

But

- we lose the right behaviour in the fluid limit...
- We can propose a convex combination of schemes A and B:

$$\hat{\mathcal{W}}_{p,l}^{n+1} = \frac{\varepsilon}{\tau + \varepsilon} \hat{\mathcal{W}}_{p,l}^{n+1,A} + \left(1 - \frac{\varepsilon}{\tau + \varepsilon} \right) \hat{\mathcal{W}}_{p,l}^{n+1,B}.$$

One-particle test: multiscale ODE framework

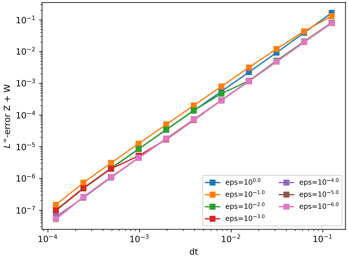
$$\dot{z}(t) = \frac{1}{\varepsilon} A z(t) + h(t,z),$$
$$\dot{\omega}(t) = \frac{1}{\tau}(\mathcal{M}(\omega, z) - \omega(t)),$$

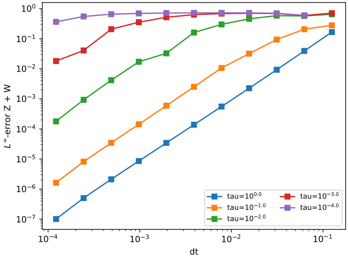
with

- $E(t,x) = ((x_1 + x_3)\cos(t), x_1 x_2 \sin(t), -x_2^2 e^{-t^2})$,
- $z(0) = (1, 1, 0, 1/2, 1/2, 3/2)$, $\omega(0) = 1$.

We plot $||z\text{-error}||_{L^\infty} + ||\omega\text{-error}||_{L^\infty}$ as a function of Δt, error compared to a reference solution.

One-particle test: $\mathcal{M}(\omega, z) = 1 + |z|^2 + e^{-\omega^2}$



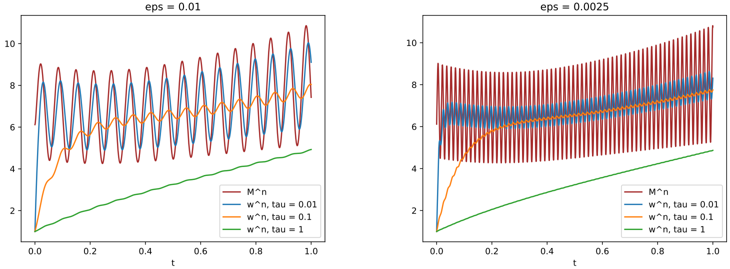


2nd-order scheme.

Left: L^∞-Error for $\tau = 1$ and different ε; UA in gyrokinetic limit.

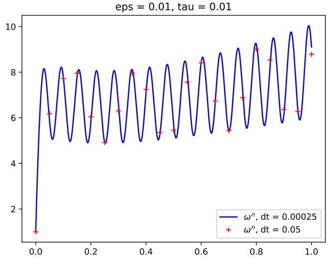
Right: L^∞-Error for $\varepsilon = 1$ and different τ; AP in fluid limit.

One-particle test: time history of $\mathcal{M}(z(t))$ and $\omega(t)$



Fluid limit: for fixed $\varepsilon > 0$, $w(t)$ converges toward $\mathcal{M}(z(t))$ when $\tau \to 0$.
Gyrokinetic limit: for fixed $\tau > 0$, $w(t)$ converges toward the average of $\mathcal{M}(z(t))$ when $\varepsilon \to 0$.

One-particle test: time history of $\omega(t)$, $\mathcal{M}(z)$-case

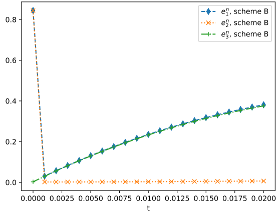


UA scheme captures the highly oscillatory behaviour, without resolving these stiffness ($\Delta t > \varepsilon$).

PDE framework: error study of scheme B

We plot 3 errors: $e_1^n = \sum_p |\omega_p^n - \mathcal{G}_p^{ex}|$, $e_2^n = \sum_p |\omega_p^n - \mathcal{G}_p^n|$ and $e_3^n = \sum_p |\mathcal{G}_p^n - \mathcal{G}_p^{ex}|$.

Parameters: $N_p = 12288000$, $\varepsilon = 10^{-8}$, $\Delta t = 10^{-3}$, $N_s = 4$, $\Delta x \approx 0.37$, $\tau = 10^{-4}$.



e_3^n increases since gyromoments are not conserved exactly (and it deteriorates $\mathcal{G}_p^n$). Better if N_p increases.
e_2^n small since scheme B is AP in the gyrofluid limit.

Conclusion and opening

- Two schemes were developed for a 3Dx-3Dv multiscale Vlasov equation involving collisions and fast oscillations.
- Asymptotic properties (UA or AP) are obtained for three limits.

- A projection technique (as in [Dimarco, Loubère 2013] or [Gamba, Tharkabhushanam 2009]) could ensure preservation of gyromoments.
- Micro-macro approach to reduce computational time.
- Coupling with Maxwell equations for a self-consistent electromagnetic field.

Thank you for your attention!