

# Numerical schemes for the collisional Vlasov equation in the finite Larmor radius regime

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Kinetic and hyperbolic equations: modeling, analysis and  
numerics

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# Context

- Consider a 3Dx-3Dv collisional Vlasov equation in the finite Larmor radius regime.
- Model and asymptotics studied by Bostan and Finot<sup>3</sup>.
- Our contribution: multiscale schemes with asymptotic properties.

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<sup>3</sup>M. Bostan, A. Finot, *Communications in Contemporary Mathematics*, 2019

# Outline

- 1 Model and asymptotics
- 2 AP/UA schemes
- 3 Numerical results

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# Presentation of the model

We consider the collisional Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = \frac{1}{\tau} Q[f]$$

with

- $(t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$  the time, space and velocity variables,
- $f(t, x, v) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$  the particle distribution function,
- $E(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  and  $B(t, x) : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  external electric and magnetic fields.

# Presentation of the model - collision term

- $Q[f]$  the BGK collision operator<sup>4</sup>:

$$Q[f] = \mathcal{M}[f] - f$$

where  $\mathcal{M}[f](t, x, v) = \frac{n}{(2\pi T)^{3/2}} e^{-\frac{|v-u|^2}{2T}}$  with

$$n(t, x) = \int_{\mathbb{R}^3} f dv, \quad nu(t, x) = \int_{\mathbb{R}^3} f v dv,$$

$$n \left( \frac{|u|^2}{2} + \frac{3}{2}T \right) (t, x) = \int_{\mathbb{R}^3} f \frac{|v|^2}{2} dv.$$

- $\tau$  the Knudsen number ( $\tau \gg 1$  few collisions,  $\tau \ll 1$  many collisions).

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<sup>4</sup>Remark: Fokker-Planck-Landau in [Bostan, Finot 2019]

## Presentation of the model - finite Larmor radius regime

- $B$  does not depend on  $t$ , is space-homogeneous and oriented along the  $x_3$ -direction only:

$$B = (0, 0, b) ,$$

- perpendicular dynamics time scale is smaller than the parallel one,
- a rescaling gives

$$\begin{aligned} \partial_t f + \frac{1}{\varepsilon} (v_1 \partial_{x_1} f + v_2 \partial_{x_2} f) + v_3 \partial_{x_3} f + E \cdot \nabla_v f \\ + \frac{1}{\varepsilon} (v_2 \partial_{v_1} f - v_1 \partial_{v_2} f) = \frac{1}{\tau} Q[f], \end{aligned}$$

with  $\varepsilon$  the scaled cyclotronic period.

# Multiscale model

$$\begin{aligned} \partial_t f + \frac{1}{\varepsilon} (v_1 \partial_{x_1} f + v_2 \partial_{x_2} f) + v_3 \partial_{x_3} f + E \cdot \nabla_v f \\ + \frac{1}{\varepsilon} (v_2 \partial_{v_1} f - v_1 \partial_{v_2} f) = \frac{1}{\tau} Q[f] \\ \iff \end{aligned}$$

$$\partial_t f + \frac{1}{\varepsilon} A z \cdot \nabla_z f + h(t, z) \cdot \nabla_z f = \frac{1}{\tau} Q[f], \quad \text{with } z = (x, v).$$

- **Red**: fast and periodic scale in the perpendicular plane  $(x_1, x_2)$ .
- **Blue**: slow scale in the parallel direction  $x_3$ .
- **Magenta**: collisional scale.

Three asymptotics are considered:

- fluid: fixed  $\varepsilon > 0$ ,  $\tau \rightarrow 0$ ,
- gyrokinetic:  $\varepsilon \rightarrow 0$ , fixed  $\tau > 0$ ,
- gyrofluid:  $\varepsilon$  and  $\tau \rightarrow 0$ ,  $\varepsilon < \tau$ .



## Fluid asymptotic: fixed $\varepsilon > 0$ , $\tau \rightarrow 0$

We get a classical hydrodynamic limit:

- $f$  converges toward a thermodynamical equilibrium given by the Gaussian function  $\mathcal{M}[f](t, x, v)$ ,
- this provides a closure condition for equations on the moments of  $f$  associated to 5 collisional invariants,
- the limit model is the classical 3D-Euler system.

## Gyrokinetic asymptotic: $\varepsilon \rightarrow 0$ , fixed $\tau > 0$

We get a highly oscillatory limit:

- strong magnetic field  $B = (0, 0, \frac{1}{\varepsilon})$  leads to fast oscillations,
- change of variable to filter out the main oscillation:  
 $Z = e^{-\frac{t}{\varepsilon}A}z$ , so that  $F(t, Z) = f(t, z)$  is solution to

$$\partial_t F + h_{filt}(t, t/\varepsilon, Z) \cdot \nabla_Z F = \frac{1}{\tau} Q_{filt}[F](t, t/\varepsilon, Z),$$

- average with respect to the fast time variable  $t/\varepsilon$   
(considering  $\langle \star \rangle = \frac{1}{2\pi} \int_0^{2\pi} \star(t, s, Z) ds$ ):

$$\partial_t F + \langle h_{filt} \rangle(t, Z) \cdot \nabla_Z F = \frac{1}{\tau} \langle Q_{filt} \rangle[F](t, Z),$$

- we obtain a collisional Vlasov equation in filtered variables.

# Gyrofluid asymptotic: first $\varepsilon \rightarrow 0$ , then $\tau \rightarrow 0$

Collision operator in the gyrokinetic model:

$$\langle Q_{filt} \rangle[F](t, Z) = \frac{1}{2\pi} \int_0^{2\pi} Q_{filt}[F](t, s, Z) ds.$$

- when  $\tau \rightarrow 0$ ,  $F$  converges toward an equilibrium of  $\langle Q_{filt} \rangle$  called a gyromaxwellian  $\mathcal{G}[F]$ ,
- in [Bostan, Finot 2019]: study of its 8 invariants, closure relation for (gyro-)moments of  $F$ ,
- the limit model is a system of 1D (in  $X_3$ ) Euler-like equations,
- important point:  $\mathcal{G}[F] \neq \frac{1}{2\pi} \int_0^{2\pi} \mathcal{M}_{filt}[F](t, s, Z) ds$ .

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# Objectives

Develop numerical schemes for this multiscale problem, which ensures asymptotic properties.

- Uniform Accuracy (UA) in gyrokinetic limit:
  - the accuracy of the scheme does not depend on  $\varepsilon$ ,
  - rich literature especially by Chartier, Crouseilles, Lemou, Méhats, Zhao (several papers since 2015).
- Asymptotic Preserving (AP) in fluid and gyrofluid limit:
  - stable and consistent scheme  $\forall \tau$ , in particular when  $\tau \rightarrow 0$ ,
  - rich literature in the fluid hydrodynamic limit, for example [Jin 1999], [Filbet, Jin 2011], [Dimarco, Pareschi 2011], [Lemou, Mieussens 2008], [Coron, Perthame 1991], [Crestetto, Crouseilles, Lemou 2012],
  - for two combined limits, the literature is less abundant: [Li, Lu 2017], [Crouseilles, Dimarco, Vignal 2016].

# Tools

Use efficient ideas from the literature:

- PIC method for the 6D phase-space semi-discretization, leading to a multiscale set of ODEs,
- scale-separation strategy<sup>5</sup> (new variable  $s$ ) + spectral method on  $s$ ,
- exponential integrator on  $t$  to get UA property in the gyrokinetic limit  $\varepsilon \rightarrow 0$ ,
- implicit scheme on weights to get AP property in the fluid limit  $\tau \rightarrow 0$ ,
- penalization method<sup>6,7</sup> to get AP property in the gyrofluid limit  $\varepsilon \rightarrow 0$  then  $\tau \rightarrow 0$ .

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<sup>5</sup>P. Chartier, N. Crouseilles, M. Lemou, F. Méhats, *Numerische Mathematik*, 2015

<sup>6</sup>F. Filbet, S. Jin, *Journal of Computational Physics*, 2010

<sup>7</sup>G. Dimarco, L. Pareschi, *SIAM Journal on Numerical Analysis*, 2011

# PIC method for $f$

Considering  $N_p$  macro-particles, of position  $x_p(t)$ , velocity  $v_p(t)$  and weight  $\omega_p(t)$ ,  $1 \leq p \leq N_p$ , we approximate

$$f(t, z) \approx f_{N_p}(t, z) = \sum_{p=1}^{N_p} \omega_p(t) \delta(z - z_p(t)), \quad \text{with } z_p = (x_p, v_p),$$

and initialization  $z_p(0) = z_{p,0}$ ,  $\omega_p(0) = f(0, z_p(0))V_{N_p}$ .

- Transport part: inserting it in the Vlasov equation and integrating gives

$$\dot{z}_p(t) = \frac{1}{\varepsilon} A z_p(t) + h(t, z_p(t)), \quad 1 \leq p \leq N_p,$$

- collisional part: weights evolve taking into account collisions

$$\dot{\omega}_p(t) = \frac{1}{\tau} (m_p(t) - \omega_p(t)), \quad 1 \leq p \leq N_p,$$

where  $m_p$  are weights associated to  $\mathcal{M}[f_{N_p}]$  after reconstruction of moments.

# PIC method in the filtered variables

As in the continuous case, the main oscillation is filtered out:

- change of variable  $Z_p(t) = e^{-\frac{t}{\varepsilon}A} z_p(t)$  gives

$$\dot{Z}_p(t) = h_{filt}(t, t/\varepsilon, Z_p(t)), \quad 1 \leq p \leq N_p,$$

- compute  $m_p(t) = \mathcal{M}_{filt}[F_{N_p}](t, t/\varepsilon, Z_p(t))V_{N_p}$  after reconstructing the moments of  $F_{N_p}$ ,
- same equation on weights:

$$\dot{\omega}_p(t) = \frac{1}{\tau}(m_p(t) - \omega_p(t)), \quad 1 \leq p \leq N_p.$$



# Scale-separation strategy

- Consider the slow time scale  $t$  and the fast periodic time scale  $s = t/\varepsilon$  as independent,
- introduce double-scale quantities  $\mathcal{Z}_p(t, s)$  and  $\mathcal{W}_p(t, s)$  satisfying

$$\mathcal{Z}_p(t, t/\varepsilon) = Z_p(t), \quad \mathcal{W}_p(t, t/\varepsilon) = \omega_p(t),$$

- additional variable gives a degree of freedom,
- use it to bound the time derivatives of  $\mathcal{Z}$  uniformly in  $\varepsilon$ : well prepared initial data.

# Spectral method on $s$ and exponential integrator in $t$

Let focus on transport equation (same strategy on weights)

$$\partial_t \mathcal{Z}_p(t, s) + \frac{1}{\varepsilon} \partial_s \mathcal{Z}_p(t, s) = \mathcal{H}_p(t, s),$$

where  $\mathcal{H}_p(t, s) = h_{filt}(t, s, \mathcal{Z}_p(t, s))$ .

- Fourier transform gives equations for modes

$$\frac{d}{dt} \hat{\mathcal{Z}}_{p,l}(t) + \frac{il}{\varepsilon} \hat{\mathcal{Z}}_{p,l}(t) = \hat{\mathcal{H}}_{p,l}(t),$$

- multiply by  $e^{\frac{il}{\varepsilon}t}$  and integrate between two times  $t^n$  and  $t^{n+1} = t^n + \Delta t$ :

$$\hat{\mathcal{Z}}_{p,l}(t^{n+1}) = e^{-\frac{il}{\varepsilon}\Delta t} \hat{\mathcal{Z}}_{p,l}(t^n) + \int_{t^n}^{t^{n+1}} e^{-\frac{il}{\varepsilon}(t^{n+1}-t)} \hat{\mathcal{H}}_{p,l}(t) dt,$$

- approximate  $\hat{\mathcal{H}}_{p,l}(t)$  by  $\hat{\mathcal{H}}_{p,l}(t^n)$  to get a first order<sup>8</sup> scheme

$$\hat{\mathcal{Z}}_{p,l}^{n+1} = e^{-\frac{il}{\varepsilon}\Delta t} \hat{\mathcal{Z}}_{p,l}^n + \frac{\varepsilon}{il} \left(1 - e^{-\frac{il}{\varepsilon}\Delta t}\right) \hat{\mathcal{H}}_{p,l}^n,$$

- reconstruct the truncated Fourier series:

$$\mathcal{Z}_p^n(s) = \sum_{l=-N_s/2}^{N_s/2-1} \hat{\mathcal{Z}}_{p,l}^n e^{ils},$$

- evaluate it on the diagonal  $s = t^n/\varepsilon$ :  $Z_p^n = \mathcal{Z}_p^n\left(\frac{t^n}{\varepsilon}\right)$  and apply the inverse change of variable  $z_p^n = e^{\frac{t^n}{\varepsilon}A} Z_p^n$ .

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<sup>8</sup>Remark: in practice we used a 2nd-order scheme as well.

# Asymptotic behaviour of scheme A

Scheme A given by

$$\hat{\mathcal{Z}}_{p,l}^{n+1} = e^{-\frac{il}{\varepsilon}\Delta t} \hat{\mathcal{Z}}_{p,l}^n + \frac{\varepsilon}{il} \left(1 - e^{-\frac{il}{\varepsilon}\Delta t}\right) \hat{\mathcal{H}}_{p,l}^n,$$

$$\hat{\mathcal{W}}_{p,l}^{n+1} = e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)\Delta t} \hat{\mathcal{W}}_{p,l}^n + \frac{\varepsilon}{il\tau + \varepsilon} \left(1 - e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)\Delta t}\right) \hat{\mathcal{M}}_{p,l}^*,$$

- is UA in the gyrokinetic limit ( $\varepsilon \rightarrow 0$ , fixed  $\tau > 0$ ), if we take constant initial datas  $\mathcal{Z}_p(0, s) = Z_p(0)$ ,  $\mathcal{W}_p(0, s) = \omega_p(0)$ <sup>9</sup>,
- is AP in the fluid limit ( $\tau \rightarrow 0$ , fixed  $\varepsilon > 0$ ), if we consider  $\hat{\mathcal{M}}_{p,l}^*$  "semi-implicit" (computed from  $\mathcal{Z}_p^{n+1}$  and  $\mathcal{W}_p^n$ ),  
remark: UA if  $\mathcal{M}$  does not depend on weights  $\mathcal{W}$ .

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<sup>9</sup>Remark: more elaborate well prepared initial datas are needed for the 2nd-order scheme.

## Penalization approach

- In the gyrofluid limit ( $\varepsilon \rightarrow 0$  then  $\tau \rightarrow 0$ ), equilibrium of collision operator  $\langle Q_{filt} \rangle[F]$  is  $\mathcal{G}[F] \neq \langle \mathcal{M}_{filt}[F] \rangle$ ,
- we will enforce the right asymptotic behaviour by modifying weights scheme, starting from

$$\partial_t \mathcal{W}_p + \frac{1}{\varepsilon} \partial_s \mathcal{W}_p = \frac{1}{\tau} (\mathcal{M}_p - \mathcal{W}_p) - \frac{1}{\tau} \mathcal{G}_p + \frac{1}{\tau} \mathcal{G}_p,$$

- Fourier transform in  $s$  + integration on  $[t^n, t^{n+1}]$  but different quadratures on

$$\frac{1}{\tau} \int_{t^n}^{t^{n+1}} e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)(t^{n+1}-t)} (\hat{\mathcal{M}}_{p,l}(t) - \hat{\mathcal{G}}_{p,l}(t)) dt$$

and

$$\frac{1}{\tau} \int_{t^n}^{t^{n+1}} e^{-\left(\frac{il}{\varepsilon} + \frac{1}{\tau}\right)(t^{n+1}-t)} \hat{\mathcal{G}}_{p,l}(t) dt,$$

- for asymptotics in  $\varepsilon \rightarrow 0$ , we focus on mode 0.

# Asymptotic behaviour of scheme B

Scheme B whose mode 0 is given by

$$\hat{\mathcal{Z}}_{p,0}^{n+1} = \hat{\mathcal{Z}}_{p,0}^n + \Delta t \hat{\mathcal{H}}_{p,0}^n,$$

$$\hat{\mathcal{W}}_{p,0}^{n+1} = e^{-\frac{\Delta t}{\tau}} \hat{\mathcal{W}}_{p,0}^n + \frac{\Delta t}{\tau} e^{-\frac{\Delta t}{\tau}} \left( \hat{\mathcal{M}}_{p,0}^n - \hat{\mathcal{G}}_{p,0}^n \right) + \left( 1 - e^{-\frac{\Delta t}{\tau}} \right) \hat{\mathcal{G}}_{p,0}^{n+1},$$

- is AP in the gyrofluid limit ( $\varepsilon \rightarrow 0$  then  $\tau \rightarrow 0$ ),
- is AP in the gyrokinetic limit ( $\varepsilon \rightarrow 0$ , fixed  $\tau > 0$ ).

But

- we lose the right behaviour in the fluid limit...
- We can propose a convex combination of schemes A and B:

$$\hat{\mathcal{W}}_{p,l}^{n+1} = \frac{\varepsilon}{\tau + \varepsilon} \hat{\mathcal{W}}_{p,l}^{n+1,A} + \left( 1 - \frac{\varepsilon}{\tau + \varepsilon} \right) \hat{\mathcal{W}}_{p,l}^{n+1,B}.$$

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# One-particle test: multiscale ODE framework

$$\begin{aligned}\dot{z}(t) &= \frac{1}{\varepsilon}Az(t) + h(t, z), \\ \dot{\omega}(t) &= \frac{1}{\tau}(\mathcal{M}(\omega, z) - \omega(t)),\end{aligned}$$

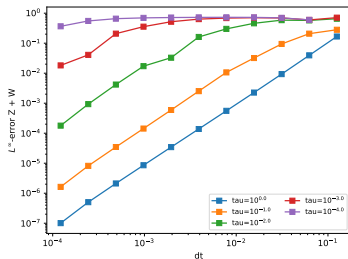
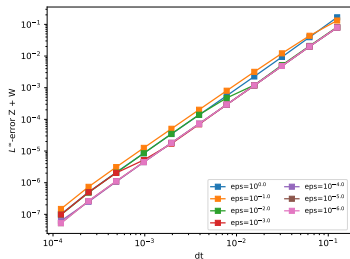
with

- $E(t, x) = ((x_1 + x_3) \cos(t), x_1 x_2 \sin(t), -x_2^2 e^{-t^2}),$
- $z(0) = (1, 1, 0, 1/2, 1/2, 3/2), \omega(0) = 1.$

We plot  $\|z\text{-error}\|_{L^\infty} + \|\omega\text{-error}\|_{L^\infty}$  as a function of  $\Delta t$ , error compared to a reference solution.



# One-particle test: $\mathcal{M}(\omega, z) = 1 + |z|^2 + e^{-\omega^2}$

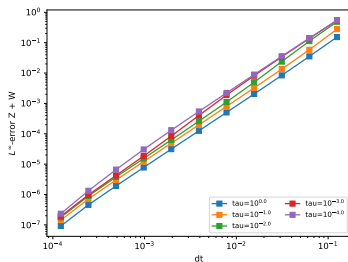


2nd-order scheme.

Left:  $L^\infty$ -Error for  $\tau = 1$  and different  $\varepsilon$ ; UA in gyrokinetic limit.

Right:  $L^\infty$ -Error for  $\varepsilon = 1$  and different  $\tau$ ; AP in fluid limit.

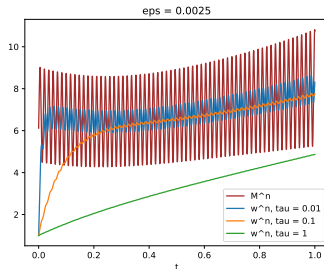
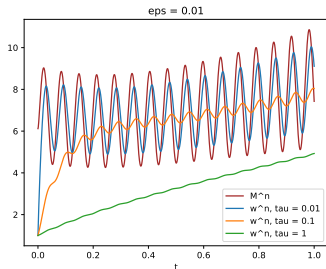
One-particle test:  $\mathcal{M}(\omega, z) = \mathcal{M}(z) = 1 + |z|^2$



2nd-order scheme.

$L^\infty$ -Error for  $\varepsilon = 1$  and different  $\tau$ ; UA in fluid limit.

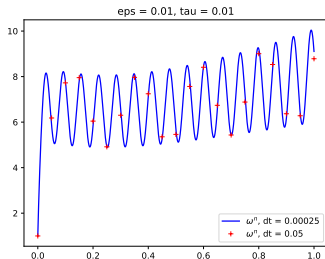
# One-particle test: time history of $\mathcal{M}(z(t))$ and $\omega(t)$



Fluid limit: for fixed  $\varepsilon > 0$ ,  $w(t)$  converges toward  $\mathcal{M}(z(t))$  when  $\tau \rightarrow 0$ .

Gyrokinetic limit: for fixed  $\tau > 0$ ,  $w(t)$  converges toward the average of  $\mathcal{M}(z(t))$  when  $\varepsilon \rightarrow 0$ .

# One-particle test: time history of $\omega(t)$ , $\mathcal{M}(z)$ -case



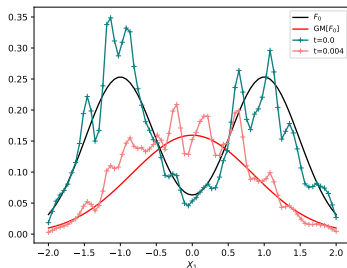
UA: scheme captures the strong relaxation and the highly oscillatory behaviour, without resolving these two stiffnesses ( $\Delta t > \varepsilon = \tau$ ).

# PDE framework: gyrofluid limit

We consider the simplified filtered model

$$\partial_t F = \frac{1}{\tau} Q_{filt}[F] = \frac{1}{\tau} (\mathcal{M}_{filt}[F] - F).$$

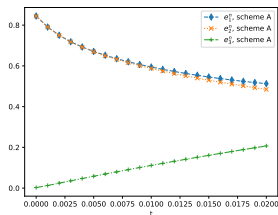
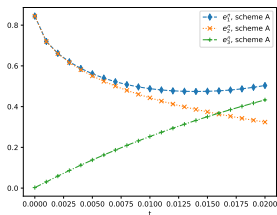
We plot (a slice of) the particles density initially and at time  $4 \times 10^{-3}$ , as well as the gyromaxwellian equilibrium.



# PDE framework: error study of scheme A

We plot 3 errors:  $e_1^n = \sum_p |\omega_p^n - \mathcal{G}_p^{ex}|$ ,  $e_2^n = \sum_p |\omega_p^n - \mathcal{G}_p^n|$  and  $e_3^n = \sum_p |\mathcal{G}_p^n - \mathcal{G}_p^{ex}|$ .

Parameters:  $N_p = 12288000$ ,  $\varepsilon = 10^{-8}$ ,  $\Delta t = 10^{-3}$ ,  $N_s = 4$ ,  $\Delta x \approx 0.37$ ,  $\tau = 10^{-4}$  (left) and  $\tau = 2 \times 10^{-3}$  (right).



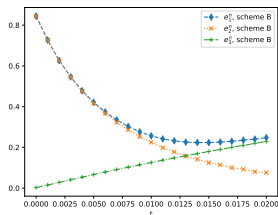
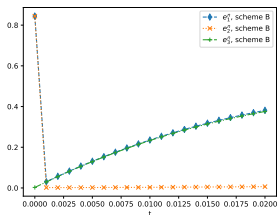
$e_3^n$  increases since gyromoments are not conserved exactly (and it deteriorates  $\mathcal{G}_p^n$ ). Better if  $N_p$  increases.

$e_2^n$  big since scheme A is not AP in the gyrofluid limit.

## PDE framework: error study of scheme B

We plot 3 errors:  $e_1^n = \sum_p |\omega_p^n - \mathcal{G}_p^{ex}|$ ,  $e_2^n = \sum_p |\omega_p^n - \mathcal{G}_p^n|$  and  $e_3^n = \sum_p |\mathcal{G}_p^n - \mathcal{G}_p^{ex}|$ .

Parameters:  $N_p = 12288000$ ,  $\varepsilon = 10^{-8}$ ,  $\Delta t = 10^{-3}$ ,  $N_s = 4$ ,  $\Delta x \approx 0.37$ ,  $\tau = 10^{-4}$  (left) and  $\tau = 2 \times 10^{-3}$  (right).



$e_3^n$  increases since gyromoments are not conserved exactly (and it deteriorates  $\mathcal{G}_p^n$ ). Better if  $N_p$  increases.

$e_2^n$  small since scheme B is AP in the gyrofluid limit.

## Conclusion and opening

- Two schemes were developed for a 3Dx-3Dv multiscale Vlasov equation involving collisions and fast oscillations.
- Asymptotic properties (UA or AP) are obtained (proofs in the submitted paper and numerical investigations) for three limits.
- A projection technique (as in [Dimarco, Loubère 2013] or [Gamba, Tharkabhushanam 2009]) could ensure preservation of gyromoments.
- Micro-macro approach to reduce computational time.
- Coupling with Maxwell equations for a self-consistent electromagnetic field.



Thank you for your attention!