A FINITE PRESENTATION OF THE MAPPING CLASS GROUP OF A PUNCTURED SURFACE

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Abstract. We give a finite presentation of the mapping class group of an oriented (possibly bounded) surface of genus greater or equal than 1, considering Dehn twists on a very simple set of curves.

Introduction and notations

Let $\Sigma_{g,n}$ be an oriented surface of genus $g \geq 1$ with n boundary components and denote by $\mathcal{M}_{g,n}$ its mapping class group, that is to say the group of orientation preserving diffeomorphisms of $\Sigma_{g,n}$ which are the identity on $\partial\Sigma_{g,n}$, modulo isotopy:

$$\mathcal{M}_{g,n} = \pi_0\big(\mathrm{Diff}^+(\Sigma_{g,n}, \partial\Sigma_{g,n})\big) .$$

For a simple closed curve α in $\Sigma_{g,n}$, denote by τ_α the Dehn twist along α. If α and β are isotopic, then the associated twists are also isotopic: thus, we shall consider curves up to isotopy. We shall use greek letters to denote them, and we shall not distinguish a Dehn twist from its isotopy class.

It is known that $\mathcal{M}_{g,n}$ is generated by Dehn twists [2, 10, 11]. Using the result of Hatcher and Thuston [6], Wajnryb gave in [12] a presentation of $\mathcal{M}_{g,1}$ and $\mathcal{M}_{g,0}$ with the minimal possible number of twist generators given by Humphries in [7]. In [3], the author gave a presentation considering either all possible Dehn twists, or just Dehn twists along non-separating curves. These two presentations appear to be very symmetric, but infinite. The aim of this article is to give a finite presentation of $\mathcal{M}_{g,n}$.

Notation. Composition of diffeomorphisms in $\mathcal{M}_{g,n}$ will be written from right to left. For two elements x, y of a multiplicative group, we will denote indifferently by x^{-1} or $\overline{x}$ the inverse of x and by $y(x)$ the conjugate $y\,x\,\overline{y}$ of x by y.

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Next, considering the curves of figure 1, we denote by $\mathcal{G}_{g,n}$ and $\mathcal{H}_{g,n}$ (we may on occasion omit the subscript "g,n" if there is no ambiguity) the following sets of curves in $\Sigma_{g,n}$:

$$\begin{aligned}\mathcal{G}_{g,n} &= \{\beta, \beta_1, \ldots, \beta_{g-1}, \alpha_1, \ldots, \alpha_{2g+n-2}, (\gamma_{i,j})_{1\leq i,j\leq 2g+n-2, i\neq j}\},\\ \mathcal{H}_{g,n} &= \{\alpha_1, \beta, \alpha_2, \beta_1, \gamma_{2,4}, \beta_2, \ldots, \gamma_{2g-4,2g-2}, \beta_{g-1}, \gamma_{1,2},\\ &\qquad \alpha_{2g}, \ldots, \alpha_{2g+n-2}, \delta_1, \ldots, \delta_{n-1}\}\end{aligned}$$

where $\delta_i = \gamma_{2g-2+i,2g-1+i}$ is the i^{th} boundary component. Note that $\mathcal{H}_{g,n}$ is a subset of $\mathcal{G}_{g,n}$.

Finally, a triple $(i,j,k)\in\{1,\ldots,2g+n-2\}^3$ will be said to be *good* when:

i) $(i,j,k)\notin\{(x,x,x)\,/\,x\in\{1,\ldots,2g+n-2\}\}$,
ii) $i\leq j\leq k$ or $j\leq k\leq i$ or $k\leq i\leq j$.

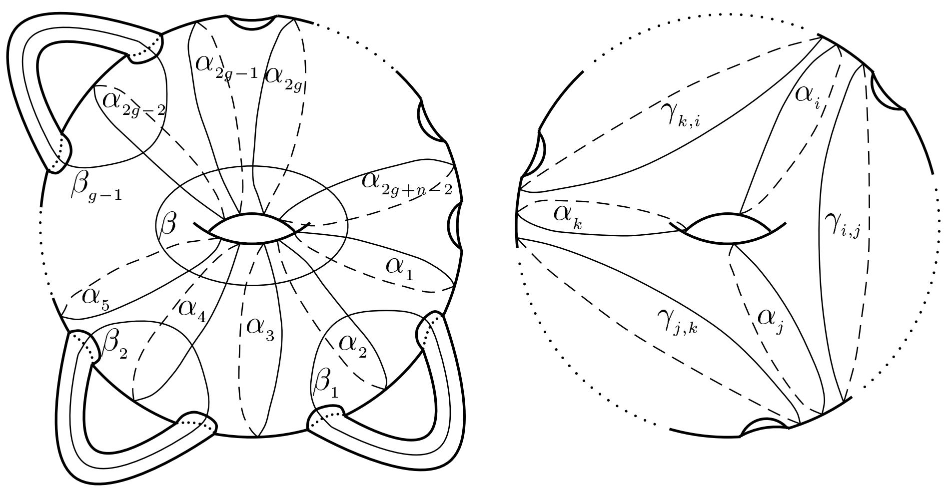

figure 1

Remark 1. For $n=0$ or $n=1$, Humphries' generators are the Dehn twists relative to the curves of $\mathcal{H}$.

We will give a presentation of $\mathcal{M}_{g,n}$ taking as generators the twists along the curves in $\mathcal{G}$. The relations will be of the following types.

The braids: If α and β are two curves in $\Sigma_{g,n}$ which do not intersect (resp. intersect in a single point), then the associated Dehn twists satisfy the relation $\tau_\alpha\tau_\beta=\tau_\beta\tau_\alpha$ (resp. $\tau_\alpha\tau_\beta\tau_\alpha=\tau_\beta\tau_\alpha\tau_\beta$).

The stars: Consider a subsurface of $\Sigma_{g,n}$ which is homeomorphic to $\Sigma_{1,3}$. Then, if α_1, α_2, α_3, β, γ_1, γ_2, γ_3 are the curves described in figure 2, one has in $\mathcal{M}_{g,n}$ the relation

$$(\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_\beta)^3 = \tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_3}\,.$$

Note that if γ_3 bounds a disc in $\Sigma_{g,n}$, then this relation becomes

$$(\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_2}\tau_\beta)^3 = \tau_{\gamma_1}\tau_{\gamma_2}\,.$$

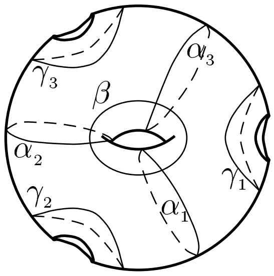

figure 2

The handles: Pasting a cylinder on two boundary components of $\Sigma_{g-1,n+2}$, the twists along these two boundary curves become equal in $\Sigma_{g,n}$.

Theorem 1. *For all* $(g,n) \in \mathbf{N}^* \times \mathbf{N}$, $(g,n) \neq (1,0)$, *the mapping class group* $\mathcal{M}_{g,n}$ *admits a presentation with generators* b, $b_1, \dots, b_{g-1}$, $a_1, \dots, a_{2g+n-2}$, $(c_{i,j})_{1\le i,j\le 2g+n-2,\, i\neq j}$ *and relations*

(A) "handles": $c_{2i,2i+1} = c_{2i-1,2i}$ *for all* i, $1 \le i \le g-1$,

(T) "braids": *for all* x, y *among the generators,* $xy = yx$ *if the associated curves are disjoint and* $xyx = yxy$ *if the associated curves intersect transversaly in a single point,*

($E_{i,j,k}$) "stars": $c_{i,j}c_{j,k}c_{k,i} = (a_i a_j a_k b)^3$ *for all good triples* (i,j,k), *where* $c_{l,l} = 1$.

Remark 2. It is clear that the handle relations are unnecessary: one has just to remove $c_{2,3}, \dots, c_{2g-2,2g-1}$ from $\mathcal{G}_{g,n}$ to eliminate them. But it is convenient for symmetry and notation to keep these generators.

Let $G_{g,n}$ denote the group with presentation given by theorem 1. Since the set of generators for $G_{g,n}$ that we consider here is parametrized by $\mathcal{G}_{g,n}$, we will consider $\mathcal{G}_{g,n}$ as a subset of $G_{g,n}$. Consequently, $\mathcal{H}_{g,n}$ will also be considered as a subset of $G_{g,n}$.

The paper is organized as follows. In section 1, we prove that $G_{g,n}$ is generated by $\mathcal{H}_{g,n}$. Section 2 is devoted to the proof of theorem 1 when $n = 1$. Finally, we conclude the proof in section 3 by proving that $G_{g,n}$ is isomorphic to $\mathcal{M}_{g,n}$.

1. Generators for $G_{g,n}$

In this section, we prove the following proposition.

Proposition 1. *$G_{g,n}$ is generated by $\mathcal{H}_{g,n}$.*

We begin by proving some relations in $G_{g,n}$.

Lemma 2. *For $i, j, k \in \{1, \ldots, 2g+n-2\}$, if $X_1 = a_i a_j$, $X_2 = bX_1b$ and $X_3 = a_k X_2 a_k$, then:*

(i) $X_pX_q = X_qX_p$ for all $p, q \in \{1, 2, 3\}$.

(ii) $(a_i a_j a_k b)^3 = X_1X_2X_3$,

(iii) $(a_i a_i a_j b)^3 = X_1^2 X_2^2 = (a_i a_j b)^4 = (a_i b\, a_j)^4$,

(iv) a_i, a_j, a_k and b commute with $(a_i a_j a_k b)^3$.

Remark 3. Combining the braid relations and lemma 2, one can see that the star relations $(E_{j,k,i})$ and $(E_{k,i,j})$ are consequences of $(E_{i,j,k})$, and that the star relation $(E_{i,j,j})$ is a consequence of $(E_{i,i,j})$ when $i \neq j$. Thus, one just needs relations $(E_{i,j,k})$ with good triples (i, j, k) such that $i \leq j \leq k$. This will be used latter for proving lemma 6.

Proof. *(i)* Using relations *(T)*, one has

$$\begin{aligned} a_i\, X_2 &= a_i\, b\, a_i\, a_j\, b \\ &= b\, a_i\, b\, a_j\, b \\ &= b\, a_i\, a_j\, b\, a_j \\ &= X_2\, a_j\,, \end{aligned}$$

and in the same way, $a_j\, X_2 = X_2\, a_i$. Thus, we get $X_1\, X_2 = X_2\, X_1$ and $X_1\, X_3 = X_3\, X_1$ since $X_1\, a_k = a_k\, X_1$.
On the other hand, the braid relations imply

$$\begin{aligned} b(X_3) &= b\, a_k\, b\, a_i\, a_j\, b\, a_k\, \overline{b} \\ &= a_k\, b\, a_k\, a_i\, a_j\, \overline{a_k}\, b\, a_k \\ &= X_3\,, \end{aligned}$$

and we get $X_2\, X_3 = X_3\, X_2$.

(ii) Using relations *(T)* and *(i)*, one obtains:

$$\begin{aligned} X_1X_2X_3 &= X_1X_3X_2 \\ &= a_i\, a_j\, a_k\, b\, a_i\, a_j\, b\, a_k\, b\, a_i\, a_j\, b \\ &= a_i\, a_j\, a_k\, b\, a_i\, a_j\, a_k\, b\, a_k\, a_i\, a_j\, b \\ &= (a_i a_j a_k b)^3. \end{aligned}$$

(iii) Replacing a_k by a_i in X_3, we get

$$X_3 = a_i\, X_2\, a_i = a_i\, a_j\, X_2 = X_1\, X_2.$$

Thus, using relations *(T)*, *(i)* and *(ii)*, one has:

$$\begin{aligned} (a_i a_i a_j b)^3 &= X_1X_2X_1X_2 = X_1^2X_2^2 \\ &= a_i\, a_j\, b\, a_i\, a_j\, b\, a_i\, a_j\, b\, a_i\, a_j\, b \;=\; (a_i a_j b)^4 \\ &= a_i\, b\, a_j\, b\, a_i\, b\, a_j\, b\, a_i\, b\, a_j\, b \\ &= a_i\, b\, a_j\, a_i\, b\, a_i\, a_j\, b\, a_i\, a_j\, b\, a_j \\ &= (a_i b\, a_j)^4. \end{aligned}$$

(iv) One has just to apply the star and braid relations. □

Lemma 3. *For all good triples (i,j,k), one has in $G_{g,n}$ the relation*

$$(L_{i,j,k}) \quad a_i\, c_{i,j}\, c_{j,k}\, a_k = c_{i,k}\, a_j\, X\, a_j\, \overline{X} = c_{i,k}\, \overline{X}\, a_j\, X\, a_j$$

where $X = b\, a_i\, a_k\, b$.

Remark 4. These relations are just the well known *lantern* relations.

Proof. If $X_1 = a_i\, a_k$ and $X_3 = a_j\, X\, a_j$, one has by lemma 2 and the star relations $(E_{i,j,k})$ and $(E_{i,k,k})$:

$$X_1\, X\, X_3 = c_{i,j}\, c_{j,k}\, c_{k,i} \quad \text{and} \quad X_1^2\, X^2 = c_{i,k}\, c_{k,i}\,.$$

From this, we get, using the braid relations, that

$$\overline{c_{k,i}}\, X_1\, X = c_{i,j}\, c_{j,k}\, \overline{X_3} = c_{i,k}\, \overline{X}\, \overline{X_1}\,,$$

that is to say, by lemma 2 and *(T)*,

$$a_i\, c_{i,j}\, c_{j,k}\, a_k = c_{i,k}\, \overline{X}\, a_j\, X\, a_j \;=\; c_{i,k}\, a_j\, X\, a_j\, \overline{X}\,.$$

□

Lemma 4. *For all i,k such that $1 \leq i \leq g-1$ and $k \neq 2i-1, 2i$, one has in $G_{g,n}$*

$$a_k \;=\; b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{c_{2i,2i-1}}\, a_{2i}\, c_{2i,k}(b_i)\,.$$

Proof. If $X = b\, a_{2i-1}\, a_{2i}\, b$, one has by the lantern relations

$$(L_{2i,k,2i-1}): \quad a_{2i}\, c_{2i,k}\, c_{k,2i-1}\, a_{2i-1} = c_{2i,2i-1}\, \overline{X}\, a_k\, X\, a_k\,,$$

which implies

$$\overline{c_{2i,2i-1}}\, a_{2i}\, c_{2i,k} = \overline{X}\, a_k\, X\, a_k\, \overline{a_{2i-1}}\, \overline{c_{k,2i-1}}\,.$$

Thus, denoting $b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{c_{2i,2i-1}}\, a_{2i}\, c_{2i,k}(b_i)$ by y, we can compute using the relations *(T)*:

$$\begin{aligned}
y &= b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{X}\, a_k\, X\, a_k\, \overline{a_{2i-1}}\, \overline{c_{k,2i-1}}(b_i) \\
&= b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{b}\, \overline{a_{2i-1}}\, \overline{a_{2i}}\, \overline{b}\, a_k\, b\, a_{2i-1}\, a_{2i}\, b\, (b_i) \\
&= b\, \overline{b_i}\, a_{2i}\, b_i\, a_k\, b\, \overline{a_k}\, \overline{b_i}\, (a_{2i}) \\
&= b\, a_k\, \overline{b_i}\, a_{2i}\, \overline{a_{2i}}(b) \\
&= b\, \overline{b}(a_k) \\
&= a_k\,.
\end{aligned}$$

□

Proof of proposition 1. If H denotes the subgroup of $G_{g,n}$ generated by $\mathcal{H}_{g,n}$, we have to prove that $\mathcal{G}_{g,n} \subset H$.

a) We first prove inductively that a_{2i-1}, a_{2i}, $c_{2i-1,2i}$ and $c_{2i,2i-1}$ are elements of H for all i, $1 \le i \le g-1$.

For $i = 1$, one obtains a_1, a_2 and $c_{1,2}$ which are in H, and the relation $(E_{1,2,2})$ gives $c_{2,1} = (a_1 a_2 a_2 b)^3 \overline{c_{1,2}} \in H$. So, suppose inductively that a_{2i-1}, a_{2i}, $c_{2i-1,2i}$, $c_{2i,2i-1}$ are elements of H $(i \le g-2)$ and let us prove that a_{2i+1}, a_{2i+2}, $c_{2i+1,2i+2}$, $c_{2i+2,2i+1}$ are also in H. Recall that by the handle relations, one has $c_{2i,2i+1} = c_{2i-1,2i} \in H$. Applying lemma 4 respectively with $k = 2i+1$ and $k = 2i+2$, we obtain

$$\begin{aligned}
a_{2i+1} &= b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{c_{2i,2i-1}}\, a_{2i}\, c_{2i,2i+1}(b_i) \in H\,, \\
a_{2i+2} &= b\, a_{2i}\, b_i\, a_{2i-1}\, b\, \overline{c_{2i,2i-1}}\, a_{2i}\, c_{2i,2i+2}(b_i) \in H\,.
\end{aligned}$$

The star relations allow us to conclude the induction as follows:

$$(E_{2i,2i+2,2i+2}): \quad c_{2i,2i+2}\, c_{2i+2,2i} = (a_{2i}\, a_{2i+2}\, b)^4,$$

which gives $c_{2i+2,2i} \in H$ $(\gamma_{2i,2i+2} \in \mathcal{H}_{g,n}$ by definition);

$$(E_{2i,2i+1,2i+2}): \quad c_{2i,2i+1} c_{2i+1,2i+2} c_{2i+2,2i} = (a_{2i} a_{2i+1} a_{2i+2} b)^3,$$

which gives $c_{2i+1,2i+2} \in H$;

$$(E_{2i+1,2i+2,2i+2}): \quad c_{2i+1,2i+2}\, c_{2i+2,2i+1} = (a_{2i+1}\, a_{2i+2}\, b)^4,$$

which gives $c_{2i+2,2i+1} \in H$.

b) By lemma 4, one has ($i = g-1$ and $k = 2g-1$)

$$a_{2g-1} = b\, a_{2g-2}\, b_{g-1}\, a_{2g-3}\, b\, \overline{c_{2g-2,2g-3}}\, a_{2g-2}\, c_{2g-2,2g-1}(b_{g-1}).$$

Recall that $c_{2g-2,2g-1} = c_{2g-3,2g-2} \in H$. Thus, combined with the case a), this relation implies $a_{2g-1} \in H$.

c) It remains to prove that $c_{i,j} \in H$ for all i, j.

$*$ By definition of H and the case a), one has $c_{i,i+1} \in H$ for all i such that $1 \leq i \leq 2g+n-3$.

$*$ Let us show that $c_{1,j}$ and $c_{j,1}$ are elements of H for all j such that $2 \leq j \leq 2g+n-2$.

We have already seen that $c_{1,2},\, c_{2,1} \in H$. Thus, suppose inductively that $c_{1,j}, c_{j,1} \in H$ ($j \leq 2g+n-3$). Using the star relations, one obtains:

$$(E_{1,j,j+1}):\ c_{1,j}\, c_{j,j+1}\, c_{j+1,1} = (a_1\, a_j\, a_{j+1}\, b)^3, \text{ which gives } c_{j+1,1} \in H,$$

$$(E_{1,j+1,j+1}):\ c_{1,j+1}\, c_{j+1,1} = (a_1\, a_{j+1}\, b)^4, \text{ which gives } c_{1,j+1} \in H.$$

$*$ Now, fix j such that $2 \leq j \leq 2g+n-2$ and let us show that $c_{i,j}, c_{j,i} \in H$ for all i, $1 \leq i < j$. Once more, the star relations allow us to prove this using an inductive argument:

$$(E_{i,i+1,j}):\ c_{i,i+1}\, c_{i+1,j}\, c_{j,i} = (a_i\, a_{i+1}\, a_j\, b)^3, \text{ which gives } c_{i+1,j} \in H,$$

$$(E_{i+1,j,j}):\ c_{i+1,j}\, c_{j,i+1} = (a_{i+1}\, a_j\, b)^4, \text{ which gives } c_{j,i+1} \in H.$$

□

2. Proof of theorem 1 for $n = 1$

Let us recall Wajnryb's result:

Theorem 2 ([12]). *$\mathcal{M}_{g,1}$ admits a presentation with generators $\{\tau_\alpha \,/\, \alpha \in \mathcal{H}\}$ and relations*

(I) $\tau_\lambda \tau_\mu \tau_\lambda = \tau_\mu \tau_\lambda \tau_\mu$ *if λ and μ intersect transversaly in a single point, and* $\tau_\lambda \tau_\mu = \tau_\mu \tau_\lambda$ *if λ and μ are disjoint.*

(II) $(\tau_{\alpha_1} \tau_\beta \tau_{\alpha_2})^4 = \tau_{\gamma_{1,2}}\, \theta$ *where* $\theta = \tau_{\beta_1} \tau_{\alpha_2} \tau_\beta \tau_{\alpha_1} \tau_{\alpha_1} \tau_\beta \tau_{\alpha_2} \tau_{\beta_1} (\tau_{\gamma_{1,2}})$.

(III) $\tau_{\alpha_2} \tau_{\alpha_1} \varphi\, \tau_{\gamma_{2,4}} = \overline{t_1}\, \overline{t_2}\, \tau_{\gamma_{1,2}}\, t_2\, t_1\, \overline{t_2}\, \tau_{\gamma_{1,2}}\, t_2\, \tau_{\gamma_{1,2}}$ *where*

$$t_1 = \tau_\beta \tau_{\alpha_1} \tau_{\alpha_2} \tau_\beta\ ,\ \ t_2 = \tau_{\beta_1} \tau_{\alpha_2} \tau_{\gamma_{2,4}} \tau_{\beta_1}\ ,$$
$$\varphi = \tau_{\beta_2} \tau_{\gamma_{2,4}} \tau_{\beta_1} \tau_{\alpha_2} \tau_\beta\, \sigma(\omega),\ \ \sigma = \overline{\tau_{\gamma_{2,4}}}\, \overline{\tau_{\beta_2}}\, \overline{t_2}(\tau_{\gamma_{1,2}})$$
$$\text{and } \omega = \overline{\tau_{\alpha_1}}\, \overline{\tau_\beta}\, \overline{\tau_{\alpha_2}}\, \overline{\tau_{\beta_1}}(\tau_{\gamma_{1,2}}).$$

Remark 5. When $g=1$, one just needs the relations *(I)*. The relations *(II)* and *(III)* appear respectively for $g=2$ and $g=3$.

Denote by $\Phi: G_{g,1} \to \mathcal{M}_{g,1}$ the map which associates to each generator a of $G_{g,1}$ the corresponding twist τ_α. Since the relations *(A)*, *(T)* and *($E_{i,j,k}$)* are satisfied in $\mathcal{M}_{g,1}$, Φ is an homomorphism.
Now, consider $\Psi : \mathcal{M}_{g,1} \to G_{g,1}$ defined by $\Psi(\tau_\alpha) = a$ for all $\alpha \in \mathcal{H}$.

Lemma 5. *Ψ is an homomorphism.*

This lemma allows us to prove the theorem 1 for $n = 1$. Indeed, since $\mathcal{M}_{g,1}$ is generated by $\{\tau_\alpha \,/\, \alpha \in \mathcal{H}_{g,1}\}$, one has $\Phi \circ \Psi = Id_{\mathcal{M}_{g,1}}$. On the other hand, $\{a \,/\, \alpha \in \mathcal{H}_{g,1}\}$ generates $G_{g,1}$ by proposition 1, so $\Psi \circ \Phi = Id_{G_{g,1}}$.

Proof of lemma 5. We have to show that the relations *(I)*, *(II)* and *(III)* are satisfied in $G_{g,1}$. Relations *(I)* are braid relations and are therefore satisfied by *(T)*. Let us look at the relation *(II)*. The star relation $(E_{1,2,2})$, together with lemma 2, gives $(a_1\, b\, a_2)^4 = c_{1,2}\, c_{2,1}$. Thus, relation *(II)* is satisfied in $G_{g,1}$ if and only if $\Psi(\theta) = c_{2,1}$. Let us compute:

$$\begin{array}{rcll}
\Psi(\theta) & = & b_1\, a_2\, b\, a_1\, a_1\, b\, a_2\, b_1(c_{1,2}) & \\
 & = & b_1\, a_2\, b\, a_1\, a_1\, b\, a_2\, \overline{c_{1,2}}(b_1) & \text{by } (T), \\
 & = & b_1\, a_2\, b\, a_1\, a_1\, b\, a_2\, (\overline{a_1}\,\overline{a_1}\,\overline{a_2}\,\overline{b})^3 c_{2,1}(b_1) & \text{by } (E_{1,1,2}), \\
 & = & b_1\, \overline{b}\,\overline{a_1}\,\overline{a_1}\,\overline{b}\,\overline{a_1}\,\overline{a_1}\, c_{2,1}(b_1) & \text{by lemma 2}, \\
 & = & b_1\, \overline{b_1}(c_{2,1}) & \text{by } (T), \\
 & = & c_{2,1}. &
\end{array}$$

Wajnryb's relation *(III)* is nothing but a lantern relation. Via Ψ, it becomes in $G_{g,1}$

$$a_2\, a_1\, f\, c_{2,4} = l\, m\, c_{1,2} \qquad (*)$$

where $m = \overline{b_1}\,\overline{a_2}\,\overline{c_{2,4}}\,\overline{b_1}(c_{1,2})$, $l = \overline{b}\,\overline{a_1}\,\overline{a_2}\,\overline{b}(m)$ and $f = b_2\, c_{2,4}\, b_1\, a_2\, b\, s(w)$, with $s = \Psi(\sigma) = \overline{c_{2,4}}\,\overline{b_2}(m)$ and $w = \Psi(\omega) = \overline{a_1}\,\overline{b}\,\overline{a_2}\,\overline{b_1}(c_{1,2})$.

In $G_{g,1}$, the lantern relation $(L_{1,2,4})$ yields

$$a_1\, c_{1,2}\, c_{2,4}\, a_4 = c_{1,4}\, \overline{X}\, a_2\, X\, a_2 \qquad (L_{1,2,4})$$

where $X = b\, a_1\, a_4\, b$. To prove that the relation $(*)$ is satisfied in $G_{g,1}$, we will see that it is exactly the conjugate of the relation $(L_{1,2,4})$ by $h = b_2\, a_4\, \overline{c_{4,1}}\,\overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1$. This will be done by proving the following seven equalities in $G_{g,1}$:

$$1)\ h(a_1) = a_2 \quad 2)\ h(c_{1,2}) = a_1 \quad 3)\ h(c_{2,4}) = f \quad 4)\ h(a_4) = c_{2,4}$$

$$5)\ h(c_{1,4}) = l \quad 6)\ h(a_2) = c_{1,2} \quad 7)\ h\overline{X}(a_2) = m.$$

1) Just applying the relations *(T)*, one obtains:

$$\begin{aligned} h(a_1) &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1(a_1) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, \overline{a_1}(b) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, \overline{b}(a_2) \\ &= a_2\,. \end{aligned}$$

2) Using the relations *(T)* again, we get

$$\begin{aligned} h(c_{1,2}) &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1(c_{1,2}) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, \overline{c_{1,2}}(b_1) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, \overline{b_1}(a_2) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, \overline{a_2}(b) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, \overline{b}(a_1) \\ &= a_1\,. \end{aligned}$$

3) The relation $(L_{2,3,4})$ yields

$$a_2\, c_{2,3}\, c_{3,4}\, a_4 = c_{2,4}\, \overline{Y}\, a_3\, Y\, a_3 \qquad \text{where } \ Y = b\, a_2\, a_4\, b.$$

Since $c_{2,3}{=}c_{1,2}$ by the handle relations, this equality implies the following one:

$$\overline{c_{2,4}}\, a_2\, c_{1,2} = \overline{Y}\, a_3\, Y\, a_3\, \overline{a_4}\, \overline{c_{3,4}} \qquad (1).$$

From this, we get:

$$\begin{aligned} h(c_{2,4}) &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1(c_{2,4}) && \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, \overline{c_{2,4}}\, c_{1,2}\, a_2(b_1) && \text{by } (T) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, \overline{Y}\, a_3\, Y\, a_3\, \overline{a_4}\, \overline{c_{3,4}}(b_1) && \text{by } (1) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, \overline{b}\, \overline{a_2}\, \overline{a_4}\, \overline{b}\, a_3\, b\, a_2\, a_4\, b(b_1) && \text{by } (T) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_1\, \overline{b_1}\, a_2\, b_1\, \overline{a_4}\, a_3\, b\, \overline{a_3}\, \overline{b_1}(a_2) && \text{by } (T) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_1\, \overline{b_1}\, a_2\, \overline{a_4}\, a_3\, \overline{a_2}(b) && \text{by } (T) \\ &= b_2\, a_4\, \overline{c_{4,1}}\, b\, a_1\, a_3\, \overline{b_2}\, b(a_4) && \text{by } (T) \\ &= b_2\, a_4\, (\overline{a_1}\, \overline{a_3}\, \overline{a_4}\, \overline{b})^3\, c_{1,3}\, c_{3,4}\, b\, a_1\, a_3\, \overline{b_2}\, b(a_4) && \text{by } (E_{1,3,4}) \\ &= b_2\, \overline{a_1}\, \overline{a_3}\, \overline{b}\, (\overline{a_1}\, \overline{a_3}\, \overline{a_4}\, \overline{b})^2\, b\, a_1\, a_3\, c_{3,4}\, b\, a_4(b_2) && \text{by } (T) \\ &= b_2\, \overline{a_1}\, \overline{a_3}\, \overline{b}\, \overline{a_1}\, \overline{a_3}\, \overline{b}\, \overline{a_4}\, \overline{b}\, b\, a_4\, \overline{b_2}(c_{3,4}) && \text{by } (T) \\ &= c_{3,4} && \text{by } (T). \end{aligned}$$

Now, if $x{=}c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2})$, one has

$$f = b_2\, c_{2,4}\, b_1\, a_2\, b\, \overline{c_{2,4}}\, \overline{b_2}\, \overline{b_1}\, \overline{a_2}\, \overline{c_{2,4}}\, \overline{b_1}(x)\,.$$

First, let us compute x:

$$\begin{array}{rcll}
x & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, c_{2,4}\, c_{1,2}\, \overline{a_1}\, \overline{b}\, \overline{a_2}(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, (a_1\, a_2\, a_4\, b)^3\, \overline{c_{4,1}}\, \overline{a_1}\, \overline{b}\, \overline{a_2}(b_1) & \text{by } (E_{1,2,4}) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, (a_1\, a_2\, a_4\, b)^2\, a_1\, a_2\, a_4\, b\, \overline{a_1}\, \overline{b}\, \overline{a_2}(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, (a_1\, a_2\, a_4\, b)^2\, a_4\, a_2\, \overline{b}\, a_1\, b\, \overline{b}\, \overline{a_2}(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, (a_1\, a_2\, a_4\, b)^2\, a_4\, \overline{b}\, \overline{a_2}\, b(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, a_1\, a_2\, a_4\, b\, a_1\, a_2\, b\, a_4\, b\, \overline{b}\, \overline{a_2}(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, a_1\, a_2\, a_4\, b\, a_1\, \overline{b}\, a_2\, b(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, b_2\, a_2\, b_1\, a_2\, a_4\, b\, a_1\, b\, \overline{b}\, a_2(b_1) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, b_2\, b_1\, a_2\, b_1\, a_4\, b\, \overline{b_1}(a_2) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, b_2\, b_1\, a_2\, a_4\, \overline{a_2}(b) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, c_{2,4}\, b_2\, \overline{b}(a_4) & \text{by } (T).
\end{array}$$

Next, using the braid relations, we prove that b_1, $c_{2,4}$, b_2 and a_2 commute with x :

$$b_1(x) = b_1\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, \overline{b}(a_4) = c_{1,2}\, b_1\, c_{1,2}\, c_{2,4}\, b_2\, \overline{b}(a_4) = x,$$

$$c_{2,4}(x) = c_{1,2}\, b_1\, c_{2,4}\, b_1\, b_2\, \overline{b}(a_4) = x,$$

$$b_2(x) = c_{1,2}\, b_1\, b_2\, c_{2,4}\, b_2\, \overline{b}(a_4) = c_{1,2}\, b_1\, c_{2,4}\, b_2\, c_{2,4}\, \overline{b}(a_4) = x,$$

$$\begin{array}{rcll}
a_2(x) & = & a_2\, c_{1,2}\, b_1\, c_{2,4}\, a_2\, b_1\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \\
 & = & c_{1,2}\, b_1\, a_2\, b_1\, c_{2,4}\, b_1\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, a_2\, c_{2,4}\, b_1\, c_{2,4}\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, a_2\, c_{2,4}\, b_1\, b_2\, c_{2,4}\, b_2\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \text{by } (T) \\
 & = & c_{1,2}\, b_1\, a_2\, c_{2,4}\, b_1\, b_2\, c_{2,4}\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{b_1}(c_{1,2}) & \text{by } (T) \\
 & = & x. &
\end{array}$$

To conclude, we get,

$$\begin{array}{rcll}
f & = & b_2\, c_{2,4}\, b_1\, a_2\, b\, \overline{c_{2,4}}\, \overline{b_2}\, \overline{b_1}\, \overline{a_2}\, \overline{c_{2,4}}\, \overline{b_1}(x) & \\
 & = & b_2\, c_{2,4}\, b_1\, a_2\, b(x) & \\
 & = & b_2\, c_{2,4}\, b_1\, a_2\, b\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, \overline{b}(a_4) & \\
 & = & b_2\, c_{2,4}\, b_1\, a_2\, c_{1,2}\, b_1\, c_{2,4}\, \overline{a_4}(b_2) & \text{by } (T) \\
 & = & b_2\, c_{2,4}\, \overline{a_4}\, \overline{b_2}\, b_1\, a_2\, c_{1,2}\, b_1(c_{2,4}) & \text{by } (T) \\
 & = & b_2\, (a_1\, a_2\, a_4\, b)^3\, \overline{c_{1,2}}\, \overline{c_{4,1}}\, \overline{a_4}\, \overline{b_2}\, b_1\, a_2\, c_{1,2}\, b_1(c_{2,4}) & \text{by } (E_{1,2,4}) \\
 & = & b_2\, (a_1\, a_2\, a_4\, b)^3\, \overline{a_4}\, \overline{c_{4,1}}\, \overline{b_2}\, \overline{c_{1,2}}\, b_1\, c_{1,2}\, a_2\, b_1(c_{2,4}) & \text{by } (T) \\
 & = & b_2\, (a_1\, a_2\, b)^2\, a_4\, b\, a_1\, a_2\, b\, \overline{c_{4,1}}\, \overline{b_2}\, \overline{c_{1,2}}\, b_1\, c_{1,2}\, a_2\, b_1(c_{2,4}) & \text{by lemma 2} \\
 & = & (a_1\, a_2\, b)^2\, b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_1\, a_2\, b\, b_1\, c_{1,2}\, \overline{b_1}\, a_2\, b_1(c_{2,4}) & \text{by } (T)
\end{array}$$

$$
\begin{array}{rcll}
& = & (a_1\, a_2\, b)^2\, b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1\, \overline{a_2}(c_{2,4}) & \text{by } \mathit{(T)} \\
& = & (a_1\, a_2\, b)^2\, h(c_{2,4}) & \\
& = & (a_1\, a_2\, b)^2 (c_{3,4}) & \\
& = & c_{3,4} & \text{by } \mathit{(T)}.
\end{array}
$$

Finally, we have proved that $h(c_{2,4}) = c_{3,4} = f$.

4) We can compute $h(a_4)$ as follows:

$$
\begin{array}{rcll}
h(a_4) & = & b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1 (a_4) & \\
& = & b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b(a_4) & \text{by } \mathit{(T)} \\
& = & b_2\, a_4\, (\overline{a_1}\, \overline{a_2}\, \overline{a_4}\, \bar{b})^3\, c_{1,2}\, c_{2,4}\, \overline{b_2}\, b\, a_2\, a_1\, b(a_4) & \text{by } (E_{1,2,4}) \\
& = & b_2\, c_{2,4}\, \overline{a_1}\, \overline{a_2}\, \bar{b}\, \overline{a_1}\, \overline{a_2}\, \overline{a_4}\, \bar{b}\, \overline{a_1}\, \overline{a_2}\, \overline{a_4}\, \bar{b}\, \overline{b_2}\, b\, a_2\, a_1\, b(a_4) & \text{by } \mathit{(T)} \\
& = & b_2\, c_{2,4}\, \overline{a_1}\, \overline{a_2}\, \bar{b}\, \overline{a_1}\, \overline{a_2}\, \bar{b}\, \overline{a_4}\, \bar{b}\, \overline{b_2}\, b(a_4) & \text{by } \mathit{(T)} \\
& = & b_2\, c_{2,4}\, \overline{a_1}\, \overline{a_2}\, \bar{b}\, \overline{a_1}\, \overline{a_2}\, \bar{b}\, \overline{a_4}\, a_4 (b_2) & \text{by } \mathit{(T)} \\
& = & b_2\, c_{2,4} (b_2) & \text{by } \mathit{(T)} \\
& = & c_{2,4} & \text{by } \mathit{(T)}.
\end{array}
$$

5) For $h(c_{1,4})$, we have:

$$
\begin{array}{rcll}
h(c_{1,4}) & = & b_2\, a_4\, \overline{c_{4,1}}\, \overline{b_2}\, b\, a_2\, a_1\, b\, b_1\, c_{1,2}\, a_2\, b_1 (c_{1,4}) & \\
& = & b_2\, a_4\, \overline{c_{4,1}}\, b\, a_2\, a_1\, b\, b_1\, a_2\, \overline{b_2}(c_{1,4}) & \text{by } \mathit{(T)} \\
& = & b_2\, a_4\, \overline{a_4}\, \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_4}\, \overline{a_2}\, \overline{a_1}\, c_{1,2}\, c_{2,4}\, b_1\, a_2\, \overline{b_2}(c_{1,4}) & \text{by } (E_{1,2,4}) \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_2\, c_{2,4}\, b_1\, \overline{a_4}\, \overline{a_1}\, a_2\, c_{1,4} (b_2) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_2\, c_{2,4}\, b_1\, c_{1,2}\, c_{2,4}\, \overline{X}\, \overline{a_2}\, X (b_2) & \text{by } (L_{1,2,4}) \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_2\, c_{2,4}\, b_1\, c_{2,4}\, \bar{b}\, \overline{a_1}\, \overline{a_4}\, \bar{b}\, \overline{a_2}\, b\, a_4 (b_2) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_2\, b_1\, c_{2,4}\, b_1\, \bar{b}\, \overline{a_1}\, \overline{a_4}\, a_2\, \bar{b}\, \overline{a_2}\, a_4 (b_2) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, b_2\, c_{2,4}\, b_1\, \bar{b}\, \overline{a_1}\, a_2\, b\, \overline{a_4}\, \bar{b} (b_2) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, b_2\, c_{2,4}\, b_1\, \bar{b}\, \overline{a_1}\, a_2\, b\, b_2 (a_4) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, c_{2,4}\, b_1\, \bar{b}\, \overline{a_1}\, a_2\, \overline{a_4} (b) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, c_{2,4}\, \bar{b}\, \overline{a_1}\, \overline{a_4}\, \bar{b}\, b_1 (a_2) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, c_{2,4}\, \bar{b}\, \overline{a_1}\, \overline{a_4}\, \bar{b}\, \overline{a_2} (b_1) & \text{by } \mathit{(T)}
\end{array}
$$

Now, by $(E_{1,2,4})$ and lemma 2, one has

$$c_{1,2}\, c_{2,4}\, c_{4,1} = a_1\, a_4\, a_2\, X\, a_2 X,$$

which gives, using the braid relations (recall that $X = b\, a_1\, a_4\, b$):

$$c_{2,4}\, \bar{b}\, \overline{a_1 a_4}\, \bar{b}\, \overline{a_2} = a_1\, a_4\, a_2\, X\, \overline{c_{1,2}}\, \overline{c_{4,1}}\,.$$

Thus, we get

$$
\begin{array}{rcll}
h(c_{1,4}) & = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, \overline{a_2}\, c_{1,2}\, b_1\, c_{2,4}\, b_2\, a_1\, a_4\, a_2\, X\, \overline{c_{1,2}}\, \overline{c_{4,1}} (b_1) & \text{by } (E_{1,2,4}) \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, c_{1,2}\, \overline{a_2}\, b_1\, a_2\, \overline{c_{1,2}}\, c_{2,4} (b_1) & \text{by } \mathit{(T)} \\
& = & \bar{b}\, \overline{a_2}\, \overline{a_1}\, \bar{b}\, c_{1,2}\, b_1\, a_2\, \overline{b_1}\, \overline{c_{1,2}}\, \overline{b_1} (c_{2,4}) & \text{by } \mathit{(T)}
\end{array}
$$

$$
\begin{array}{rll}
= & \overline{b}\,\overline{a_2}\,\overline{a_1}\,\overline{b}\,c_{1,2}\,b_1\,a_2\,\overline{c_{1,2}}\,\overline{b_1}\,\overline{c_{1,2}}(c_{2,4}) & \text{by } (T) \\
= & \overline{b}\,\overline{a_2}\,\overline{a_1}\,\overline{b}\,\overline{b_1}\,c_{1,2}\,b_1\,a_2\,\overline{b_1}(c_{2,4}) & \text{by } (T) \\
= & \overline{b}\,\overline{a_2}\,\overline{a_1}\,\overline{b}\,\overline{b_1}\,c_{1,2}\,\overline{a_2}\,b_1\,a_2(c_{2,4}) & \text{by } (T) \\
= & \overline{b}\,\overline{a_2}\,\overline{a_1}\,\overline{b}\,\overline{b_1}\,\overline{a_2}\,\overline{c_{2,4}}\,c_{1,2}(b_1) & \text{by } (T) \\
= & \overline{b}\,\overline{a_2}\,\overline{a_1}\,\overline{b}\,\overline{b_1}\,\overline{a_2}\,\overline{c_{2,4}}\,\overline{b_1}(c_{1,2}) & \text{by } (T) \\
= & l\,. &
\end{array}
$$

6) By the relations *(T)*, one has

$$
\begin{array}{rcl}
h(a_2) & = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,a_1\,b\,b_1\,c_{1,2}\,a_2\,b_1(a_2) \\
& = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,a_1\,b\,b_1\,c_{1,2}\,a_2\,\overline{a_2}(b_1) \\
& = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,a_1\,b\,b_1\,\overline{b_1}(c_{1,2}) \\
& = & c_{1,2}\,.
\end{array}
$$

7) Using the braid relations, one gets

$$
\begin{array}{rcl}
h(b) & = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,a_1\,b\,b_1\,c_{1,2}\,a_2\,b_1(b) \\
& = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,a_1\,b\,b_1\,\overline{b}(a_2) \\
& = & b_2\,a_4\,\overline{c_{4,1}}\,\overline{b_2}\,b\,a_2\,\overline{a_2}(b_1) \\
& = & b_1\,.
\end{array}
$$

Thus, one has $h\overline{X}(a_2) = \overline{b_1}\,\overline{a_2}\,\overline{c_{2,4}}\,\overline{b_1}(c_{1,2}) = m$.

This conclues the proof of lemma 5.

□

3. Proof of theorem 1

We will proceed by induction on n. Thus, suppose that $g \geq 1$, $n \geq 2$, and consider the exact sequence (see [8][1] and [9]):

$$
1 \longrightarrow \mathbf{Z} \times \pi_1(\Sigma_{g,n-1}, p) \xrightarrow{f_1} \mathcal{M}_{g,n} \xrightarrow{f_2} \mathcal{M}_{g,n-1} \longrightarrow 1
$$

where, f_2 is defined by collapsing δ_n with a disc centred at p and by extending each map over the disc by the identity. One has $f_1(k) = \tau_{\delta_n}^k$ for all $k \in \mathbf{Z}$, and, if α is the homotopy class of a simple closed curve in $\Sigma_{g,n-1}$, $f_1(\alpha)$ is equal to the spin map $\tau_{\alpha'}\tau_{\alpha''}^{-1}$, where α' and α'' are the two boundary components of an annulus on $\Sigma_{g,n-1}$ which contains the collapsed disc (see [9] for the details).

Let us denote by $a'_1, \ldots, a'_{2g+n-3}, b', b'_1, \ldots, b'_{g-1}, (c'_{i,j})_{1 \leq i \neq j \leq 2g+n-3}$ the generators of $G_{g,n-1}$ corresponding to the curves in $\mathcal{G}_{g,n-1}$. We define $g_2 : G_{g,n} \to G_{g,n-1}$ by

[1] Johnson asserts that, if $g \geq 2$, the kernel of f_2 is isomorphic to the fundamental group of $U\Sigma_{g,n-1}$, the unit tangent bundle of $\Sigma_{g,n-1}$. Actually, his argument still works when $g = 1$ and $n \geq 2$ since in this case, $\pi_1(\Sigma_{g,n-1}, p)$ is centerless (see [8] and [4] for the details).

$$\begin{array}{rcll} g_2(a_i) & = & a'_i & \text{for all } i \neq 2g+n-2 \\ g_2(a_{2g+n-2}) & = & a'_1 & \\ g_2(b) & = & b' & \\ g_2(b_i) & = & b'_i & \text{for } 1 \leq i \leq g-1 \\ g_2(c_{i,j}) & = & c'_{i,j} & \text{for } 1 \leq i,j \leq 2g+n-3 \\ g_2(c_{i,2g+n-2}) & = & c'_{i,1} & \text{for } 2 \leq i \leq 2g+n-3 \\ g_2(c_{2g+n-2,j}) & = & c'_{1,j} & \text{for } 2 \leq j \leq 2g+n-3 \\ g_2(c_{1,2g+n-2}) & = & (a'_1\, b'\, a'_1)^4 & \\ g_2(c_{2g+n-2,1}) & = & 1\,. & \end{array}$$

Lemma 6. *For all* $(g,n) \in \mathbf{N}^* \times \mathbf{N}^*$, g_2 *is an homomorphism.*

Proof. We have to prove that the relations in $G_{g,n}$ are satisfied in $G_{g,n-1}$ via g_2. Since for all i such that $1 \leq i \leq g-1$, one has $g_2(c_{2i,2i+1}) = c'_{2i,2i+1}$ and $g_2(c_{2i-1,2i}) = c'_{2i-1,2i}$, this is clear for the handle relations.

So, let λ, μ be two elements of $\mathcal{G}_{g,n}$ which do not intersect (resp. intersect transversaly in a single point). If l and m are the associated elements of $G_{g,n}$, we have to prove that

$$(\bullet)\left\{\begin{array}{c} g_2(l)g_2(m) = g_2(m)g_2(l) \\ \Big(\text{resp. } \; g_2(l)g_2(m)g_2(l) = g_2(m)g_2(l)g_2(m)\Big). \end{array}\right.$$

When λ and μ are distinct from $\gamma_{2g+n-2,1}$ and $\gamma_{1,2g+n-2}$, these relations are precisely braid relations in $G_{g,n-1}$. If not, λ and μ do not intersect in a single point. Thus, it remains to consider the cases where $\lambda = \gamma_{1,2g+n-2}$ or $\gamma_{2g+n-2,1}$ and $\mu \in \mathcal{G}_{g,n}$ is a curve disjoint from λ. For $\lambda = \gamma_{2g+n-2,1}$, one has $g_2(l) = 1$ and the relation $(\bullet)$ is satisfied in $G_{g,n-1}$. So, suppose that $\lambda = \gamma_{1,2g+n-2}$. Then, we have $g_2(l) = (a'_1\, b'\, a'_1)^4$. The curves in $\mathcal{G}_{g,n}$ which are disjoint from λ are $\beta, \beta_1, \ldots, \beta_{g-1}, \alpha_1, \alpha_{2g+n-2}, \gamma_{2g+n-2,1}$ and $(\gamma_{i,j})_{1 \leq i < j \leq 2g+n-2}$. Let us look at the different cases:

- By lemma 2, $b' = g_2(b)$ and $a'_1 = g_2(a_1) = g_2(a_{2g+n-2})$ commute with $(a'_1\, b'\, a'_1)^4 = g_2(l)$.
- For all i, $1 \leq i \leq g-1$, $b'_i = g_2(b_i)$ commutes with $(a'_1\, b'\, a'_1)^4$ by the braid relations in $G_{g,n-1}$.
- For all i, j such that $1 \leq i < j \leq 2g+n-2$, one has $g_2(c_{i,j}) = c'_{i,j}$ if $j \neq 2g+n-2$, and $g_2(c_{i,j}) = c'_{i,1}$ otherwise. In all cases, one has that $g_2(c_{i,j})g_2(l) = g_2(l)g_2(c_{i,j})$ by the braid relations in $G_{g,n-1}$.

Now, let us look at the star relations. For $i,j,k \neq 2g+n-2$, $(E_{i,j,k})$ is sent by g_2 to $(E'_{i,j,k})$, the star relation in $G_{g,n-1}$ involving the same curves. For all i,j such that $2 \leq i \leq j < 2g+n-2$, $(E_{i,j,2g+n-2})$ is sent to $(E'_{i,j,1})$. Next, for $2 \leq j < 2g+n-2$, $(E_{1,j,2g+n-2})$ is sent to $(E'_{1,1,j})$. Finally, since $g_2(c_{2g+n-2,1}) = 1$ and $g_2(c_{1,2g+n-2}) = (a'_1 b' a'_1)^4$, the relation $(E_{1,1,2g+n-2})$ is satisfied in $G_{g,n-1}$ via g_2 by lemma 2. This concludes the proof by remark 3.

□

Since the relations *(T)*, *(A)* and *($E_{i,j,k}$)* are satisfied in $\mathcal{M}_{g,n}$ (see [3]), one has an homomorphism $\Phi_{g,n} : G_{g,n} \to \mathcal{M}_{g,n}$ which associates to each $a \in \mathcal{G}_{g,n}$ the corresponding twist τ_α. Since we view $\Sigma_{g,n}$ as a subsurface of $\Sigma_{g,n-1}$, we have $\Phi_{g,n-1} \circ g_2 = f_2 \circ \Phi_{g,n}$. Thus, we get the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \ker g_2 \arrow[r] \arrow[d, "h_{g,n}"] & G_{g,n} \arrow[r, "g_2"] \arrow[d, "\Phi_{g,n}"] & G_{g,n-1} \arrow[r] \arrow[d, "\Phi_{g,n-1}"] & 1 \\
1 \arrow[r] & \mathbf{Z} \times \pi_1(\Sigma_{g,n-1}, p) \arrow[r, "f_1"] & \mathcal{M}_{g,n} \arrow[r, "f_2"] & \mathcal{M}_{g,n-1} \arrow[r] & 1
\end{tikzcd}$$

where $h_{g,n}$ is induced by $\Phi_{g,n}$.

Proposition 7. *$h_{g,n}$ is an isomorphism for all $g \geq 1$ and $n \geq 2$.*

In order to prove this proposition, we will first give a system of generators for $\ker g_2$. Thus, we consider the following elements of $\ker g_2$:

$$x_0 = a_1 \overline{a_{2g+n-2}}, \quad x_1 = b(x_0), \quad x_2 = a_2(x_1), \quad x_3 = b_1(x_2),$$

$$\text{for } 2 \leq i \leq g-1, \quad x_{2i} = c_{2i-2,2i}(x_{2i-1}) \quad \text{and} \quad x_{2i+1} = b_i(x_{2i}),$$

$$\text{and for } 2g \leq k \leq 2g+n-3, \quad x_k = a_k(x_1)\,.$$

Remark 6. If $g=1$, one has just to concider $x_0, x_1, x_2, \ldots, x_{n-1}$.

Lemma 8. *For all $(g,n) \in \mathbf{N}^* \times \mathbf{N}^*$, $\ker g_2$ is normally generated by d_n and x_0.*

Proof. Let us denote by K the subgroup of $G_{g,n}$ normally generated by d_n and x_0. Since $g_2(d_n) = 1$ and $g_2(a_{2g+n-2}) = g_2(a_1)$, one has $K \subset \ker g_2$. In order to prove the equality, we shall prove that g_2 induces a monomorphism $\widetilde{g_2}$ from $G_{g,n}/K$ to $G_{g,n-1}$.

Define $k : G_{g,n-1} \to G_{g,n}/K$ by

$$\begin{array}{rcll} k(b') & = & \widetilde{b} & \\ k(b'_i) & = & \widetilde{b}_i & \text{for } 1 \leq i \leq g-1 \\ k(a'_i) & = & \widetilde{a}_i & \text{for all } i,\ 1 \leq i \leq 2g+n-3 \\ k(c'_{i,j}) & = & \widetilde{c_{i,j}} & \text{for all } i \neq j,\ 1 \leq i,j \leq 2g+n-3 \end{array}$$

where, for $x \in G_{g,n}$, $\widetilde{x}$ denote the class of x in $G_{g,n}/K$. Pasting a pair of pants to $\gamma_{2g+n-3,1}$ allows us to view $\Sigma_{g,n-1}$ as a subsurface of $\Sigma_{g,n}$, and $\mathcal{G}_{g,n-1}$ as a subset of $\mathcal{G}_{g,n}$. Thus, k appears to be clearly a morphism. Let us prove that $k \circ \widetilde{g_2} = Id$.
Denote by H the subgroup of $G_{g,n}/K$ generated by $\{\widetilde{b}, \widetilde{b}_1, \ldots, \widetilde{b}_{g-1}, \widetilde{a}_1, \ldots, \widetilde{a}_{2g+n-3}, (\widetilde{c}_{i,j})_{1 \leq i \neq j \leq 2g+n-3}\}$. Since, by definition of g_2 and k, one has $k \circ g_2(\widetilde{x}) = \widetilde{x}$ for all $\widetilde{x} \in H$, we just need to prove that $G_{g,n}/K = H$. We know that $G_{g,n}/K$ is generated by $\{\widetilde{x} \,/\, x \in \mathcal{G}_{g,n}\}$; thus, the following computations allow us to conclude.

- $\widetilde{a}_{2g+n-2} = \widetilde{a}_1$.
- $\widetilde{c}_{2g+n-2,1} = \widetilde{d}_n = 1$.
- By the star relation $(E_{1,1,2g+n-2})$, one has

  $$\widetilde{c}_{1,2g+n-2} = (\widetilde{a}_1\, \widetilde{a}_1\, \widetilde{a}_{2g+n-2}\, \widetilde{b})^{-3}\, \widetilde{c}_{2g+n-2,1} \;=\; (\widetilde{a}_1\, \widetilde{a}_1\, \widetilde{a}_1\, \widetilde{b})^{-3}.$$

- For $2 \leq i \leq 2g+n-3$, one has by the lantern relation $(L_{2g+n-2,1,i})$:

  $$a_{2g+n-2}\, c_{2g+n-2,1}\, c_{1,i}\, a_i = c_{2g+n-2,i}\, a_1\, X\, a_1\, \overline{X}$$

  where $X = b\, a_{2g+n-2}\, a_i\, b$. This relation implies the following one by *(T)*:

  $$\begin{array}{rcl} c_{2g+n-2,i} & = & c_{1,i}\, a_i\, X\, \overline{a_1}\, \overline{X}\, \overline{a_1}\, a_{2g+n-2}\, c_{2g+n-2,1} \\ & = & c_{1,i}\, X\, \overline{x_0}\, \overline{X}\, \overline{x_0}\, d_n\ , \end{array}$$

  which yields $\widetilde{c}_{2g+n-2,i} = \widetilde{c}_{1,i}$.
- In the same way, using the lantern relation $(L_{i,2g+n-2,1})$, one proves that $\widetilde{c}_{i,2g+n-2} = \widetilde{c}_{i,1}$ for $2 \leq i \leq 2g+n-3$.

□

Lemma 9. *For all* $(g,n) \in \mathbf{N}^* \times \mathbf{N}^*$, $\ker g_2$ *is generated by* $d_n = c_{2g+n-2,1}$ *and* $x_0, \ldots, x_{2g+n-3}$.

Proof. By lemma 8, $\ker g_2$ is normally generated by d_n and x_0. Furthermore, by the braid relations, d_n is central in $G_{g,n}$. Thus, denoting by K the subgroup generated by $d_n, x_0, \ldots, x_{2g+n-3}$, we have to prove that $gx_0g^{-1}\in K$ for all $g\in G_{g,n}$. To do this, it is enough to show that K is a normal subgroup of $G_{g,n}$.

By proposition 1, $G_{g,n}$ is generated by $\mathcal{H}_{g,n} = \{a_1, b, a_2, b_1, \dots, b_{g-1}, c_{2,4}, \dots, c_{2g-4,2g-2}, c_{1,2}, a_{2g}, \dots, a_{2g+n-2}, d_1, \dots, d_{n-1}\}$. Since, by the braid relations, $d_1, \dots, d_{n-1}$ are central in $G_{g,n}$, we have to prove that $y(x_k)$ and $\overline{y}(x_k)$ are elements of K for all k, $0 \le k \le 2g+n-3$, and all $y\in\mathcal{E}$ where $\mathcal{E} = \mathcal{H}_{g,n}\setminus\{d_1, \dots, d_{n-1}\}$.

$*$ <u>Case 1: $k=0$</u>.

- $b(x_0) = x_1$.
- We prove, using relations *(T)*, that $\overline{b}(x_0) = x_0\,\overline{x_1}\,x_0$:

$$
\begin{aligned}
x_0\,\overline{x_1}\,x_0 &= a_1\,\overline{a_{2g+n-2}}\,b\,a_{2g+n-2}\,\overline{a_1}\,\overline{b}\,a_1\,\overline{a_{2g+n-2}}\\
&= a_1\,b\,a_{2g+n-2}\,\overline{b}\,b\,\overline{a_1}\,\overline{b}\,\overline{a_{2g+n-2}}\\
&= \overline{b}\,a_1\,b\,\overline{b}\,\overline{a_{2g+n-2}}\,b\\
&= \overline{b}(x_0)\,.
\end{aligned}
$$

- For $y\in\mathcal{E}\setminus\{b\}$, one has $y(x_0) = \overline{y}(x_0) = x_0$ by the braid relations.

$*$ <u>Case 2: $k=1$</u>.

- $a_1(x_1) = a_1\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_1} = b\,a_1\,b\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_1}$
  $= b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,a_{2g+n-2}\,\overline{a_1} = x_1\,\overline{x_0}\,,$

  $\overline{a_1}(x_1) = \overline{a_1}\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,a_1 = b\,a_1\,\overline{b}\,\overline{a_{2g+n-2}}\,\overline{b}\,a_1$
  $= b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_{2g+n-2}}\,a_1 = x_1\,x_0\,.$

- $$
\begin{aligned}
a_{2g+n-2}(x_1) &= a_{2g+n-2}\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_{2g+n-2}}\\
&= a_{2g+n-2}\,b\,a_1\,\overline{b}\,\overline{a_{2g+n-2}}\,\overline{b}\\
&= a_{2g+n-2}\,\overline{a_1}\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b} = \overline{x_0}\,x_1\,,
\end{aligned}
$$

  $$
\begin{aligned}
\overline{a_{2g+n-2}}(x_1) &= \overline{a_{2g+n-2}}\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,a_{2g+n-2}\\
&= \overline{a_{2g+n-2}}\,b\,a_1\,b\,\overline{a_{2g+n-2}}\,\overline{b}\\
&= \overline{a_{2g+n-2}}\,a_1\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b} = x_0\,x_1\,.
\end{aligned}
$$

- One has $\overline{b}(x_1) = x_0$, and by the braid relations, $b(x_1) = x_1\,\overline{x_0}\,x_1$:

$$
\begin{aligned}
x_1\,\overline{x_0}\,x_1 &= b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_1}\,a_{2g+n-2}\,b\,a_1\,\overline{a_{2g+n-2}}\,\overline{b}\\
&= b\,\overline{a_{2g+n-2}}\,\overline{b}\,\overline{a_1}\,b\,\overline{b}\,a_{2g+n-2}\,b\,a_1\,\overline{b}\\
&= b\,b\,\overline{a_{2g+n-2}}\,\overline{b}\,b\,a_1\,\overline{b}\,\overline{b}\\
&= b(x_1).
\end{aligned}
$$

– For $i \in \{2, 2g, 2g+1, \ldots, 2g+n-3\}$, we have $a_i(x_1) = x_i$ and $\overline{a_i}(x_1) = x_1\, \overline{x_i}\, x_1$:

$$\begin{array}{rcll} x_1\, \overline{x_i}\, x_1 & = & b\, x_0\, \overline{b}\, a_i\, b\, \overline{x_0}\, \overline{b}\, \overline{a_i}\, b\, x_0\, \overline{b} & \\ & = & b\, x_0\, a_i\, b\, \overline{a_i}\, \overline{x_0}\, a_i\, \overline{b}\, \overline{a_i}\, x_0\, \overline{b} & \text{by } (T) \\ & = & b\, a_i\, x_0\, b\, \overline{x_0}\, \overline{b}\, x_0\, \overline{a_i}\, \overline{b} & \text{by case 1} \\ & = & b\, a_i\, x_0\, \overline{x_1}\, x_0\, \overline{a_i}\, \overline{b} & \\ & = & b\, a_i\, \overline{b}\, x_0\, b\, \overline{a_i}\, \overline{b} & \text{by case 1} \\ & = & \overline{a_i}\, b\, a_i\, x_0\, \overline{a_i}\, \overline{b}\, a_i & \text{by } (T) \\ & = & \overline{a_i}(x_1) & \text{by case 1.} \end{array}$$

– Each $y \in \{b_1, \ldots, b_{g-1}, c_{2,4}, \ldots, c_{2g-4,2g-2}, c_{1,2}\}$ commutes with x_1 by the braid relations, so $y(x_1) = \overline{y}(x_1) = x_1$.

∗ <u>Case 3: $k \in \{2, 2g, \ldots, 2g+n-3\}$</u>.

– By the braid relations and the preceeding cases, we have:

$$a_1(x_k) = a_k\, a_1(x_1) = a_k\, x_1\, \overline{x_0}\, \overline{a_k} = x_k\, \overline{x_0}\,,$$

$$\overline{a_1}(x_k) = a_k\, \overline{a_1}(x_1) = a_k\, x_1\, x_0\, \overline{a_k} = x_k\, x_0\,,$$

$$a_{2g+n-2}(x_k) = a_k\, a_{2g+n-2}(x_1) = a_k\, \overline{x_0}\, x_1\, \overline{a_k} = \overline{x_0}\, x_k\,,$$

$$\overline{a_{2g+n-2}}(x_k) = a_k\, \overline{a_{2g+n-2}}(x_1) = a_k\, x_0\, x_1\, \overline{a_k} = x_0\, x_k\,.$$

– It follows from the braid relations and the case 2 that

$$b(x_k) = b\, a_k\, b(x_0) = a_k\, b\, a_k(x_0) = a_k\, b(x_0) = x_k\,,$$

and we get also $\overline{b}(x_k) = x_k$.

– For $k \neq 2$, one has $b_1(x_k) = \overline{b_1}(x_k) = x_k$ by the braid relations. When $k=2$, we get $b_1(x_2) = x_3$ and $\overline{b_1}(x_2) = x_2\, \overline{x_3}\, x_2$:

$$\begin{array}{rcll} x_2\, \overline{x_3}\, x_2 & = & a_2\, x_1\, \overline{a_2}\, b_1\, a_2\, \overline{x_1}\, \overline{a_2}\, \overline{b_1}\, a_2\, x_1\, \overline{a_2} & \\ & = & a_2\, x_1\, b_1\, a_2\, \overline{b_1}\, \overline{x_1}\, b_1\, \overline{a_2}\, \overline{b_1}\, x_1\, \overline{a_2} & \text{by } (T) \\ & = & a_2\, b_1\, x_1\, \overline{x_2}\, x_1\, \overline{b_1}\, \overline{a_2} & \text{by case 2} \\ & = & a_2\, b_1\, \overline{a_2}\, x_1\, a_2\, \overline{b_1}\, \overline{a_2} & \text{by case 2} \\ & = & \overline{b_1}\, a_2\, b_1\, x_1\, \overline{b_1}\, \overline{a_2}\, b_1 & \text{by } (T) \\ & = & \overline{b_1}(x_2) & \text{by case 2}\,. \end{array}$$

– Each $y \in \{b_2, \ldots, b_{g-1}, c_{2,4}, \ldots, c_{2g-4,2g-2}, c_{1,2}\}$ commutes with x_k for $k = 2, 2g, \ldots, 2g+n-3$ by the braid relations. Therefore, we get $y(x_k) = \overline{y}(x_k) = x_k$.

– Let $i\in\{2,2g,\dots,2g+n-3\}$. Suppose first that $i\geq k$. Then, if $m_k=\overline{x_1}(a_k)$, we have

$$a_i(x_k)=a_i\,a_k\,x_1\,\overline{a_k}\,\overline{a_i}=a_i\,x_1\,m_k\,\overline{a_i}\,\overline{a_k}\,.$$

By the braid relations, one has

$$m_k=b\,\overline{a_1}\,a_{2g+n-2}\,\overline{b}(a_k)=b\,\overline{a_1}\,a_{2g+n-2}\,a_k(b)=b\,a_{2g+n-2}\,a_k\,b(a_1)$$

and the lantern relation $(L_{2g+n-2,1,k})$ says that

$$a_{2g+n-2}\,c_{2g+n-2,1}\,c_{1,k}\,a_k=c_{2g+n-2,k}\,a_1\,Y\,a_1\,\overline{Y}$$

where $Y=b\,a_{2g+n-2}\,a_k\,b$. Thus, we get

$$m_k=Y(a_1)=\overline{a_1}\,\overline{c_{2g+n-2,k}}\,a_{2g+n-2}\,c_{2g+n-2,1}\,c_{1,k}\,a_k\,,$$

which implies by the braid relations $m_k a_i=a_i m_k$ since $i\geq k$. From this, one obtains

$$a_i(x_k)=a_i\,x_1\,\overline{a_i}\,m_k\,\overline{a_k}=a_i\,x_1\,\overline{a_i}\,\overline{x_1}\,a_k\,x_1\,\overline{a_k}=x_i\,\overline{x_1}\,x_k\,.$$

In particular, we have $x_k=x_1\,\overline{x_i}\,a_i\,x_k\,\overline{a_i}$ and so:

$$\begin{array}{rcll}\overline{a_i}(x_k)&=&\overline{a_i}\,x_1\,\overline{x_i}\,a_i\,x_k\,\overline{a_i}\,a_i&\\ &=&\overline{a_i}\,x_1\,a_i\,\overline{a_i}\,\overline{x_i}\,a_i\,x_k&\\ &=&x_1\,\overline{x_i}\,x_1\,\overline{x_1}\,x_k&\text{by case 2}\\ &=&x_1\,\overline{x_i}\,x_k\,.&\end{array}$$

Conclusion: $\left\{\begin{array}{lll}a_i(x_k)=x_i\,\overline{x_1}\,x_k\,, & \overline{a_i}(x_k)=x_1\,\overline{x_i}\,x_k & \text{if } i\geq k,\\ a_i(x_k)=x_k\,\overline{x_1}\,x_i\,, & \overline{a_i}(x_k)=x_1\,\overline{x_k}\,x_i & \text{if } i\leq k.\end{array}\right.$

∗ Case 4: $k=3$.

– By the braid relations and the preceeding cases, we have:

$$a_1(x_3)=b_1\,a_1(x_2)=b_1\,x_2\,\overline{x_0}\,\overline{b_1}=x_3\,\overline{x_0}\,,$$

$$\overline{a_1}(x_3)=b_1\,\overline{a_1}(x_2)=b_1\,x_2\,x_0\,\overline{b_1}=x_3\,x_0\,,$$

$$a_{2g+n-2}(x_3)=b_1\,a_{2g+n-2}(x_2)=b_1\,\overline{x_0}\,x_2\,\overline{b_1}=\overline{x_0}\,x_3\,,$$

$$\overline{a_{2g+n-2}}(x_3)=b_1\,\overline{a_{2g+n-2}}(x_2)=b_1\,x_0\,x_2\,\overline{b_1}=x_0\,x_3\,.$$

– The relations *(T)* and the case 3 prove that

$$b(x_3)=b\,b_1(x_2)=b_1(x_2)=x_3=\overline{b}(x_3),$$

and

$$a_2(x_3)=a_2\,b_1\,a_2(x_1)=b_1\,a_2\,b_1(x_1)=b_1\,a_2(x_1)=x_3=\overline{a_2}(x_3).$$

– One has $\overline{b_1}(x_3)=x_2$. On the other hand, we get

$$\begin{array}{rcll} b_1(x_3) & = & b_1\,x_2\,\overline{b_1}\,\overline{x_2}\,b_1\,x_2\,\overline{b_1} & \text{by case 3} \\ & = & x_3\,\overline{x_2}\,x_3 . & \end{array}$$

– Using the braid relations and the case 3, we get $\overline{c_{2,4}}(x_3)=x_3\,\overline{x_4}\,x_3$:

$$\begin{array}{rcl} x_3\,\overline{x_4}\,x_3 & = & b_1\,x_2\,\overline{b_1}\,c_{2,4}\,b_1\,\overline{x_2}\,\overline{b_1}\,\overline{c_{2,4}}\,b_1\,x_2\,\overline{b_1} \\ & = & b_1\,x_2\,c_{2,4}\,b_1\,\overline{c_{2,4}}\,\overline{x_2}\,c_{2,4}\,\overline{b_1}\,\overline{c_{2,4}}\,x_2\,\overline{b_1} \\ & = & b_1\,c_{2,4}\,x_2\,\overline{x_3}\,x_2\,\overline{c_{2,4}}\,\overline{b_1} \\ & = & b_1\,c_{2,4}\,\overline{b_1}\,x_2\,b_1\,\overline{c_{2,4}}\,\overline{b_1} \\ & = & \overline{c_{2,4}}\,b_1\,c_{2,4}\,x_2\,\overline{c_{2,4}}\,\overline{b_1}\,c_{2,4} \\ & = & \overline{c_{2,4}}(x_3). \end{array}$$

On the other hand, we have $c_{2,4}(x_3)=x_4$.

– The braid relations assure that $y(x_3)=\overline{y}(x_3)=x_3$ for all $y\in\{b_2,\dots,b_{g-1},c_{4,6},\dots,c_{2g-4,2g-2}\}$.

– For each $i\in\{2g,\dots,2g+n-3\}$, one has by the case 3

$$a_i(x_3)=b_1\,a_i(x_2)=b_1\,x_i\,\overline{x_1}\,x_2\,\overline{b_1}=x_i\,\overline{x_1}\,x_3$$

and

$$\overline{a_i}(x_3)=b_1\,\overline{a_i}(x_2)=b_1\,x_1\,\overline{x_i}\,x_2\,\overline{b_1}=x_1\,\overline{x_i}\,x_3 .$$

– Finally, we shall prove that $c_{1,2}(x_3)=x_3\,\overline{x_2}\,x_1\,\overline{x_0}\,d_n$.

The lantern relation $(L_{2g+n-2,1,2})$ says

$$a_{2g+n-2}\,c_{2g+n-2,1}\,c_{1,2}\,a_2=c_{2g+n-2,2}\,\overline{X}\,a_1\,X\,a_1=c_{2g+n-2,2}\,a_1\,X\,a_1\,\overline{X}$$

where $X=b\,a_2\,a_{2g+n-2}\,b$, that is to say $(d_n=c_{2g+n-2,1})$:

$$a_{2g+n-2}\,c_{1,2}\,\overline{a_1}=c_{2g+n-2,2}\,\overline{a_2}\,\overline{d_n}\,\overline{X}\,a_1\,X \quad (\star)$$

and

$$c_{2g+n-2,2}\,\overline{c_{1,2}}=X\,\overline{a_1}\,\overline{X}\,\overline{a_1}\,a_2\,d_n\,a_{2g+n-2} \quad (\star\star).$$

Then, one can compute

$$\begin{array}{rcll} \overline{x_3}(c_{1,2}) & = & b_1\,a_2\,b\,a_{2g+n-2}\,\overline{a_1}\,\overline{b}\,\overline{a_2}\,\overline{b_1}(c_{1,2}) & \\ & = & b_1\,a_2\,b\,a_{2g+n-2}\,c_{1,2}\,\overline{a_1}\,\overline{b}\,\overline{a_2}(b_1) & \text{by } (T) \\ & = & b_1\,a_2\,b\,c_{2g+n-2,2}\,\overline{a_2}\,\overline{d_n}\,\overline{X}\,a_1\,X\,\overline{b}\,\overline{a_2}(b_1) & \text{by } (\star) \\ & = & b_1\,a_2\,b\,c_{2g+n-2,2}\,\overline{a_2}\,\overline{d_n}\,\overline{X}\,a_1\,b\,a_{2g+n-2}(b_1) & \\ & = & b_1\,c_{2g+n-2,2}\,\overline{b}\,a_2\,b\,\overline{X}(b_1) & \text{by } (T) \\ & = & b_1\,c_{2g+n-2,2}\,\overline{b}\,a_2\,b\,\overline{b}\,\overline{a_2}\,\overline{a_{2g+n-2}}\,\overline{b}(b_1) & \\ & = & b_1\,\overline{b_1}(c_{2g+n-2,2}) & \text{by } (T) \\ & = & c_{2g+n-2,2} . & \end{array}$$

Thus, we get

$$\begin{array}{rcll}
c_{1,2}(x_3) & = & c_{1,2}\, x_3\, \overline{c_{1,2}} & \\
& = & x_3\, \overline{x_3}\, c_{1,2}\, x_3\, \overline{c_{1,2}} & \\
& = & x_3\, c_{2g+n-2,2}\, \overline{c_{1,2}} & \\
& = & x_3\, X\, \overline{a_1}\, \overline{X}\, \overline{a_1}\, a_2\, a_{2g+n-2}\, d_n & \text{by } (\star\star) \\
& = & x_3\, b\, a_2\, a_{2g+n-2}\, b\, \overline{a_1}\, \overline{b}\, \overline{a_2}\, \overline{a_{2g+n-2}}\, \overline{b}\, a_2\, \overline{x_0}\, d_n & \\
& = & x_3\, b\, a_{2g+n-2}\, a_2\, \overline{a_1}\, \overline{b}\, a_1\, \overline{a_2}\, \overline{a_{2g+n-2}}\, \overline{b}\, a_2\, \overline{x_0}\, d_n & \\
& = & x_3\, b\, \overline{x_0}\, \overline{b}\, \overline{a_2}\, b\, x_0\, \overline{b}\, a_2\, \overline{x_0}\, d_n & \text{by } (T) \\
& = & x_3\, \overline{x_1}\, \overline{a_2}\, x_1\, a_2\, \overline{x_0}\, d_n & \\
& = & x_3\, \overline{x_1}\, x_1\, \overline{x_2}\, x_1\, \overline{x_0}\, d_n & \text{by case 2} \\
& = & x_3\, \overline{x_2}\, x_1\, \overline{x_0}\, d_n\, . &
\end{array}$$

It follows from this that

$$\overline{c_{1,2}}(x_3) = \overline{c_{1,2}}\, c_{1,2}\, x_3\, \overline{c_{1,2}}\, \overline{d_n}\, x_0\, \overline{x_1}\, x_2\, c_{1,2} = x_3\, \overline{d_n}\, x_0\, \overline{x_1}\, x_2\, .$$

$*$ <u>Case 5: $k \in \{4, 5, \dots, 2g-1\}$</u>.

In order to simplify the notation, let us denote

$$e_3 = b_1\,,\quad e_4 = c_{2,4}\,,\quad e_5 = b_2\,,\quad \dots\,,\quad e_{2g-2} = c_{2g-4,2g-2}\,,\quad e_{2g-1} = b_{g-1}\,,$$

so that, for $i \in \{3, \dots, 2g-1\}$, $x_i = e_i(x_{i-1})$.

– Then, one has by the braid relations and the case 4:

$$a_1(x_k) = e_k\, e_{k-1} \cdots e_4\, a_1(x_3) = e_k \cdots e_4 x_3\, \overline{x_0}\, \overline{e_4} \cdots \overline{e_k} = x_k\, \overline{x_0}\, .$$

Likewise, we get

$$\overline{a_1}(x_k) = x_k\, x_0\,,\quad a_{2g+n-2}(x_k) = \overline{x_0}\, x_k\,,\quad \overline{a_{2g+n-2}}(x_k) = x_0\, x_k\,,$$

$$\text{and}\quad b(x_k) = \overline{b}(x_k) = x_k = a_2(x_k) = \overline{a_2}(x_k)\, .$$

– For $i \in \{3, 4, \dots, 2g-1\}$, $i < k$, one obtains, using the braid relations, $e_i(x_k) = \overline{e_i}(x_k) = x_k$:

$$\begin{array}{rl} e_i(x_k) = & e_k \cdots e_i\, e_{i+1}\, e_i \cdots e_3(x_2) \;=\; e_k \cdots e_{i+1}\, e_i\, e_{i+1} \cdots e_3(x_2) \\ = & e_k \cdots e_3(x_2) = x_k\, . \end{array}$$

For $i > k+1$, e_i commutes with $e_k, \dots, e_4$ and x_3, thus we also have

$$e_i(x_k) = \overline{e_i}(x_k) = x_k \quad (i > k+1) \quad (*).$$

– One has $e_{k+1}(x_k) = x_{k+1}$. Let us prove by induction on k that $\overline{e_{k+1}}(x_k) = x_k\, \overline{x_{k+1}}\, x_k$. We have seen in case 4 that this equaliy is satisfied at the rank $k=3$. Suppose it is true at the rank $k-1$, $4\leq k\leq 2g-2$. Then, we get:

$$\begin{array}{rcll} x_k\,\overline{x_{k+1}}\,x_k & = & e_k\,x_{k-1}\,\overline{e_k}\,e_{k+1}\,e_k\,\overline{x_{k-1}}\,\overline{e_k}\,\overline{e_{k+1}}\,e_k\,x_{k-1}\,\overline{e_k} & \\ & = & e_k\,x_{k-1}\,e_{k+1}\,e_k\,\overline{e_{k+1}}\,\overline{x_{k-1}}\,e_{k+1}\,\overline{e_k}\,\overline{e_{k+1}}\,x_{k-1}\,\overline{e_k} & \text{by } (T) \\ & = & e_k\,e_{k+1}\,x_{k-1}\,e_k\,\overline{x_{k-1}}\,\overline{e_k}\,x_{k-1}\,\overline{e_{k+1}}\,\overline{e_k} & \text{by } (*) \\ & = & e_k\,e_{k+1}\,x_{k-1}\,\overline{x_k}\,x_{k-1}\,\overline{e_{k+1}}\,\overline{e_k} & \\ & = & e_k\,e_{k+1}\,\overline{e_k}\,x_{k-1}\,e_k\,\overline{e_{k+1}}\,\overline{e_k} & \text{by inductive hypothesis} \\ & = & \overline{e_{k+1}}\,e_k\,e_{k+1}\,x_{k-1}\,\overline{e_{k+1}}\,\overline{e_k}\,e_{k+1} & \text{by } (T) \\ & = & \overline{e_{k+1}}\,e_k\,x_{k-1}\,\overline{e_k}\,e_{k+1} & \text{by } (*) \\ & = & \overline{e_{k+1}}(x_k). & \end{array}$$

– This last relation implies $x_k = x_{k-1}\,\overline{e_k}\,\overline{x_{k-1}}\,e_k\,x_{k-1}$. Thus, we get

$$e_k(x_k) = e_k\,x_{k-1}\,\overline{e_k}\,\overline{x_{k-1}}\,e_k\,x_{k-1}\,\overline{e_k} = x_k\,\overline{x_{k-1}}\,x_k\,.$$

On the other hand, one has $\overline{e_k}(x_k) = x_{k-1}$.

– For $i\in\{2g,\dots,2g+n-3\}$, we have, by the braid relations and the cases 2, 3 and 4:

$$a_i(x_k) = e_k\cdots e_4\,a_i(x_3) = e_k\cdots e_4\,x_i\,\overline{x_1}\,x_3\,\overline{e_4}\cdots\overline{e_k} = x_i\,\overline{x_1}\,x_k\,,$$

and likewise, we get $\overline{a_i}(x_k) = x_1\,\overline{x_i}\,x_k$.

– Finally, since $c_{1,2}(x_3) = x_3\,\overline{x_2}\,x_1\,\overline{x_0}\,d_n$, it follows from the braid relations and the preceeding cases that $c_{1,2}(x_k) = x_k\,\overline{x_2}\,x_1\,\overline{x_0}\,d_n$. In the same way, we get $\overline{c_{1,2}}(x_k) = x_k\,\overline{d_n}\,x_0\,\overline{x_1}\,x_2$.

□

Proof of proposition 7. If $\pi : \mathbf{Z}\times\pi_1(\Sigma_{g,n-1},p)\to\pi_1(\Sigma_{g,n-1},p)$ denotes the projection, the loops $\pi\circ h_{g,n}(x_0),\dots,\pi\circ h_{g,n}(x_{2g+n-3})$ form a basis of the free group $\pi_1(\Sigma_{g,n-1},p)$. Thus, F, the subgroup of $\ker g_2$ generated by $x_0,\dots,x_{2g+n-3}$ is free of rank $2g+n-2$ and the restriction of $\pi\circ h_{g,n}$ to this subgroup is an isomorphism.
Now, for every element x of $\ker g_2$, there are, by lemma 9, an integer k and an element f of F such that $x = d_n^k\,f$ (d_n is central in $\ker g_2$). Then, one has $h_{g,n}(x) = \big(k,\pi\circ h_{g,n}(x)\big)$ and therefore, $h_{g,n}$ is one to one. But $h_{g,n}$ is also onto. This concludes the proof.

□

Proof of theorem 1. In section 2, we proved that $\Phi_{g,1}$ is an isomorphism. Thus, by the five-lemma, proposition 7 and an inductive argument, $\Phi_{g,n}$ is an isomorphism for all $n\geq 1$. In order to conclude the proof, it remains to look at the case $n=0$.

Consider once more the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker g_2 & \longrightarrow & G_{g,1} & \xrightarrow{g_2} & G_{g,0} & \longrightarrow & 1 \\ & & \downarrow h_{g,1} & & \approx \downarrow \Phi_{g,1} & & \downarrow \Phi_{g,0} & & \\ 1 & \longrightarrow & \ker f_2 & \longrightarrow & \mathcal{M}_{g,1} & \xrightarrow{f_2} & \mathcal{M}_{g,0} & \longrightarrow & 1 \end{array}$$

Wajnryb proved in [12] that $\ker f_2$ is normally generated by τ_{δ_1} and $\tau_{\alpha_1}\tau_{\alpha_{2g-1}}^{-1}$. Thus, since $\ker g_2$ is normally generated by d_1 and $a_1\,\overline{a_{2g-1}}$ (lemma 8), we conclude that $h_{g,1}$ is still an isomorphism. So, we get that $\Phi_{g,0}$ is an isomorphism.

□

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